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The cone percolation model on Galton–Watson and on spherically symmetric trees

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Abstract. We study a rumor model from a percolation theory and branching process point of view. It is defined according to the following rules: (1) at time zero, only the root (a fixed vertex of the tree) is declared informed, (2) at time $n+1$, an ignorant vertex gets the information if it is, at a graph distance, at most R_v of some its ancestral vertex v, previously informed. We present relevant lower and upper bounds for the probability of that event, according to the distribution of the random variables that defines the radius of influence of each individual. We work with (homogeneous and non-homogeneous) Galton–Watson branching trees and spherically symmetric trees which includes homogeneous and k-periodic trees. We also present bounds for the expected size of the connected component in the subcritical case for homogeneous trees and homogeneous Galton–Watson branching trees.

1 Introduction and basic definitions

Lebensztayn and Rodriguez (2008), introduced a disk percolation model on general graphs where a reaction chain starting from the origin of the graph, based on independent copies of a geometric random variables, may lead to the existence of a giant component, a connected set o vertices of a given graph representing a positive fraction of the entire graph's set of vertices.

This line of research was continued by Junior *et al.* (2011) and (2014), focusing on $\mathbb{N}$ and on the homogeneous tree respectively, studying a family of dependent long range (not necessarily homogeneous) percolation model. They studied the criticality of each model, presenting sufficient conditions under which the processes reach a giant component with positive probability. Besides they presented bounds for the probability of having a giant component based on the radius of influence of each vertex of $\mathbb{N}$.

Gallo *et al.* (2014) computed precisely the probability of having a giant component for the homogeneous version of one of the models proposed in Junior *et al.* (2011), and obtained information about the distribution of the range of the cluster of the origin when it is finite. Besides that, they obtained a law of large numbers and a central limit theorem for the proportion of the cluster of the origin in a range of size n as n diverges. The key step of of proofs presented in Gallo *et al.* (2014) is to show that, in each model, the vertices belonging to the cluster of the origin can be related to a suitably chosen discrete renewal process. Related results have been obtained recently by Bertacchi and Zucca (2013). All these research papers are, to a different degree, stimulated by the seminal work of Benjamini and Schram (1996) when they proposed the study of percolation theory beyond the nearest neighbor independent setup on $\mathbb{Z}^d$.

A graph $\mathbb{T}$ is a *tree* if for any pair of its vertices there is one and only one self-avoiding path (a subset of edges) connecting them. In this paper, we study a process where the *radii of influence*, to be assigned to each vertex v of a tree $\mathbb{T}$, are given by independent copies of

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R, a non-negative integer random variable. To make formulas neater, we define $p_k = \mathbb{P}(R = k)$ for $k = 0, 1, \dots$ besides that we assume $p_0 \in (0, 1)$, avoiding trivialities.

We define the *Cone Percolation Model* on $\mathbb{T}$ according to the following rules: (1) at time zero, only the root (a fixed vertex of $\mathbb{T}$) is declared informed, (2) at time $n+1$, an ignorant vertex gets the information if it is, at a graph distance, at most R_v of some of its ancestral vertex v, previously informed. Informed vertices remain informed forever. Here we focus on Galton–Watson, homogeneous, periodic and spherically symmetric trees.

By $|A|$ we denote the cardinality of A. The *degree* of a vertex is the cardinality of its set of neighbors. For two vertices u, v let $d(u, v)$, be the distance between u and v, that is the number of edges the shortest path from u to v has.

Consider a tree $\mathbb{T}$ and its set of vertices $\mathcal{V}(\mathbb{T})$. Single out one vertex from $\mathcal{V}(\mathbb{T})$ and call this $\mathcal{O}$, the origin of $\mathcal{V}(\mathbb{T})$. For each two vertices $u, v \in \mathcal{V}(\mathbb{T})$, we say that $u \le v$ if u belongs to the path connecting $\mathcal{O}$ to v.

For a tree $\mathbb{T}$ and $n \ge 1$ we define

$$T^u := \{v \in \mathcal{V} : u \le v\}, \qquad T_n^u := \{v \in T^u : d(u, v) \le n)\}$$

and

$$M_n(u) := |\partial T_n^u| := |\{v \in T^u : d(v, \mathcal{O}) = d(u, \mathcal{O}) + n\}|. \tag{1.1}$$

Definition 1.1 (The Cone Percolation Model on $\mathbb{T}$). Let $\{R_v\}_{\{v \in \mathcal{V}(\mathbb{T})\}}$ and R be a set of independent and identically distributed random variables. Furthermore, for each $u \in \mathcal{V}(\mathbb{T})$, we define the random sets

$$B_u = \{v \in \mathcal{V}(\mathbb{T}) : v \ge u \text{ and } d(u, v) \le R_u\}. \tag{1.2}$$

With these sets, we define the *Cone Percolation Model* on $\mathbb{T}$, the non-decreasing sequence of random sets $C_0 \subset C_1 \subset \cdots$ defined as $C_0 = \{\mathcal{O}\}$ and inductively $C_{n+1} = \bigcup_{u \in C_n} B_u$ for all $n \ge 0$.

Definition 1.2 (The Cone Percolation Model survival). Consider $C = \bigcup_{n \ge 0} C_n$ be the connected component of the origin of $\mathbb{T}$. Under the rumor process interpretation, C is the set of vertices which heard the rumor. We say that the process *survives* if $|C| = \infty$, referring to the surviving event as V.

As in Junior *et al.* (2014), we say that the process *survives* if the number of vertices belonging to the cluster of the origin is infinite. Otherwise we say the process *dies out*. They obtain results on homogeneous trees concerning whether the process has positive probability of involving an infinite set of individuals and present relevant lower and upper bounds for the probability of that event, according to the distribution of the random variables that defines the radius of influence of each individual. Bertacchi and Zucca (2013) study this process on $\mathbb{Z}$ and homogeneous Galton–Watson branching trees.

Our main contribution here is the extension of some results from (2013) and (2014) to a class of non homogeneous trees (including Galton–Watson branching trees). We provide bounds to the probability of existence of a giant component for Cone Percolation Model on non-homogeneous trees. Besides that, we provide bounds to the expected size of clusters in the subcritical case for Cone Percolation Model on homogeneous trees and homogeneous Galton–Watson branching trees.

The main tool to study survival condition is to define a sequence of branching processes where each one of them is dominated by the *Cone Percolation Model*. In order to obtain upper and lower bounds for the survival probability, besides information about the phase transition parameter, we define proper dominant branching processes (upper and lower). The domination is obtained by controlling the number of vertices that gets informed in each step of the process evolution.

The paper is organized as follows. Sections 2, 3, 4 and 5 present the main results and specific setups and distributions for the Cone Percolation Model on homogeneous trees, periodic trees, spherically symmetric trees and Galton–Watson trees, respectively. Section 6 brings the proofs for the main results presented along Sections 2, 3, 4 and 5 together with auxiliary lemmas and useful inequalities.

2 Homogeneous trees

In Junior *et al.* (2014) the authors study phase transition properties for the *Cone Percolation Model* on $\mathbb{T}_d$. Besides that, they present sharp lower and upper bounds for the probability of the existence of a giant component, according to the distribution of the random variables defining the radius of influence of each individual. Next, we present results for the expected size of the component containing the origin of $\mathbb{T}_d$.

Definition 2.1 (Rooted homogeneous trees). We say that a tree, $\mathbb{T}_d$, is *homogeneous*, if each one of its vertices has degree $d+1$. From $\mathbb{T}_d$, we define $\mathbb{T}_d^+$, a *rooted homogeneous tree*. Pick a $u \in \mathcal{V}(\mathbb{T}_d)$ such that $d(\mathcal{O}, u) = 1$, consider

$$\mathbb{T}_d^+(u) = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v\}$$

and finally

$$\mathbb{T}_d^+ := \mathbb{T}_d \backslash \mathbb{T}_d^+(u).$$

Consider $\mathbb{P}_+$ and $\mathbb{P}$ the probability measures associated to the processes on $\mathbb{T}_d^+$ and $\mathbb{T}_d$ (R is dropped from notations unless it can cause confusion). By a coupling argument one can see that for a fixed distribution of R

$$\mathbb{P}_+(V) \leq \mathbb{P}(V). \tag{2.1}$$

Furthermore, by the definition of $\mathbb{T}_d^+$ and its relation with $\mathbb{T}_d$ we have that for a fixed distribution of R

$$\mathbb{P}_+(V) = 0 \quad \text{if and only if} \quad \mathbb{P}(V) = 0. \tag{2.2}$$

Computing the distribution of the random set C, the connected component of the origin seems to be very difficult. Even computing the expectation of C is a hard task in many cases. That is why next result is a useful one as it presents bounds for it.

Theorem 2.2. *Consider a Cone Percolation Model on $\mathbb{T}_d$. Then, for R and d such that $\mathbb{E}(d^R) < 2 - \frac{1}{d}$, we have*

$$\frac{\mathbb{E}(d^R) + d - p_0}{d[1 - \mathbb{E}(d^R) + p_0]} \leq \mathbb{E}(|C|) \leq \frac{\mathbb{E}(d^R) + d - 2}{d[2 - 1/d - \mathbb{E}(d^R)]}.$$

From Theorem 2.2, one can see that $\mathbb{E}(d^R) < 2 - \frac{1}{d}$ is a sufficient condition for subcritical behaviour meaning no giant components almost surely. Next, we present a few examples in order to highlight the usefulness and the fact that these bounds can be sharp in interesting cases.

Example 2.3. Consider $R \sim \mathcal{B}(p)$, a radius of influence satisfying

$$\mathbb{P}(R = 1) = p = 1 - \mathbb{P}(R = 0),$$

with $pd < 1$. Then, as the lower and upper bounds are equal, we have

$$\mathbb{E}(|C|) = \frac{1+p}{1-dp}.$$

Example 2.4. Consider $R \sim \mathcal{G}(1-p)$, a radius of influence satisfying

$$\mathbb{P}(R = k) = (1-p)p^k, \quad k = 0, 1, 2, \ldots$$

and assume also $pd < \frac{1}{2}$. So we have

$$\frac{1 - dp + p - p^2}{1 - 2dp + dp^2} \le \mathbb{E}(|C|) \le \frac{1 - dp + p}{1 - 2dp}.$$

That gives us a fairly sharp bound even when we pick p and d such that pd is very close to $\frac{1}{2}$ as, for example, $p = 10^{-6}$ and $d = 499{,}000$. For these parameters we get $250.438 \le \mathbb{E}(|C|) \le 250.501$.

Example 2.5. For $R \sim \mathcal{P}(\lambda)$, a radius of influence satisfying

$$\mathbb{P}(R = k) = \frac{\exp(-\lambda)\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

and $\lambda < \ln(\sqrt[d-1]{2 - \frac{1}{d}})$, we have

$$\frac{e^{(d-1)\lambda} + d - e^{-\lambda}}{d[1 - e^{(d-1)\lambda} + e^{-\lambda}]} \le \mathbb{E}(|C|) \le \frac{e^{(d-1)\lambda} + d - 2}{d[2 - 1/d - e^{(d-1)\lambda}]}.$$

In particular, if $d = 1000$ and $\lambda = 6 \times 10^{-4}$, we find $5.61 \le \mathbb{E}(|C|) \le 5.62$.

3 Periodic trees

Periodic trees, defined next, are simple examples of non homogeneous trees.

Definition 3.1 (k-periodic trees). We define a *k-periodic tree* with degree $\tilde{d} = (d_1, \ldots, d_k)$, $d_i \ge 2$ for all $i = 1, 2, \ldots, k$, as a tree such that any of its vertices, whose distance to the origin is $nk + i - 1$ for some $n \in \mathbb{N}$, has degree $d_i + 1$. We refer to this tree as $\mathbb{T}_{\tilde{d}}$.

In this section, we consider the cone percolation model on periodic trees. It is clear how to obtain lower and upper bounds for the survival probability for a fixed periodic tree and a fixed distribution for R, the radius of influence, by considering what is shown in (2014). Next, we present significantly sharper results.

In order to understand the results of this section, we need to define the following quantities

$$d_{(i)} = \text{the } i\text{th smallest value in } \tilde{d},$$

$$G = G(\tilde{d}) := \sqrt[k]{\prod_{j=1}^{k} d_j},$$

$$c_0 := 1 \quad \text{and} \quad c_i := \frac{\prod_{j=1}^{i} d_{(j)}}{\sqrt[k]{\prod_{j=1}^{k} (d_j)^i}} = \frac{\prod_{j=1}^{i} d_{(j)}}{G^i}, \quad i = 1, \ldots, k-1;$$

$$\bar{c_0} := 1 \quad \text{and} \quad \bar{c_i} := \frac{\prod_{j=k+1-i}^{k} d_{(j)}}{\sqrt[k]{\prod_{j=1}^{k}(d_j)^i}} = \frac{\prod_{j=k+1-i}^{k} d_{(j)}}{G^i}, \quad i = 1, \dots, k-1.$$

Definition 3.2. For $i = 1, \dots, k$ and R, the radius of influence, we define

$$I_i(R) = \begin{cases} 1 & \text{if } R = nk + i \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Besides, we define

$$\underline{x}_{n,i} := \left(\prod_{j=1}^{k} d_j\right)^n \prod_{j=1}^{i} d_{(j)} \quad \text{for } i \neq 0, \qquad \underline{x}_{n,0} := \left(\prod_{j=1}^{k} d_j\right)^n \quad \text{and} \quad \underline{x}_{-1,i} := 0,$$

$$\bar{x}_{n,i} := \left(\prod_{j=1}^{k} d_j\right)^n \prod_{j=1}^{i} d_{(k+1-j)} \quad \text{for } i \neq 0 \quad \text{and} \quad \bar{x}_{n,0} := \left(\prod_{j=1}^{k} d_j\right)^n.$$

and

$$h_i(R) = \left[\sum_{m=0}^{\lfloor \frac{R-i}{k} \rfloor - 1} \sum_{j=0}^{k-1} (\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1} (\underline{x}_{\lfloor \frac{R-i}{k} \rfloor, j})^{-1}\right] G^R.$$

Analogously to definition 2.1, we consider the *Cone Percolation Model* on $\mathbb{T}^+_{\tilde{d}}$. Relations analogous to (2.1) and (2.2) also holds between $\mathbb{T}_{\tilde{d}}$ and $\mathbb{T}^+_{\tilde{d}}$.

Theorem 3.3. *Consider the Cone Percolation Model on $\mathbb{T}^+_{\tilde{d}}$ with radius of influence R*

(I) *If*

$$\sum_{i=0}^{k-1} c_i \mathbb{E}\big(G^R I_i(R)\big) > 1 + p_0$$

then, $\mathbb{P}_+(V) > 0$,

(II) *If*

$$\sum_{i=0}^{k-1} \bar{c_i} \mathbb{E}\big(h_i(R) I_i(R)\big) \le 1$$

then, $\mathbb{P}_+(V) = 0$.

We point out that Theorem 2.1 of (2014) and part of Theorem 5.2 of (2013) applied to an homogeneous trees may also be seen as a corollary of Theorem 3.3 as follows.

Corollary 3.4. *Consider the Cone Percolation Model on $\mathbb{T}^+_d$ (the d-dimensional rooted homogeneous tree) with radius of influence R*

(I) *If* $(1 - p_0)d > 1$, *then* $\mathbb{P}_+(V) > 0$,
(II) *If* $(1 - p_0)d \le 1$ *and* $\mathbb{E}(d^R) > 1 + p_0$, *then* $\mathbb{P}_+(V) > 0$,
(III) *If* $\mathbb{E}(d^R) \le 2 - \frac{1}{d}$, *then* $\mathbb{P}_+(V) = 0$.

Let ρ and ψ be, respectively, the smallest non-negative root of the equations

$$\sum_{i=0}^{k-1} \mathbb{E}\big(\rho^{c_i G^R} I_i(R)\big) + (1-\rho)p_0 = \rho, \tag{3.1}$$

$$\sum_{i=0}^{k-1} \mathbb{E}\big(\psi^{\lfloor \bar{c}_i h_i(R) \rfloor} I_i(R)\big) = \psi, \tag{3.2}$$

Theorem 3.5. *Consider the Cone Percolation Model on $\mathbb{T}^{+}_{\tilde{d}}$. Then*

$$1-\rho \le \mathbb{P}_{+}(V) \le 1-\psi.$$

Theorem 3.6. *For the Cone Percolation Model on $\mathbb{T}_{\tilde{d}}$ with radius of influence R, it holds that*

$$1-\sum_{i=0}^{k-1} \mathbb{E}\big(\rho^{M_R(\mathcal{O})} I_i(R)\big) \le \mathbb{P}(V) \le 1-\sum_{i=0}^{k-1} \mathbb{E}\big(\psi^{|T_R^{\mathcal{O}}|} I_i(R)\big).$$

We observe that Theorem 2.3 of (2014) may also be presented as the next corollary of Theorem 3.6.

Corollary 3.7. *For the Cone Percolation Model on $\mathbb{T}_d$ (the d-dimensional homogeneous tree) with radius of influence R, it holds that*

$$1-\big(1-\rho^{\frac{d+1}{d}}\big)p_0 - \mathbb{E}\big(\rho^{\frac{(d+1)}{d} d^R}\big) \le \mathbb{P}(V) \le 1-\mathbb{E}\big(\psi^{\frac{(d+1)}{d-1}(d^R-1)}\big),$$

where ρ and ψ are the smallest non-negative root of the equations (3.1) and (3.2), respectively.

Example 3.8. Consider a *Cone Percolation Model* in $\mathbb{T}_{\tilde{d}}$, $\tilde{d} = (4, 9)$ assuming $R \sim \mathcal{G}(1-p)$. From Theorem 3.3 and observation (2.2)

$$0.078542 \le \inf\{p : \mathbb{P}(V) > 0\} \le 0.097374.$$

Example 3.9. Consider a *Cone Percolation Model* in $\mathbb{T}_{\tilde{d}}$, with $\tilde{d} = (12, 15, 16)$. Assuming $R \sim \mathcal{B}(3, 0.1)$, from Theorem 3.6, we have

$$0.266557 \le \mathbb{P}(V) \le 0.266894.$$

4 Spherically symmetric trees

Periodic trees are a subclass of spherically symmetric trees and therefore the results of this section will also apply to periodic trees. In the previous section, we obtained stronger results using particular properties of periodic trees.

Definition 4.1. We say that a tree, $\mathbb{T}_S$, is *spherically symmetric*, if any pair of vertices at the same distance from the origin, have the same degree.

From Definition 1.1, we consider the *Cone Percolation Model* on $\mathbb{T}_S$.

Definition 4.2. Let us define for an infinite, locally finite tree $\mathbb{T}$

$$\dim\inf \partial\mathbb{T} := \lim_{n\to\infty} \min_{v\in\mathcal{V}} \frac{1}{n} \ln M_n(v).$$

The above limit is easily seen to exist by a standard superadditive argument (Fekete's Lemma). More details on this can be found in Lyons and Peres (2016).

Observe that

$$\dim\inf \partial\mathbb{T}_d = \ln d.$$

Theorem 4.3. *For a Cone Percolation Model in $\mathbb{T}_S$ and R, the radius of influence, $\mathbb{P}(V) > 0$ if*

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-\dim\inf \partial\mathbb{T}_S},$$

where

$$\rho_n := \prod_{k=0}^{n-1}\left[1 - \prod_{i=0}^{k} \mathbb{P}(R < i+1)\right].$$

Lemma 6.7 shows that ρ_n is as a lower bound of the probability that the process starting from any vertex v reaches the vertices at ∂T_n^v,

Corollary 4.4. *For a Cone Percolation Model in $\mathbb{T}_S$ and R, a radius of influence satisfying $\mathbb{P}(R \le k) = 1$ for some $k \in \mathbb{N}$, $\mathbb{P}(V) > 0$ if*

$$\dim\inf \partial\mathbb{T}_S > \ln\left[\frac{1}{1 - \prod_{j=1}^{k} \mathbb{P}(R < j)}\right].$$

Corollary 4.5. *For a Cone Percolation Model in $\mathbb{T}_S$ and R, a radius of influence satisfying*

$$\mathbb{P}(R = k) = \frac{Z_\alpha}{(k+1)^\alpha}, \quad k = 1, 2, \dots$$

if $\dim\inf \partial\mathbb{T}_S > 0$, *then* $\mathbb{P}(V) > 0$.

Example 4.6. Consider a *Cone Percolation Model* in $\mathbb{T}_S$ with $R \sim \mathcal{B}(p)$.

- If $\dim\inf \partial\mathbb{T}_S > -\ln p$, then $\mathbb{P}(V) > 0$,
- If $\mathbb{T}_S = \mathbb{T}_{\tilde{d}}$ and $G(\tilde{d}) > \frac{1}{p}$, then $\mathbb{P}(V) > 0$.

5 Galton–Watson branching trees

Bertacchi and Zucca (2013) present a model that encompass the cone percolation model on homogeneous Galton Watson trees. Here we study the cone percolation model on nonhomogeneous trees giving a sufficient condition for the existence of a giant component. Besides that we revisit the Bertacchi and Zucca (2013) framework presenting lower and upper bounds for the expected size of the connected component of the origin of Galton–Watson branching trees.

5.1 Nonhomogeneous Galton–Watson branching trees

Consider a supercritical Galton–Watson branching process starting from a single progenitor such that each individual whose distance from the progenitor is n has a random number of offspring (independent of everything else) with generating function $f_n(s) = \sum_{k=0}^{\infty} q_n(k)s^k$.

Let us define $F = \{(f_n, d_n)\}_{n\in\mathbb{N}}$ where $d_n = f_n'(1) \in (0, \infty)$. This Galton–Watson branching process yields a random family tree $\mathbb{T}_F$. We are particularly interested in a supercritical Galton–Watson tree, on the event of non extinction (infinite trees). Sufficient condition for that are provided in Bertacchi et al. ().

Definition 5.1. For a supercritical Galton-Watson tree on $\mathbb{T}_F$, let us define

$$D(\mathbb{T}_F) := \lim_{n\to\infty} \min_{i\in\mathbf{N}} \frac{1}{n} \ln\left[\prod_{l=i}^{i+n-1} d_l\right].$$

In particular, if F is a sequence of generating functions of degenerated random variables $\{X_n\}_{n\geq 0}$ such that $X_n = a_n$ we have that $\mathbb{T}_F$ equals to a spherically symmetric tree $\mathbb{T}_S$ with probability 1. Then, with probability 1

$$D(\mathbb{T}_F) = \dim\inf \partial\mathbb{T}_S.$$

Theorem 5.2. *For a Cone Percolation Model on $\mathbb{T}_F$ with a radius of influence R, $\mathbb{P}(V) > 0$ if*

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-D(\mathbb{T}_F)},$$

where

$$\rho_n := \prod_{k=0}^{n-1}\left[1 - \prod_{i=0}^{k} \mathbb{P}(R < i+1)\right].$$

5.2 Homogeneous Galton–Watson branching trees

Consider a supercritical Galton–Watson branching process starting from a single progenitor such that each individual has a random number of offspring (independent of everything else) whose average is $d > 1$. This process yields a random infinite family tree, known as a supercritical Galton–Watson tree $\mathbb{T}_F$, where $d_n = d$ for all $n \in \mathbb{N}$, on the event of non extinction. Our next result brings a contribution (item (IV)) to a former result presented in Bertacchi and Zucca (2013), as we present lower and upper bounds to the expected size of the cluster of the origin. For items (I) to (III) we present a proof alternative to theirs.

Theorem 5.3. *Consider the Cone Percolation Model on a homogeneous supercritical Galton–Watson branching tree with radius of influence R.*

(I) *If $(1-p_0)d > 1$, then $\mathbb{P}[V] > 0$,*
(II) *If $(1-p_0)d \leq 1$ and $\mathbb{E}(d^R) > 1 + p_0$, then $\mathbb{P}[V] > 0$,*
(III) *If $\mathbb{E}(d^R) \leq 2 - \frac{1}{d}$, then $\mathbb{P}[V] = 0$.*
(IV) *For $\mathbb{E}(d^R) < 2 - \frac{1}{d}$ we have*

$$\frac{1}{1 - \mathbb{E}(d^R) + p_0} \leq \mathbb{E}(|C|) \leq \frac{d-1}{2d - 1 - d\mathbb{E}(d^R)}.$$

6 Proofs

6.1 Homogeneous trees

Proof of Theorem 2.2. Let us define now two auxiliary branching process. For the first, $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$, each individual has a number of offspring distributed as the random variable X, assuming values in $\{0, d, d^2, \dots\}$ such that

$$\mathbb{P}[X = 0] = p_o, \qquad \mathbb{P}[X = d] = p_1, \dots, \mathbb{P}[X = d^k] = p_k \quad \text{for all } k = 1, 2, \dots$$

In the second auxiliary process, $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, each individual has a number of offspring distributed as the random variable Y, assuming values in $\{0, d, d+d^2, \dots, \sum_{i=1}^{k} d^i, \dots\}$ such that

$$\mathbb{P}[Y=0]=p_o, \qquad \mathbb{P}[Y=d]=p_1, \dots, \mathbb{P}\left[Y=\sum_{i=1}^{k} d^i\right]=p_k \quad \text{for all } k=1,2,\dots$$

These two processes provide convenient minorization ($\{\mathcal{X}_n\}_{n\in\mathbb{N}}$) and majorization ($\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$) for the process on $\mathbb{T}_d^+$. Suppose that $R_v = r$ for a fixed site v. Then the set of vertices activated by v is T_r^v, whose cardinality is $\sum_{i=1}^{r} d^i$ vertices. The activation process will go on. The process $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ will only count on those d^r vertices which are at distance r from v (the set ∂T_r^v). On the other hand, the process $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ counts activation that will be made by all of them (T_r^v), in addition to disregarding the fact that some vertex will experience multiple activations from sites belonging to T_r^v.

For these processes the average number of offspring are respectively, $\mu_X = \mathbb{E}(d^R) - p_0$ and $\mu_Y = \frac{d}{d-1}[\mathbb{E}(d^R) - 1]$. As $\mu_X < 1$ and $\mu_Y < 1$ by hypothesis, the expected values for the total number of individuals are respectively

$$\frac{1}{1-\mu_X} = \frac{1}{1+p_0-\mathbb{E}(d^R)}$$

and

$$\frac{1}{1-\mu_Y} = \frac{d-1}{2d-1-d\mathbb{E}(d^R)}.$$

Using the fact that the root has degree $d+1$ we can modify the processes $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ such that the offspring distributions for the first generation are respectively,

$$\begin{aligned}\mathbb{P}[X=0]&=p_0,\\ \mathbb{P}\big[X=(d+1)d^{k-1}\big]&=p_k \quad \text{for } k=1,2,\dots\end{aligned}$$

and

$$\mathbb{P}\left[Y=\frac{(d+1)(d^k-1)}{d-1}\right]=p_k \quad \text{for } k=0,1,2,\dots$$

For these modified processes the total expected number of individuals are respectively,

$$\mathbb{E}(|C_x|) = \sum_{k=1}^{\infty}\left(\frac{(d+1)d^{k-1}}{1+p_0-\mathbb{E}(d^R)}+1\right)p_k + p_0 = \frac{d+\mathbb{E}(d^R)-p_0}{d(1-\mathbb{E}(d^R)+p_0)}$$

and

$$\begin{aligned}\mathbb{E}(|C_y|) &= \sum_{k=0}^{\infty}\left(\left[\frac{(d+1)(d^k-1)}{(d-1)}\right]\left[\frac{(d-1)}{2d-1-d\mathbb{E}(d^R)}\right]+1\right)p_k\\ &= \frac{\mathbb{E}(d^R)+d-2}{2d-1-d\mathbb{E}(d^R)}.\end{aligned}$$

Since the arguments that justified the lower and upper bounds at the beginning of the proof are valid with this modification for the process on $\mathbb{T}_d$, we have that $\mathbb{E}(|C_x|) \le \mathbb{E}(|C|) \le \mathbb{E}(|C_y|)$ and the result follows. □

6.2 Periodic trees

Consider a k-periodic tree whose degrees are $d_1+1, d_2+1, \ldots, d_k+1$. Let $\mathbb{K}=\{1,2,\ldots,k\}$. For $i=1,\ldots,k-1$ let

$$J_i = \left\{(j_1,\ldots,j_i)\in\mathbb{K}^i : j_m \neq j_n \text{ for } m\neq n\right\}.$$

Let us define for $n\in\mathbb{N}$

$$A_{nk} = \left(\prod_{j=1}^{k} d_j\right)^n,$$

$$A_{nk+i} = \left\{\left(\prod_{j=1}^{k} d_j\right)^n \prod_{l=1}^{i} d_{j_l}, (j_1,\ldots,j_i)\in J_i\right\} \quad \text{for } i=1,\ldots,k-1.$$

We claim that for all $n\in\mathbb{N}$, $k\in\mathbb{N}$ and $v\neq\mathcal{O}$ that

$$\min_{J_i} A_{nk+i} = \underline{x}_{n,i}, \tag{6.1}$$

$$\max_{J_i} A_{nk+i} = \bar{x}_{n,i}, \tag{6.2}$$

$$M_{nk+i}(v)\in A_{nk+i}. \tag{6.3}$$

Let

$$y_{n,i} := \sum_{m=0}^{n-1}\sum_{j=0}^{k-1}(\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1}(\underline{x}_{m,j})^{-1}$$

Lemma 6.1. *Consider a k-periodic tree whose degrees are $d_1+1, d_2+1, \ldots, d_k+1$, $d_i\geq 2$ for all $i=1,2,\ldots k$. Consider a vertex $v\neq\mathcal{O}$. Then*

$$|T^v_{nk+i}| \leq \lfloor y_{n,i}\cdot\bar{x}_{n,i}\rfloor.$$

Proof of Lemma 6.1. Consider first the following set up: $d(\mathcal{O},v)=mk$ for $m\in\mathbb{N}$ and $\mathbb{T}_{\tilde{d}}$ such that $d_i=d_{(i)}$ ($\{d_i\}$ is non-decreasing) for all $i=1,\ldots,k$. Then

$$|T^v_k| = \left|\bar{x}_{1,0} + \frac{\bar{x}_{1,0}}{d_k} + \frac{\bar{x}_{1,0}}{d_k d_{k-1}} + \cdots + \frac{\bar{x}_{1,0}}{\prod_{j=2}^k d_j}\right|$$

and for $n\in\mathbb{N}$

$$\begin{aligned}|T^v_{nk}| &= \left|\bar{x}_{n,0} + \frac{\bar{x}_{n,0}}{d_k} + \frac{\bar{x}_{n,0}}{d_k d_{k-1}} + . + \frac{\bar{x}_{n,0}}{\prod_{j=1}^k d_j}\right| \\ &\quad + \left|\frac{\bar{x}_{n,0}}{(\prod_{j=1}^k d_j)d_k} + \cdots + \frac{\bar{x}_{n,0}}{(\prod_{j=1}^k d_j)^2}\right| + \cdots \\ &\quad + \left|\frac{\bar{x}_{n,0}}{(\prod_{j=1}^k d_j)^{n-1}d_k} + \cdots + \frac{\bar{x}_{n,0}}{(\prod_{j=1}^k d_j)^{n-1}\prod_{j=2}^k d_j}\right|.\end{aligned}$$

Observe now that on any tree, for any v_r such that $d(\mathcal{O},v_r)=r$

$$|T^{v_r}_m| = \sum_{j=1}^{m} M_j(v_r) = M_m(v_r) + \sum_{j=1}^{m-1} M_m(v_r).\left[\prod_{i=1}^{j} M_1(v_{r+m-i})\right]^{-1}.$$

From (6.1), (6.2) and (6.3), for $n \in \mathbb{N}$ and $i = 1, \dots, k-1$, it follows that (without the restriction, $d_i = d_{(i)}$ for all $i = 1, \dots, k$)

$$\begin{aligned}
|T^v_{nk+i}| &\le \Bigg\lfloor \bar{x}_{n,i} + \frac{\bar{x}_{n,i}}{d_{(1)}} + \frac{\bar{x}_{n,i}}{d_{(1)}d_{(2)}} + \cdots + \frac{\bar{x}_{n,i}}{\prod_{j=1}^{k} d_{(j)}} + \cdots \\
&\quad + \frac{\bar{x}_{n,i}}{(\prod_{j=1}^{k-1} d_{(j)})^n d_{(1)}} + \cdots + \frac{\bar{x}_{n,i}}{(\prod_{j=1}^{k} d_{(j)})^n \prod_{j=1}^{i} d_{(j)}} \Bigg\rfloor \\
&= \lfloor y_{n,i} \cdot \bar{x}_{n,i} \rfloor.
\end{aligned}$$

□

Let us define two auxiliary branching process, being the first one $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. This process is represented by a random variable X, assuming values in $\{\underline{x}_{n,i}, i = 0, \dots, k-1, \text{ and } n = 0, 1, \dots, (n,i) \neq (0,0)\} \cup \{0\}$ such that

$$\begin{aligned}
&\mathbb{P}[X = 0] =: p_0, \\
&\mathbb{P}[X = \underline{x}_{n,i}] =: p_{nk+i} \quad \text{for } i = 0, \dots, k-1, \text{ and } n = 0, 1, \dots, (n,i) \neq (0,0)
\end{aligned}$$

Its expected value is given by the following lemma.

Lemma 6.2.

$$\mathbb{E}[X] = \sum_{i=0}^{k-1} c_i \mathbb{E}\big[G^R I_i(R)\big] - p_0$$

Proof of Lemma 6.2.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{\infty} \underline{x}_{n,0} p_{nk} + \sum_{i=1}^{k-1}\sum_{n=0}^{\infty} \underline{x}_{n,i} p_{nk+i} \\
&= \sum_{i=0}^{k-1} c_i \sum_{n=0}^{\infty} \prod_{j=1}^{k} (\sqrt[k]{d_j})^{nk+i} p_{nk+i} - p_0 \\
&= \mathbb{E}\big[G^R I_0(R)\big] + \sum_{i=1}^{k-1} c_i \mathbb{E}\big[G^R I_i(R)\big] - p_0 \\
&= \sum_{i=0}^{k-1} c_i \mathbb{E}\big[G^R I_i(R)\big] - p_0.
\end{aligned}$$

□

The following lemma gives an explicit expression for the generating function of X.

Lemma 6.3.

$$\varphi_X(s) = \sum_{i=0}^{k-1} \mathbb{E}\big[s^{c_i G^R} I_i(R)\big] + (1-s)p_0.$$

Proof of Lemma 6.3.

$$\begin{aligned}
\varphi_X(s) &= p_0 + \sum_{n=1}^{\infty} s^{\underline{x}_{n,0}} p_{nk} + \sum_{i=1}^{k-1}\sum_{n=0}^{\infty} s^{\underline{x}_{n,i}} p_{nk+i} \\
&= p_0 + \sum_{n=1}^{\infty} s^{G^{nk}} p_{nk} + \sum_{i=1}^{k-1}\sum_{n=0}^{\infty} s^{c_i G^{nk+i}} p_{nk+i}
\end{aligned}$$

$$= p_0 - sp_0 + \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} s^{c_i G^{nk+i}} p_{nk+i}$$

$$= (1-s)p_0 + \sum_{i=0}^{k-1} \mathbb{E}\big[s^{c_i G^R} I_i(R)\big].$$

□

The second auxiliary process is $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$, a branching process represented by a random variable Y, assuming values on $\{\lfloor y_{n,i}\bar{x}_{n,i}\rfloor, i=0,\dots,k-1,$ and $n=0,1,\dots\}$ such that

$$\mathbb{P}\big[Y = \lfloor y_{n,i}\bar{x}_{n,i}\rfloor\big] = p_{nk+i} \quad \text{for } i=0,1,\dots,k-1 \text{ and } n=0,1,\dots$$

Its expected value satisfies the following lemma.

Lemma 6.4.

$$\mathbb{E}[Y] \le \sum_{i=0}^{k-1} \bar{c}_i \mathbb{E}\big[h_i(R)I_i(R)\big].$$

Proof of Lemma 6.4.

$$\mathbb{E}(Y) \le \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} y_{n,i}\bar{x}_{n,i}p_{nk+i}$$

$$= \sum_{i=0}^{k-1}\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n-1}\sum_{j=0}^{k-1}(\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1}(\underline{x}_{m,j})^{-1}\right]\bar{x}_{n,i}p_{nk+i}$$

$$= \sum_{i=0}^{k-1}\bar{c}_i\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n-1}\sum_{j=0}^{k-1}(\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1}(\underline{x}_{m,j})^{-1}\right]\prod_{j=1}^{k}\big(\sqrt[k]{d_j}\big)^{nk+i}p_{nk+i}$$

$$= \sum_{i=0}^{k-1}\bar{c}_i\mathbb{E}\big[h_i(R)I_i(R)\big].$$

□

The following lemma gives an explicit expression for the generating function of Y.

Lemma 6.5.

$$\varphi_Y(s) = \sum_{i=0}^{k-1}\mathbb{E}\big[s^{\lfloor\bar{c}_i h_i(R)\rfloor}I_i(R)\big]. \tag{6.4}$$

Proof of Lemma 6.5.

$$\varphi_Y(s) = \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} s^{\lfloor y_{n,i}\bar{x}_{n,i}\rfloor}p_{nk+i}$$

$$= \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} s^{\lfloor y_{n,i}G^{nk+i}\bar{c}_i\rfloor}p_{nk+i}$$

$$= \sum_{i=0}^{k-1}\mathbb{E}\big[s^{\lfloor\bar{c}_i h_i(R)\rfloor}I_i(R)\big].$$

□

Proof of Theorem 3.3. By a coupling argument one can see that our process dominates (by (6.1) and (6.3)) $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. This process survives as long as $\mathbb{E}[X] > 1$. Therefore from Lemma

6.2 our process survives if

$$\sum_{i=0}^{k-1} c_i \mathbb{E}[G^R I_i(R)] > 1 + \mathbb{P}(R=0),$$

proving (I).

On the other hand, also by a coupling argument, our process is dominated (by (6.2) and (6.3)) by $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. That process dies out provided $\mathbb{E}[Y] \le 1$ therefore from Lemma 6.4 our process dies out if

$$\sum_{i=0}^{k-1} \bar{c}_i \mathbb{E}(h_i(R) I_i(R)) \le 1,$$

proving (II). □

Proof of Theorem 3.5. In order to find the extinction probability of $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ (Grimmett and Stirzaker 2001, p. 173), let us consider the smallest non-negative root of the equation $\rho = \varphi_X(\rho)$. Therefore from Lemma 6.3

$$\sum_{i=0}^{k-1} \mathbb{E}[\rho^{c_i G^R} I_i(R)] + (1-\rho) p_0 = \rho$$

and by construction of the processes, as $\mathbb{P}_+[V^c] \le \rho$, we have that

$$1 - \rho \le \mathbb{P}_+(V).$$

In order to find the extinction probability of $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ (Grimmett and Stirzaker 2001, p. 173), let us consider the smallest non-negative root of the equation $\psi = \varphi_Y(\psi)$. Therefore from Lemma 6.5

$$\sum_{i=0}^{k-1} \mathbb{E}[\psi^{\lfloor \bar{c}_i h_i(R) \rfloor} I_i(R)]) = \psi$$

and by the construction of the processes, as $\mathbb{P}_+[V^c] \ge \psi$, we have that

$$\mathbb{P}_+(V) \le 1 - \psi. \qquad \square$$

Proof of Theorem 3.6. Observe that assuming $R_\mathcal{O} = nk + i$, the probability for the process to survive is greater or equal than the probability of the process to survive from at least one of the $M_{nk+i}(\mathcal{O})$ trees that have as root the furthest infected vertices. Now note that, still assuming $R_\mathcal{O} = nk + i$, the probability for the process to survive on $\mathbb{T}_{\tilde{d}}$ is smaller or equal than the probability for the process to survive from at least one of the $|T^\mathcal{O}_{nk+i}|$ vertices which are in the radius of influence ($R_\mathcal{O}$) of the origin of the tree as if each one had its own tree. Then

$$\mathbb{P}(V | R_\mathcal{O} = nk + i) \ge 1 - (1 - \mathbb{P}_+(V))^{M_{nk+i}(\mathcal{O})} \ge 1 - \rho^{M_{nk+i}(\mathcal{O})}$$

and

$$\mathbb{P}(V | R_\mathcal{O} = nk + i) \le 1 - (1 - \mathbb{P}_+(V))^{|T^\mathcal{O}_{nk+i}|} \le 1 - \psi^{|T^\mathcal{O}_{nk+i}|}.$$

Then,

$$\mathbb{P}(V) = \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \mathbb{P}(V | R_\mathcal{O} = nk + i) p_{nk+i}$$

$$\geq \sum_{i=0}^{k-1}\sum_{n=0}^{\infty}[1-\rho^{M_{nk+i}(\mathcal{O})}]p_{nk+i}$$

$$=1-\sum_{i=0}^{k-1}\mathbb{E}[\rho^{M_R(\mathcal{O})}I_i(R)]$$

and

$$\mathbb{P}(V)=\sum_{i=0}^{k-1}\sum_{n=0}^{\infty}\mathbb{P}(V|R_{\mathcal{O}}=nk+i)\mathbb{P}(R_{\mathcal{O}}=nk+i)$$

$$\leq \sum_{i=0}^{k-1}\sum_{n=0}^{\infty}[1-\psi^{|T_{nk+i}^{\mathcal{O}}|}]p_{nk+i}$$

$$=1-\sum_{i=0}^{k-1}\mathbb{E}\big(\psi^{|T_R^{\mathcal{O}}|}I_i(R)\big).$$

□

6.3 Spherically symmetric trees

Suppose we have a set of independent random variables $\{R_v\}_{\{v\in\mathcal{V}(\mathbb{T}_S)\}}$ distributed as R. Assume $\mathbb{P}(R=0)<1$.

For $u\leq v\in\mathcal{V}(\mathbb{T}_S)$, consider the event

$V_{u,v}$: *The process starting from u reaches v.*

For a fixed integer n, let $X_0^n=\{\mathcal{O}\}$. Besides, for $j=1,2,\dots$ consider

$$X_j^n=\bigcup_{u\in X_{j-1}^n}\{v\in\partial T_n^u : V_{u,v}\ occurs\}.$$

Again, for all $j=1,2,\dots$ consider

$$Z_j^n=|X_j^n|.$$

So, for any fixed positive integer n, $\{Z_j^n\}_{j\geq 0}$ is a branching process dominated by the number of vertices $v\in\partial T_{jn}^{\mathcal{O}}$ which are activated.

Lemma 6.6. *Assume n fixed. For μ_j, the mean number of offspring of one individual of generation j for the process $\{Z_j^n\}_{j\geq 0}$, it holds that*

$$\mu_j:=\mu_j^n=M_n(u)\rho_j^{(n)},$$

where $\rho_j^{(n)}=\mathbb{P}(V_{u,v})$, for any fixed pair $u\leq v$ such that $d(\mathcal{O},u)=jn$ and $d(\mathcal{O},v)=(j+1)n$.

Proof of Lemma 6.6. For fixed j and n, consider a vertex u such that $d(\mathcal{O},u)=jn$, $\partial T_n^u=\{v_1,v_2,\dots,v_{M_n(u)}\}$. So we can write the number of offspring of u as $\sum_{i=1}^{M_n(u)}I_{\{V_{u,v_i}\}}$. Taking expectation finishes the proof. □

Lemma 6.7. *Assume n fixed and $\rho_j^{(n)}=\mathbb{P}(V_{u,v})$, for any fixed pair $u\leq v$ such that $d(\mathcal{O},u)=jn$ and $d(\mathcal{O},v)=(j+1)n$,*

$$\rho_j^{(n)}\geq\prod_{k=0}^{n-1}\Bigg[1-\prod_{i=0}^{k}\mathbb{P}(R<i+1)\Bigg].$$

Proof of Lemma 6.7. For any fixed pair $u \le v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$ we have that

$$V_{u,v} = \bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^{k} \{R_{u(i)} \ge k+1-i\} \right]$$

where $u(i)$ is the vertex from the path connecting u to v such that $d(\mathcal{O}, u(i)) = jn + i$. From this follows

$$\begin{aligned} \rho_j^{(n)} &= \mathbb{P}\left(\bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^{k} \{R_{u(i)} \ge k+1-i\} \right] \right) \\ &\ge \prod_{k=0}^{n-1} \mathbb{P}\left(\bigcup_{i=0}^{k} \{R_{u(i)} \ge k+1-i\} \right). \end{aligned}$$

The inequality is a consequence of the FKG inequality (N. Alon and J. Spencer 2008, p. 89). □

Proof of Theorem 4.3. Assume that $\dim\inf \partial\mathbb{T}_S > 0$. Then, for all $\alpha \in (0, \dim\inf \partial\mathbb{T}_S)$ there exists $N = N(\alpha)$ such that for all $n \ge N$

$$\min_{v \in \mathcal{V}} \frac{1}{n} \ln M_n(v) > \alpha,$$

where

$$M_n(v) \ge e^{\alpha n} \quad \text{for all } v \in \mathcal{V} \text{ and } n \ge N.$$

From Souza and Biggins (1992, p. 40) a branching process in varying environments is *uniformly supercritical* if there exists constants $a > 0$ and $c > 1$ such that

$$\prod_{k=i}^{j+i-1} \mu_k \ge ac^j \quad \text{for all } i \ge 0 \text{ and } j \ge 0.$$

Observe that the above condition holds if

$$\liminf_{j \to \infty} \mu_j > 1.$$

From Lemma 6.6, we have that for $n \ge N$

$$\liminf_{j \to \infty} \mu_j \ge e^{\alpha n} \rho_n = \left(e^{\alpha} \sqrt[n]{\rho_n}\right)^n.$$

Now note that we can write

$$Z_{j+1} = \sum_{i=1}^{Z_j} Y_{j,i}^n,$$

where $Y_{j,i}^n$ are i.i.d. copies of Y_j^n, being the number of offspring from the ith individual of the jth generation. By considering Lemma 6.6, we have for all j that

$$\frac{Y_j^n}{\mu_j} \le \frac{M_n(u)}{\mu_j} = \frac{1}{\rho_j^{(n)}} \le \left(\mathbb{P}[R > 0]\right)^{-n},$$

where $\rho_j^{(n)} = \mathbb{P}(V_{u,v})$, for any fixed pair $u \le v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

So, from Theorem 1 in Souza and Biggins (1992, p. 40), we conclude that the cone percolation process has a giant component with positive probability if

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-\alpha}.$$

As this hold for every $\alpha \in (0, \dim\inf \partial\mathbb{T}_S)$, the condition

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-\dim\inf \partial\mathbb{T}_S}$$

guarantees the survival of the process with positive probability. □

Proof of Corollary 4.4.

$$\begin{aligned}
&\sqrt[n]{\prod_{i=0}^{n-1}\left[1-\prod_{j=0}^{i}\mathbb{P}(R<j+1)\right]} \\
&\quad = \left[1-\prod_{j=1}^{k}\mathbb{P}(R<j)\right]\sqrt[n]{\frac{\prod_{i=0}^{k-1}[1-\prod_{j=0}^{i}\mathbb{P}(R<j+1)]}{(1-\prod_{j=1}^{k}\mathbb{P}(R<j))^k}} \\
&\quad \to 1-\prod_{j=1}^{k}\mathbb{P}(R<j), \ \text{when } n\to\infty.
\end{aligned}$$

□

Proof of Corollary 4.5. Observe that

$$\begin{aligned}
\rho_n \geq \mathbb{P}(R\geq n) &= \sum_{k=n}^{\infty}\frac{Z_\alpha}{(k+1)^\alpha} \\
&\geq \int_{n+1}^{\infty}\frac{Z_\alpha}{x^\alpha}\,dx = \frac{Z_\alpha}{(\alpha-1)(n+1)^{\alpha-1}}
\end{aligned}$$

The above inequality follows from the integral test.

Now observe that if $\dim\inf \partial\mathbb{T}_S > 0$, we have that

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} \geq \lim_{n\to\infty} \sqrt[n]{\frac{Z_\alpha}{(\alpha-1)}\frac{1}{(n+1)^{\alpha-1}}} = 1 > e^{-\dim\inf \partial\mathbb{T}_S}$$

Theorem 4.3 guarantees the desired result. □

6.4 Galton–Watson branching trees

6.4.1 *Non homogeneous Galton–Watson branching trees.* Suppose we have a set of independent random variables $\{R_{n,m}\}_{\{n,m\in\mathbb{N}\}}$ distributed as R. Assume $\mathbb{P}(R=0)<1$. For each tree $\mathbb{T}_f$ on $\mathbb{T}_F$ we associate each of its existing vertices to a pair $u=(n,m)$ so that $R_{n,m}$ is its radius of influence. In what follows, n stands for the distance from a set of $k(n)$ vertices to the tree progenitor while $m=1,\dots,k(n)$ stands for an enumeration on the set of the existing vertices at level n.

For each tree $\mathbb{T}_f$ on $\mathbb{T}_F$ and $u\leq v\in\mathcal{V}(\mathbb{T}_f)$, consider the event

$$V_{u,v}: \textit{The process starting from } u \textit{ reaches } v.$$

Let

$$\Omega = \left\{(\mathbb{T}_f; \{r_{n,m}\}_{\{n,m\in\mathbb{N}\}}); \mathbb{T}_f\in\mathbb{T}_F; \{r_{n,m}\}_{\{n,m\in\mathbb{N}\}}\in\mathbb{N}^{\mathbb{N}\times\mathbb{N}}\right\}$$

Take $\omega = (\mathbb{T}_f; \{r_{n,m}\}_{\{n,m\in\mathbb{N}\}})$. For a fixed integer n, let $X_0^n(\omega) = \{\mathcal{O}\}$. Besides, for $j = 1, 2, \dots$ consider

$$X_j^n(\omega) = \bigcup_{u\in X_{j-1}^n(\omega)} \{v \in \partial T_n^u(\omega) : I_{V_{u,v}}(\omega) = 1\}.$$

The definition for $\partial T_n^u(\omega)$ is analogous to (1.1). Again, for all $j = 1, 2, \dots$ consider

$$Z_j^n = |X_j^n|.$$

So, for any fixed positive integer n, $\{Z_j^n\}_{j\geq 0}$ is a branching process dominated by the number of vertices $v \in \partial T_{jn}^{\mathcal{O}}$ which are activated.

Lemma 6.8. *Assume n fixed. For μ_j, the mean number of offspring of one individual of generation j for the process $\{Z_j^n\}_{j\geq 0}$, it holds that*

$$\mu_j := \mu_j^n = \left[\prod_{i=jn+1}^{jn+n} d_i\right]\rho_j^{(n)},$$

where $\rho_j^{(n)} = \mathbb{P}(V_{u,v})$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

Proof of Lemma 6.8. For fixed j and n, consider for some u such that $d(\mathcal{O}, u) = jn$, $\partial T_n^u = \{v_1, v_2, \dots, v_{M_n(u)}\}$. So we can write the number of offspring of u as $\sum_{i=1}^{M_n(u)} I_{\{V_{u,v_i}\}}$, where $M_n(u)$ is a random quantity. Note that $\mathbb{E}[M_n(u)] = \prod_{i=jn+1}^{jn+n} d_j$. Taking expectation and using principle of substitution finishes the proof. □

Proof of Theorem 5.2. Assume that $D(\mathbb{T}_F) > 0$. Then, for all $\alpha \in (0, D(\mathbb{T}_F))$ there exists $N = N(\alpha)$ such that for all $n \geq N$

$$\min_{i\in\mathbf{N}} \frac{1}{n} \ln\left[\prod_{j=i+1}^{i+n} d_j\right] > \alpha,$$

where

$$\prod_{j=i+1}^{i+n} d_j \geq e^{\alpha n} \quad \text{for all } i \in \mathbf{N} \text{ and } n \geq N. \tag{6.5}$$

Now we write

$$Z_{j+1} = \sum_{i=1}^{Z_j} Y_{j,i}^n,$$

where $Y_{j,i}^n$ are i.i.d. copies of Y_j^n, being the number of offspring from the ith individual of the jth generation. By considering Lemma 6.6, we have for all j that

$$\mathbb{E}\left[\frac{Y_j^n}{\mu_j}\right] = \frac{1}{\rho_j^{(n)}} \leq \left(\mathbb{P}[R > 0]\right)^{-n},$$

where $\rho_j^{(n)} = \mathbb{P}(V_{u,v})$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

Besides, by (6.5), Lemma 6.7 and Lemma 6.8

$$\text{if } \lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-\alpha} \quad \text{then } \liminf_{j\to\infty} \mu_j > 1.$$

So, from Theorem 1 in Souza and Biggins (1992, p. 40), we conclude that the cone percolation process has a giant component with positive probability if

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-\alpha}.$$

As this holds for every $\alpha \in (0, D(\mathbb{T}_F))$, the condition

$$\lim_{n\to\infty} \sqrt[n]{\rho_n} > e^{-D(\mathbb{T}_F)}$$

guarantees the survival of the process with positive probability. □

6.4.2 *Homogeneous Galton–Watson branching trees*.

Proof of Theorem 5.3. Let us define two auxiliary branching process, being the first one $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. For this process,

$$\mathbb{E}(\mathcal{X}) = \sum_{n=0}^{\infty} \mathbb{P}(R=n)\mathbb{E}(\mathcal{X}|R=n),$$

where

$$\mathbb{E}(\mathcal{X}|R=0) = 0,$$
$$\mathbb{E}(\mathcal{X}|R=n) = d^n, \quad \text{for } n = 1, 2, \dots.$$

Note that

$$\mathbb{E}(\mathcal{X}) = \mathbb{E}[d^R] - p_0. \tag{6.6}$$

The second auxiliary process is $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. For this process

$$\mathbb{E}(\mathcal{Y}) = \sum_{n=0}^{\infty} \mathbb{P}(R=n)\mathbb{E}(\mathcal{X}|R=n),$$

where

$$\mathbb{E}(\mathcal{Y}|R=n) = d + d^2 + \cdots + d^n.$$

Note that

$$\mathbb{E}(\mathcal{Y}) = \frac{d}{d-1}\left(\mathbb{E}[d^R] - 1\right). \tag{6.7}$$

Firstly we can assure (I) by a comparison with a supercritical branching process. In order to prove (II) and the left-hand side one can see that our process dominates $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. This process survives as long as $\mathbb{E}[X] > 1$ therefore from (6.6) our process survives if $\mathbb{E}[d^R] > 1 + p_0$.

Secondly, also by a coupling argument, our process is dominated by $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. That process dies out provided $\mathbb{E}[Y] \le 1$ and $\mathbb{P}[Y=1] \ne 1$, therefore from (6.7) our process dies out if $\mathbb{E}[d^R] \le 2 - \frac{1}{d}$, proving (III).

The proof of (IV) follows from the fact that

$$\frac{1}{1-\mathbb{E}[\mathcal{X}]} \le \mathbb{E}[|C|] \le \frac{1}{1-\mathbb{E}[\mathcal{Y}]}.$$ □

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