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Weighted graphs and complex Gaussian free fields*

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Abstract

We prove a combinatorial statement about the distribution of directed currents in a complex "loop soup" and use it to give a new proof of the isomorphism, which relates loop measures and complex Gaussian free fields.

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1 Introduction

Loops and related measures are useful tools in the analysis of random walks. They have come under study in [9] as a discrete analogue of the Brownian loop soup introduced in [10], which itself was motivated by the study of the Schramm-Loewner evolution [8]. Loop measures were explored by Le Jan in a continuous setting [4], where he established a connection (isomorphism) between the Gaussian free field and the occupation field of a Poissonian ensemble of Markov loops. Le Jan's isomorphism can be viewed as an extension of the Dynkin's isomorphism theorem [1]. Le Jan also extended this isomorphism to relate certain non-symmetric Markov processes and complex Gaussian fields in [3].

In [7] and [5] the authors proved a version of the isomorphism theorem using the discrete time loop soup without introducing a Markov chain to analyze loop measures. They observed that a random walk on a finite graph can be fully described by a sub-stochastic transition matrix $\boldsymbol{Q}$. Any event is essentially a union of chain trajectories, and its probability is additive on sets of trajectories. Even if $\boldsymbol{Q}$ takes complex values, we can still build objects with probabilistic analogues, such as loop soups, by putting potentially complex weights on paths.

The complex Gaussian free field is introduced in [7] as a pair of real Gaussian free fields with potentially negative correlations between fields and within each field. A version of the isomorphism theorem is formulated and proved there by comparing the Laplace transforms of a complex Gaussian field squared and a continuous occupation field of a complex loop soup. There is a combinatorial proof of the isomorphism in [5],

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which is discussed under the assumption that weights correspond to a certain probability space.

This note continues the work of [7] and [5]. We extend some of the results of [7] to a wider range of weights and adapt the arguments from [5] to the complex setting. We derive the exact distribution on directed currents, induced by the random walk loop soup at intensity 1 (Proposition 3.3), and we also prove the isomorphism theorem (Theorem 3.2), which connects the continuous occupation field of the loop soup and the absolute value of a complex Gaussian free field squared. The theorem that we prove is a special case of the Theorem 7.6 of [2], but our approach is combinatorial in nature and does not involve the calculation of Laplace transforms. The isomorphism theorems from [7] and [5] are special cases of Theorem 3.2.

This note is structured as follows. We introduce the setup and basic notations in section 2. In section 3 we state the main results. The proofs are postponed till section 4.

2 Basic definitions

Consider a finite complete digraph (V, E) with $N = |V|$ vertices. We pick any order on the set of vertices: $V = (v_j)_{j=1}^N = (v_1, v_2, \ldots, v_N)$. (We use $(\cdot)$, $[\cdot]$ and $\{\cdot\}$ to denote sequences, matrices and unordered sets, respectively.) Directed edges $E \cong V \times V$ are identified with ordered pairs of vertices; note that we allow self-edges.

Fix $U \subseteq V$. A *path* ω of *length* $|\omega| = k$ in U is a sequence of $k+1$ vertices in U:

$$\omega = (\omega^0, \omega^1, \ldots, \omega^k) = (\omega^j)_{j=0}^k, \qquad \{\omega^j\}_{j=0}^k \subseteq U.$$

If $\{x, y\} \subseteq U$, we let $\mathcal{P}_U(x, y)$ denote the set of paths in U starting at x and ending at y. Paths $\mathcal{P}_U(x) := \mathcal{P}_U(x, x)$ are called *loops rooted at* x in U. We call paths of zero length *trivial loops*. We use $\mathcal{P}$ to denote all paths $\cup_{x,y\in V}\mathcal{P}_V(x, y)$.

Suppose that $\boldsymbol{Q} = [Q_{xy}]_{x,y\in V}$ is a *weight* on edges, that is, any complex-valued function on E. We assume that edge weights are *integrable*, that is, $\rho(|\boldsymbol{Q}|) < 1$, where ρ denotes the spectral radius operator, and $|\boldsymbol{Q}| := \big[|Q_{xy}|\big]_{x,y\in V}$. We associate $\boldsymbol{Q}$ with a function $q : \mathcal{P} \mapsto \mathbb{C}$ as follows:

$$q(\omega) := \prod_{j=1}^{|\omega|} Q_{\omega^{j-1}\omega^j} \quad \text{for} \quad |\omega| \geq 1, \quad q(\omega) = 1 \ \text{ for trivial loops } \ \omega; \qquad \omega \in \mathcal{P}.$$

Note that q is a *(complex) measure* on $\mathcal{P}$, because $\boldsymbol{Q}$ is integrable.

We define the *vertex local time* $\boldsymbol{n}(\omega) = \big(n_x(\omega)\big)_{x\in V}$ and the *(directed) edge local time* $\boldsymbol{c}(\omega) = [c_{xy}(\omega)]_{x,y\in V}$ as integer-valued functions on paths $\omega \in \mathcal{P}$, where

$$n_x(\omega) := \sum_{j=1}^{|\omega|} \mathbb{1}\{\omega^j = x\} \qquad \text{and} \qquad c_{xy}(\omega) := \sum_{j=1}^{|\omega|} \mathbb{1}\{\omega^{j-1} = x, \omega^j = y\}.$$

We call $\boldsymbol{C} = [C_{xy}]_{x,y\in V}$ with entries in $\mathbb{N} = \{0, 1, 2, \ldots\}$ a *(directed) current*, if

$$\sum_{x\in V} C_{xy} = \sum_{x\in V} C_{yx}, \qquad \forall y \in V.$$

If $U \subseteq V$, we use $\mathcal{C}_U$ to denote the set of currents restricted to U, that is, such that $C_{xy} = 0$ if either x or y is in $V \setminus U$. If ω is a rooted loop, then $\boldsymbol{c}(\omega) \in \mathcal{C}_V$ and

$$n_x(\omega) = \frac{1}{2} \sum_{y\in V} \big[c_{xy}(\omega) + c_{yx}(\omega)\big], \qquad \forall x \in V. \tag{2.1}$$

We extend the definition of q to $\mathcal{C}_V$. Let $q(\boldsymbol{C}) = q(\omega)$, where $\omega \in \mathcal{P}$ is any loop with $\boldsymbol{c}(\omega) = \boldsymbol{C} \in \mathcal{C}_V$. Equivalently, we can set $q(\boldsymbol{C}) = \prod_{x,y\in V} Q_{xy}^{C_{xy}}$.

An *(oriented) unrooted loop* is an equivalence class of rooted nontrivial loops under cyclic permutations:

$$(\omega^0, \omega^1, \dots, \omega^k = \omega^0) \sim (\omega^1, \dots \omega^{k-1}, \omega^0, \omega^1) \sim \dots \sim (\omega^{k-1}, \omega^0, \omega^1, \dots, \omega^{k-1}).$$

The set of unrooted loops is denoted by $\mathcal{L}$. If a rooted loop ω represents a class $l \in \mathcal{L}$, we will write $\omega \in l$. We extend the definitions of q, $\boldsymbol{n}$ and $\boldsymbol{c}$ from $\mathcal{P}$ to $\mathcal{L}$ by taking any rooted representative:

$$q(l) := q(\omega), \quad \boldsymbol{n}(l) := \boldsymbol{n}(\omega), \quad \boldsymbol{c}(l) := \boldsymbol{c}(\omega), \quad \text{if} \quad \omega \in l; \qquad \forall l \in \mathcal{L}.$$

If $\mathcal{X}$ is any countable set, we let $\mathbb{N}_{\text{fin}}^{\mathcal{X}}$ stand for finite multisets of elements from $\mathcal{X}$, that is, the set of functions from $\mathcal{X}$ to $\mathbb{N}$, which are supported on a finite set. Local times $\boldsymbol{n}$ and currents $\boldsymbol{c}$ can be viewed as functions on $\mathbb{N}_{\text{fin}}^{\mathcal{L}}$:

$$\boldsymbol{n}(\boldsymbol{s}) := \sum_{l\in\mathcal{L}} \boldsymbol{n}(l)\, s_l, \quad \boldsymbol{c}(\boldsymbol{s}) := \sum_{l\in\mathcal{L}} \boldsymbol{c}(l)\, s_l; \qquad \boldsymbol{s} \in \mathbb{N}_{\text{fin}}^{\mathcal{L}}.$$

3 Main results

In this section we define the loop soup occupation field, the complex Gaussian free field, state our isomorphism theorem and introduce some constructions that help to prove it. Since some of the constructions might seem ad hoc, we also discuss how the presented definitions and statements work in a more familiar setting, when $\boldsymbol{Q}$ is nonnegative and serves as a transition matrix of a certain submarkovian chain.

3.1 Loop soup, Gaussian field and the isomorphism

Suppose that $\boldsymbol{Q}$ is integrable. We define the *unrooted loop measure* m as

$$m(l) := \sum_{\omega\in l} \frac{q(\omega)}{|\omega|} = \frac{q(l)}{d(l)}, \tag{3.1}$$

where $d(l)$ is the largest integer d such that every $\omega \in l$ is a concatenation of d identical rooted loops. Let the *(random walk) loop soup (at intensity 1)* be the following measure on finite multisets of unrooted loops

$$\nu_m\{\boldsymbol{s}\} := \prod_{l\in\mathcal{L}} \frac{e^{-m(l)}\, m(l)^{s_l}}{s_l!} = e^{-m(\mathcal{L})} \prod_{l\in\mathcal{L}} \frac{m(l)^{s_l}}{s_l!}, \qquad \forall \boldsymbol{s} \in \mathbb{N}_{\text{fin}}^{\mathcal{L}}. \tag{3.2}$$

Remark 3.1. Note that $e^{-m(\mathcal{L})} = \det(\boldsymbol{I} - \boldsymbol{Q})$; e.g., see Lemma 3.1 in [7].

We define the *(continuous) occupation field* ν_n as a measure with the following density with respect to Lebesgue measure on $\mathbb{R}_+^N = (0,\infty)^N$,

$$d\nu_n := \sum_{\boldsymbol{s}\in\mathbb{N}_{\text{fin}}^{\mathcal{L}}} \left[\nu_m\{\boldsymbol{s}\} \prod_{x\in V} \frac{r_x^{n_x(\boldsymbol{s})}\, e^{-r_x}\, dr_x}{n_x(\boldsymbol{s})!} \right], \qquad \boldsymbol{r} \in \mathbb{R}_+^N. \tag{3.3}$$

Now suppose that $\boldsymbol{Q}$ is Hermitian and $\rho(\boldsymbol{Q}) < 1$, which holds for integrable $\boldsymbol{Q}$. We associate it with a *Green's function* $\boldsymbol{G} := (\boldsymbol{I} - \boldsymbol{Q})^{-1}$ (which is a positive definite Hermitian matrix) and a *(discrete centered) complex Gaussian free field* $\Phi = (\Phi_x)_{x\in V}$ on V. The latter is a circularly-symmetric complex normal distribution with covariance $\boldsymbol{G}$, that is, a random complex vector in $\mathbb{C}^N$ with density

$$f_\Phi(\boldsymbol{z}) := \frac{\exp\{-\langle \boldsymbol{z}, \boldsymbol{G}^{-1}\boldsymbol{z}\rangle\}}{\pi^N \det \boldsymbol{G}}, \qquad \boldsymbol{z} \in \mathbb{C}^N \tag{3.4}$$

with respect to the Lebesgue measure on $\mathbb{C}^N$; here $\langle \cdot, \cdot \rangle$ denotes the dot product.

We can decompose the Green's function into real and imaginary parts: $\boldsymbol{G} = \boldsymbol{G}^R + i\boldsymbol{G}^I$. Since $\boldsymbol{G}$ is Hermitian, $\boldsymbol{G}^R$ is symmetric and $\boldsymbol{G}^I$ is antisymmetric. The field Φ can be viewed as a pair of identically distributed, correlated distributions on $\mathbb{R}^N$. Indeed, let

$$(\Phi^R, \Phi^I) := (\Phi^R_x)_{x\in V} \oplus (\Phi^I_x)_{x\in V} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \boldsymbol{G}^R & -\boldsymbol{G}^I \\ \boldsymbol{G}^I & \boldsymbol{G}^R \end{bmatrix}\right), \tag{3.5}$$

where $\oplus$ concatenates sequences. We have the following decomposition then (see, for example, Proposition 4.5 in [7]):

$$\Phi \overset{\text{law}}{=} (\Phi^R + i\Phi^I)/\sqrt{2}, \tag{3.6}$$

that is, the probability distributions of these complex random vectors are the same.

Thanks to the remark at the end of Section 4 in [7], we expect the occupation field at intensity 1 to have the same density with respect to Lebesgue measure on $\mathbb{R}^N_+$ as the square of the absolute value of a complex Gaussian free field squared. The following generalizes the isomorphism theorems as stated in [7] and [5].

Theorem 3.2 (Isomorphism)**.** *If $\boldsymbol{Q}$ is integrable and Hermitian, then the associated occupation field ν_n is a probability distribution, identical to that of $|\Phi|^2 := (\bar{\Phi}_x \Phi_x)_{x\in V}$, where Φ is the complex Gaussian free field associated with $\boldsymbol{Q}$.*

We can interpret this differently in view of (3.6). If Φ^R and Φ^I are two correlated Gaussian fields as in (3.5), then ν_n has the same distribution as $(|\Phi^R|^2 + |\Phi^I|^2)/2$.

3.2 Distribution on currents and the bubble soup

According to (2.1), the vertex local time induced by a loop soup is determined by the currents induced by this loop soup. We define the *(directed) current field* ν_c on $\mathcal{C}_V$ as a pushforward of ν_m under $\boldsymbol{c}$:

$$\nu_c\{\boldsymbol{C}\} := \sum_{\boldsymbol{s}\in\mathbb{N}^{\mathcal{L}}_{\text{fin}} : \boldsymbol{c}(\boldsymbol{s}) = \boldsymbol{C}} \nu_m\{\boldsymbol{s}\}, \qquad \boldsymbol{C} \in \mathcal{C}_V.$$

It turns out, we can write an explicit formula for ν_c.

Proposition 3.3 (Current distribution)**.** *For any $\boldsymbol{C} \in \mathcal{C}_V$ and $\boldsymbol{Q} \in \mathbb{C}^{N\times N}$,*

$$\nu_c\{\boldsymbol{C}\} = \det(\boldsymbol{I} - \boldsymbol{Q})\, q(\boldsymbol{C}) \prod_{x\in V} \binom{n_x(\boldsymbol{C})}{\{C_{xy}\}_{y\in V}}. \tag{3.7}$$

The proof of this fact is combinatorial in nature and revolves around the identity (4.7), which can be viewed as a useful result on its own. Thanks to this result, we can rewrite the density of the occupation field from (3.3):

$$\begin{aligned} d\nu_n &= \sum_{\boldsymbol{C}\in\mathcal{C}_V} \left[\nu_c\{\boldsymbol{C}\} \prod_{x\in V} \frac{r_x^{n_x(\boldsymbol{C})}\, e^{-r_x} dr_x}{n_x(\boldsymbol{C})!} \right] \\ &= \det(\boldsymbol{I} - \boldsymbol{Q}) \sum_{\boldsymbol{C}\in\mathcal{C}_V} \left[q(\boldsymbol{C}) \prod_{x\in V} \frac{r_x^{n_x(\boldsymbol{C})}\, e^{-r_x} dr_x}{\prod_{y\in V} C_{xy}!} \right], \quad \boldsymbol{r} \in \mathbb{R}^N_+. \end{aligned} \tag{3.8}$$

To prove Proposition 3.3, we introduce a certain auxiliary measure which induces the same measure on currents as the loop soup. We let $V_k := (v_j)_{j=k}^N$ for $k \in [N] := \{1, 2, \ldots, N\}$ (in particular, $V_1 = V$) and define the *bubble soup* ν_b as a measure on N-tuples of loops, which we refer to as *bubbles* (for a given order on V):

$$\nu_b\{\boldsymbol{\omega}\} := \det(\boldsymbol{I} - \boldsymbol{Q}) \prod_{j=1}^N q(\omega_j), \quad \boldsymbol{\omega} = (\omega_j)_{j=1}^N, \text{ where } \omega_j \in \mathcal{P}_{V_j}(v_j) \quad \forall j \in [N]. \tag{3.9}$$

We let $\tilde{\nu}_c$ denote the pushforward of ν_b under $\boldsymbol{c}$:

$$\tilde{\nu}_c\{\boldsymbol{C}\} = \sum_{\boldsymbol{\omega} \to \boldsymbol{C}} \nu_b\{\boldsymbol{\omega}\}, \qquad \boldsymbol{C} \in \mathcal{C}_V,$$

where the sum is over all bubbles $\boldsymbol{\omega} = (\omega_j)_{j=1}^N$ such that $\sum_{j=1}^N \boldsymbol{c}(\omega_j) = \boldsymbol{C}$.

Lemma 3.4 (Bubble representation). *For any ordering of V and any $\boldsymbol{C} \in \mathcal{C}_V$, we have $\tilde{\nu}_c\{\boldsymbol{C}\} = \nu_c\{\boldsymbol{C}\}$.*

This follows immediately from the Proposition 5.8 of [5], when $\boldsymbol{Q}$ is a strictly substochastic matrix. A similar result was established for intensity one and positive weights combinatorially in the Proposition 9.4.1 of [6]. Unfortunately, there was a misstatement in Exercise 9.1, which was part of the proof, which is why we redo that part of the proof here.

3.3 Probability setting

We now turn our attention to a special class of edge weights with a natural probabilistic interpretation. Suppose that

$$\boldsymbol{Q} \in \mathbb{R}_+^{N\times N} \quad \text{is a symmetric matrix such that} \quad \rho(\boldsymbol{Q}) < 1. \tag{3.10}$$

We construct discrete-time Markov chain $(X_j)_{j\geq 0}$ on a state space $V \cup \{0\}$:

$$\mathbb{P}^x\{X_1 = y\} := Q_{xy}, \qquad \mathbb{P}^x\{X_1 = 0\} := 1 - \sum_{y\in V} Q_{xy}, \qquad \mathbb{P}^0\{X_1 = 0\} := 1,$$

where $x, y \in V$, and $\mathbb{P}^x$ is the distribution of the chain started at x. Here state 0 can be interpreted as a sink: once the chain reaches 0, it stays at 0 forever. The probability that the trajectory of the chain starts with $\omega \in \mathcal{P}$ is equal to $q(\omega)$.

The assumption (3.10) implies that $(X_j)_{j\geq 0}$ is a symmetric Markov chain and, if τ is the lifespan of the chain, that is, $\tau := \min\{j \geq 0 : X_j = 0\}$, then $\mathbb{P}^x\{\tau < \infty\} = 1$ for any $x \in V$. We can define the Green's function of the chain $(X_j)_{j\geq 0}$ and see that it coincides with the definition from subsection 3.1:

$$G(x,y) = \sum_{j\geq 0} \mathbb{P}^x\{X_j = y; \tau > j\} = \sum_{\omega\in\mathcal{P}(x,y)} q(\omega) = \left[\sum_{j\geq 0} \boldsymbol{Q}^j\right]_{x,y} = \left[(\boldsymbol{I} - \boldsymbol{Q})^{-1}\right]_{x,y}.$$

The loop soup ν_m from (3.2) is in fact a collection of countably many independent Poisson random variables, where the variable with index $l \in \mathcal{L}$ has mean $m(l)$. If we sample a finite multiset of loops from ν_m and count how often it visits each vertex, we get a random integer-valued vector $\boldsymbol{N} = (N_x)_{x\in V}$. Finally, we replace N_x with a gamma random variable with rate 1 and mean $N_x + 1$ independently at each vertex $x \in V$ to get the occupation field ν_n defined in (3.3).

Note that ν_n is the same probability measure as the occupation field of the Markovian loop soup introduced in [4] for a continuous-time Markov chain. We introduce a standard exponential random variable to each vertex (in addition to the ones that come with visits from the loop soup) to account for the trivial loops that are present in Le Jan's setup, but are not present in ours.

If Φ^R and Φ^I are the real fields defined in (3.5), then for real edge weights the Green's function has no imaginary part, and Φ^R and Φ^I are two independent identically distributed fields with covariance $\boldsymbol{G}$. In this case Theorem 3.2 implies that $(\Phi^R)^2/2$ has the same distribution as the occupation field at intensity $1/2$, even if $\boldsymbol{G}$ has negative entries. In this setting the result of Theorem 3.2 is equivalent to Le Jan's isomorphism (e.g., see Theorem 2 in [4]).

4 Proofs

4.1 Proof of Lemma 3.4

The goal is to prove that

$$\sum_{\boldsymbol{\omega}\to \boldsymbol{C}} \left[\prod_{j=1}^{N} q(\omega_j)\right] = \sum_{\boldsymbol{s}\in \mathbb{N}^{\mathcal{L}}_{\text{fin}} : \boldsymbol{c}(\boldsymbol{s})=\boldsymbol{C}} \left[\prod_{l\in\mathcal{L}} \frac{m(l)^{s_l}}{s_l!}\right], \tag{4.1}$$

where the first sum is over all such bubbles $\boldsymbol{\omega} = (\omega_j)_{j=1}^N$, that $\omega_j \in \mathcal{P}_{V_j}(v_j)$ for $j \in [N]$, and $\sum_{j=1}^N \boldsymbol{c}(\omega_j) = \boldsymbol{C}$.

We let $\mathcal{L}_j$ be the set of unrooted loops that go through v_j and stay in V_j for every $j \in [N]$. Since $\mathcal{L} = \sqcup_{j=1}^N \mathcal{L}_j$, any multiset $\boldsymbol{s} \in \mathbb{N}^{\mathcal{L}}_{\text{fin}}$ can be uniquely decomposed into a sum of multisets $\{\boldsymbol{s}^j\}_{j=1}^N$, where $\boldsymbol{s}^j \in \mathbb{N}^{\mathcal{L}_j}_{\text{fin}}$ for each $j \in [N]$. Using (3.1), we rewrite (4.1) as

$$\sum_{\boldsymbol{\omega}\to \boldsymbol{C}} \left[\prod_{j=1}^{N} q(\omega_j)\right] = \sum_{\boldsymbol{s}\in \mathbb{N}^{\mathcal{L}}_{\text{fin}} : \boldsymbol{c}(\boldsymbol{s})=\boldsymbol{C}} \left[\prod_{j=1}^{N}\prod_{l\in\mathcal{L}_j} \frac{q(l)^{s_l}}{s_l!\, d(l)^{s_l}}\right]. \tag{4.2}$$

The terms involving q can be factored out as $q(\boldsymbol{C})$ on both sides, so either $q(\boldsymbol{C}) = 0$ and (4.2) holds trivially, or all the terms involving q can be removed from the expression.

In the second case we fix $j \in [N]$, let $x = v_j$, $L = \mathcal{L}_j$ and $P = \mathcal{P}_{V_j}(v_j)$ for brevity. We take any $\boldsymbol{s} \in \mathbb{N}^L_{\text{fin}}$ and order the unrooted loops in it arbitrarily. For each unordered loop, we choose a representative loop in P uniformly at random from all the possibilities. We then concatenate all the rooted loops in the order they were produced into a rooted loop ω. If $\boldsymbol{o}$ is the combination of ordering and choice of rooted loops, then we define $\psi(\boldsymbol{s}, \boldsymbol{o})$ to be the resulting loop $\omega \in P$. Let $O(\boldsymbol{s})$ denote the set of all the possible choices $\boldsymbol{o}$ for the multiset $\boldsymbol{s}$. According to this definition,

$$|O(\boldsymbol{s})| = \frac{S_{\boldsymbol{s}}!}{\prod_{l\in L} s_l!} \prod_{l\in L} \left(\frac{n_x(l)}{d(l)}\right)^{s_l} = S_{\boldsymbol{s}}! \prod_{l\in L} \frac{n_x(l)^{s_l}}{s_l!\, d(l)^{s_l}}, \tag{4.3}$$

where $S_{\boldsymbol{s}} = \sum_{l\in L} s_l$. We now see from (4.2) and (4.3), that it is sufficient to prove that for any $\omega \in P$ with $n_0 := n_x(\omega) \geq 1$,

$$1 = \sum_{(\boldsymbol{s},\boldsymbol{o})\to\omega} \frac{1}{S_{\boldsymbol{s}}! \prod_{l\in L} n_x(l)^{s_l}}, \tag{4.4}$$

where the sum is over all pairs $(\boldsymbol{s}, \boldsymbol{o})$ with $\boldsymbol{o} \in O(\boldsymbol{s})$ and $\psi(\boldsymbol{s}, \boldsymbol{o}) = \omega$.

There is a natural bijection between ψ and finite sequences of positive integers $(n_j)_{j=1}^k$ with $\sum_{j=1}^k n_j = n_0$, which we call $\text{seq}(k, n_0)$. Now (4.4) can be rewritten as follows:

$$1 = \sum_{k=1}^{\infty} \sum_{\text{seq}(k,n_0)} \frac{1}{k!\, \prod_{j=1}^k n_j}. \tag{4.5}$$

To see that the goal expression (4.5) holds, note that for all $c \in (-1, 1)$,

$$\sum_{j=0}^{\infty} c^j = (1-c)^{-1} = \exp\{-\log(1-c)\} = \exp\left\{\sum_{j=1}^{\infty} \frac{c^j}{j}\right\} = \sum_{k=1}^{\infty} \left[\frac{1}{k!}\Big(\sum_{j=1}^{\infty} \frac{c^j}{j}\Big)^k\right].$$

By comparing the coefficients in front of c^{n_0} on both sides, we finally establish (4.5).

4.2 Proof of Proposition 3.3

First note, that if $q(\boldsymbol{C}) = 0$, then there must be $x, y \in V$ such that $C_{xy} \neq 0$ and $Q_{xy} = 0$, but then the loop soup measure of loops that induce $\boldsymbol{C}$ is also zero. For the rest of the proof we assume that $q(\boldsymbol{C}) \neq 0$. In view of Lemma 3.4, the goal is then to prove the following combinatorial identity:

$$\big|\{\text{bubbles } \boldsymbol{\omega} \text{ such that } \boldsymbol{c}(\boldsymbol{\omega}) = \boldsymbol{C}\}\big| = \prod_{x \in V} \binom{n_x(\boldsymbol{C})}{\{C_{xy}\}_{y \in V}}. \tag{4.6}$$

We prove (4.6) by induction on the number of vertices $N = |V|$. If $V = \{x\}$ is a singleton, then $n_x(\boldsymbol{C}) = C_{xx}$ and (4.6) holds trivially.

Now suppose that we have $N \geq 2$ vertices. Note that if $\boldsymbol{c}(\boldsymbol{\omega}) = \boldsymbol{C}$ for some bubble $\boldsymbol{\omega} = (\omega_j)_{j=1}^N$, then $\boldsymbol{C} = \boldsymbol{C}^0 + \boldsymbol{C}^+$ with $\boldsymbol{C}^0 = \sum_{j=2}^N \boldsymbol{c}(\omega_j) \in \mathcal{C}_{V_2}$ and $\boldsymbol{C}^+ = \boldsymbol{c}(\omega_1)$. We define

$$L(\boldsymbol{C}) := \big\{\omega \in \mathcal{P}_V(v_1) : \boldsymbol{c}(\omega) = \boldsymbol{C}\big\}, \qquad P(\boldsymbol{C}) := \big\{(\boldsymbol{C}^+, \boldsymbol{C}^0) : \boldsymbol{C}^0 \in \mathcal{C}_{V_2},\, \boldsymbol{C}^0 + \boldsymbol{C}^+ = \boldsymbol{C}\big\}.$$

To pick a bubble $\boldsymbol{\omega}$ that induces $\boldsymbol{C}$, we can first choose $\omega_1 \in L(\boldsymbol{C})$ so that $\boldsymbol{C}^0 := \boldsymbol{C} - \boldsymbol{c}(\omega_1) \in \mathcal{C}_{V_2}$, and then choose $\omega_j \in \mathcal{P}_{V_j}(v_j)$ for $j \in \{2, \dots, N\}$ so that $\sum_{j=2}^N \boldsymbol{c}(\omega_j) = \boldsymbol{C}^0$. The induction assumption is that the number of ways to do the latter for a fixed $\boldsymbol{C}^0$ is given by (4.6) with V replaced by V_2, so what we really want to prove is the following:

$$\prod_{x \in V} \binom{n_x(\boldsymbol{C})}{\{C_{xy}\}_{y \in V}} = \sum_{(\boldsymbol{C}^+, \boldsymbol{C}^0) \in P(\boldsymbol{C})} \left[\big|L(\boldsymbol{C}^+)\big| \prod_{x \in V_2} \binom{n_x(\boldsymbol{C}^0)}{\{C^0_{xy}\}_{y \in V_2}} \right], \quad \boldsymbol{C} \in \mathcal{C}_V. \tag{4.7}$$

For every $x \in V$, we let $N_x = n_x(\boldsymbol{C})$, and define $\mathcal{A}^x(\boldsymbol{C})$ as the set of N_x-tuples $\boldsymbol{a}^x = (a_1^x, \dots, a_{N_x}^x)$ in V^{N_x} that contain C_{xy} elements y for every $y \in V$. Let $\boldsymbol{A}(\boldsymbol{C}) := \big(\mathcal{A}^x(\boldsymbol{C})\big)_{x \in V}$ be the collection of such sequences.

Note that the left-hand side of (4.7) is equal to $|\boldsymbol{A}(\boldsymbol{C})| = \prod_{x \in V} |\mathcal{A}^x(\boldsymbol{C})|$. To show that we have the same quantity on the right-hand side of (4.7), let

$$\boldsymbol{A}'(\boldsymbol{C}) := \bigcup_{(\boldsymbol{C}^+, \boldsymbol{C}^0) \in P_{\boldsymbol{C}}} \big[L(\boldsymbol{C}^+) \times \boldsymbol{A}(\boldsymbol{C}^0)\big],$$

and note that it suffices to give a bijection between $\boldsymbol{A}(\boldsymbol{C})$ and $\boldsymbol{A}'(\boldsymbol{C})$ to finish the proof.

Suppose $(\boldsymbol{a}^x)_{x \in V} \in \boldsymbol{A}(\boldsymbol{C})$ are given. To map $\boldsymbol{A}(\boldsymbol{C})$ to $\boldsymbol{A}'(\boldsymbol{C})$, we define $\omega \in \mathcal{P}_V(v_1)$ by means of an algorithm.

- Set $\omega = (x)$. If $N_x = 0$, stop and output the trivial loop.
- Otherwise, let $\omega = (x, a_1^x)$, remove a_1^x from $\boldsymbol{a}^x$ and reset $N_x \to N_x - 1$.

For $j = 1, 2, \dots$, we do the following.

- If $\omega^j = x$ and $N_x = 0$, stop and output $\omega = (\omega^0, \dots, \omega^j)$ and $(\boldsymbol{a}^y)_{y \in V_2}$.
- Otherwise, if $\omega^j = y \neq x$, let ω^{j+1} equal a_1^y, remove a_1^y from $\boldsymbol{a}^y$, and reset $N_y \to N_y - 1$.

The correctness of the algorithm follows from the current property of $\boldsymbol{C} \in \mathcal{C}_V$: we cannot encounter a situation where $\omega^j = y \neq x$ and $N_y = 0$, because it would imply that

$$\sum_{z \in V} C_{yz} = |\boldsymbol{a}^y| < \sum_{z \in V} \sum_{k=1}^{|\boldsymbol{a}^z|} \mathbb{1}\{y = a_k^z\} = \sum_{z \in V} C_{zy}.$$

Once the algorithm terminates, we have $\omega \in L(\boldsymbol{C}^+)$ for $\boldsymbol{C}^+ = \boldsymbol{c}(\omega)$, also $\boldsymbol{C}^0 := \boldsymbol{C} - \boldsymbol{C}^+ \in \mathcal{C}_{V_2}$ and $(\boldsymbol{a}^x)_{x \in V} \in \boldsymbol{A}(\boldsymbol{C}^0)$.

There is a natural inverse mapping from $\boldsymbol{A}'(\boldsymbol{C})$ to $\boldsymbol{A}(\boldsymbol{C})$, because this algorithm can be run in reverse. Instead of "reading" a loop $\omega \in L(\boldsymbol{C}^+)$ from a tuple $(\boldsymbol{a}^x)_{x \in V}$, we can "record" it, and then concatenate resulting tuple it with an element of $\boldsymbol{A}(\boldsymbol{C}^0)$.

4.3 Proof of Theorem 3.2

To avoid cumbersome notation, we identify vertices with integers: $V = (1, 2, \ldots, N)$. We take the density from (3.4) and switch to "polar" coordinates $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{r}, \boldsymbol{\theta})$ by letting $z_j = \sqrt{r_j} e^{i\theta_j}$ for $j \in [N]$. We then calculate the Jacobian:

$$\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \;\text{ with }\; \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N \Rightarrow \det \left[\frac{\partial(\boldsymbol{x}, \boldsymbol{y})}{\partial(\boldsymbol{r}, \boldsymbol{\theta})} \right] = \prod_{j=1}^{N} \det \begin{bmatrix} \frac{\cos\theta_j}{2\sqrt{r_j}} & -\sqrt{r_j}\sin\theta_j \\ \frac{\sin\theta_j}{2\sqrt{r_j}} & \sqrt{r_j}\cos\theta_j \end{bmatrix} = 2^{-N}.$$

To get the marginal density $f_{|\Phi|^2}(\boldsymbol{r})$ for $\boldsymbol{r} \in \mathbb{R}_+^N$, we apply the change of variables to $f_\Phi(\boldsymbol{z})$ and integrate it over $\boldsymbol{\theta} \in T := [0, 2\pi)^N$:

$$f_{|\Phi|^2}(\boldsymbol{r}) = g(\boldsymbol{r}) \int_T d\boldsymbol{\theta} \exp\left\{ \sum_{j,k=1}^{N} \sqrt{r_j r_k}\, Q_{jk}\, e^{i(\theta_k - \theta_j)} \right\}, \tag{4.8}$$

where

$$g(\boldsymbol{r}) := \exp\left\{ -\sum_{j=1}^{N} r_j \right\} \frac{\det(\boldsymbol{I} - \boldsymbol{Q})}{(2\pi)^N}, \qquad \boldsymbol{r} \in \mathbb{R}_+^N.$$

Next we find the density of the occupation field using its current representation (3.8). Suppose that we have a matrix $\boldsymbol{C} \in \mathbb{N}^{N \times N}$. Then

$$\boldsymbol{C} \in \mathcal{C}_V \iff \sum_{j=1}^{N} (C_{jk} - C_{kj}) = 0 \qquad \forall k \in [N].$$

Since the right-hand side is always an integer for $\boldsymbol{C} \in \mathbb{N}^{N \times N}$, we see that

$$\begin{aligned} \mathbb{1}\{\boldsymbol{C} \in \mathcal{C}_V\} &= \prod_{j=1}^{N} \int_0^{2\pi} \frac{d\theta_j}{2\pi} \exp\left\{ i\theta_j \sum_{k=1}^{N} (C_{kj} - C_{jk}) \right\} \\ &= \int_T \frac{d\boldsymbol{\theta}}{(2\pi)^N} \left[\prod_{j,k=1}^{N} e^{iC_{jk}(\theta_k - \theta_j)} \right] \end{aligned} \tag{4.9}$$

We rewrite the density of ν_n using (2.1) and (3.8):

$$\frac{d\nu_n}{d\boldsymbol{r}} = g(\boldsymbol{r})\,(2\pi)^N \sum_{\boldsymbol{C} \in \mathcal{C}_V} \left[\prod_{j,k=1}^{N} \frac{(r_j r_k)^{C_{jk}/2} Q_{jk}^{C_{jk}}}{C_{jk}!} \right].$$

To see that this density is equal to (4.8) and finish the proof, we use (4.9):

$$
\begin{aligned}
&(2\pi)^N \sum_{\boldsymbol{C}\in\mathcal{C}_V} \left(\prod_{j,k=1}^{N} \frac{(r_j r_k)^{C_{jk}/2}\, Q_{jk}^{C_{jk}}}{C_{jk}!} \right) \\
&= \sum_{\boldsymbol{C}\in\mathbb{N}^{N\times N}} \int_T d\boldsymbol{\theta} \left[\prod_{l,m=1}^{N} e^{iC_{lm}(\theta_m-\theta_l)} \right] \left[\prod_{j,k=1}^{N} \frac{(\sqrt{r_j r_k}\, Q_{jk})^{C_{jk}}}{C_{jk}!} \right] \\
&= \int_T d\boldsymbol{\theta} \sum_{\boldsymbol{C}\in\mathbb{N}^{N\times N}} \left[\prod_{l,m=1}^{N} e^{iC_{lm}(\theta_m-\theta_l)} \right] \left[\prod_{j,k=1}^{N} \frac{(\sqrt{r_j r_k}\, Q_{jk})^{C_{jk}}}{C_{jk}!} \right] \\
&= \int_T d\boldsymbol{\theta} \left[\prod_{j,k=1}^{N} \sum_{C_{jk}\geq 0} \frac{\left(\sqrt{r_j r_k} Q_{jk} \exp\{i(\theta_k-\theta_j)\}\right)^{C_{jk}}}{C_{jk}!} \right] \\
&= \int_T d\boldsymbol{\theta} \left[\prod_{j,k=1}^{N} \exp\left\{ \sqrt{r_j r_k}\, Q_{jk}\, e^{i(\theta_k-\theta_j)} \right\} \right].
\end{aligned}
$$

References

[1] E. B. Dynkin (1983). Local times and quantum fields. *Seminar on Stochastic Processes*, 64-84. Birkhauser. MR-0902412.

[2] A. Kassel, T. Lévy (2016). Covariant Symanzik identities. *Preprint available:* arXiv:1607.05201.

[3] Y. Le Jan (2008). Dynkin's isomorphism without symmetry. *Stochastic analysis in mathematical physics*. ICM 2006 Satellite conference in Lisbon. 43-53 World Scientific. MR-2406020.

[4] Y. Le Jan (2011). *Markov Paths, Loops and Fields*, Lecture Notes in Mathematics **2026**, Springer-Verlag. MR-2815763.

[5] G. F. Lawler (2018). Topics in loop measures and the loop-erased walk. *Probability Surveys* **15**, 28-101. MR-3770886.

[6] G. F. Lawler, V. Limic (2010). *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. MR-2677157.

[7] G. F. Lawler, J. Perlman (2015). Loop measures and the Gaussian free field, in *Random Walks, Random Fields, and Disordered Systems*, Lecture Notes in Mathematics **2144**, M. Biskup, J. Černý, R. Kotecký, ed., Springer-Verlag, 211-235. MR-3382175.

[8] G. F. Lawler, O. Schramm, W. Werner (2003). Conformal restriction: the chordal case (electronic). J. Am. Math. Soc. **16**(4), 917-955. MR-1992830.

[9] G. F. Lawler, J. A. Trujillo Ferreras (2007). Random walk loop soup. *Trans. Amer. Math. Soc.*, **359**(2): 767-787 (electronic). MR-2255196.

[10] G. F. Lawler, W. Werner (2004). The Brownian loop soup. *Probab. Theory Related Fields*, **128**(4): 565-588. MR-2045953.

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