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# Option pricing with bivariate risk-neutral density via copula and heteroscedastic model: A Bayesian approach

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**Abstract.** Multivariate options are adequate tools for multi-asset risk management. The pricing models derived from the pioneer Black and Scholes method under the multivariate case consider that the asset-object prices follow a Brownian geometric motion. However, the construction of such methods imposes some unrealistic constraints on the process of fair option calculation, such as constant volatility over the maturity time and linear correlation between the assets. Therefore, this paper aims to price and analyze the fair price behavior of the call-on-max (bivariate) option considering marginal heteroscedastic models with dependence structure modeled via copulas. Concerning inference, we adopt a Bayesian perspective and computationally intensive methods based on Monte Carlo simulations via Markov Chain (MCMC). A simulation study examines the bias, and the root mean squared errors of the posterior means for the parameters. Real stocks prices of Brazilian banks illustrate the approach. For the proposed method is verified the effects of strike and dependence structure on the fair price of the option. The results show that the prices obtained by our heteroscedastic model approach and copulas differ substantially from the prices obtained by the model derived from Black and Scholes. Empirical results are presented to argue the advantages of our strategy.

## 1 Introduction

An option is a financial derivative which the investor acquires the right, but not the obligation, to buy or sell a particular asset for a predetermined price and time, where that price is known as the exercise price. Thus, a put option may be interpreted as an auto insurance policy, where it allows the investor to recover a pre-established value for the asset, even if it has devalued. Regarding the call option, it is compared to the signal paid in the purchase of a house, as it guarantees the fixed price and also the preference in the purchase.

The elaboration of models with the purpose of pricing options began with the authors Black and Scholes (1973) and Merton (1973). The model proposed by the authors uses Brownian motion techniques to obtain the fair price of an option in the univariate case. In the multivariate case, there are several methodologies for achieving the fair price of the options, one of them being the model of Black

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and Scholes multivariate, where this approach consists of the use of Brownian geometric movement for $n$ assets considering the volatility constant over time.

Tools that accommodate the co-movements between its underlying processes are needed to understand the price behavior of a multivariate option. A primary tool that is widely used by the methods derived from the traditional Black and Scholes model is the multivariate normal distribution modeling. However, the use of this approach implies in linear associations as a measure of dependence between the assets, and empirical evidence shows that a real association between financial series is much more complex (Lopes and Pessanha, 2018).

The works of Margrabe (1978), Johnson and David (1987), Nelsen (2006) and Shimko (1994) used the linear correlation coefficient to analyze and capture dependence among the underlying assets. However, Embrechts, McNeil and Straumann (2002) and Forbes and Rigobon (2002) criticize the use of this tool, where the authors highlight the stylized facts in finance, such as the heavy tails of returns distributions, their autocorrelations, groupings of volatilities over time and non-normality.

As an alternative, the use of the copulas theory allows the joint modeling of the assets in which there is a separation of the structure of dependence between the variables and their marginal distributions, where this dependence can be linear, nonlinear and even dependence on the tails. Therefore, Rosenberg (2002) and Cherubini and Luciano (2002) used the copula theory in an attempt to capture the dependency among the assets in the derivative pricing process.

Besides, many models use the premise of constant volatility over time, which may not be observed in finance series (French, William and Stambaugh, 1987; Franses and Van Dijk, 2000). Thus, to make the pricing process more realistic, Duan (1995) explored the concept of option pricing considering the heteroscedasticity of the assets, where the author proposed to follow a modification of the GARCH process.

Therefore, this paper aims to price and analyze the fair price behavior of bivariate call-on-max option considering marginal heteroscedastic models and the dependence structure modeled via copulas. Besides, the results found will be compared with the values obtained by the classic extended models of Black and Scholes, known as Stulz Closed-form for a call-on-max option.

This work differs from the others found in the literature in two aspects: no studies are comparing the heteroscedastic approach with the classical one (derivations from the Black and Scholes model) for the bivariate case and, furthermore, there are no studies with this methodology considering the Brazilian stock market.

The structure of this paper is divided as follows. Section 2 presents the classical models and the heteroscedastic approach for pricing call-on-max option. Section 3 gives the Bayesian inference procedure. Section 4 presents a simulation study. Section 5 shows the application of the methodology in real data of the Brazilian stock market. Finally, Section 6 gives some final remarks on this work.

## 2 Copula functions

Copulas are useful tools in constructing joint distributions (Sharifonnasabi, Alamatsaz and Kazemi, 2018). That is, copula is a multidimensional distribution function in which the marginal distributions are uniform in [0, 1]. A bivariate copula is a function that satisfies $C : I^2 \longrightarrow I \in [0, 1]$ that satisfies the following conditions

$$C(x_1, 0) = C(0, x_1) = 0 \quad \text{and} \quad C(x_1, 1) = C(1, x_1) = x_1, \quad x_1 \in I,$$

and the 2-increasing condition

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0,$$

for all $u_1, u_2, v_1$ and $v_2 \in [0, 1]$ such $u_1 \leq u_2$ and $v_1 \leq v_2$.

One of the most famous theorems in copula theory is the Sklar theorem. According to Sklar's theorem (Sklar, 1959), any bivariate cumulative distribution $H_{S_1,S_2}$ can be represented as a function of the marginal distributions $F_{S_1}$ and $F_{S_2}$. Besides, if the marginal distributions are continuous, the copula exists, is unique and is given by

$$H_{S_1,S_2}(x_1, x_2) = C\big(F_{S_1}(x_1), F_{S_2}(x_2)\big),$$

which $C(u, v) = P(U \leq u, V \leq v)$, $U = F_{S_1}(x_1)$ and $V = F_{S_2}(x_2)$.

In the case of continuous and differentiable marginal distributions, the joint density function of the copula is given by

$$f(x_1, x_2) = f_{S_1}(x_1) f_{S_2}(x_2) c\big(F_{S_1}(x_1), F_{S_2}(x_2)\big),$$

which $f_{S_1}(x_1)$ and $f_{S_2}(x_2)$ are the density for the distribution function $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, respectively, and

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial uv},$$

is the density of copula. For further details about copulas, see Nelsen (2006) and Sanfins and Valle (2012). In this work we will use the Normal, $t$-Student, Gumbel, Frank and Joe copulas. Details are given in the Appendix.

## 3 Conceptual framework and model formulation

In this section, we introduce the Stulz (1982) model, which is an extension of the Black and Scholes model for the bivariate case for the call-on-max option and the Duan (1995) model, where the author considers the heteroskedasticity of the underlying assets of the option. Besides, we will introduce how to use the copula theory to model the joint distribution of assets, to capture non-linear dependence between the assets.

### 3.1 Call-on-max option

A European option call on the maximum of two risky assets (call-on-max) is defined based on the maximum price between two assets. The payoff function of this option is given by

$$g\big(S(T)\big) = \max\big[\max\big(S_1(T), S_2(T)\big) - K, 0\big],$$

where $S_i(T)$ is the price of the $i-th$ asset ($i = 1, 2$) at the maturity date $T$ and $K$ is the strike price or exercise price. In this work will be discussed two approaches for obtaining $g(S(T))$. In the first one, we will use marginal heteroscedastic processes to modeling $S_1$ and $S_2$ and structure of copulas functions to analyze the dependence between the assets. The second approach, with the objective of backtesting, will use the model proposed by Stulz (1982), where the author proposed a derivation of the Black and Scholes model for the bivariate case, in which the main premises derive from Brownian geometric motion.

### 3.2 First approach: Duan model and copulas

To introduce heteroscedasticity, we will use the fundamental theorem of asset pricing described by Delbaen and Schachermayer (1994). This theorem states that since the stock price $S_i(T)$ ($i = 1, 2$) is free from arbitrage and present in a complete market (Hull, 1992), there exists a measure of probability $\mathbb{Q}$ such that the discounted price of the stock, $e^{-r(T-t)}S_i(T)$, is a martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is equivalent to the real world probability measure $\mathbb{P}$.

The fair price of the call-on-max option depends on the dependency structure among the object assets since its price is defined as an expected value (by definition and ownership of a martingale measure, for more details, see Madan and Milne (1991)). Therefore, we define the following definition to perform the pricing.

**Definition 1.** Let $S_1$ and $S_2$ be two stocks traded in a complete and free arbitrary market. In addition, be $t$ the present date, $T$ the maturity date and r the fixed risk-free rate yield, then the option price considering the payoff function $g(S_1, S_2) = \max[\max(S_1(T), S_2(T)) - K, 0]$ is

$$\begin{aligned} &v(t, S_1, S_2) \\ &\quad = e^{-r(T-t)} E^{\mathbb{Q}}\big[\max\big[\max\big(S_1(T), S_2(T)\big) - K, 0\big] | F_t\big] && (3.1) \\ &\quad = e^{-r(T-t)} \int_0^\infty \int_0^\infty \max\big[\max\big(S_1(T), S_2(T)\big) - K, 0\big] \\ &\qquad \times f^{\mathbb{Q}}_{S_1,S_2}(x_1, x_2)\, dx_1\, dx_2, && (3.2) \end{aligned}$$

which $f^{\mathbb{Q}}_{S_1,S_2}$ is the the joint density function of the two measures under neutral risk probability $\mathbb{Q}$, which in this work will be modeled by copula functions, and $F_t$ is a filtering containing all information about the assets up to time $t$.

Thus, we will express the joint density function using the marginal densities $f_{S_1}(x_1)$ e $f_{S_2}(x_2)$ by means of copula functions as follows

$$f^{\mathbb{Q}}_{S_1,S_2} = c^{\mathbb{Q}}(F^{\mathbb{Q}}_{S_1}, F^{\mathbb{Q}}_{S_2}) f^{\mathbb{Q}}_{S_1}(x_1) f^{\mathbb{Q}}_{S_2}(x_2),$$

which $c^{\mathbb{Q}} = \frac{\partial^2 C^{\mathbb{Q}}(x_1,x_2)}{\partial x_1 \partial x_2}$, where $c^{\mathbb{Q}}(\cdot)$ is the copula density and $C^{\mathbb{Q}}(\cdot)$ is a copula function.

Therefore, to construct a joint process of neutral risk for the bivariate distribution of the option, the marginal processes are derived first. Duan (1995) defined an option pricing model considering that the variance of the asset-object is not constant over time.

**Definition 2.** Let $r$ a fixed risk-free interest rate and $\lambda > 0$. Under the Duan GARCH process (DGARCH) the log returns, $x_t = \log(\frac{S_t}{S_{t-1}}) = \log(s_t) - \log(s_{t-1})$, for $t = 1, \dots, n$, are given by

$$x_t = r + \lambda\sqrt{h_t} + \frac{1}{2}h_t + \sqrt{h_t}\epsilon_t, \quad \epsilon_t \sim N(0,1), \tag{3.3}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \epsilon^2_{t-j} h_{t-j} + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{3.4}$$

which the parameters $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$ and $\sum_{j=1}^{q} \alpha_j + \sum_{j=1}^{p} \beta_j < 1$, which the latter condition guarantees that the process variance will not explode, that is, to maintain the stationarity of the process. The parameter $\lambda$ can be interpreted as the risk premium.

To apply the DGARCH model in the option pricing process, Duan (1995) defined the concept of *locally risk-neutral valuation relationship* (LRNVR), where it transforms the model of Equation (3.4) into a neutral risk measure $\mathbb{Q}$. For more details on the transformation of the real-world measure $\mathbb{P}$ to the neutral risk measure $\mathbb{Q}$, see Duan (1995).

**Definition 3.** A measure $\mathbb{Q}$ satisfies the LRNVR if a measure $\mathbb{Q}$ is absolutely continuous in respect to the measure $\mathbb{P}$ (real world). Under $\mathbb{Q}$ we have

$$E^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}|F_t\right] = e^r \quad \text{and} \quad \mathrm{Var}^{\mathbb{Q}}(x_t|F_t) = \mathrm{Var}^{\mathbb{P}}(x_t|F_t).$$

This definition shows that the conditional variance is the same for both measures so that we can use the parameters of equation (3.4) under $\mathbb{P}$. With this definition, Duan showed that under local measurement of neutral risk $\mathbb{Q}$, the previously defined DGARCH process becomes

$$x_t = r - \frac{1}{2}h_t + \sqrt{h_t}\epsilon^*, \quad \epsilon^* \sim N(0,1), \tag{3.5}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \left(\epsilon_{t-j}^* - \lambda\sqrt{h_{t-j}}\right)^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{3.6}$$

and in this work, as in Duan (1995) and Zhang and Guegan (2008), the orders $p = 1$ and $q = 1$ will be used. The construction and derivation of the Duan model is based on the premise of normality of the errors, but it is possible to consider other distributions, as in Fonseca, Migon and Ferreira (2012). These extensions are being studied in a different manuscript.

When the concept of *locally risk-neutral valuation relationship* is present, the futures prices of the individual assets can be expressed by

$$S_i(T) = S_i(0) \exp\left[rT - 0.5 \sum_{t=1}^{T} h_{i,t} + \sum_{t=1}^{T} \sqrt{h_{i,t}} \epsilon_{i,t}^*\right],$$

which $S_i(0)$ is the last price of the period under analysis for each $i = 1, 2$.

To obtain the expected value of the continuous function given by equation 3.1 of a bivariate vector $(S_1, S_2)$ with cumulative distribution function $H(x_1, x_2)$, we will use Monte Carlo integration expressed by

$$E\left[g(S_1, S_2)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(S_1, S_2)\, dH(x_1, x_2),$$

which can be approximated by following the algorithm below:

1. Generate $n$ observations of bivariate random vector $(\boldsymbol{S}_1, \boldsymbol{S}_2)$;
2. For each observation $i$, calculate $g_i = g(x_{1i}, x_{2i})$, for $i = 1, 2, \dots, n$;
3. $E[g(S_1, S_2)] \approx \frac{1}{n} \sum_{i=1}^{n} g_i$.

To generate $n$ samples of the specific copula we will use the algorithms proposed by Schmidt (2007) and Nelsen (2006). Therefore, under the probability measure of neutral risk $\mathbb{Q}$, the fair price of the option with payoff function $g(\cdot)$ at the maturity time $T$ is given by

$$v(t, S_1, S_2) = \frac{e^{-r(T-t)}}{N} \sum_{i=1}^{N} g\left(S_{1,i}(T), S_{2,i}(T)\right). \tag{3.7}$$

In order to compare the consistency of the results obtained by the duan model and copulas approach, we will examine the prices generated by applying the closed formula of Stulz (1982), where it is a derivation of the Black and Scholes model for the bivariate case, where the author considers that the active objects follow a geometric Brownian motion, as in Black and Scholes (1973) and Merton (1973).

### 3.3 Second approach: Stulz closed-form solution

The closed formula proposed by Stulz (1982) has two significant limitations, being that the volatility of the asset-object is considered constant throughout the time of maturity and the joint distribution is a bivariate normal, which implies a linear correlation between the assets. The fair price for the call-on-max option is set by

$$c_{\max}(S_1, S_2, K, T) = S_1 e^{-rT} M(y_1, d; \rho_1) + S_2 e^{-rT} M(y_2, -d + \sigma\sqrt{T}; \rho_2) - K e^{-rT} * \left[1 - M(-y_1 + \sigma_1\sqrt{T}, -y_2 + \sigma_2\sqrt{T}; \rho)\right],$$

where

$$d = \frac{\ln(S_1/S_2) + (\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad y_1 = \frac{\ln(S_1/K) + (\sigma_1^2/2)T}{\sigma_1\sqrt{T}},$$
$$y_2 = \frac{\ln(S_2/K) + (\sigma_2^2/2)T}{\sigma_2\sqrt{T}},$$
$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \qquad \rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma} \quad \text{and} \quad \rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma},$$

where $S_i$ is the price of stock $i$, $K$ the strike price, $T$ the time for the option to expire in years, $r$ the risk-free interest rate, $\sigma_i$ the stock volatility of asset $i$, $\rho$ the linear correlation between the two assets, $N(x)$ the cumulative function of the standard normal distribution and $M(a, b; \rho)$ the cumulative function of the bivariate normal distribution in $(a, b)$ with linear correlation coefficient $\rho$.

## 4 Bayesian inference

Given a 2-dimensional copula, $C(u_1, u_2)$, and two univariate distributions, $F_{S_1}(x_1)$ and $F_{S_2}(x_2)$, the joint density function is given by

$$f(x_1, x_2) = c\big(F_{S_1}(x_1), F_{S_2}(x_2)\big) \prod_{i=1}^{2} f_{S_i}(x_i),$$

where $f_{S_i}$ represents the marginal density functions and $c$ is the density function of the copula which is given by

$$c(u_1, u_2) = \frac{f(F_{S_1}^{-1}(u_1), F_{S_2}^{-1}(u_2))}{\prod_{i=1}^{2} f_{S_i}(F_{S_i}^{-1}(u_i))}.$$

The marginal distribution for each $x_{it}$ is given by $u_{it} = F_{S_i}(x_{it}) = F_{\epsilon_i}([x_{it} - \mu_{it}]/\sqrt{h_{it}})$, where $F_{\epsilon_i}(\cdot)$ denotes the univariate distribution function of $\epsilon_{it}$ (Ausin and Lopes, 2010; Rossi, Ehlers and Andrade, 2012). Therefore, the joint density of $\boldsymbol{x}_t$ is then given by,

$$f(x_{1t}, x_{2t}) = c(u_{1t}, u_{2t}) \prod_{i=1}^{2} f_{S_i}(x_{it}) = c(u_{1t}, u_{2t}) \prod_{i=1}^{2} \frac{1}{\sqrt{h_{it}}} f_{\epsilon_i}\left(\frac{x_{it} - \mu_{it}}{\sqrt{h_{it}}}\right),$$

where $f_{\epsilon_i}(\cdot)$ is the marginal density function of each $\epsilon_{it}$ and $\mu_{it}$ is the mean of duan process.

Now, given a bivariate density function $f(\cdot)$ with joint distribution function $F(\cdot)$ and corresponding marginal densities $f_{S_i}(\cdot)$ the copula density is obtained and then,

$$f(x_{1t}, x_{2t}) = \frac{f(F_{S_1}^{-1}(u_{1t}), F_{S_2}^{-1}(u_{2t}))}{\prod_{i=1}^{2} f_{S_i}(F_{S_i}^{-1}(u_{it}))} \prod_{i=1}^{2} \frac{1}{\sqrt{h_{it}}} f_{\epsilon_i}\left(\frac{x_{it} - \mu_{it}}{\sqrt{h_{it}}}\right).$$

In this work we will use Bayesian inference, which is an approach that describes the model parameters by probability distributions. It offers a natural way to introduce parameter uncertainty in the estimation of volatilities. We design here a two-step Bayesian algorithm, for more details in Ausin and Lopes (2010). In the first step, we estimate each marginal series independently considering a univariate Duan GARCH model under measure $\mathbb{P}$ given in equation (3.3), where $x_{it}|h_{it} \sim N(r + \lambda\sqrt{h_t} - 1/2h_t, h_t)$, for $i = 1, 2$. For each marginal series, we have four parameters to estimate $\boldsymbol{\theta}_i = (\alpha_{0,i}, \alpha_{1,i}, \beta_i, \lambda_i)$, for $i = 1, 2$, and the log-likelihood is given by

$$l(\boldsymbol{\theta}_i|x_t) = -\frac{n}{2}\left[\log(2\pi) + \frac{1}{n}\sum_{t=1}^{n}\left[\log(h_t) + \frac{(x_t - r - \lambda\sqrt{h_t} + 1/2h_t)^2}{h_t}\right]\right].$$

Therefore, we define an MCMC algorithm for sample from the posteriori distribution of $\boldsymbol{\theta}_i$ for each series with a Gibbs sampling scheme, where each parameter is updated using a Metropolis–Hastings. For each element of the Monte Carlo sample of size $N$, we can obtain a set of residuals,

$$\alpha_{0,i}^{(n)}, \alpha_{1,i}^{(n)}, \beta_i^{(n)}, \lambda_i^{(n)} \quad \Longrightarrow \quad \epsilon_{it}^{(n)} = \frac{x_{it} - \mu_i^{(n)}}{\sqrt{h_{it}^{(n)}}},$$

for $t = 1, \ldots, T$, and for $n = 1, \ldots, N$, where $\mu_t = r + \lambda\sqrt{h_t} - 1/2h_t$ denote the mean process.

Thus, we can estimate the residual for each time $t$ for each series as follows,

$$\hat{\epsilon}_{it} = \frac{1}{N}\sum_{n=1}^{N} \epsilon_{it}^{(n)},$$

for $i = 1, 2$.

To estimate the copula parameters, $\boldsymbol{\rho}_c$, we plug in these estimations in the likelihood of specific copula using

$$\hat{U}_{it} = F^{-1}\big(F(\hat{\epsilon}_{it})\big),$$

and obtaining the following likelihood functions for $\boldsymbol{\rho}_c$,

$$l(\boldsymbol{\rho}_c|x_t) = \sum_{i=1}^{n} \ln c_{\boldsymbol{\rho}}(\hat{U}_{it}),$$

where $c_{\boldsymbol{\rho}}$ is the density of the copula displayed in annex, $\boldsymbol{\rho}$ is a vector of the parameters of the copula and $\hat{U}_i$ refers to the pseudo uniform sample.

Now, we construct another Markov Chain to sample from the posterior distribution of $\boldsymbol{\rho_c}$ using Metropolis–Hasting steps, as in Ausin and Lopes (2010) and Rossi, Ehlers and Andrade (2012).

## 4.1 Prior distributions

In the Bayesian approach, we need to specify prior distributions for the vector of parameters which define the marginal Duan GARCH model, that is, $\alpha_{0,i}, \alpha_{1,i}, \beta_i$ and $\lambda_i$, $i = 1, 2$ plus the parameters in the copula functions, that is, $\rho_i$ in the Normal, Gumbel, Frank and Joe Copulas and $\rho_i$ and $\nu_i$ in the $t$ copula. Following Ausin and Lopes (2010), for each parameter we assume a uniform prior over their respective domains imposing the stationary condition, that is, $\alpha_{1,i} + \beta_i \leq 1$. We shall adopt these prior choices in the simulation studies of Section 4.

## 4.2 Selection criteria for marginal and joint models

In order to verify if the distribution of the residues follows a standard normal distribution, the Kolmogorov–Smirnov and Shapiro–Wilk tests will be used for a random sample. The Kolmogorov–Smirnov (KS) test for a random sample is used to compare a dataset through its distribution function $F(\mathbf{x})$ with a known cumulative function $G(\mathbf{x})$. The null hypothesis is that $\boldsymbol{x} \sim G$, and the KS statistic is defined by $D_{KS} = \max(|F(\boldsymbol{x}) - G(\boldsymbol{x})|)$. The Shapiro–Wilk (SW) test statistic is $W = (\sum_{i=1}^{n} a_i x_{(i)})^2 / \sum_{i=1}^{n} (x_i - \bar{x})^2$, where $x_{(i)}$ is the $i$th order statistic, $\bar{x}$ is the sample mean and the constants $a_i$ is given by $(a_1, \dots, a_n) = m^T V^{-1} / (m^T V^{-1} V^{-1} m)^{0.5}$, where $m = (m_1, \dots, m_n)^T$, and $m_1, \dots, m_n$ are the expected values of the order statistics of independent and identically distributed random variables sampled from the standard normal distribution, and $V$ is the covariance matrix of those order statistics.

The Ljung–Box test (LB) will be performed to test whether residuals from marginal distributions have independent increments. Considering the null hypothesis that the residuals do not have autocorrelation, the Ljung-box test statistic is given by $Q = N(N+2) \sum_{k=1}^{M} \frac{\rho_k^2}{N-k}$, which $N$ is the sample size, $M$ is the number of autocorrelated lags and $\rho_k$ is the autocorrelation in lag $k$. Moreover, under the null hypothesis, the test statistic follows asymptotically a distribution $\chi^2(M)$.

In order to make the choice of the best copula model in the bivariate distribution fitted, the Expected Akaike Information Criteria (EAIC), Expected Bayesian Information Criterion (EBIC), Deviance Information Criteria (DIC) and and Log-Predictive Score (LPS) will be adopted. These are given by $\text{EAIC} = E[D(\boldsymbol{\theta}_M)] + 2np_M$, $\text{EBIC} = E[D(\boldsymbol{\theta}_M)] + \log(n)np_M$, $\text{DIC} = 2E[D(\boldsymbol{\theta}_M)] - D(E[\boldsymbol{\theta}_M])$ and Log-Predictive Score $(\text{LPS}) = \frac{-1}{T} \sum_{t=1}^{T} \log p(y_t | \boldsymbol{y}_{t-1}, \boldsymbol{\theta}_M)$ (Delatola and Griffin, 2011; Abanto-Valle, Lachos and Dey, 2015; Leão, Abanto-Valle and Chen, 2017) respectively, where $np_M$ represents the number of parameters in model $M$, $\boldsymbol{\theta}_M$ is the set os parameters in model $M$, $n$ is the sample size and $D(\cdot)$ is the deviance function defined as minus twice the log-likelihood function. For more details see, Spiegelhalter et al. (2002).

## 5 Simulation study

In this section, we illustrate the proposed methodology with artificial time series. The simulation study main concern is to assess the bias, mean squared error (MSE) and coverage probabilities of the posterior means for the parameters of marginals and copula obtained by two-step Bayesian algorithm described previously.

First, we simulate the innovation distribution ($\epsilon_{1t}$ and $\epsilon_{2t}$) through a copula with a fix parameter $\rho$. For show the proposed simulation study will be used the Frank copula, where it obtains good fitted to financial series in several works in the literature (Klugman and Parsa, 1999; Cherubini and Luciano, 2002; Hürlimann, 2004). Then we simulate bivariate time series Duan GARCH processes with these copula-dependent innovations for each sample size ($n = 250, 500$ and $1000$) and fixed interest rate $r$ at 7% per annum with the following univariate models,

$$
\begin{aligned}
x_{1t} &= r - \frac{1}{2}h_{1t} + \sqrt{h_{1t}}\epsilon_{1t}, \quad \epsilon_{1t} \sim N(0, 1), \\
x_{2t} &= r - \frac{1}{2}h_{2t} + \sqrt{h_{2t}}\epsilon_{2t}, \quad \epsilon_{2t} \sim N(0, 1), \\
h_{1t} &= 0.012 + 0.17(\epsilon_{1t-1} - 0.12\sqrt{h_{1t-1}})^2 + 0.81h_{1t-1}, \\
h_{2t} &= 0.01 + 0.15(\epsilon_{2t-1} - 0.1\sqrt{h_{2t-1}})^2 + 0.8h_{2t-1},
\end{aligned}
$$

and the Frank copula parameter is fixed in $\rho = 2$. The arbitrary choice of the copula parameter value was based on a positive relationship between the underlying assets, representing a moderately correlated market.

The priors were chosen following Ausin and Lopes (2010), as described in the previous subsection. For each setup, we generated 500 (replication) bivariate time series. The proposed two-stage MCMC algorithm is run for 20,000 iterations with first 10,000 as burn-in iterations. The code was made in R.

The Table 1 presents the true values, posterior mean, posterior median, highest posterior density (HPD) interval 95%, size of HPD interval, bias, MSE and coverage probabilities for each model parameter obtained from MCMC outputs. Observe that, the bias and MSEs decrease tending to zero when the sample size increases. We also noticed that the posterior means are very close to the posterior medians. Furthermore, the amplitude of the HPD interval tends to decrease as the sample size increases. The coverages are closer to the nominal ones for increasing sample sizes. Therefore, through this simulation study, the asymptotic properties of the model are satisfactorily verified.

**Table 1** *Simulation results to $n = 250, 500$ and 1000*

| Sample size | Parameter | True value | Posterior mean | Posterior median | HPD 95% | Size of HPD | Bias | MSE | C.P |
|---|---|---|---|---|---|---|---|---|---|
| $n = 250$ | $\alpha_{01}$ | 0.012 | 0.0593 | 0.0525 | [0.0082; 0.1291] | 0.1209 | −0.0473 | 0.0022 | 0.9266 |
| | $\alpha_{11}$ | 0.170 | 0.2093 | 0.2024 | [0.0941; 0.3361] | 0.2420 | −0.0393 | 0.0015 | 0.9725 |
| | $\beta_1$ | 0.810 | 0.6560 | 0.6690 | [0.4337; 0.8449] | 0.4112 | 0.1540 | 0.0237 | 0.8716 |
| | $\lambda_1$ | 0.120 | 0.1364 | 0.1337 | [0.0299; 0.2430] | 0.2131 | −0.0164 | 0.0003 | 0.9358 |
| | $\alpha_{02}$ | 0.010 | 0.0441 | 0.0411 | [0.0081; 0.0877] | 0.0796 | −0.0341 | 0.0012 | 0.7982 |
| | $\alpha_{12}$ | 0.150 | 0.1989 | 0.1900 | [0.0700; 0.3453] | 0.2752 | −0.0489 | 0.0024 | 0.9725 |
| | $\beta_2$ | 0.800 | 0.5677 | 0.5757 | [0.2938; 0.8141] | 0.5203 | 0.2323 | 0.0540 | 0.7890 |
| | $\lambda_2$ | 0.100 | 0.1120 | 0.1084 | [0.0172; 0.2112] | 0.1940 | −0.0120 | 0.0001 | 0.9633 |
| | $\rho$ | 2.000 | 1.9317 | 1.9314 | [1.1733; 2.7030] | 1.5297 | 0.0683 | 0.0047 | 0.9639 |
| $n = 500$ | $\alpha_{01}$ | 0.012 | 0.0232 | 0.0217 | [0.0065; 0.0429] | 0.0363 | −0.0112 | 0.0001 | 0.9662 |
| | $\alpha_{11}$ | 0.170 | 0.1878 | 0.1849 | [0.1177; 0.2639] | 0.1462 | −0.0178 | 0.0003 | 0.9595 |
| | $\beta_1$ | 0.810 | 0.7655 | 0.7698 | [0.6712; 0.8504] | 0.1792 | 0.0445 | 0.0020 | 0.9189 |
| | $\lambda_1$ | 0.120 | 0.1193 | 0.1183 | [0.0368; 0.2012] | 0.1644 | 0.0007 | 0.0000 | 0.9595 |
| | $\alpha_{02}$ | 0.010 | 0.0191 | 0.0195 | [0.0048; 0.0368] | 0.0320 | −0.0091 | 0.0001 | 0.9797 |
| | $\alpha_{12}$ | 0.150 | 0.1749 | 0.1706 | [0.0929; 0.2643] | 0.1714 | −0.0249 | 0.0006 | 0.9527 |
| | $\beta_2$ | 0.800 | 0.7755 | 0.7345 | [0.5777; 0.8547] | 0.2770 | 0.0245 | 0.0006 | 0.9392 |
| | $\lambda_2$ | 0.100 | 0.1024 | 0.1010 | [0.0268; 0.1784] | 0.1516 | −0.0024 | 0.0000 | 0.9257 |
| | $\rho$ | 2.000 | 1.9413 | 1.9412 | [1.4021; 2.4781] | 1.0760 | 0.0587 | 0.0035 | 0.9502 |
| $n = 1000$ | $\alpha_{01}$ | 0.012 | 0.0149 | 0.0145 | [0.0066; 0.0241] | 0.0176 | −0.0029 | 0.0000 | 0.9548 |
| | $\alpha_{11}$ | 0.170 | 0.1757 | 0.1747 | [0.1322; 0.2195] | 0.0872 | −0.0057 | 0.0000 | 0.9582 |
| | $\beta_1$ | 0.810 | 0.8017 | 0.8031 | [0.7539; 0.8474] | 0.0936 | 0.0083 | 0.0001 | 0.9481 |
| | $\lambda_1$ | 0.120 | 0.1209 | 0.1210 | [0.0338; 0.1511] | 0.1173 | −0.0009 | 0.0000 | 0.9502 |
| | $\alpha_{02}$ | 0.010 | 0.0138 | 0.0130 | [0.0056; 0.0226] | 0.0170 | −0.0038 | 0.0000 | 0.9409 |
| | $\alpha_{12}$ | 0.150 | 0.1606 | 0.1582 | [0.1045; 0.2211] | 0.1166 | −0.0106 | 0.0001 | 0.9620 |
| | $\beta_2$ | 0.800 | 0.7980 | 0.7929 | [0.6787; 0.8505] | 0.1718 | 0.0020 | 0.0000 | 0.9492 |
| | $\lambda_2$ | 0.100 | 0.0933 | 0.0928 | [0.0342; 0.1495] | 0.1154 | 0.0067 | 0.0000 | 0.9591 |
| | $\rho$ | 2.000 | 1.9861 | 1.9845 | [1.5515; 2.3128] | 0.7612 | 0.0139 | 0.0002 | 0.9481 |

**Table 2** *Summary descriptive statistics of the daily log returns*

| | Min. | Median | Mean | Max. | S.D. | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| Banco do Brasil | −0.2378 | 0.0000 | 0.0008 | 0.1342 | 0.0322 | −0.2236 | 7.6770 |
| Itau | −0.0909 | 0.0003 | 0.0007 | 0.1036 | 0.0215 | 0.2432 | 4.7957 |


**Figure 1** *Original time series of prices and log-returns.*

## 6 Application to Brazilian stock market data

In this section, our methodology is illustrated on real Brazilian stock market data, specifically the stock price of Banco do Brasil and Itau, where the prices of the option will be compared with the results of the methodology proposed by Stulz (1982) presented in Chapter 2. The data is from 03/07/2014 to 03/22/2017, containing 754 daily observations. Data was collected on the Google Finance website. Table 2 presents the descriptive statistics of log-return data, where it is given by $x_{it} = \log(S_{it}/S_{it-1}) = \log(S_{it}) - \log(S_{it-1})$ for $t = 1, \dots, n$ and $i = 1, 2$.

As expected, the mean returns of the two stocks are close to zero, means are close to medians, and the returns have kurtosis greater than 3. The skewness presents a different result for the series, where the Banco do Brasil obtained left (negative) asymmetry and the Itau right asymmetry (positive). Figure 1 show the behavior of the original series and the log-returns, respectively. Similar variability becomes apparent, as indicated by the standard deviation (S.D.) in the descriptive

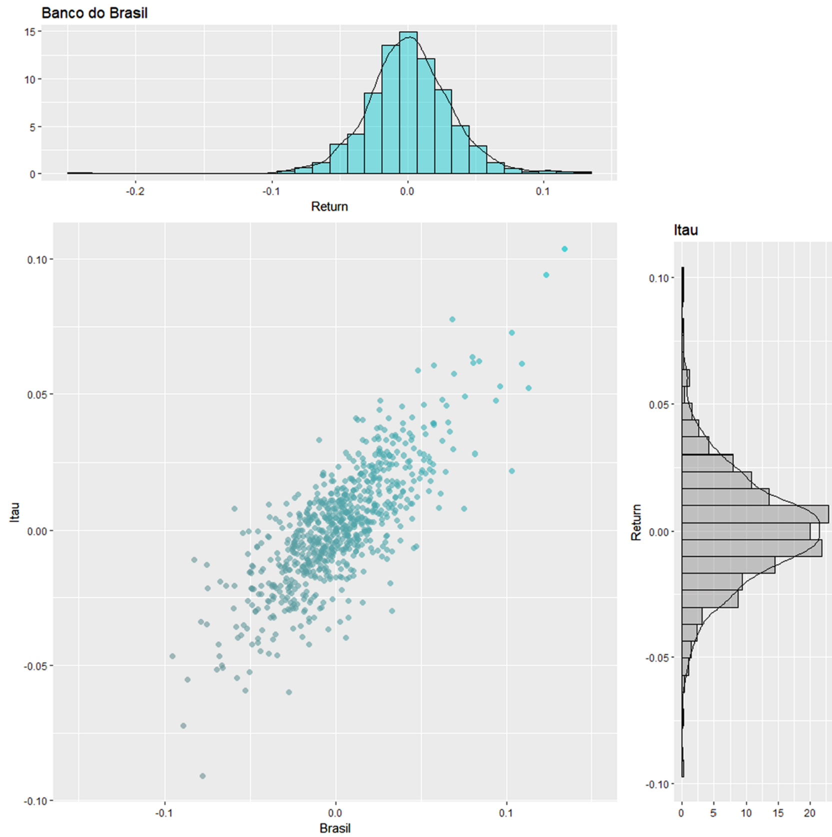


**Figure 2** *Histograms and scatterplot of the log-returns.*

statistics table. This result is expected, given that the two companies are from the same sector industry.

The scatterplot and histograms provide us a visual analysis of the log-returns dispersions and are shown in Figure 2. Concerning the joint dispersion of the log-returns, we observed the greatest agglomeration around the point of origin $(0, 0)$ and a smaller concentration, but not insignificant, in the tails, which is corroborated by the histograms.

Prior distributions for marginals and joint distributions were equal to ones specified in the simulation study. We considered two chains of 100 000 iterations and the first 40,000 were ignored to avoid the influence of first value, i.e., as burn-in. The resulting samples are checked for absence of convergence using the test and the graphics analysis proposed by Geweke (1992).

Table 3 shows the values of the Geweke's statistic for each parameter obtained for marginals. Using statistical convergence diagnostics, we cannot prove convergence, but these provide evidence for no lack of convergence, since, if the samples are drawn from the stationary distribution of the chain, the Geweke's statistic has

**Table 3** *Values of the Geweke's statistic for each parameter obtained*

| | $\alpha_{01}$ | $\alpha_{11}$ | $\beta_1$ | $\lambda_1$ | $\alpha_{02}$ | $\alpha_{12}$ | $\beta_2$ | $\lambda_2$ |
|---|---|---|---|---|---|---|---|---|
| Chain 1 | −0.2564 | 1.0583 | −0.4843 | 1.3618 | −0.7262 | −1.4010 | 1.2620 | −0.6024 |
| Chain 2 | −1.3027 | −0.3860 | 1.5931 | 0.7212 | 1.3068 | 0.8917 | −1.1425 | −0.9654 |

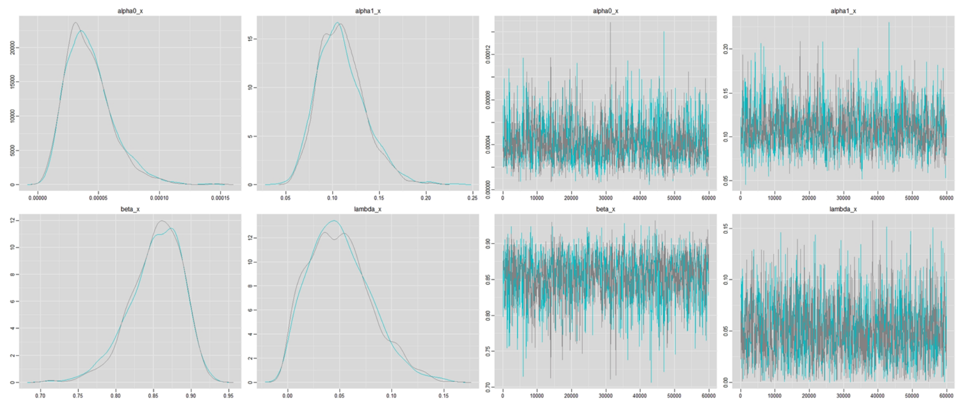


**Figure 3** *Densities and convergence diagrams of the posterior samples of each parameter for the Banco do Brasil.*

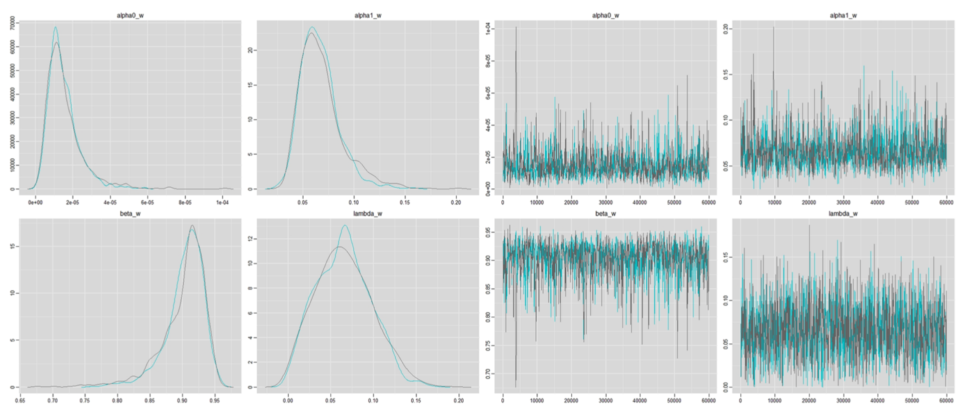


**Figure 4** *Densities and convergence diagrams of the posterior samples of each parameter for the Itau.*

an asymptotically standard normal distribution. Also, Figure 3 and Figure 4 shows the traces of the posterior samples of each model parameter. These indicate a good mixing performance of the Markov chain as it moves fluidly through all possible states.

**Table 4** *Parameter estimation results*

| Parameter | Posterior mean | S.D. | HPD 95% |
|---|---|---|---|
| $\alpha_{01}$ | 0.00004 | 0.00001 | [0.00001; 0.00008] |
| $\alpha_{11}$ | 0.10940 | 0.02495 | [0.06025; 0.15737] |
| $\beta_1$ | 0.85390 | 0.03123 | [0.77720; 0.91061] |
| $\lambda_1$ | 0.05035 | 0.02944 | [0.00018; 0.10332] |
| $\alpha_{02}$ | 0.00001 | 0.00000 | [0.00000; 0.00003] |
| $\alpha_{12}$ | 0.06698 | 0.02022 | [0.03591; 0.10247] |
| $\beta_2$ | 0.90230 | 0.03278 | [0.84367; 0.95212] |
| $\lambda_2$ | 0.06529 | 0.03260 | [0.00871; 0.12427] |

**Table 5** *Kolmogorov–Smirnov, Shapiro–Wilk and Ljung–Box test*

| Test | Banco do Brasil | Itau |
|---|---|---|
| KS statistic ($p$-value) | 0.0325 ($p$-value = 0.4553) | 0.0428 ($p$-value = 0.2792) |
| Shapiro statistic ($p$-value) | 0.9464 ($p$-value = 0.3193) | 0.8639 ($p$-value = 0.1613) |
| LB statistic ($p$-value) | 2.7249 ($p$-value = 0.1249) | 0.0791 ($p$-value = 0.7502) |

Table 4 presents the posterior means together with their 95% HPD credibility intervals for the marginals process and their standard deviation *(s.d)*.

The DGARCH model assumes that the residues follow a standard normal distribution and that they have independent increments. Table 5 shows the results of KS, Shapiro–Wilk and LB test for the significance level of 5%. As we can see, we do not reject the null hypothesis that the residues follow a normal distribution and have independent increments.

Figure 5 shows the normal quantile plot for the standardized residuals of the fitted DGARCH(1, 1) model for each series. In this particular case, the Gaussian assumption is not perfect, but acceptable for this paper.

Therefore, we conclude that there was a good fit of the DGARCH(1, 1) model for both series and, thus, we can follow in the joint modeling through the copulas theory. Table 6 presents the posterior means (mean), *s.d*, 95% HPD credibility intervals and their corresponding EAIC, EBIC, DIC and LPS criteria for copulas fitted.

Table 6 shows that the best copula according to the selection criteria was Frank, followed by $t$-student, Normal, Gumbel and Joe copulas. The Geweke criterion for Frank's copula obtained the values of 0.2482 and 0.6438 for the first and second chain, respectively, showing that there is no evidence of non-convergence. Figure 6 shows the density and convergence diagram of its parameter.

For the sake of space, we omit here the tables and figures with the same results considering the marginal processes but changing the copula structure, which we obtained the same satisfactory results presented for the Frank copula.

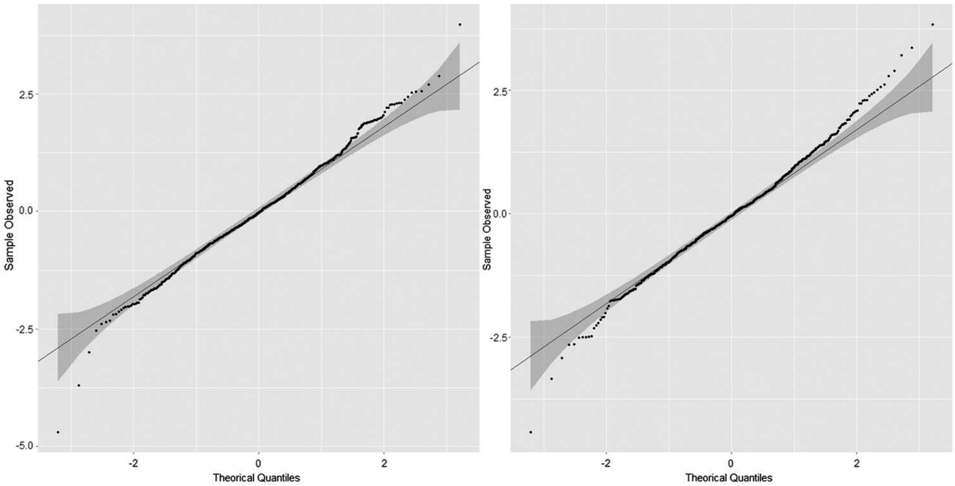


**Figure 5** *QQ-plot for the standardized residuals—Banco do Brasil (left) and Itau (right).*

**Table 6** *Copulas fitted*

| Copula | Mean | S.D. | HPD 95% | EAIC | EBIC | DIC | LPS |
|---|---|---|---|---|---|---|---|
| Normal | 0.7596 | 0.0123 | [0.7354; 0.7843] | 16,839.35 | 16,838.51 | 16,836.48 | 1.4525 |
| $t$-student ($\nu$) | 0.7724 | 0.0146 | [0.7435; 0.7996] | 16,672.18 | 16,676.8 | 16,670.18 | 1.4380 |
| | [4.9530] | [0.5216] | [3.9518; 5.9857] | | | | |
| Gumbel | 2.1183 | 0.0661 | [2.0729; 2.1387] | 17,928.41 | 17,937.65 | 17,926.3 | 1.5466 |
| Frank | 7.2524 | 0.3146 | [6.6251; 7.8433] | 16,652.83 | 16,657.45 | 16,651.82 | 1.4365 |
| Joe | 2.2792 | 0.0796 | [2.0294; 2.5283] | 19,743.26 | 19,742.41 | 19,740.39 | 1.6937 |

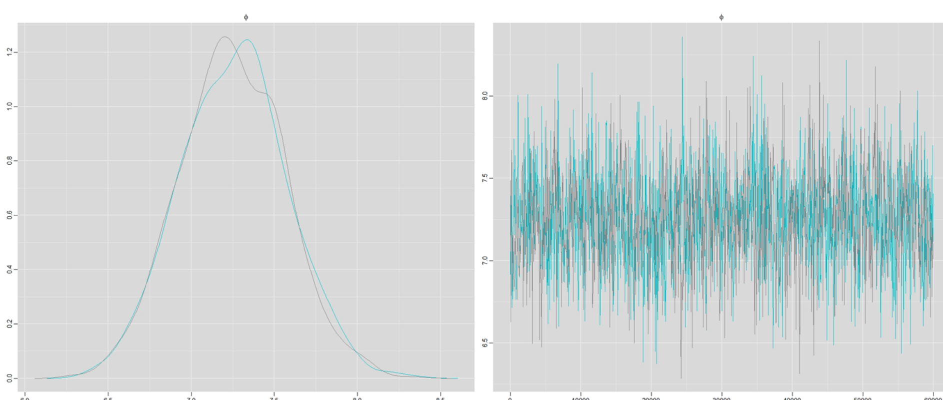

**Figure 6** *Density and convergence diagram of the posterior samples of Frank copula.*

**Table 7** *Classification of moneyness*

| Classification | Call option | Put option |
|---|---|---|
| ITM | $\text{Min}(S_1, S_2) > \text{Strike}$ | $\text{Max}(S_1, S_2) < \text{Strike}$ |
| ATM | $\text{Max}(S_1, S_2) = \text{Strike}$ | $\text{Max}(S_1, S_2) = \text{Strike}$ |
| OTM | $\text{Max}(S_1, S_2) < \text{Strike}$ | $\text{Min}(S_1, S_2) > \text{Strike}$ |

### 6.1 Fixed parameters used in black and Scholes models

To make a comparison of the results of the methodology discussed in this work with the classic Stulz model, we will define some necessary parameters presented in Chapter 2. The parameters, their interpretations, and their values are given below.

1. *Interest Rate*: An annualized rate expresses the annual interest rate takes into account the effect of compound interest. That is, the average daily interest rate, annualized based on 252 traded days. The value 7% per year was chosen in an attempt to standardize the rate over the maturity period according to the SELIC rate presented by Central Bank of Brazil.
2. $S_i$: Stock price of asset $i$, $i = 1, 2$. And we observe $S_1 = R\$33.05$ and $S_2 = R\$38.05$.
3. $T$: Time of maturity. That is, 1/2 means half a year. We adopt one year.
4. $\sigma_i$: The annualized volatility of the stock $i$. Volatility is the annualized expression of the average variability of the stock return. As the returns were calculated on a daily basis, to obtain volatility, the standard deviation obtained by multiplying the squared root of the annual term used, which in this work is 252 days, should be annualized. We calculate and obtained $\sigma_1 = 43.44\%$ and $\sigma_2 = 30.19\%$.
5. $\rho$: The coefficient of linear correlation between the returns of the two assets in the last year. We calculate $\rho = 0.7374$.
6. $K$: The strike price of the option. The chosen had as a criterion the use of ATM (moneyness) defined below.

Moneyness is the difference between the strike price and the asset value and is classified into three categories: in-the-money (ITM), at-the-money (ATM) and out-the-money (OTM). The more out-the-money the option is, the less likely it is to exercise on the part of the holder and consequently the more in-the-money, the more likely it is to exercise. Let $S_1$ be the market price of asset 1 and $S_2$ the market price of asset 2, Table 7 shows which classification will be used from now on.

As we are interested in calculating a call option, we have that an option will be ATM when striking (K) $= R\$38.05$. This extrapolation of the concepts of moneyness to the bivariate case aims to analyze the effect of its classification on the final prices of the options.

**Table 8** *Option pricing call-on-max (R\$) with $K = R\$38.05$ and $T = 1$ year*

| Model | Option price |
|---|---|
| Stulz model | R\$ 5.755644 |
| Normal copula | R\$ 5.563650 |
| *t* copula | R\$ 5.593955 |
| Gumbel copula | R\$ 5.666805 |
| Frank copula | R\$ 5.600540 |
| Joe copula | R\$ 5.875227 |

### 6.2 Comparison of methodologies

The purchase and sale of multivariate options are traded over the counter, that is, from individual to individual. Moreover, for this reason, there is no series in which we can check their prices for comparison of fit of models concerning their errors. However, the comparison of models with different assumptions is and can be performed, to efficiently price an option with realistic characteristics.

In this paper, we will perform the fitted of several models and compare them, mainly in relation to the models from Stulz (1982), where the latter is the most widely used, widespread and with more credibility in the literature and the market for call-on-max option, because it is a derivation of the model of Black and Scholes (Zhang and Guegan, 2008). Table 8 shows the price of the options considering the call-on-max option defined in Chapter 2 and the parameters presented previously, considering all the copulas and the Stulz model. Moreover, 100,000 Monte Carlo simulations were performed to obtain the fair price of the option in Equation (3.7).

Two strong arguments to give credibility to the results obtained by the copulas are: (1) the marginals process and copulas obtained good joint fitted of the series, and (2) the dependencies derived from these models take into account the non-linear dependence between the observations, which is inherent in the universe of finance. Therefore, the difference obtained between these models and Stulz model brings with it these two arguments that make modeling more realistic.

To analyze the effect of strike price, Table 9 shows the values of the call-on-max option for all models varying strike from R\$ 31.00 to R\$ 42.00. We verified the same behavior in all the models, that is when we increase the strike price the value of the option decreases. This result was expected because, according to the logic of the options contract, if we expect to buy an option for a higher price on the maturity date, the price of its premium tends to be lower (Chiou and Tsay, 2008). Figure 7 shows the same results in graphic form.

Note that Joe copula obtained the highest values when we varied the variable strike (K). The result is justified because this copula obtained the worst fitted according to the criteria of selection of models used. Besides, the classical Stulz model also obtained discrepant results from the others, a finding that can be based on the constant volatility over time of the option and the modeling through the

**Table 9** *Option prices call on max (R$) varying strike*

| Strike | Stulz | Normal | *t* | Gumbel | Frank | Joe |
|---|---|---|---|---|---|---|
| 31 | 9.77469 | 10.98429 | 11.02635 | 11.03995 | 11.01208 | 11.36508 |
| 32 | 9.10702 | 10.10753 | 10.14828 | 10.16287 | 10.13688 | 10.48244 |
| 33 | 8.47138 | 9.25741 | 9.29584 | 9.31168 | 9.28815 | 9.62346 |
| 34 | 7.86822 | 8.43997 | 8.47662 | 8.49324 | 8.47240 | 8.79502 |
| 35 | 7.29764 | 7.66072 | 7.69669 | 7.71314 | 7.69550 | 8.00356 |
| 36 | 6.75950 | 6.92642 | 6.96056 | 6.97811 | 6.96281 | 7.25351 |
| 37 | 6.25337 | 6.24172 | 6.27292 | 6.29216 | 6.27855 | 6.54970 |
| 38 | 5.77858 | 5.60880 | 5.63776 | 5.65781 | 5.64567 | 5.89643 |
| 39 | 5.33430 | 5.03038 | 5.05693 | 5.07829 | 5.06646 | 5.29689 |
| 40 | 4.91953 | 4.50697 | 4.53031 | 4.55341 | 4.54263 | 4.75068 |
| 41 | 4.53313 | 4.03815 | 4.05868 | 4.08241 | 4.07271 | 4.25778 |
| 42 | 4.17390 | 3.62216 | 3.63895 | 3.66288 | 3.65373 | 3.81750 |

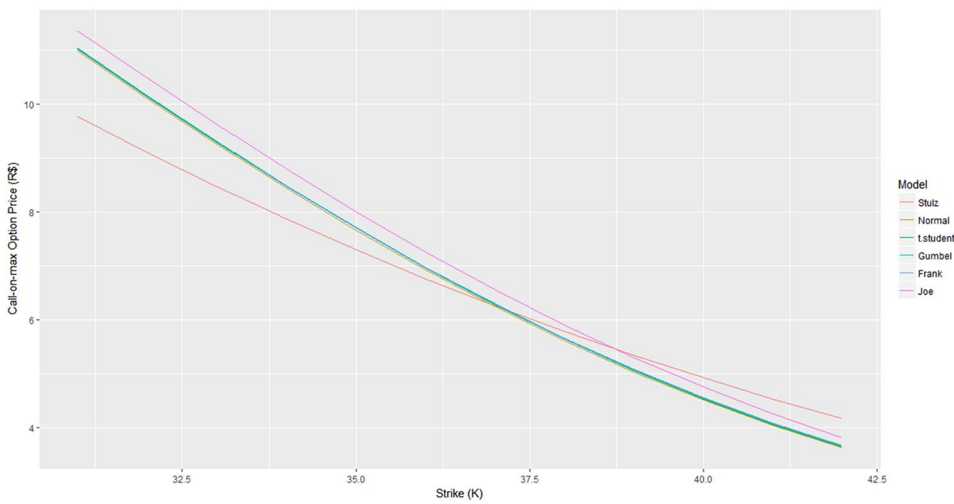


**Figure 7** *Option prices call on max (R$) varying strike.*

normal bivariate distribution. The normal, $t$-student, Gumbel and Frank copulas obtained results very close to each other, an expected result because these distributions obtained very close results in EBIC, EAIC, DIC and LPS metrics.

It should be noted that at the beginning of the graph, when the strike price is at R$ 31.00, there is the most significant difference between the models and that, with the increasing strike, this difference becomes smaller. The finds of Hull and White (1987), Johnson and David (1987) and Kang and Brorsen (1993) corroborate this result, where the authors empirically demonstrated that, in the in-the-money options, the Black and Scholes (BS) model (Stulz model is derived from BS Model)

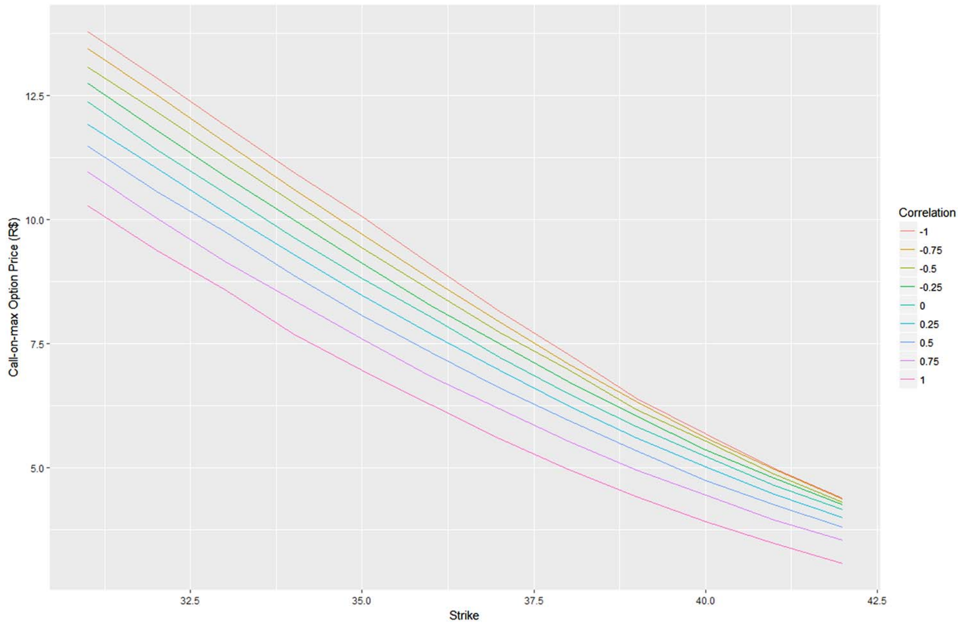


**Figure 8** *Option prices call on max (R$) vs. correlation.*

underestimates the options. This result highlights the importance of the joint distribution to capture the dependency structure in the pricing process.

To analyze the impact of the dependency parameter on the final price of the option we adopted the following criterion. The choice of copula for this analysis was based on the good fitted shown in Table 6 and on the selection of a copula that presents negative and positive dependence. So, the analysis of this subsection is based on the $t$-student copula with 4.9530 degrees of freedom. Besides, authors such as Zhang and Guegan (2008) and Lopes and Pessanha (2018) have found empirical results that $t$-student copula has a good fit and functional characteristics in the joint modeling of stock returns.

Figure 8 emphasized that when the dependence is negative, the values are higher than with positive dependence. This result corroborates with those found by Chiou and Tsay (2008) for the call-on-max option using the American and Taiwanese indices. An intuitive interpretation is that the values of this option tend to be smaller when the underlying assets move in the same direction as when in opposite directions.

Furthermore, this figure represents the importance of copula selection to represent the joint structure and especially the importance of a good inferential approach, where a high discrepancy between the values is observed, varying the correlation coefficient of the $t$-student copula, which is between $-1$ and 1.

## 7 Final remarks

In this paper, we proposed to accommodate the heteroskedasticity of the assets-objects through the marginal model Duan GARCH and to capture the structure of dependency between them through copulas functions. To compare and analyze the method proposed in this work with Stulz's already consolidated model, we price the call-on-max option for two Brazilian companies stock prices.

As a result, there was evidence that the DGARCH(1, 1) model fitted well to the data of the Banco do Brasil and Itau stocks prices, as we did not reject the normality of residues using the KS and SW tests and its increments were not autocorrelated through the Ljung-box test. These results are prerequisites for transforming the data into uniform distribution to adjust the copula functions.

Besides, we verified the good fit of the copula functions, especially Frank and $t$-student, and these two copulas obtained the values closest to each other. Another result shown in this paper, which corroborates with the options literature, is the effect of a strike at the fair price. Besides, we illustrate the impact of choosing the dependency structure on the final options prices.

We can point out the following contributions of this paper. It is possible to use the same tooling described in Section 3 to obtain the fair price for various payoff functions, this is not verified in the case of extensions of the Black and Scholes model, as presented in Haug (2007), because for each option one formula is required. The heteroscedastic model approach and the capture of dependence via copulas bring more realistic support for the modeling of financial assets and consequently more credibility. Due to the good marginal and joint fit, in addition to the values obtained concerning the Stulz consolidated model, there are arguments to believe that the differences obtained between the best models through the copulas and the extension of the classical method are improvements in the calculation of fair value. It is an empirical study providing evidence and corroborating the use of techniques that consider the modeling of non-normality in financial markets, especially considering this approach in emerging markets.

Finally, a vast field of research in this area is emphasized, as follows. The use of non-parametric copulas, copulas with dependency parameter varying in time, other processes for $h_t$, trivariate and/or multivariate case using Vine copulas, other tests of suitability of marginal and joint fitted, comparison of different sectors stocks, other assets-objects, the use of the predictive density to calculate the option price and so on.

## Appendix section

### Normal copula

The normal copula or commonly known as Gaussian copula receives this name because it comes from the normal density function for $d \geq 2$. A bivariate Normal copula is expressed by

$$C(u,v)=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{t_1^2-2\rho t_1t_2+t_2^2}{2(1-\rho^2)}\right)dt_1^2\,dt_2^2,$$

which $x_1=\Phi^{-1}(u)$, $x_2=\Phi^{-1}(v)$, where $\Phi(\cdot)$ denotes the cumulative function of the $N(0,1)$ and $-1\leq\rho\leq 1$.

This type of copula has no dependence on the tails of the distributions and is symmetric.

### *t*-Student copula

A $t$-student copula coincides with the bivariate $t$-student's distribution function, where its form is expressed by

$$C(u,v)=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{t_1^2-2\rho t_1t_2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2}dt_1\,dt_2,$$

which $\nu$ represents the degrees of freedom of *t-student*. As in the case of normal copula, the marginal copula *t-student* bivariate coincide with the *t-student* standard, being $x_1=t_\nu^{-1}(u)$ e $x_2=t_\nu^{-1}(v)$.

This type of copula does not have independence in the tails, which favors its use in extreme events, such as, for example, unplanned oscillations in the stock market. However, given the symmetry of the function, the degree of dependence on the upper tail is equal to the lower tail.

### Gumbel copula

The Gumbel copula is characterized by the dependence only on the upper tail and is represented by

$$C(u,v)=\exp(-[(-\ln(u))^\rho+(-\ln(v))^\rho]^{1/\rho}),$$

which $\rho\in[1,\infty]$. When $\rho\longrightarrow\infty$ dependence is perfectly positive and independent when $\rho=1$.

### Frank copula

The form of a Frank copula is expressed through

$$C(u,v)=-\frac{1}{\rho}\ln\left(1+\frac{[\exp(-\rho u)-1][\exp(-\rho v)-1]}{\exp(-\rho)-1}\right),$$

which $\rho\neq 0$. When $\rho\longrightarrow\infty$ we have perfect positive dependence and we have the case of independence when we $\rho\longrightarrow 0$. This copula has the same dependence on both function tails, such as elliptic copulas.

## Joe copula

Copula Joe is expressed by

$$C(u,v) = 1 - \left([1-u]^{\rho} + [1-v]^{\rho} - [1-u]^{\rho}[1-v]^{\rho}\right)^{1/\rho},$$

which $1 \leq \rho \leq \infty$. When $\rho = 1$ we have the case of independence and the case of perfect positive dependence when $\rho \longrightarrow \infty$.

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