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Subsequential tightness of the maximum of two dimensional Ginzburg-Landau fields

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Abstract

We prove the subsequential tightness of centered maxima of two-dimensional Ginzburg-Landau fields with bounded elliptic contrast.

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1 Introduction

Let $V \in C^2(\mathbb{R})$ satisfy

$$V(x) = V(-x), \tag{1.1}$$

$$0 < c_- \le V''(x) \le c_+ < \infty, \tag{1.2}$$

where c_-, c_+ are positive constants. The ratio $\kappa = c_+/c_-$ is called the *elliptic contrast* of V. We assume (1.1) and (1.2) throughout this note without further mentioning it.

We treat V as a nearest neighbor potential for a two dimensional Ginzburg-Landau gradient field. Explicitly, let $D_N := [-N,N]^2 \cap \mathbb{Z}^2$ and let the boundary ∂D_N consist of the vertices in D_N that are connected to $\mathbb{Z}^2 \setminus D_N$ by an edge. The Ginzburg-Landau field on D_N with zero boundary condition is a random field denoted by $\phi^{D_N,0}$, whose distribution is given by the Gibbs measure

$$d\mu_N = Z_N^{-1} \exp\left[-\sum_{v\in D_N}\sum_{i=1}^{2} V(\nabla_i \phi(v))\right] \prod_{v\in D_N\setminus\partial D_N} d\phi(v) \prod_{v\in\partial D_N} \delta_0(\phi(v)), \tag{1.3}$$

where $\nabla_i \phi(v) = \phi(v+e_i) - \phi(v)$, $e_1 = (1,0)$ and $e_2 = (0,1)$, and we set $\phi(v) = 0$ for all $v \in \mathbb{Z}^2 \setminus D_N$. Here Z_N is the normalizing constant ensuring that μ_N is a probability measure, i.e. $\mu_N(\mathbb{R}^{|D_N|}) = 1$. We denote expectation with respect to μ_N by $\mathbb{E}_N$, or simply by $\mathbb{E}$ when no confusion can occur.

Ginzburg-Landau fields with convex potentials, which are natural generalizations of the standard lattice Gaussian free field corresponding to quadratic V (DGFF), have been extensively studied since the seminal works [9, 10, 13]. Of particular relevance

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to this paper is Miller's coupling, described in Section 2.2 below, which shows that certain multi-scale decompositions that hold for the Gaussian case continue to hold, approximately, for the Ginzburg-Landau model.

In this paper, we study the maximum of Ginzburg-Landau fields. Given $U \subset D_N$, let

$$M_U := \max_{x \in U} \phi^{D_N,0}(x),$$

and set $M_N = M_{D_N}$. For the Gaussian case, we write M_N^G for M_N. Much is known about M_N^G, following a long succession of papers starting with [4]. In particular, see [5] and [2], $M_N^G - m_N^G$ converges in distribution to a randomly shifted Gumbel, with $m_N^G = c_1 \log N - c_2 \log\log N$ and explicit constants c_1, c_2.

Much less is known concerning the extrema in the Ginzburg-Landau setup, even though linear statistics of such fields converge to their Gaussian counterparts [13]. A first step toward the study of the maximum was undertaken in [1], where the following law of large numbers is proved:

$$\frac{M_{D_N}}{\log N} \to 2\sqrt{g} \text{ in } L^2, \text{ for some } g = g(V) > 0. \tag{1.4}$$

Moreover, g is bounded above and below by strictly positive functions of the constants c_+, c_- from (1.2).

In this note we prove that the fluctuations of M_{D_N} around its mean are tight, at least along some (deterministic) subsequence.

Theorem 1.1. *There is a deterministic sequence $\{n_k\}$ with $n_k \to_{k\to\infty} \infty$ such that the sequence of random variables $\left\{M_{D_{n_k}} - \mathbb{E}M_{D_{n_k}}\right\}$ is tight.*

As will be clear from the proof, the sequence $\{n_k\}$ can be chosen with density arbitrarily close to 1. Theorem 1.1 is the counterpart of an analogous result for the Gaussian case proved in [3], building on a technique introduced by Dekking and Host [7]. The Dekking-Host technique is also instrumental in the proof of Theorem 1.1. However, due to the fact that the Ginzburg-Landau field does not possess good decoupling properties near the boundary, significant changes need to be made. Additional crucial ingredients in the proof are Miller's coupling and a decomposition in differences of harmonic functions introduced in [1].

Recall that the extreme value statistics for log correlated Gaussian fields (including the 2D discrete Gaussian free field) are universal up to a random shift, see [8]. We conjecture that the Ginzburg-Landau field belongs to that universality class. More explicitly, we conjecture that the expected maximum has the asymptotic expansion

$$\mathbb{E}M_{D_N} = 2\sqrt{g}\log N - \frac{3}{4} g \log\log N + O(1),$$

and that $\{M_{D_N} - \mathbb{E}M_{D_N}\}$ converges in distribution to a randomly shifted Gumbel random variable.

2 Preliminaries

2.1 The Brascamp-Lieb inequality

One can bound the variances and exponential moments with respect to the Ginzburg-Landau measure by those with respect to the Gaussian measure, using the following Brascamp-Lieb inequality. Let ϕ be a sample from the Gibbs measure (1.3). Given $\eta \in \mathbb{R}^{D_N}$, set

$$\langle \phi, \eta \rangle := \sum_{v \in D_N} \phi_v \eta(v).$$

Lemma 2.1 (Brascamp-Lieb inequalities [6]). *Assume that $V \in C^2(\mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}} V''(x) \geq c_- > 0$. Let $\mathbb{E}_{GFF}$ and Var_{GFF} denote the expectation and variance with respect to the (standard)* DGFF *measure (that is, (1.3) with $V(x) = x^2/2$). Then for any $\eta \in \mathbb{R}^{D_N}$,*

$$\mathit{Var}\langle\phi,\eta\rangle \leq c_-^{-1}\mathit{Var}_{GFF}\langle\phi,\eta\rangle, \tag{2.1}$$

$$\mathbb{E}\left[\exp\left(\langle\phi,\eta\rangle - \mathbb{E}\langle\phi,\eta\rangle\right)\right] \leq \exp\left(\frac{1}{2}c_-^{-1}\mathit{Var}_{GFF}\langle\phi,\eta\rangle\right). \tag{2.2}$$

2.2 Approximate harmonic coupling

By their definition, the Ginzburg-Landau measures satisfy the domain Markov property: conditioned on the values on the boundary of a domain, the field inside the domain is again a gradient field with boundary condition given by the conditioned values. For the discrete GFF, there is in addition a nice orthogonal decomposition. More precisely, the conditioned field inside the domain is the discrete harmonic extension of the boundary value to the whole domain plus an *independent* copy of a *zero boundary* discrete GFF.

While this exact decomposition does not carry over to general Ginzburg-Landau measures, the next result due to Jason Miller, see [12], provides an approximate version.

Theorem 2.2 ([12]). *Let $D \subset \mathbb{Z}^2$ be a simply connected domain of diameter R, and denote $D^r = \{x \in D : \mathit{dist}(x,\partial D) > r\}$. Let Λ be such that $f : \partial D \to \mathbb{R}$ satisfies $\max_{x\in\partial D}|f(x)| \leq \Lambda|\log R|^{\Lambda}$. Let ϕ be sampled from the Ginzburg-Landau measure (1.3) on D with zero boundary condition, and let ϕ^f be sampled from Ginzburg-Landau measure on D with boundary condition f. Then there exist constants $c,\gamma,\delta' \in (0,1)$, that only depend on V, so that if $r > cR^{\gamma}$ then the following holds. There exists a coupling (ϕ,ϕ^f), such that if $\hat{\phi} : D^r \to \mathbb{R}$ is discrete harmonic with $\hat{\phi}|_{\partial D^r} = \phi^f - \phi|_{\partial D^r}$, then*

$$\mathbb{P}\left(\phi^f = \phi + \hat{\phi} \text{ in } D^r\right) \geq 1 - c(\Lambda)R^{-\delta'}.$$

Here and in the sequel of the paper, for a set $A \subset \mathbb{Z}^2$ and a point $x \in \mathbb{Z}^2$, we use $\mathrm{dist}(x,A)$ to denote the (lattice) distance from x to A.

2.3 Pointwise tail bound

We also recall the pointwise tail bound for the Ginzburg-Landau field (1.3), proved in [1].

Theorem 2.3. *[1, Proposition 1.3] Let g be the constant as in (1.4). For all $u > 0$ large enough and all $v \in D_N$ we have*

$$\mathbb{P}(\phi(v) \geq u) \leq \exp\left(-\frac{u^2}{2g\,\mathit{dist}(v,\partial D_N)} + o(u)\right). \tag{2.3}$$

This allows us to conclude that the maximum of $\phi^{D_N,0}$ does not occur within a thin layer near the boundary.

Lemma 2.4. *Given $\delta < 1$, there exists $\delta' > 0$ such that*

$$\mathbb{P}\left(M_{A_{N,N^\delta}} > (2\sqrt{g} - \delta')\log N\right) \leq N^{\frac{\delta-1}{2}},$$

where

$$A_{N,N^\delta} := \left\{v \in D_N : \mathit{dist}(x,\partial D_N) < N^\delta\right\}.$$

Proof. Let $\Delta = \mathrm{dist}(x,\partial D_N)$. For δ' small enough, applying Theorem 2.3 with $u =$

$\left(2\sqrt{g}-\delta'\right)\log N$ yields

$$\begin{aligned} P\left(\phi_v \geq (2\sqrt{g}-\delta')\log N\right) &\leq \exp\left(-2\frac{(\log N)^2}{\log\Delta}+\frac{2\delta'}{\sqrt{g}}\frac{(\log N)^2}{\log\Delta}+o\left(\log N\right)\right) \\ &\leq N^{-2+2\delta'/\sqrt{g}+o(1)}, \quad \text{for all } v\in A_{N,N^\delta}. \end{aligned}$$

Therefore a union bound yields

$$P\left(M_{A_{N,N^\delta}} \geq (2\sqrt{g}-\delta')\log N\right) \leq N^{\delta-1+2\delta'/\sqrt{g}+o(1)}.$$

It suffices to take δ' such that $2\delta'/\sqrt{g} < \frac{1-\delta}{2}$. □

3 The recursion and proof of Theorem 1.1

We prove Theorem 1.1 by establishing a recursion similar to the one in [3]. It is natural to look for such recursion on dyadic boxes. However, unlike the Gaussian case, the fine field and the harmonic function obtained by applying Theorem 2.2 are not independent. This makes it difficult to control these harmonic functions and make the recursion work for dyadic boxes. Instead, we prove a recusion for some random variable M_{Y_N}, where $Y_N \subset D_N$ is a specific subset that interpolates the εN interior and N^γ interior at one side of D_N (with γ as in Theorem 2.2), such that the harmonic functions obtained from that theorem can be explicitly controlled (with their expected maximum uniformly bounded, see Lemma 3.3 below).

Denote by $T_N = [-N,N]\times\{N\} \subset D_N$ the top part of the boundary of D_N. For fixed $\varepsilon > 0$, define

$$Y_N = \{v\in D_N : \text{dist}(v,\partial D_N) \geq \varepsilon N\} \cup \{v\in D_N : \text{dist}(v,\partial D_N) = \text{dist}(v,T_N)\}.$$

For $\delta\in(0,1)$, we also define $Y_{N,\delta}\subset Y_N$ as

$$Y_{N,\delta} = \left\{v\in Y_N : \text{dist}(v,T_N) > N^{1-\delta}\right\},$$

see Figure 1. We will later choose $1>\delta>\gamma$ according to Lemma 3.2 below.

Lemma 3.1. *For the constant $g=g(V)$ in (1.4), we have*

$$\frac{M_{Y_{N,\delta}}}{\log N} \to 2\sqrt{g} \text{ in } L^2. \tag{3.1}$$

Moreover, for any $\delta'\in\left(0,2\sqrt{g}\right)$ there exists $\beta>0$, such that

$$\mathbb{P}\left(M_{Y_{N,\delta}} \leq (2\sqrt{g}-\delta')\log N\right) \leq N^{-\beta}. \tag{3.2}$$

Proof. Let $D_N^\varepsilon := \{v\in D_N : \text{dist}(v,\partial D_N)\geq \varepsilon N\}$. Since

$$\frac{M_{D_N^\varepsilon}}{\log N} \leq \frac{M_{Y_{N,\delta}}}{\log N} \leq \frac{M_{D_N}}{\log N},$$

the claim (3.1) follows from [1], since the upper control on $M_{D_N}/\log N$ follows from (1.4) while the estimate for $\mathbb{P}\left(M_{D_N^\varepsilon} \leq \left(2\sqrt{g}-\delta'\right)\log N\right)$ (and therefore the lower control on $M_{D_N^\varepsilon}/\log N$) follows from the display below (5.19) in [1]. The latter also yields (3.2). □

We now switch to dyadic scales. For $n\in\mathbb{N}$, set $N=2^n$ and $m_n := M_{Y_{2^n,\delta}}$. We set up a recursion for m_n. The starting point of the recursion is the following inequality, which we will prove,

$$\mathbb{E}m_{n+2} = \mathbb{E}M_{Y_{4N,\delta}} \geq \mathbb{E}\max\left\{\max_{v\in Y^{(1)}_{N,\delta}} \phi_v^{D_{4N},0}, \max_{v\in Y^{(2)}_{N,\delta}} \phi_v^{D_{4N},0}\right\} - o_N(1),$$

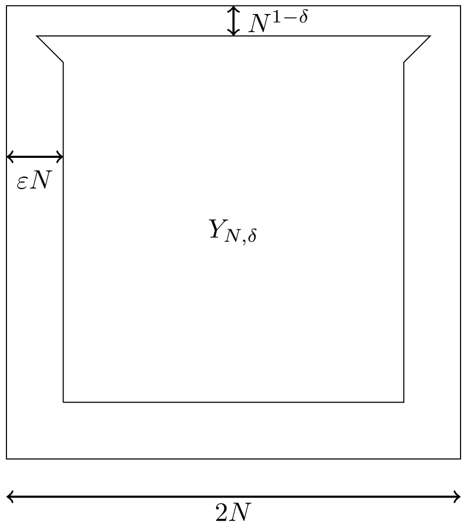

Figure 1: The domain $Y_{N,\delta}$.

where $Y_{N,\delta}^{(i)}$ are the translations of $Y_{N,\delta}$, defined by $Y_{N,\delta}^{(1)} = Y_{N,\delta} + (-1.1N, 3N)$, $Y_{N,\delta}^{(2)} = Y_{N,\delta} + (1.1N, 3N)$, see Figure 2 Here and in the sequel, we write $\phi_v^{A,0}$ for various sets As to emphasize the 0 boundary conditions on the boundary of A.

The next two lemmas will allow us to control the difference between $\phi^{D_{4N},0}$ and $\phi^{D_N,0}$ (and as a consequence, between m_{n+2} and m_n). Set $D_N^{(1)} = D_N + (-1.1N, 3N)$, $D_N^{(2)} = D_N + (1.1N, 3N)$.

Lemma 3.2. *There exist $\delta', 1 > \delta > \gamma > 0$, such that the following statement holds. Let $D_N^{\gamma,(i)} := \left\{ v \in D_N^{(i)} : \text{dist}(v, \partial D_N^{(i)}) \geq N^\gamma \right\}$. Then there exists a coupling $\mathbb{P}$ of $\left(\phi^{D_{4N},0}, \phi^{D_N^{(1)},0}, \phi^{D_N^{(2)},0}\right)$ and an event $\mathcal{G}$ with $\mathbb{P}(\mathcal{G}^c) \leq N^{-\delta'}$, such that $\phi^{D_N^{(1)},0}$ and $\phi^{D_N^{(2)},0}$ are independent and, with $h_v^{(i)}$ being harmonic functions in $D_N^{(i)}$ with boundary conditions $\phi^{D_{4N},0} - \phi^{D_N^{(i)},0}$, on the event $\mathcal{G}$, we have*

$$\phi_v^{D_{4N},0} = \phi_v^{D_N^{(i)},0} + h_v^{(i)}, \text{ for all } v \in Y_{N,\gamma}^{(i)}, \text{ for } i = 1, 2.$$

Moreover, there is a constant $C_0 = C_0(\delta)$, such that, for any $1 > \delta > \gamma$,

$$\max_{\substack{i=1,2 \\ v \in Y_{N,\delta}^{(i)}}} \text{Var}\left(h_v^{(i)}\right) \leq C_0(\delta).$$

Lemma 3.3. *With notation as in Lemma 3.2, there exists a constant $C_1 < \infty$, such that*

$$\mathbb{E} \min_i \min_{v \in Y_{N,\delta}^{(i)}} h_v^{(i)} = -\mathbb{E} \max_i \max_{v \in Y_{N,\delta}^{(i)}} h_v^{(i)} \geq -C_1.$$

The proof of Lemmas 3.2 and 3.3 are postponed to Section 4. In the rest of this section, we bring the proof of Theorem 1.1.

Proof of Theorem 1.1. Denote by m_n^* an independent copy of m_n. In order to compare $\mathbb{E}m_{n+2}$ and $\mathbb{E}m_n$, we first consider a thickening of $Y_{4N,\delta}$, defined by

$$\tilde{Y}_{4N,\delta} = \left\{ v \in Y_{4N} : \text{dist}(v, T_N) > N^{1-\delta} \right\},$$

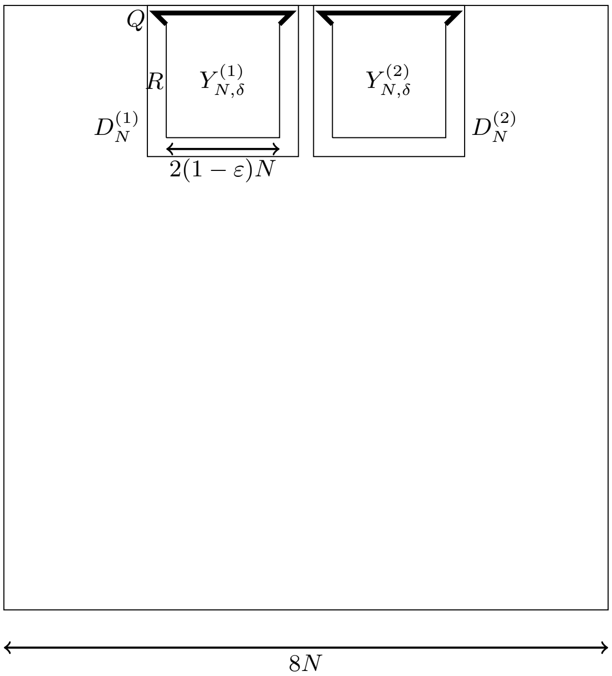

Figure 2: The domains $Y_{N,\delta}^{(i)}$, with the boundary pieces R, Q.

so that $Y_{N,\delta}^{(1)} \cup Y_{N,\delta}^{(2)} \subset \tilde{Y}_{4N,\delta}$. Let $\tilde{m}_{n+2} := M_{\tilde{Y}_{4N,\delta}}$. We have

$$\begin{aligned} \mathbb{E}\tilde{m}_{n+2} - \mathbb{E}m_{n+2} &\leq \mathbb{E}\left(\tilde{m}_{n+2} - m_{n+2}\right) 1_{\tilde{m}_{n+2} > m_{n+2}} \\ &\leq 2\left(\mathbb{E}\tilde{m}_{n+2}^2\right)^{1/2} \mathbb{P}\left(\tilde{m}_{n+2} > m_{n+2}\right)^{1/2}. \end{aligned}$$

To see that the last quantity goes to zero as $n \to \infty$, use (3.1) of Lemma 3.1 and $m_{n+2} \leq \tilde{m}_{n+2} \leq M_{D_{4N}}$ together with (1.4) to obtain

$$\left(\mathbb{E}\tilde{m}_{n+2}^2\right)^{1/2} = O\left(\log N\right). \tag{3.3}$$

On the other hand, the same argument as Lemma 2.4 (that uses Theorem 2.3 and a union bound) implies the existence of some $\delta' > 0$, such that

$$\mathbb{P}\left(\max_{v \in \tilde{Y}_{4N,\delta} \setminus Y_{4N,\delta}} \phi\left(v\right) > \left(2\sqrt{g} - \delta'\right) \log N\right) \leq N^{\frac{\delta-1}{2}}$$

Together with the lower tail in Lemma 3.1 we see that

$$\mathbb{P}\left(\tilde{m}_{n+2} > m_{n+2}\right)^{1/2} \leq \left(N^{\frac{\delta-1}{2}} + N^{-\beta}\right)^{1/2}.$$

Therefore $\mathbb{E}\tilde{m}_{n+2} - \mathbb{E}m_{n+2} \to 0$ as $n \to \infty$. We also notice that it follows from (3.3) and Lemma 3.2 that

$$\mathbb{E}\tilde{m}_{n+2} 1_{\mathcal{G}^c} \leq \left(\mathbb{E}\tilde{m}_{n+2}^2\right)^{1/2} \mathbb{P}\left(\mathcal{G}^c\right)^{1/2} \leq CN^{-\delta'/2} \log N \to 0.$$

We combine Lemmas 3.2 and 3.3 and the estimates above, to conclude

$$\begin{aligned}\mathbb{E}m_{n+2} &\geq \mathbb{E}\tilde{m}_{n+2} - o_N(1) \geq \mathbb{E}\tilde{m}_{n+2}1_{\mathcal{G}} - o_N(1)\\ &\geq \mathbb{E}\left[1_{\mathcal{G}} \max_i \max_{v \in Y_{N,\delta}^{(i)}} \left(\phi_v^{D_N^{(i)},0} + h_v^{(i)}\right)\right] - o_N(1)\\ &\geq \mathbb{E}\max\{m_n, m_n^*\} + 2\mathbb{E}\min_i \min_{v \in Y_{N,\delta}^{(i)}} h_v^{(i)} - 2\mathbb{E}\left[1_{\mathcal{G}^c} m_n\right] - o_N(1).\end{aligned}$$

We apply (1.4) to conclude that

$$\mathbb{E}\left[1_{\mathcal{G}^c} m_n\right] \leq \mathbb{P}\left(\mathcal{G}^c\right)^{1/2} \mathbb{E}\left[m_n^2\right]^{1/2} \leq C\frac{\log N}{N^{\delta'/2}}$$

Thus for all large n, we can apply Lemma 3.3 to get

$$\mathbb{E}m_{n+2} \geq \mathbb{E}\max\{m_n, m_n^*\} - 3C_1.$$

Using $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ in the first inequality and and Jensen's inequality in the second', we obtain

$$\mathbb{E}m_{n+2} - \mathbb{E}m_n \geq \frac{1}{2}\mathbb{E}\left|m_n - m_n^*\right| - 3C_1 \geq \frac{1}{2}\mathbb{E}\left|m_n - \mathbb{E}m_n^*\right| - 3C_1. \tag{3.4}$$

We need the following lemma.

Lemma 3.4. *There exists a sequence $\{n_k\}$ and a constant $K < \infty$ such that*

$$\mathbb{E}m_{n_k+2} \leq \mathbb{E}m_{n_k} + K.$$

Proof of Lemma 3.4. Take $K > 4\sqrt{g}$. Suppose that no such sequence $\{n_k\}$ exits. Then, for all n large enough, $\mathbb{E}m_{n+2} > \mathbb{E}m_n + K$. This implies

$$\liminf_{n\to\infty} \frac{\mathbb{E}m_{2n}}{2n} \geq \frac{K}{2} > 2\sqrt{g},$$

which contradicts with (3.1). □

Remark 3.5. A variation of this proof shows that the sequence $\log_2 n_k$ can be chosen with density $\alpha(K)$, with $\alpha(K) \to_{K\to\infty} 1$.

We continue with the proof of Theorem 1.1. Using the subsequence $\{n_k\}$ from Lemma 3.4, we have from (3.4) that

$$\mathbb{E}\left|m_{n_k} - \mathbb{E}m_{n_k}^*\right| \leq 2K + 6C_1,$$

which implies that $\left\{m_{n_k} - \mathbb{E}m_{n_k}^*\right\}$ is tight. It follows that the sequence of random variables

$$\bar{M}_{D_{N_k}^\delta} := \max\left\{\phi_v^{D_{N_k},0} : v \in D_{N_k}, \mathrm{dist}(v, \partial D_{N_k}) \geq N_k^{1-\delta}\right\}$$

is tight around its mean. Indeed, since $\bar{M}_{D_{N_k}^\delta}$ is the maximum of 4 rotated (possibly dependent) copies of m_{n_k}, there exists $C_2 < \infty$, such that

$$\begin{aligned}\mathbb{E}\left|\bar{M}_{D_{N_k}^\delta} - \mathbb{E}m_{n_k}\right| &= \mathbb{E}\left|\max_{i=1}^4 (m_{n_k}^{(i)} - \mathbb{E}m_{n_k})\right|\\ &\leq \mathbb{E}\sum_{i=1}^4 \left|m_{n_k}^{(i)} - \mathbb{E}m_{n_k}\right| \leq 4\mathbb{E}\left|m_{n_k} - \mathbb{E}m_{n_k}\right| \leq C_2.\end{aligned}$$

Jensen's inequality implies

$$\left|\mathbb{E}\bar{M}_{D_{N_k}^\delta} - \mathbb{E}m_{n_k}\right| \leq C_2,$$

and therefore

$$\mathbb{E}\left|\bar{M}_{D^{\delta}_{N_k}} - \mathbb{E}\bar{M}_{D^{\delta}_{N_k}}\right| \leq 2C_2.$$

Finally, combining the lower tail estimate in Lemma 3.1 and Lemma 2.4, we obtain

$$\mathbb{P}\left(M_{D_{N_k}} > \bar{M}_{D^{\delta}_{N_k}}\right) \leq \left(2^{n_k(\delta-1)} + 2^{-\beta n_k}\right)^{1/2}, \text{ for some } \beta > 0,$$

so that

$$\begin{aligned}
\mathbb{E}M_{D_{N_k}} - \mathbb{E}\bar{M}_{D^{\delta}_{N_k}} &\leq \mathbb{E}\left(M_{D_{N_k}} - \bar{M}_{D^{\delta}_{N_k}}\right)1_{\left\{M_{D_{N_k}} > \bar{M}_{D^{\delta}_{N_k}}\right\}} \\
&\leq \mathbb{P}\left(M_{D_{N_k}} > \bar{M}_{D^{\delta}_{N_k}}\right)^{1/2}\left[\left(\mathbb{E}M^2_{D_{N_k}}\right)^{1/2} + \left(\mathbb{E}\bar{M}^2_{D^{\delta}_{N_k}}\right)^{1/2}\right] \\
&\leq 2\cdot\left(2^{n_k(\delta-1)} + 2^{-\beta n_k}\right)^{1/2} O\left(\log N_k\right) \to 0.
\end{aligned}$$

We conclude that the sequence $\left\{M_{D_{N_k}} - \mathbb{E}M_{D_{N_k}}\right\}$ is tight. □

4 Proof of Lemma 3.2 and 3.3

Proof of Lemma 3.2. The existence of the harmonic decomposition is implied by the Markov property and Theorem 2.2 (with δ', γ taken as the constants in Theorem 2.2). It thus suffices to obtain an upper bound for $\mathrm{Var}\left(h^{(i)}_v\right)$. Write $h^{(i)}_v = \hat{h}^{(i)}_v - \tilde{h}^{(i)}_v$, where $\hat{h}^{(i)}_v$ is the harmonic function in $D^{\gamma,(i)}_N$ with boundary value $\phi^{D_{4N},0}$, and $\tilde{h}^{(i)}_v$ is the harmonic function in $D^{\gamma,(i)}_N$ with boundary value $\phi^{D^{(i)}_N,0}$. Without loss of generality we set $i = 1$. Notice that $h^{(1)}_v$ is a linear functional of $\phi^{D^{(1)}_N,0}$:

$$h^{(1)}_v = \sum_{z \in \partial D^{\gamma,(1)}_N} H_{\partial D^{\gamma,(1)}_N}(v,z)\,\phi^{D^{(1)}_N,0}(z),$$

where $H_{\partial D^{\gamma,(1)}_N}(v,\cdot)$ is the harmonic measure of $D^{\gamma,(1)}_N$ seen at v. Applying the Brascamp-Lieb inequality (2.1) with $\eta = H_{\partial D^{\gamma,(1)}_N}(v,\cdot)$ we get

$$\mathrm{Var}\left(h^{(1)}_v\right) \leq c_-^{-1}\mathrm{Var}_{\mathrm{GFF}}\left(h^{(1)}_v\right).$$

The orthogonal decomposition for GFF implies

$$\begin{aligned}
\mathrm{Var}_{\mathrm{GFF}}\left(\hat{h}^{(1)}_v\right) &= \mathrm{Var}_{\mathrm{GFF}}\left(\mathbb{E}_{\mathrm{GFF}}\left[\phi^{D_{4N},0}_v \middle| \mathcal{F}_{\partial D^{\gamma,(1)}_N}\right]\right) \\
&= \mathrm{Var}_{\mathrm{GFF}}\left[\phi^{D_{4N},0}_v\right] - \mathrm{Var}_{\mathrm{GFF}}\left[\phi^{D^{\gamma,(1)}_N,0}_v\right]
\end{aligned}$$

and

$$\mathrm{Var}_{\mathrm{GFF}}\left(\tilde{h}^{(1)}_v\right) = \mathrm{Var}_{\mathrm{GFF}}\left[\phi^{D^{(1)}_N,0}_v\right] - \mathrm{Var}_{\mathrm{GFF}}\left[\phi^{D^{\gamma,(1)}_N,0}_v\right].$$

Take some $\delta \in (\gamma, 1)$. We now estimate the last two expressions for different regions of $v \in Y^{(1)}_{N,\delta}$. First of all, it suffices to control $h^{(1)}_v$ for $v \in \partial Y^{(1)}_{N,\delta}$. Let

$$\begin{aligned}
Q &:= \left\{v \in \partial Y^{(1)}_{N,\delta} : \mathrm{dist}(v, \partial D_N) = \mathrm{dist}(v, T)\right\}, \\
R &:= \left\{v \in \partial Y^{(1)}_{N,\delta} : \mathrm{dist}(v, \partial D_N) = \varepsilon N\right\}.
\end{aligned}$$

We first show that

$$\max_{v\in R} \mathsf{Var}_{\mathrm{GFF}}\left(\hat{h}_v^{(1)}\right) \leq C(\varepsilon),$$
$$\max_{v\in Q\cup R} \mathsf{Var}_{\mathrm{GFF}}\left(\tilde{h}_v^{(1)}\right) \leq C_0 N^{\gamma-\delta}. \tag{4.1}$$

Indeed, standard asymptotics for the lattice Green's function (following e.g. from [11, Proposition 1.6.3]) give, for some constant g_0,

$$\begin{aligned}
&\mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_{4N},0}\right] - \mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_N^{\gamma,(1)},0}\right] \\
= \ & g_0\left(\log \mathsf{dist}(v,\partial D_{4N}) - \log \mathsf{dist}(v,\partial D_N^{\gamma,(1)})\right) + o_N(1) \\
\leq \ & g_0 \log \frac{4N}{\varepsilon N - N^\gamma} + o_N(1) \leq C(\varepsilon),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_N^{(1)},0}\right] - \mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_N^{\gamma,(1)},0}\right] \\
= \ & g_0\left(\log \mathsf{dist}(v,\partial D_N^{(1)}) - \log \mathsf{dist}(v,\partial D_N^{\gamma,(1)})\right) + O\left(N^{-1}\right) \\
\leq \ & g_0 \log \frac{N^\delta}{N^\delta - N^\gamma} + O\left(N^{-1}\right) \leq C_0 N^{\gamma-\delta}.
\end{aligned}$$

To conclude the proof, we also claim that

$$\max_{v\in Q} \mathsf{Var}_{\mathrm{GFF}}\left(\hat{h}_v^{(1)}\right) \leq C N^{\gamma-\delta}. \tag{4.2}$$

(Recall that $\delta \in (\gamma, 1)$.) Indeed, denote by T_γ the top boundary of D_N^γ, we apply asymptotics for lattice Green's function to obtain

$$\begin{aligned}
&\mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_{4N},0}\right] - \mathsf{Var}_{\mathrm{GFF}}\left[\phi_v^{D_N^\gamma,0}\right] \\
&= g_0\left(\log \mathsf{dist}(v,\partial D_{4N}) - \log \mathsf{dist}(v,\partial D_N^\gamma)\right) + O\left(N^{-1}\right) \\
&= g_0\left(\log \mathsf{dist}(v,T) - \log \mathsf{dist}(v,T_\gamma)\right) + O\left(N^{-1}\right).
\end{aligned}$$

Since

$$\log \frac{\mathsf{dist}(v,T)}{\mathsf{dist}(v,T_\gamma)} \leq \log \frac{N^\delta}{N^\delta - N^\gamma} \leq C N^{\gamma-\delta},$$

we obtain (4.2). □

Proof of Lemma 3.3. Recall that $h_v^{(i)} = \hat{h}_v^{(i)} - \tilde{h}_v^{(i)}$. We will prove that there exist $C_0 < \infty$ and $\alpha > 0$, such that for all $C_1 > C_0$,

$$\mathbb{P}\left(\max_{v\in Q} \hat{h}_v^{(1)} > C_1\right) \leq e^{-\alpha C_1}, \tag{4.3}$$
$$\mathbb{P}\left(\max_{v\in R} \hat{h}_v^{(1)} > C_1\right) \leq e^{-\alpha C_1}, \tag{4.4}$$
$$\mathbb{P}\left(\min_{v\in Q\cup R} \tilde{h}_v^{(1)} < -C_1\right) \leq e^{-\alpha C_1}. \tag{4.5}$$

Indeed, (4.3) follows from (4.2) and the exponential Brascamp-Lieb inequality (2.2) (with $\eta = H_{\partial D_N^{\gamma,(1)}}(v,\cdot)$):

$$\begin{aligned}\mathbb{P}\left(\max_{v\in Q}\hat{h}_v^{(1)} > C_1\right) &\leq |Q|\max_{v\in Q}\mathbb{P}\left(\hat{h}_v^{(1)} > C_1\right)\\ &\leq C_3 N\exp\left(-\frac{C_1^2}{C_2\mathrm{Var}_{\mathrm{GFF}}\left(\hat{h}_v^{(1)}\right)}\right)\\ &\leq C_3 N\exp\left(-\frac{C_1^2}{C_2}N^{\delta-\gamma}\right),\end{aligned}$$

where C_2, C_3 are some fixed constants. The same argument using (4.1) gives (4.5).

We now prove (4.4) using chaining. Omitting the superscripts (1) in $\hat{h}^{(1)}$ and $D_N^{\gamma,(1)}$, we claim that there exists $K<\infty$, such that for $u,v\in R$,

$$\mathrm{Var}_{\mathrm{GFF}}\left[\hat{h}_u - \hat{h}_v\right] \leq K\frac{|u-v|}{\varepsilon N}. \tag{4.6}$$

Applying the orthogonal decomposition of the DGFF we obtain

$$\phi_u^{D_{4N},0} - \phi_v^{D_{4N},0} = \phi_u^{D_N^\gamma,0} - \phi_v^{D_N^\gamma,0} + \hat{h}_u - \hat{h}_v,$$

and therefore, by the independence of $\phi_u^{D_N^\gamma,0} - \phi_v^{D_N^\gamma,0}$ and $\hat{h}_u - \hat{h}_v$ under the DGFF measure,

$$\mathrm{Var}_{\mathrm{GFF}}\left[\hat{h}_u - \hat{h}_v\right] = \mathrm{Var}_{\mathrm{GFF}}\left[\phi_u^{D_{4N},0} - \phi_v^{D_{4N},0}\right] - \mathrm{Var}_{\mathrm{GFF}}\left[\phi_u^{D_N^\gamma,0} - \phi_v^{D_N^\gamma,0}\right]. \tag{4.7}$$

We now apply the representation of the lattice Green's function, see, e.g., [11, Proposition 1.6.3],

$$G^{D_N}(u,v) = \sum_{y\in\partial D_N} H_{\partial D_N}(u,y)a(y-v) - a(u-v),$$

where $H_{\partial D_N}(u,\cdot)$ is the harmonic measure of D_N seen at u and a is the potential kernel on $\mathbb{Z}^2$ which satisfies the asymptotics

$$a(x) = \frac{2}{\pi}\log|x| + D_0 + O\left(|x|^{-2}\right),$$

where D_0 is an explicit constant (see e.g. [11, Page 39] for a slightly weaker result which nevertheless is sufficient for our needs). Substituting into (4.7), we see that

$$\begin{aligned} &\mathrm{Var}_{\mathrm{GFF}}\left[\phi_u^{D_{4N},0} - \phi_v^{D_{4N},0}\right] - \mathrm{Var}_{\mathrm{GFF}}\left[\phi_u^{D_N^\gamma,0} - \phi_v^{D_N^\gamma,0}\right]\\ = \;& G^{D_{4N}}(u,u) + G^{D_{4N}}(v,v) - 2G^{D_{4N}}(u,v)\\ &-\left(G^{D_N^\gamma}(u,u) + G^{D_N^\gamma}(v,v) - 2G^{D_N^\gamma}(u,v)\right)\\ = \;& \sum_{z\in\partial D_{4N}} H_{\partial D_{4N}}(u,z)\,a(u-z) + \sum_{z\in\partial D_{4N}} H_{\partial D_{4N}}(v,z)\,a(v-z)\\ &\quad -2\sum_{z\in\partial D_{4N}} H_{\partial D_{4N}}(u,z)\,a(v-z)\\ &-\sum_{z\in\partial D_N^\gamma} H_{\partial D_N^\gamma}(u,z)\,a(u-z) - \sum_{z\in\partial D_N^\gamma} H_{\partial D_N^\gamma}(v,z)\,a(v-z)\\ &\quad +2\sum_{z\in\partial D_N^\gamma} H_{\partial D_N^\gamma}(u,z)\,a(v-z)\\ : \;& = A_{D_{4N}} - A_{D_N^\gamma}\end{aligned}$$

We now apply the Harnack inequality, see [11, Theorem 1.7.1],

$$\left|H_{\partial D_{4N}}(u,z)-H_{\partial D_{4N}}(v,z)\right|\le\frac{|u-v|}{4N}$$

to obtain

$$\begin{aligned}A_{D_{4N}} &= \sum_{z\in\partial D_{4N}} H_{\partial D_{4N}}(u,z)\left(a(u-z)-a(v-z)\right)\\ &\quad+\sum_{z\in\partial D_{4N}}\left(H_{\partial D_{4N}}(v,z)-H_{\partial D_{4N}}(u,z)\right)a(v-z)\\ &\le \frac{|u-v|}{4N}\sum_{z\in\partial D_{4N}} H_{\partial D_{4N}}(u,z)\\ &\quad+\sum_{z\in\partial D_{4N}}\left(H_{\partial D_{4N}}(v,z)-H_{\partial D_{4N}}(u,z)\right)\left(a(v-z)-\frac{2}{\pi}\log N-D_0\right)\\ &\le K\frac{|u-v|}{N},\quad\text{for some } K<\infty.\end{aligned}$$

The same argument gives $\left|A_{D_N^\gamma}\right|\le K\frac{|u-v|}{\varepsilon N}$, thus (4.6) is proved.

Now fix a large k_0. For $k\ge k_0$ let P_k be subsets of R that plays the role of dyadic approximations: P_k contains $O\left(2^k\right)$ vertices that are equally spaced and the graph distance between adjacent points is $\varepsilon N 2^{-k}$. For $v\in R$, denote by $P_k(v)$ the k^{th} dyadic approximation of v, namely the vertex in P_k that is closest to v. Then for $v\in R$,

$$\hat{h}_v=P_{k_0}(v)+\sum_{k\ge k_0}\hat{h}_{P_{k+1}(v)}-\hat{h}_{P_k(v)}.$$

We now apply the exponential Brascamp-Lieb inequality (2.2) with

$$\eta=H_{\partial D_N^{\gamma,(1)}}\left(P_{k+1}(v),\cdot\right)-H_{\partial D_N^{\gamma,(1)}}\left(P_k(v),\cdot\right),$$

(4.6), and a union bound to obtain

$$\begin{aligned}&\mathbb{P}\left(\max_{v\in R}\left[\hat{h}_{P_{k+1}(v)}-\hat{h}_{P_k(v)}\right]>\sqrt{K\left(\frac{3}{2}\right)^{-k}}\right)\\ \le&\; C_3 2^k\exp\left(-K\left(\frac{3}{2}\right)^{-k}\frac{C_4}{2\cdot 2^{-k}}\right)\\ \le&\; C_3 2^k\exp\left(-C_4\left(\frac{4}{3}\right)^k\frac{K}{2}\right),\end{aligned}$$

for some constant C_4. Since both $\sqrt{K\left(\frac{3}{2}\right)^{-k}}$ and the tail probability are summable in k, we conclude that (4.4) holds. □

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