

# HITTING TIMES OF INTERACTING DRIFTED BROWNIAN MOTIONS AND THE VERTEX REINFORCED JUMP PROCESS

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Consider a negatively drifted one-dimensional Brownian motion starting at positive initial position, its first hitting time to 0 has the inverse Gaussian law. Moreover, conditionally on this hitting time, the Brownian motion up to that time has the law of a three-dimensional Bessel bridge. In this paper, we give a generalization of this result to a family of Brownian motions with interacting drifts, indexed by the vertices of a conductance network. The hitting times are equal in law to the inverse of a random potential that appears in the analysis of a self-interacting process called the vertex reinforced jump process (*Ann. Probab.* **45** (2017) 3967–3986; *J. Amer. Math. Soc.* **32** (2019) 311–349). These Brownian motions with interacting drifts have remarkable properties with respect to restriction and conditioning, showing hidden Markov properties. This family of processes are closely related to the martingale that plays a crucial role in the analysis of the vertex reinforced jump process and edge reinforced random walk (*J. Amer. Math. Soc.* **32** (2019) 311–349) on infinite graphs.

**1. Introduction.** We first recall some classic facts about hitting times of standard Brownian motion. Let  $(B_t)_{t \geq 0}$  be a standard Brownian motion and

$$X(t) = \theta + B(t),$$

be a Brownian motion starting from initial position  $\theta > 0$ . It is well known that the first hitting time of 0:

$$(1.1) \quad T = \inf\{t \geq 0, X(t) = 0\}$$

has the law of the inverse of a Gamma random variable with parameter (shape, rate) =  $(\frac{1}{2}, \frac{\theta^2}{2})$ . Moreover, conditionally on  $T$ ,  $(X_t)_{0 \leq t \leq T}$  has the law of a three-dimensional Bessel bridge from  $\theta$  to 0 on time interval  $[0, T]$  (see Chapter XI, Section 3 of [15]). More generally, if

$$(1.2) \quad X(t) = \theta + B(t) - \eta t,$$

is a drifted Brownian motion with negative drift  $-\eta < 0$  starting at  $\theta > 0$ , then  $T$  has the inverse Gaussian distribution with parameters  $(\frac{\theta}{\eta}, \theta^2)$ , that is,  $T$  has density

$$f(t) = \frac{\theta}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2}\left(\frac{\theta^2}{t} + \eta^2 t - 2\eta\theta\right)\right) \mathbb{1}_{t>0} dt.$$

Moreover, conditionally on  $T$ ,  $(X_t)_{0 \leq t \leq T}$  has the law of a three-dimensional Bessel bridge from  $\theta$  to 0 on time interval  $[0, T]$ . (See [24], Theorem 3.1, or [15], p. 317, Corollary 4.6, and [14, 23] for complements.)

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This paper aims at giving a generalization of these statements on a conductance network, namely for a family of Brownian motions with interacting drifts indexed by the vertices of the network. The distribution of hitting times of these processes will be given by a multivariate exponential family of distributions introduced by Sabot, Tarrès and Zeng [18], and generalized in [9, 10], which appeared in the context of self-interacting processes and random Schrödinger operators. This family of distributions is also intimately related to the supersymmetric hyperbolic sigma model introduced by Zirnbauer [25] and investigated by Disertori, Spencer and Zirnbauer [6, 7], and plays a crucial role in the analysis of the edge reinforced random walk (ERRW) and the vertex reinforced jump process (VRJP) [5, 17, 19].

The generalization of the one-dimensional statement presented in this Introduction was hinted by the martingales that appear in [19]. This martingale has played an important role in the analysis of the ERRW and the VRJP on infinite graphs. In Section 2.3, we explain the relations between the stochastic differential equations (SDEs) defined in this paper and the VRJP and in Section 9 we relate the martingales that appear in the study of VRJP to the SDEs.

Note that the computations done in this paper seem to have many similarities with computations done for exponential functional of the Brownian motion in dimension one (see, in particular, Matsumoto, Yor [11–13]). More precisely, it would be possible to write an analogue of the Lamperti transformation that changes the SDE  $\mathcal{E}_V^{W, \theta, \eta}(Y)$  presented below in its exponential functional counterpart with  $\mu = \frac{1}{2}$  (see the Matsumoto Yor opposite drift theorem [11]): the counterpart of the representation of Theorem 1 would correspond to a representation of the SDE with a Brownian motion with opposite drifts as in [11]. In fact, in dimension one (i.e., one vertex), the inverse Gaussian distribution corresponds to  $\mu = \frac{1}{2}$ , and the Generalized Inverse Gaussian (GIG) distribution corresponds to general  $\mu \in \mathbb{R}$ ; see [1] and [23]. On a conductance network (i.e., multidimensional), the case  $\mu = \frac{1}{2}$  can be carried out by explicit computation; for general  $\mu$ , one will have to use Bessel K functions as normalizing constant. We plan to develop these aspects in a further work.

It might not be a coincidence that the GIG distribution was initially called generalized hyperbolic distribution, and the distribution we considered here stems from a supersymmetric hyperbolic sigma model, where one considered spin systems with spins taking values on a super hyperbolic space. Interested readers can check [1] and [22] for more details.

Another related direction goes back to Vallois, where GIG is conceived as the exit law of some one-dimensional diffusion. Matrix version of geometric drifted Brownian motions are studied in [16] and such matrix process is shown to have interesting properties which seems related to our model. In [3], Chhaibi explicitly computed the exit law of certain hypoelliptic Brownian motion on a solvable Lie group where, for example, he recovered the Matsumoto Yor opposite drift theorem, by taking the group to be  $\mathfrak{sl}_2$ . It is very likely that there is a connection with our work. Note also that the integral of a geometric Brownian motion is closely related to the study of Asian option. At last, some related open questions are listed in Section 4.5 of [10].

## 2. Statement of the main results.

2.1. *The multivariate generalization of inverse Gaussian law: The random potential associated with the VRJP.* Let  $N$  be a positive integer and  $V = \{1, \dots, N\}$ . Given a symmetric matrix

$$W = (W_{i,j})_{i,j=1,\dots,N}$$

with nonnegative coefficients  $W_{i,j} = W_{j,i} \geq 0$  (in particular, we allow  $W_{i,i} \neq 0$ ). We denote by  $\mathcal{G} = (V, E)$  the associated graph with

$$V = \{1, \dots, N\} \quad \text{and} \quad E = \{\{i, j\}, i \neq j, W_{i,j} > 0\}.$$

We always assume that the matrix  $W$  is irreducible, that is, the graph  $\mathcal{G}$  is connected. If  $(\beta_i)_{i \in V}$  is a vector indexed by the vertices, we set

$$(2.1) \quad H_\beta = 2\beta - W,$$

where  $2\beta$  represents the operator of multiplication by the vector  $(2\beta_i)$  (or equivalently the diagonal matrix with diagonal coefficients  $(2\beta_i)_{i \in V}$ ). We always write  $H_\beta > 0$  to mean that  $H_\beta$  is positive definite. Remark that when  $H_\beta > 0$ , all the entries of  $(H_\beta)^{-1}$  are positive (since  $\mathcal{G}$  is connected and  $H_\beta$  is a M-matrix; see, e.g., [2], Proposition 3).

The following distribution was introduced in [18], and generalized in [9, 10].

LEMMA A. *Let  $(\theta_i)_{i \in V} \in (\mathbb{R}_+^*)^V$  be a positive<sup>1</sup> vector indexed by  $V$ . Let  $(\eta_i)_{i \in V} \in (\mathbb{R}_+)^V$  be a nonnegative vector indexed by  $V$ . The measure*

$$(2.2) \quad \nu_V^{W,\theta,\eta}(d\beta) := \mathbb{1}_{H_\beta > 0} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\left(-\frac{1}{2}\langle \theta, H_\beta \theta \rangle - \frac{1}{2}\langle \eta, H_\beta^{-1} \eta \rangle + \langle \eta, \theta \rangle\right) \\ \times \frac{\prod_{i \in V} \theta_i}{\sqrt{\det H_\beta}} d\beta$$

is a probability distribution on  $\mathbb{R}^V$ , where  $\mathbb{1}_{H_\beta > 0}$  is the indicator function that the operator  $H_\beta$  (defined in (2.1)) is positive definite,  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^V$ , and  $d\beta = \prod_{i \in V} d\beta_i$ . When  $\eta = 0$ , we simply write  $\nu_V^{W,\theta}$  for  $\nu_V^{W,\theta,0}$ .

Moreover, the Laplace transform of (2.2) is explicitly given by

$$(2.3) \quad \int e^{-\langle \lambda, \beta \rangle} \nu_V^{W,\theta,\eta}(d\beta) = e^{-\frac{1}{2}\langle \sqrt{\theta^2 + \lambda}, W \sqrt{\theta^2 + \lambda} \rangle + \frac{1}{2}\langle \theta, W \theta \rangle + \langle \eta, \theta - \sqrt{\theta^2 + \lambda} \rangle} \\ \times \prod_{i \in V} \frac{\theta_i}{\sqrt{\theta_i^2 + \lambda_i}}$$

for all  $(\lambda_i)_{i \in V}$  such that  $\lambda_i + \theta_i^2 > 0, \forall i \in V$ .

REMARK 1. The probability distribution  $\nu_V^{W,\theta,\eta}$  was initially defined in [18] in the case  $\eta = 0$ . In [9, 10], Letac gave a shorter proof of the fact that  $\nu_V^{W,\theta}$  is a probability and remarked that the family can be generalized to the family  $\nu_V^{W,\theta,\eta}$  above. It appears (see the forthcoming Lemma C) that the general family  $\nu_V^{W,\theta,\eta}$  can be obtained from the family  $\nu_V^{W,\theta}$  by taking marginal laws.

REMARK 2. The definition of  $\nu_V^{W,\theta}$  is not strictly the same as  $\nu_V^{W,\theta}$  in [18]. First, compared with the definition of [18], the parameter  $\theta_i$  above corresponds to  $\sqrt{\theta_i}$  in [18]. It is in fact simpler to write the formula as in (2.3) since the quadratic form  $\langle \theta, H_\beta \theta \rangle$  appears naturally in the density and since  $\theta_i$  will play the role of the initial value in the forthcoming SDE. Second, we do not assume here that the diagonal coefficients of  $W$  are zero. It is obvious that the two definitions are equivalent up to a translation of  $\beta_i$  by  $W_{i,i}$ . It will be more convenient here to allow this generality.

NOTATION 1. To simplify notation, in the sequel, for any function  $\zeta : V \mapsto \mathbb{R}$  and any subset  $U \subset V$ , we write  $\zeta_U$  for the restriction of  $\zeta$  to the subset  $U$ . We write  $d\beta_U = \prod_{i \in U} d\beta_i$  to denote integration on variables in  $\beta_U$ . Similarly, if  $A$  is a  $V \times V$  matrix and  $U \subset V$ ,

<sup>1</sup>Our convention is  $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$  and  $\mathbb{R}_+^* = \{x \in \mathbb{R}, x > 0\}$ .

$U' \subset V$ , we write  $A_{U,U'}$  for its restriction to the block  $U \times U'$ . Note also that when  $(\xi_i)_{i \in V}$  is in  $\mathbb{R}^V$ , we sometimes simply write  $\xi$  for the operator of multiplication by  $\xi$  (i.e., the diagonal matrix with diagonal coefficients  $(\xi_i)_{i \in V}$ ), as it is done in formula (2.1). It will be clear from the context and considerations of dimension if it denotes a vector or the operator of multiplication. Finally, we write  $\nu_U^{W,\theta,\eta}$  for  $\nu_U^{W_{U,U},\theta_U,\eta_U}$  when  $U \subset V$  is a subset of  $V$  and  $W$  (resp.,  $\theta, \eta$ ) is a  $V \times V$  matrix (resp., vectors in  $\mathbb{R}^V$ ).

We state the counterpart of Proposition 1 of [18] in the context of the measure  $\nu_V^{W,\theta,\eta}$ .

**COROLLARY B.** *Under the probability distribution  $\nu_V^W(d\beta)$ :*

- (i) *the random variable  $\frac{1}{2\beta_i - W_{i,i}}$  follows an inverse Gaussian distribution with parameters  $(\frac{\theta_i}{\eta_i + \sum_{j \neq i} W_{i,j}\theta_j}, \theta_i^2)$ , for all  $i \in V$ ,*
- (ii) *the random vector  $(\beta_i)$  is 1-dependent, that is, for any subsets  $V_1 \subset V$ , and  $V_2 \subset V$  such that the distance in the graph  $\mathcal{G}$  between  $V_1$  and  $V_2$  is strictly larger than 1, then the random variables  $\beta_{V_1}$  and  $\beta_{V_2}$  are independent.*

The following lemma was proved independently in the 3rd arXiv version of [18] and in [10]. (The result is stated in the case of  $\theta = 1$  in [18], Lemma 4, but it can be easily extended to the case of general  $\theta$ ; see Section 3.)

**LEMMA C.** *Let  $U \subset V$ . Under the probability distribution  $\nu_V^{W,\theta,\eta}(d\beta)$ :*

- (i)  *$\beta_U$  is distributed according to  $\nu_U^{W,\theta,\hat{\eta}}$  (i.e.,  $\nu_U^{W_{U,U},\theta_U,\hat{\eta}}$ , cf. Notation 1) where*
- (2.4) 
$$\hat{\eta} = \eta_U + W_{U,U^c}(\theta_{U^c}).$$
- (ii) *conditionally on  $\beta_U, \beta_{U^c}$  is distributed according to  $\nu_{U^c}^{\tilde{W},\theta,\tilde{\eta}}$  where*

$$\begin{aligned} \tilde{W} &= W_{U^c,U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}W_{U,U^c}, \\ \tilde{\eta} &= \eta_{U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}(\eta_U). \end{aligned}$$

**2.2. Brownian motions with interacting drifts: Main results.** Let  $t^0 = (t_i^0)_{i \in V} \in (\mathbb{R}_+)^V$  be a nonnegative vector. We set

(2.5) 
$$K_{t^0} = \text{Id} - t^0 W,$$

where  $t^0$  denotes the operator of multiplication by  $t^0$  (or equivalently the diagonal matrix with diagonal coefficients  $(t_i^0)$ ). Note that when  $t_i^0 > 0, \forall i \in V$ , we have  $K_{t^0} = t^0(H_{\frac{1}{2t^0}})$ , with notation (2.1) and  $\frac{1}{2t^0} = (\frac{1}{2t_i^0})_{i \in V}$ .

For  $T = (T_i)_{i \in V} \in (\mathbb{R}_+ \cup \{\infty\})^V$  and  $t \in \mathbb{R}_+$ , we write  $t \wedge T$  for the vector  $(t \wedge T_i)_{i \in V}$ , where for reals  $x, y, x \wedge y = \min(x, y)$ .

The following lemma introduces the processes which are the main objects of study of this paper as solution to a SDE.

**LEMMA 1.** *Let  $\theta = (\theta_i)_{i \in V} \in (\mathbb{R}_+^*)^V$  and  $\eta = (\eta_i)_{i \in V} \in (\mathbb{R}_+)^V$  be nonnegative vectors. Denote  $|V| = N$ , let  $(B_i(t))_{i \in V}$  be a standard  $N$ -dimensional Brownian motion.*

(i) *The following stochastic differential equation is well defined for all  $t \geq 0$  and has a unique pathwise solution:*

$$\begin{aligned}
 (\mathcal{E}_V^{W,\theta,\eta}(Y)) \quad Y_i(t) &= \theta_i + \int_0^t \mathbb{1}_{s < T_i} dB_i(s) - \int_0^t \mathbb{1}_{s < T_i} (W\psi(s))_i ds \\
 &\quad \forall i \in V,
 \end{aligned}$$

where  $T = (T_i)_{i \in V}$  is the random vector of stopping times defined by

$$T_i = \inf\{t \geq 0; Y_i(t) - t\eta_i = 0\} \quad \forall i \in V,$$

and where, for all  $t \geq 0$ ,

$$(2.6) \quad \psi(t) := K_{t \wedge T}^{-1} Y(t)$$

and  $K_{t \wedge T}$  is positive definite. Moreover,  $T_i < +\infty$  a.s. for all  $i \in V$ , and  $K_T$  is a.s. positive definite.

(ii) *Denote  $X(t) = Y(t) - (t \wedge T)\eta$ . The previous SDE is equivalent to the following:*

$$\begin{aligned}
 (\mathcal{E}_V^{W,\theta,\eta}(X)) \quad X_i(t) &= \theta_i + \int_0^t \mathbb{1}_{s < T_i} dB_i(s) - \int_0^t \mathbb{1}_{s < T_i} ((W\psi)(s) + \eta)_i ds \\
 &\quad \forall i \in V,
 \end{aligned}$$

with

$$(2.7) \quad \psi(t) = K_{t \wedge T}^{-1} (X(t) + (t \wedge T)\eta)$$

and  $T_i$  is identified to be the first hitting time of 0 by  $X_i(t)$ .

(iii) *The process  $\psi(t)$  is a continuous vectorial martingale, it can be written as (recall that  $\mathbb{1}_{s < T}$  is the operator of multiplication by  $\mathbb{1}_{s < T_i}$ ):*

$$(\mathcal{E}_V^{W,\theta,\eta}(\psi)) \quad \psi(t) = \theta + \int_0^t K_{s \wedge T}^{-1} (\mathbb{1}_{s < T} dB(s)).$$

Moreover, the quadratic variation of  $\psi(t)$  is given by, for all  $t \geq 0$  (with convention that  $\frac{1}{\infty} = 0, \frac{1}{0} = \infty$ ),

$$\langle \psi, \psi \rangle_t = (H_{\frac{1}{2(t \wedge T)}})^{-1}.$$

REMARK 3. One can combine (iii) and (i) of Lemma 1 and write

$$Y_t = K_{t \wedge T} \left( \theta + \int_0^t K_{s \wedge T}^{-1} (\mathbb{1}_{s < T} dB_s) \right), \quad t \geq 0.$$

This defines the solution of  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  directly as a stochastic integral. It is easy to check “informally” that the previous equation is indeed a solution of  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  by Itô formula, but it is not obvious that the previous expression is well defined for all time  $t \geq 0$ : indeed,  $K_{t,0}$  defined in (2.5) is not invertible for all values of  $t^0 \in (\mathbb{R}_+)^V$ . It is the main difficulty of the lemma to prove that the solution of  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  can be defined for all  $t \geq 0$ .

It may not seem obvious at this point why we call these processes “Brownian motions with interacting drifts.” The explanation will come at the end of this section as a consequence of the Abelian property Theorem 2: under the condition that the diagonal terms of  $W$  are null, we will show that the marginals  $(X_i(t))_{t \geq 0}$  are Brownian motions with constant negative drift stopped at their first hitting time of 0; see Corollary 1.

Our first main result concerns the distribution of its hitting time.

THEOREM 1. Let  $\theta \in (\mathbb{R}_+^*)^V$ ,  $\eta \in (\mathbb{R}_+)^V$  and  $Y(t)$ ,  $X(t)$ ,  $T$  be as in Lemma 1.

- (i) The random vector  $\frac{1}{2T}$  has law  $\nu_V^{W, \theta, \eta}$ .
- (ii) Conditionally on  $T$ ,  $(X_i(t))_{i \in V, 0 \leq t \leq T_i}$  are independent three-dimensional Bessel bridges started at  $\theta_i$  and ending at 0 over the time intervals  $[0, T_i]$ .

Remark that when  $V = \{1\}$  is a single point and  $W_{1,1} = 0$ , then  $X_1(t) = Y_1(t) - t\eta_1$  is a drifted Brownian motion with initial value  $\theta_1 > 0$  and negative drift  $-\eta_1$  stopped at its first hitting time of 0. Hence, it corresponds to the problem presented in (1.2); in particular,  $\eta_1 = 0$  corresponds to (1.1).

When  $V = \{1\}$  and  $W_{1,1} > 0$ ,  $(Y_1(t))_{t \geq 0}$  is the solution of the SDE

$$(2.8) \quad dY_1(t) = \mathbb{1}_{t < T_1} \left( dB_1(t) - \frac{W_{1,1}}{1 - tW_{1,1}} Y(t) dt \right)$$

with initial condition  $Y_1(0) = \theta_1$ . It follows that  $Y_1(t) - t\eta_1$  has the law of a drifted Brownian bridge from  $\theta_1$  to 0 on time interval  $[0, 1/W_{1,1}]$  with constant negative drift  $-\eta_1$ , and stopped at its first hitting of 0. By drifted Brownian bridge from  $\theta_1$  to 0 on time interval  $[0, 1/W_{1,1}]$  with constant negative drift  $-\eta_1$ , we mean the process  $Z_t - t\eta_1$  where  $(Z_t)_{t \in [0, 1/W_{1,1}]}$  is the Brownian bridge. (It may also be viewed as a Brownian bridge from  $\theta_1$  to  $-\frac{\eta_1}{W_{1,1}}$  on time interval  $[0, 1/W_{1,1}]$ .) Consequently,  $Y_1(t)$  has the same law as  $(1 - tW_{1,1})[\theta + B_1(\frac{t}{1-tW_{1,1}})]$  up to time  $T_1$  (see, e.g., [15], p. 154) and  $T_1$  has the same law as  $\frac{1}{1+\tau W_{1,1}}$  where  $\tau$  is the first hitting time of 0 by a Brownian motion with drift  $-\eta_1$ . Therefore,  $\frac{1}{T_1 - W_{1,1}}$  follows an inverse Gaussian law with parameters  $(\frac{\theta_1}{\eta_1}, \theta_1^2)$ , and it is coherent with the expression of marginal law of  $\beta_i$  in Corollary B.

REMARK 4. As pointed out by a referee, in dimension  $d = 1$ , the time reversal of the process is closely related to the law of the first and last hitting time of the drifted Brownian motion; see, for example, [20]. Indeed, in the case  $d = 1$ , if  $X(t) := \theta + B(t) - \eta t$  is a drifted Brownian motion with drift  $-\eta$  starting from  $\theta$ , then  $-tX(1/t) = \eta - tB(1/t) - t\theta$  is a drifted Brownian motion starting from  $\eta$  and with drift  $-\theta$ . The first hitting time  $T$  of 0 by  $X$  satisfies

$$\frac{1}{T} = \sup\{s > 0; -sX(1/s) = 0\},$$

and  $T$  is the last hitting time of 0 by the drifted Brownian motion  $-tX(1/t)$ . This proves that the last hitting time of 0 by a drifted Brownian motion with drift  $-\theta$  and initial value  $\eta$  has the law of the inverse of an inverse Gaussian r.v. with parameters  $(\frac{\theta}{\eta}, \theta^2)$ . It is not clear how to give a counterpart of this relation in the multidimensional case presented in this paper. Indeed, our process is only defined up to the hitting times of 0 and it is not clear how to continue it after the hitting time, but it is an interesting question.

The next result shows some ‘‘abelianity’’ of the process, in the sense that times on each coordinates can be run somehow independently. The first two statements are counterparts of the two statements of Lemma C.

THEOREM 2 (Abelian properties). Let  $(X(t))$  be the solution of  $\mathcal{E}_V^{W, \theta, \eta}(X)$ . Denote  $\beta = \frac{1}{2T}$ .

(i) (Marginal) Let  $U \subset V$ . Then  $(X_U(t))$  has the same law as the solution of  $E_U^{W_{U,U}, \theta_U, \hat{\eta}}(X)$ , where

$$\hat{\eta} = \eta_U + W_{U,U^c}(\theta_{U^c}).$$

(ii) (Conditioning on a subset) Let  $U \subset V$ . Then, conditionally on  $(X_U(t))_{t \geq 0}$ ,  $(X_{U^c}(t))_{t \geq 0}$  has the law of the solutions of the SDE  $E_{U^c}^{\tilde{W}, \theta_{U^c}, \tilde{\eta}}(X)$ , where

$$\begin{aligned} \tilde{W} &= W_{U^c,U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1} W_{U,U^c}, \\ \tilde{\eta} &= \eta_{U^c} + W_{U^c,U}((H_\beta)_{U,U})^{-1}(\eta_U). \end{aligned}$$

(iii) (Markov property) Consider  $t^0 = (t_i^0)_{i \in V} \in (\mathbb{R}_+)^V$ . Denote by

$$\mathcal{F}^X(t^0) = \sigma\{(X_k(s))_{s \leq t_k^0}, k \in V\},$$

the filtration generated by the past of the trajectories before time  $(t_k^0)_{k \in V}$ . Then consider for  $t \geq 0$ ,

$$\tilde{X}(t) = X(t^0 + t) \quad (= (X_i(t_i^0 + t))_{i \in V}),$$

the process shifted by times  $(t_i^0)_{i \in V}$ . (Note that the shift in time is not necessarily the same for each coordinate). Conditionally on  $\mathcal{F}^X(t^0)$ , the process  $(\tilde{X}(t))_{t \geq 0}$  has the same law as the solution of the equation  $E_V^{\tilde{W}^{(t^0)}, X(t^0), \tilde{\eta}^{(t^0)}}(X)$ , with

$$\tilde{W}^{(t^0)} = W(K_{t^0 \wedge T})^{-1}, \quad \tilde{\eta}^{(t^0)} = \eta + \tilde{W}^{(t^0)}((t^0 \wedge T)\eta),$$

where in the second expression,  $t^0 \wedge T$  denotes the operator of multiplication by  $(t_i^0 \wedge T_i)$ . In particular, if  $V(t^0) = \{i \in V, T_i > t_i^0\}$ , conditionally on  $\mathcal{F}(t^0)$ ,  $(\frac{1}{T_i - t_i^0})_{i \in V(t^0)}$  has the law  $E_{V(t^0)}^{\tilde{W}^{(t^0)}, X(t^0), \tilde{\eta}^{(t^0)}}$ .

(iv) (Strong Markov property) Let  $T^0 = (T_i^0)_{i \in V} \in (\mathbb{R}_+ \cup \{\infty\})^V$  be a ‘‘multistopping time,’’ that is, for all  $t^0 \in (\mathbb{R}_+)^V$ , the event  $\{T^0 \leq t^0\} := \bigcap_{i \in V} \{T_i^0 \leq t_i^0\}$  is  $\mathcal{F}^X(t^0)$ -measurable. Denote by

$$\mathcal{F}^X(T^0) = \{A \in \mathcal{F}^X(\infty), \forall t^0 \in (\mathbb{R}_+)^V, A \cap \{T^0 \leq t^0\} \in \mathcal{F}^X(t^0)\}$$

the filtration of events anterior to  $T^0$ . Define for  $t \geq 0$ ,

$$\tilde{X}(t) = X(T^0 + t)$$

the process shifted at times  $(T_i^0)_{i \in V}$ . On the event  $\{T_i^0 < \infty, \forall i \in V\}$ , conditionally on  $T^0$  and  $\mathcal{F}^X(T^0)$ , the process  $\tilde{X}(t)$  has the same law as the solution of the SDE  $E_V^{\tilde{W}^{(T^0)}, X(T^0), \tilde{\eta}^{(T^0)}}(X)$ , where

$$\tilde{W}^{(T^0)} = W(K_{T^0 \wedge T})^{-1}, \quad \tilde{\eta}^{(T^0)} = \eta + \tilde{W}^{(T^0)}((T^0 \wedge T)\eta),$$

where in the second expression,  $T^0 \wedge T$  denotes the operator of multiplication by  $(T_i^0 \wedge T_i)$ .

REMARK 5. Assertions (i) and (ii) of the theorem are direct consequences of Theorem 1 and Lemma C. The assertion (iii) is more involved. The extension to the strong Markov property (iv) follows rather standard arguments. See the proofs in Section 8.

REMARK 6. In all these statements, the restricted (or conditioned) process that appears is not in general solution of the SDE with the original shifted Brownian motion, but with a different one, which is a priori not a Brownian motion in the original filtration. Nevertheless, when all the  $t_0^i$  are equal to the same real  $s$ , then it is the case:  $(X(t + s))_{t \geq 0}$  is solution of the SDE with the shifted Brownian motion  $(B(s + t))_{t \geq 0}$ ; cf. forthcoming Proposition 1. The result in the latter case is much simpler and is a consequence of a plain computation, whereas the general case uses the representation of Theorem 1.

Note that this allows to identify the law of marginals and conditional marginals.

COROLLARY 1. Consider  $(X(t))_{t \geq 0}$  solution of  $\mathcal{E}_V^{W, \theta, \eta}(X)$ . Fix  $i_0 \in V$ .

(i) If  $W_{i_0, i_0} = 0$  (resp.,  $W_{i_0, i_0} > 0$ ), the marginal  $(X_{i_0}(t))_{t \geq 0}$  has the law of a drifted Brownian motion starting at  $\theta_{i_0}$  (resp., drifted Brownian bridge from  $\theta_{i_0}$  to 0 on time interval  $[0, \frac{1}{W_{i_0, i_0}}]$ , with the meaning given in the discussion of equation (2.8)) with constant drift

$$-\widehat{\eta}_{i_0} = -\left(\eta_{i_0} + \sum_{j \neq i_0} W_{i_0, j} \theta_j\right)$$

and stopped at its first hitting time of 0.

(ii) Conditionally on  $((X_k(t))_{t \geq 0})_{k \neq i_0}$ , the process  $(X_{i_0}(t))_{t \geq 0}$  has the law of a drifted Brownian bridge from  $\theta_{i_0}$  to 0 on time interval  $[0, \frac{1}{\widetilde{W}_{i_0, i_0}}]$  with constant drift  $-\check{\eta}_{i_0}$  and stopped at its first hitting time of 0, where, with  $U = V \setminus \{i_0\}$ ,

$$\begin{aligned} \check{W}_{i_0, i_0} &= W_{i_0, i_0} + W_{i_0, U}((H_\beta)_{U, U})^{-1} W_{U, i_0}, \\ \check{\eta}_{i_0} &= \eta_{i_0} + W_{i_0, U}((H_\beta)_{U, U})^{-1}(\eta_U). \end{aligned}$$

PROOF. Apply Theorem 2(i) to the case  $U = \{i_0\}$  for (i) and Theorem (2)(ii) to  $U = \{i_0\}^c$  for (ii), and the considerations following Theorem 1.  $\square$

In particular, it means that the marginal  $(X_{i_0}(t))_{t \geq 0}$  is a diffusion process, as well as the (conditional) marginal  $(X_{i_0}(t))_{t \geq 0}$  conditioned on  $((X_k(t))_{t \geq 0})_{\{k: k \neq i_0\}}$ . This Markov property is not obvious in the initial equation  $\mathcal{E}_V^{W, \theta, \eta}(X)$ . Indeed, the process  $(X_{i_0}(u))_{u \leq s}$  before time  $s$  affects the drifts of  $(X_{\{k: k \neq i_0\}}(u))_{u \leq s}$ , and so the values  $X_{\{k: k \neq i_0\}}(s)$ , which themselves affect the drift of  $X_{i_0}(s)$ .

More generally, there are hidden Markov properties in the restricted process  $(X_U(t))_{t \geq 0}$ . Indeed, the law of the future path  $(X_U(t))_{t \geq s}$  only depends on the past of  $(X_U(u))_{u \leq s}$  through the values of  $X_U(s)$  and  $(s \wedge T)_U$ . This is not obvious from the initial equation  $\mathcal{E}_V^{W, \theta, \eta}(X)$ . The same is true for the process  $(X_{U^c}(t))_{t \geq 0}$  conditioned on  $(X_U(t))_{t \geq 0}$ .

2.3. *Relation with the vertex reinforced jump process.* Let us describe the VRJP in its “exchangeable” time scale introduced in [17]. We consider the VRJP with a general initial local time, as in [18], Section 3.1. The VRJP, with initial local time  $(\theta_i)_{i \in V}$ , is the self-interacting process  $(Z_t)_{t \geq 0}$  that, conditionally on its past at time  $t$ , jumps from a vertex  $i$  to  $j$  with rate

$$W_{i, j} \frac{\sqrt{\theta_j + \ell_j^Z(t)}}{\sqrt{\theta_i + \ell_i^Z(t)}},$$



where  $\ell_i^Z(t) = \int_0^t \mathbb{1}_{Z_s=i} ds$  denotes the local time of  $Z$  at site  $i$ . In [17], it was proved that this process is a mixture of Markov jump processes and that the mixing law can be represented by a marginal of a supersymmetric  $\sigma$ -field investigated by Disertori, Spencer and Zirnbauer in [6, 7, 25]. In [18], it was related to the random potential  $\beta$  of Lemma A.

**THEOREM D** ([17] Theorem 2, [18] Theorem 3). *Let  $\delta \in V$  where  $V$  is finite, and  $U = V \setminus \{\delta\}$ . Let  $(\theta_i)_{i \in V} \in (\mathbb{R}_+^*)^V$  be a positive vector. Consider  $\beta = (\beta_j)_{j \in V}$  sampled with distribution  $\nu_V^{W,\theta}$ . Define  $(\psi_j)_{j \in V}$  as the unique solution of*

$$\begin{cases} \psi(\delta) = 1, \\ H_\beta(\psi)|_U = 0. \end{cases}$$

*Then the VRJP starting at vertex  $\delta$  and initial local times  $(\theta_i)_{i \in V}$  is a mixture of Markov jump processes with jumping rates from  $i$  to  $j$  equal to*

$$(2.9) \quad \frac{1}{2} W_{i,j} \frac{\psi_j}{\psi_i}.$$

*More precisely, it means that*

$$\mathbb{P}_\delta^{\text{VRJP},\theta}(\cdot) = \int P_\delta^\psi(\cdot) \nu_V^{W,\theta}(d\beta),$$

*where  $\mathbb{P}_\delta^{\text{VRJP},\theta}$  is the law of the VRJP starting at vertex  $\delta$  and initial local times  $(\theta_i)_{i \in V}$  and  $P_\delta^\psi$  is the law of the Markov jump process with jumping rates (2.9) starting at vertex  $\delta$ .*

Remark that the random variables  $(\beta_j)_{j \in U}$  appear as asymptotic holding times of the VRJP. Indeed, let  $N_i(t)$  be the number of visits of vertex  $i$  by  $Z$  before time  $t$ . Then, by Theorem D, the empirical holding times converge  $\mathbb{P}_\delta^{\text{VRJP},\theta}$  a.s., that is, the following limit exists a.s.,

$$\lim_{t \rightarrow \infty} \frac{N_i(t)}{\ell_i^Z(t)} = \frac{1}{2} \sum_{j \sim i} W_{i,j} \frac{\psi_j}{\psi_i} = \beta_i \quad \forall i \in U,$$

and, by Lemma C(i),  $\beta_U$  has law  $\nu_U^{W,\theta,\eta}$  where  $\eta = W_{U,\delta}\theta_\delta$ . Moreover, conditionally on  $\beta_U$ , the VRJP is a Markov jump process with jump rates given by (2.9).

Consider now the SDE  $\mathcal{E}_U^{W,U,\theta_U,\eta}(Y)$  with same parameters. From Theorem 1, the law  $(\frac{1}{T_i})_{i \in U}$  coincides with that of  $\beta_U$ . Moreover, if we set

$$\psi_j(\infty) := \lim_{t \rightarrow \infty} \psi_j(t) \quad \forall j \in U,$$

then  $\psi(\infty) = ((H_{\frac{1}{2T}})_{U,U})^{-1}\eta$ . Hence, it means that  $\psi(\infty)$  coincides with the  $\psi$  of Theorem D if we identify  $\beta_U$  and  $\frac{1}{2T}$ . Hence,  $(\beta_U, \psi)$  of Theorem D has the same law as  $(\frac{1}{2T}, \psi)$  arising in the SDE  $\mathcal{E}_U^{W,\theta,\eta}(Y)$ .

There are remarkable similarities between Theorem 1 and Theorem D. First,  $(\beta_i)_{i \in U}$  are homogeneous to the inverse of time, and have same distribution in both cases. Second, in both cases, a type of exchangeability appears in the sense that, conditionally on the limiting holding times or hitting times, the processes are simpler: in the case of the VRJP, it becomes Markov; in the case of the SDE, the marginals are independent and diffusion processes (in fact Bessel bridges).

In Section 9, we push forward this relation, by explaining the martingale property that appears in [17], and the exponential martingale property that extends it in [4], by Theorem 1 and the Abelian properties of Theorem 2.

However, we do not yet clearly understand the relation between the VRJP and the SDE  $\mathcal{E}_V^{W,\theta,\eta}$  beyond these remarks.

2.4. *Organization of the paper.* In Section 3, we prove the properties related to the distribution  $\nu_V^{W,\theta,\eta}$ , Lemma A, Lemma C and Corollary B. In Section 4, we present some simple key computations that are used several times in the proofs. In Section 5, we prove the results concerning existence and uniqueness of pathwise solution of the SDE, Lemma 1 and state and prove Proposition 1 mentioned in Remark 5 above. Section 7 is devoted to the proof of the main Theorem 1. In Section 8, we prove the Abelian properties of Theorem 2. Finally, in Section 9, we explain the relation between the Abelian properties of Theorem 2 and the martingale that appears in [19]. In Section 10, we illustrate some of the results in the case of the graph with 2 vertices.

**3. Proof of the results concerning the distribution  $\nu_V^{W,\theta,\eta}$ : Lemma A, Lemma C and Corollary B.** Lemma A and Lemma C are proved in [19] (third arXiv version) in the case  $\theta_i = 1$  for all  $i \in V$ ; see Lemma 3 and Lemma 4 therein (see also [10]). The case of general  $\theta$  can be deduced from the special case  $\theta = 1$  by a change of variables. More precisely, setting  $\beta'_i = \theta_i^2 \beta_i$ ,  $W'_{i,j} = \theta_i \theta_j W_{i,j}$  and  $\eta'_i = \theta_i \eta_i$ , then we have

$$\langle \theta, H_\beta \theta \rangle = \langle 1, H_{\beta'} 1 \rangle, \quad \langle \eta, H_\beta^{-1} \eta \rangle = \langle \eta', (H_{\beta'})^{-1} \eta' \rangle, \quad \langle \eta, \theta \rangle = \langle \eta', 1 \rangle,$$

where  $H_{\beta'} = 2\beta' - W'$ , so that  $\beta \sim \nu_V^{W,\theta,\eta}$  if and only if  $\beta' \sim \nu_V^{W',1,\eta'}$ .

Corollary B is a direct consequence of the expression of the Laplace transform. Indeed, under  $\nu_V^{W,\theta,\eta}$ , the Laplace transform of the marginal  $\beta_i - \frac{1}{2} W_{i,i}$  is given for  $\zeta \in \mathbb{R}_+$  by

$$\begin{aligned} & \int \exp\left(-\zeta\left(\beta_i - \frac{1}{2} W_{i,i}\right)\right) \nu_V^{W,\theta,\eta}(d\beta) \\ &= \frac{\theta_i}{\sqrt{\theta_i^2 + \zeta}} \exp\left(-\left(\sqrt{\theta_i^2 + \zeta} - \theta_i\right)\left(\eta_i + \sum_{j \neq i} W_{i,j} \theta_j\right)\right). \end{aligned}$$

It coincides with the Laplace transform of the inverse of the inverse Gaussian density. More precisely, by changing the parameter of inverse Gaussian distribution, we have

$$\begin{aligned} & \int_0^\infty \exp\left(-\frac{\zeta}{2x}\right) \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) dx \\ &= \frac{\sqrt{\lambda}}{\sqrt{\zeta + \lambda}} \exp\left(-\sqrt{\frac{\lambda}{\mu^2}}(\sqrt{\zeta + \lambda} - \sqrt{\lambda})\right). \end{aligned}$$

It means that the law of  $2\beta_i - W_{i,i}$  coincides with the law of the inverse of an inverse Gaussian random variable with parameters  $(\lambda, \mu)$  such that  $\lambda = \theta_i^2$  and  $\sqrt{\frac{\lambda}{\mu^2}} = \eta_i + \sum_{j \neq i} W_{i,j} \theta_j$ .

**4. Simple key formulas.** Let us start with a remark. If  $(t_i) \in (\mathbb{R}_+)^V$  and  $K_t > 0$ , then the operator  $H_{\frac{1}{2t}}^{-1}$  is well defined even when some of the  $t_i$ 's vanish: indeed, using the identity

$$H_{\frac{1}{2t}}^{-1} = K_t^{-1} t,$$

the right-hand side is perfectly well defined when  $K_t$  is invertible. In all of the sequel, we will implicitly consider that  $H_{\frac{1}{2t}}^{-1}$  is defined by this formula when some of the  $t_i$ 's vanish.

We prove below some simple formulas that will be key tools in forthcoming computations.

LEMMA 2. *Let  $(t_i^0)_{i \in V}$  and  $(t_i^1)_{i \in V}$  be vectors in  $\mathbb{R}_+^V$  such that  $K_{t^0+t^1} > 0$ .*

(i) We have

$$(4.1) \quad K_{t^0+t^1} = \tilde{K}_{t^1} K_{t^0},$$

with

$$\tilde{K}_{t^1} = \text{Id} - t^1 \tilde{W} \quad \text{where } \tilde{W} = W K_{t^0}^{-1}.$$

Hence, we also have, with  $\tilde{H}_{\frac{1}{2t^1}} = \frac{1}{t^1} - \tilde{W}$  (where  $|H| := \det H$ )

$$(4.2) \quad \frac{|H_{\frac{1}{2(t^0+t^1)}}|}{|\tilde{H}_{\frac{1}{2t^1}}|} = \left( \prod_{i \in V} \frac{t_i^1}{t_i^0 + t_i^1} \right) |K_{t^0}|.$$

(ii) Let

$$\tilde{\eta} = \eta + \tilde{W}(t^0 \eta),$$

then

$$(4.3) \quad \tilde{\eta} = (t^0)^{-1} H_{\frac{1}{2t^0}}^{-1} \eta,$$

and

$$(4.4) \quad \langle \tilde{\eta}, (\tilde{H}_{\frac{1}{2t^1}})^{-1} \tilde{\eta} \rangle = \langle \eta, (H_{\frac{1}{2(t^0+t^1)}})^{-1} \eta \rangle - \langle \eta, (H_{\frac{1}{2t^0}})^{-1} \eta \rangle.$$

REMARK 7. One should not confuse  $\tilde{W}$  in Lemma 2 (which is deterministic) with  $\tilde{W}^{(t^0)}$  in Theorem 2, which should be considered as a process.

PROOF OF LEMMA 2. (i) We can write

$$K_{t^0+t^1} = K_{t^0} - t^1 W = (\text{Id} - t^1 W K_{t^0}^{-1}) K_{t^0} = \tilde{K}_{t^1} K_{t^0}.$$

(ii) Formula (4.3) follows from

$$\tilde{\eta} = (t^0)^{-1} (\text{Id} + t^0 W K_{t^0}^{-1}) t^0 \eta = (t^0)^{-1} K_{t^0}^{-1} t^0 \eta = (t^0)^{-1} H_{\frac{1}{2t^0}}^{-1} \eta.$$

Turning to Formula (4.4), using (4.1), we have

$$K_{t^0+t^1}^{-1} = K_{t^0}^{-1} \tilde{K}_{t^1}^{-1}$$

and

$$(4.5) \quad \begin{aligned} \tilde{H}_{\frac{1}{2t^1}}^{-1} &= K_{t^0} K_{t^0+t^1}^{-1} t^1 \\ &= K_{t^0} K_{t^0+t^1}^{-1} (t^0 + t^1) \left( \frac{1}{t^0} - \frac{1}{t^0 + t^1} \right) t^0 \\ &= t^0 H_{\frac{1}{2t^0}} H_{\frac{1}{2(t^0+t^1)}}^{-1} \left( \frac{1}{t^0} - \frac{1}{t^0 + t^1} \right) t^0 \\ &= t^0 H_{\frac{1}{2t^0}} H_{\frac{1}{2(t^0+t^1)}}^{-1} (H_{\frac{1}{2t^0}} - H_{\frac{1}{2(t^0+t^1)}}) t^0 \\ &= t^0 H_{\frac{1}{2t^0}} H_{\frac{1}{2(t^0+t^1)}}^{-1} H_{\frac{1}{2t^0}} t^0 - t^0 H_{\frac{1}{2t^0}} t^0. \end{aligned}$$

Now, (4.3) implies

$$\tilde{H}_{\frac{1}{2t^1}} \tilde{\eta} = t^0 H_{\frac{1}{2t^0}} H_{\frac{1}{2(t^0+t^1)}}^{-1} \eta - t^0 \eta.$$

Since  $H_{\frac{1}{2t^0}}$  is symmetric, we get (4.4) by (4.3).  $\square$

**5. Proof of basic properties of the SDE  $\mathcal{E}_V^{W,\theta,\eta}$ : Proof of Lemma 1.** Remark that (i) and (ii) of Lemma 1 are equivalent since  $dX(t) = dY(t) - \eta dt$ . In order to prove the existence and uniqueness of the pathwise solution of  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  (or equivalently  $\mathcal{E}_V^{W,\theta,\eta}(X)$ ), we first consider a nonstopped version of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(Y)$ , for which the existence and uniqueness is simpler.

LEMMA 3. *Let  $(\theta_i)_{i \in V} \in \mathbb{R}_+^V$ . Let  $h > 0$  be the smallest positive real such that  $\det(K_h) = 0$ . Then the following SDE is well defined on time interval  $[0, h)$  and has a unique pathwise solution*

$$(5.1) \quad \tilde{Y}_i(t) = \theta_i + B_i(t) - \int_0^t (WK_s^{-1}\tilde{Y}(s))_i ds \quad \forall i \in V.$$

Moreover, there exists a time  $\tau < h$  such that  $\tilde{Y}_i(\tau) = \tau \eta_i$  for some vertex  $i \in V$ .

PROOF. As  $WK_t^{-1}$  is bounded on time interval  $[0, h - \epsilon)$  for all  $\epsilon > 0$ , (5.1) is a linear SDE with bounded coefficients which has a unique pathwise solution, with continuous sample paths, by standard existence and uniqueness theorems on SDE.

We denote

$$\mathcal{K}(t) = \exp\left(\int_0^t WK_s^{-1} ds\right).$$

Note that  $WK_s^{-1}$  is symmetric with positive coefficients for  $s \in [0, h)$ , and commutes with any  $WK_{s'}^{-1}$  for any  $s, s' \in [0, h)$ . Therefore, it commutes with  $\mathcal{K}(t)$  for  $t \in [0, h)$ . Besides,  $\mathcal{K}(t)$  has positive coefficients and

$$(5.2) \quad \lim_{t \rightarrow h} \min_{i,j} (\mathcal{K}(t))_{i,j} = +\infty.$$

Indeed, since  $W$  is symmetric with positive coefficients and irreducible, by the Perron–Frobenius theorem, it has a unique maximal eigenvalue  $\lambda_1$  associated with an eigenfunction with strictly positive coefficients. By construction, since  $K_s = \text{Id} - sW$  and  $h$  is the smallest positive real such that  $\det(K_h) = 0$ , we have  $\lambda_1 = 1/h$ . Hence, denoting  $\lambda_1 = 1/h > \lambda_2 > \dots$  the eigenvalues of  $W$  and  $p_1, p_2, \dots$  the corresponding spectral projectors, we have

$$K_s^{-1} = \sum \frac{1}{1 - s\lambda_k} p_k,$$

and  $\lim_{s \rightarrow h} (1 - s/h)K_s^{-1} = p_1$ , which has strictly positive coefficients. This implies (5.2), since the integral in the definition of  $\mathcal{K}(t)$  is a matrix with positive coefficients and it diverges when  $t \rightarrow h, t < h$ .

Define

$$\mathcal{H}(t) := \mathcal{K}(t)\tilde{Y}(t),$$

we have, using the commutation above,  $d\mathcal{H}(t) = \mathcal{K}(t)dB(t)$ , hence

$$\mathcal{H}(t) = \theta + \int_0^t \mathcal{K}(s)dB(s).$$

Therefore,  $\mathcal{H}(t)$  is a Gaussian process and its first coordinate  $(\mathcal{H}_1(t))_{0 \leq t < h}$  is a time-changed Brownian motion, more precisely, since  $\mathcal{K}(s)$  is symmetric,

$$\mathcal{H}_1(t) \stackrel{\text{law}}{=} \theta_1 + B_1\left(\sqrt{\left(\int_0^t \mathcal{K}^2(s) ds\right)_{1,1}}\right).$$

Since the time change tends to  $+\infty$  when  $t$  tends to  $h$ , it implies that  $\mathcal{H}_1(t)$  becomes negative on time interval  $[0, h)$ .

Assume by contradiction that  $\tilde{Y}_i(t) \geq 0$  for all  $i \in V$  and all  $t \in [0, h)$ . Since  $\mathcal{K}(t)$  has positive coefficients and by definition of  $\mathcal{H}(t)$ , it follows that  $\mathcal{H}(t)$  has nonnegative coordinates for all  $t$  in  $[0, h)$ , which leads to a contradiction. Hence,  $\tilde{Y}_i(t)$  hits 0 for some  $i \in V$  on time interval  $[0, h)$ . In particular, as the drift  $\eta$  has nonnegative coefficients,  $\tilde{Y}_i(t)$  must hit the line  $y = t\eta_i$  at some time  $\tau < h$  for some vertex  $i$ .  $\square$

PROOF OF LEMMA 1(I). We prove it by recurrence on the size of  $V$ . We will gradually define  $Y(t)$ , solution to the equation  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  and  $X(t) = Y(t) - t\eta$ . Consider

$$\tau = \inf\{t \geq 0, \exists i \in V \text{ such that } X_i(t) = 0\}$$

and denote by  $i_0$  the vertex in  $V$  such that  $X_{i_0}(\tau) = 0$ . Up to time  $\tau$ , equation  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  is equivalent to equation (5.1), hence equation  $\mathcal{E}_V^{W,\theta,\eta}(Y)$  is well defined and has unique pathwise solution up to time  $\tau$  and  $\tau < \infty$  a.s. Moreover,  $T_{i_0} = \tau$ . Now we set  $U = \{i_0\}^c$  and

$$(\tilde{T}_i)_{i \in V} = (T_i - \tau)_{i \in V},$$

$$\tilde{W} = WK_\tau^{-1}, \quad \tilde{K}_s = \text{Id} - s\tilde{W}, \quad \tilde{\eta} = \eta + \tilde{W}(\tau\eta)$$

and use that, by (4.1) applied to  $t_i^0 = \tau$  for all  $i$ , and  $t^1 = s \wedge \tilde{T}$ ,

$$K_{(\tau+s) \wedge T}^{-1} = K_\tau^{-1} \tilde{K}_{s \wedge \tilde{T}}^{-1}.$$

We set

$$\tilde{X}(s) = X(\tau + s), \quad \tilde{B}(s) = B(\tau + s).$$

Hence, we have that

$$(\tau + s) \wedge T = \tau + s \wedge \tilde{T}, \quad WK_{(\tau+s) \wedge T}^{-1} = \tilde{W} \tilde{K}_{s \wedge \tilde{T}}^{-1}$$

and after time  $\tau$ ,  $(X_{\tau+t})_{t \geq 0}$  is solution of  $\mathcal{E}_V^{W,\theta,\eta}(X)$  if and only if  $\tilde{X}(s)$  is solution of

$$(5.3) \quad d\tilde{X}(s) = \mathbb{1}_{s < \tilde{T}} d\tilde{B}(s) + \mathbb{1}_{s < \tilde{T}} (\tilde{W} \tilde{K}_{s \wedge \tilde{T}}^{-1} (\tilde{X}(s) + \tau\eta + (s \wedge \tilde{T})\eta) + \eta) ds.$$

Using that

$$\begin{aligned} & \tilde{W} \tilde{K}_{s \wedge \tilde{T}}^{-1} (\tilde{X}(s) + \tau\eta + (s \wedge \tilde{T})\eta) \\ &= \tilde{W} \tilde{K}_{s \wedge \tilde{T}}^{-1} (\tilde{X}(s) + \tilde{K}_{s \wedge \tilde{T}}(\tau\eta) + (s \wedge \tilde{T})\tilde{W}(\tau\eta) + (s \wedge \tilde{T})\eta) \\ &= \tilde{W} \tilde{K}_{s \wedge \tilde{T}}^{-1} (\tilde{X}(s) + (s \wedge \tilde{T})\tilde{\eta}) + \tilde{W}(\tau\eta) \end{aligned}$$

we see that (5.3) is equivalent to the fact that  $\tilde{X}$  is solution of  $\mathcal{E}_V^{\tilde{W},X(\tau),\tilde{\eta}}(\tilde{X})$ . Since,  $X_{i_0}(\tau) = 0$  is it equivalent to the fact that  $\tilde{X}_U$  is solution of  $\mathcal{E}_U^{\tilde{W},X(\tau),\tilde{\eta}}(\tilde{X})$ . Hence, we conclude by the recurrence hypothesis applied to  $U$ , which implies that  $\mathcal{E}_U^{\tilde{W},X(\tau),\tilde{\eta}}(\tilde{X})$  has a unique pathwise solution.  $\square$

PROOF OF LEMMA 1(III). Remark first that

$$\frac{\partial}{\partial t} K_{t \wedge T}^{-1} = K_{t \wedge T}^{-1} \mathbb{1}_{t < T} WK_{t \wedge T}^{-1}.$$

Differentiating  $\psi(t) = K_{t \wedge T}^{-1}(Y(t))$ , we get

$$\begin{aligned} d\psi_i(t) &= (K_{t \wedge T}^{-1}(dY(t)))_i + (K_{t \wedge T}^{-1} \mathbb{1}_{t < T} W K_{t \wedge T}^{-1}(Y(t)))_i dt \\ &= (K_{t \wedge T}^{-1}(\mathbb{1}_{t < T} dB(t)))_i. \end{aligned}$$

Moreover, the quadratic variation of  $\psi_i(t)$  and  $\psi_j(t)$  is given by

$$\begin{aligned} \langle \psi_i, \psi_j \rangle_t &= \sum_{l \in V} \int_0^t (K_{s \wedge T}^{-1})_{i,l} \mathbb{1}_{s < T_l} (K_{s \wedge T}^{-1})_{j,l} ds \\ &= \sum_{l \in V} \int_0^t (H_{\frac{1}{2(s \wedge T)}}^{-1})_{i,l} \left( \frac{1}{s \wedge T_l} \right)^2 \mathbb{1}_{s < T_l} (H_{\frac{1}{2(s \wedge T)}}^{-1})_{l,j} ds \\ &= \int_0^t \frac{\partial}{\partial s} (H_{\frac{1}{2(s \wedge T)}}^{-1})_{i,j} ds \\ &= (H_{\frac{1}{2(t \wedge T)}}^{-1})_{i,j}, \end{aligned}$$

where in the second equality, we used  $H_\beta$  is a symmetric matrix and  $H_{\frac{1}{2(t \wedge T)}}^{-1} = K_{t \wedge T}^{-1}(t \wedge T)$ , and so that  $H_{\frac{1}{2(t \wedge T)}}^{-1} = (t \wedge T)(K_{t \wedge T}^{-1})^t$ . In the last equality, we used that  $H_{\frac{1}{2(t \wedge T)}}^{-1}$  is well defined and null for  $t = 0$ .  $\square$

**6. Stationarity property.**

PROPOSITION 1 (Stationarity). *If  $(X(t))_{t \geq 0}$  is the solution of  $\mathcal{E}_V^{W,\theta,\eta}(X)$  and  $s \geq 0$ , then  $(X(t+s))_{t \geq 0}$  is solution of the SDE  $\mathcal{E}_V^{\tilde{W}^{(s)}, X(s), \tilde{\eta}^{(s)}}(X)$  directed by the shifted Brownian motion  $(B(t+s))_{t \geq 0}$ , and with*

$$\tilde{W}^{(s)} = W K_{s \wedge T}^{-1}, \quad \tilde{\eta}^{(s)} = \eta + \tilde{W}^{(s)}((s \wedge T)\eta).$$

REMARK 8. Proposition 1 corresponds to Theorem 2(iii) in the case where all the coordinates of  $(t_i^0)$  are equal to  $s$ , except that in this case the equation is directed by the shifted Brownian motion, which is not the case when coordinates are not all equal. The proof in this case is based on elementary computations and do not rely on the representation given in Theorem 1. The result can be interpreted as a dynamic evolution of the parameters along the trajectory: conditioned on the past, the future of the trajectory is in the same family of SDEs with deformed parameters.

PROOF OF PROPOSITION 1. Set  $(\tilde{X}(t))_{t \geq 0} := (X(t+s))_{t \geq 0}$ ,  $(\tilde{B}(t))_{t \geq 0} := (B(t+s))_{t \geq 0}$ , and  $\tilde{T}^{(s)} = T - s \wedge T$ . Remark that by Lemma 2

$$(s+t) \wedge T = s \wedge T + t \wedge \tilde{T}^{(s)}, \quad W(K_{(s+t) \wedge T})^{-1} = \tilde{W}^{(s)}(\tilde{K}_{t \wedge \tilde{T}^{(s)}}^{(s)})^{-1}$$

with  $\tilde{W}^{(s)}$  defined in Proposition 1 and  $\tilde{K}_{t \wedge \tilde{T}^{(s)}}^{(s)} = \text{Id} - (t \wedge \tilde{T}^{(s)})\tilde{W}^{(s)}$ . The SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$  after time  $s$  is thus equivalent to

$$\begin{aligned} d\tilde{X}_i(t) &= \mathbb{1}_{t < \tilde{T}_i^{(s)}} d\tilde{B}_i(t) \\ &\quad - \mathbb{1}_{t < \tilde{T}_i^{(s)}} (\tilde{W}^{(s)}(\tilde{K}_{t \wedge \tilde{T}^{(s)}}^{(s)})^{-1} (\tilde{X}(t) + (s \wedge T)\eta + (t \wedge \tilde{T}^{(s)})\eta) + \eta)_i dt \\ &\quad \forall i \in V. \end{aligned}$$

By Lemma 2, we have that

$$\begin{aligned} & \widetilde{W}^{(s)}(\widetilde{K}_{t \wedge \widetilde{T}}^{(s)})^{-1}(\widetilde{X}(t) + (s \wedge T)\eta + (t \wedge \widetilde{T}^{(s)})\eta) \\ &= \widetilde{W}^{(s)}(\widetilde{K}_{t \wedge \widetilde{T}}^{(s)})^{-1}(\widetilde{X}(t) + \widetilde{K}_{t \wedge \widetilde{T}}^{(s)}((s \wedge T)\eta) \\ &\quad + (t \wedge \widetilde{T}^{(s)})\widetilde{W}^{(s)}((s \wedge T)\eta) + (t \wedge \widetilde{T}^{(s)})\eta) \\ &= \widetilde{W}^{(s)}(\widetilde{K}_{t \wedge \widetilde{T}}^{(s)})^{-1}(\widetilde{X}(t) + (t \wedge \widetilde{T}^{(s)})\widetilde{\eta}^{(s)}) + \widetilde{W}^{(s)}(s \wedge T)\eta. \end{aligned}$$

Hence,  $\widetilde{X}(t)$  is solution of

$$\begin{aligned} d\widetilde{X}(t) &= \mathbb{1}_{t < \widetilde{T}_i^{(s)}} d\widetilde{B}_i(t) \\ &\quad - \mathbb{1}_{t < \widetilde{T}_i^{(s)}} (\widetilde{W}^{(s)}(\widetilde{K}_{t \wedge \widetilde{T}}^{(s)})^{-1}(\widetilde{X}(t) + (t \wedge \widetilde{T}^{(s)})\widetilde{\eta}^{(s)}) + \widetilde{\eta}^{(s)})_i dt \\ &\quad \forall i \in V. \end{aligned}$$

Since,  $\widetilde{X}(0) = X(s)$ , we have the result.  $\square$

**7. Proof of Theorem 1.** We provide below a convincing but incomplete argument for the proof of Theorem 1. We do not know yet how to turn this argument into a rigorous alternative proof, even though we think that it should be possible. The rigorous proof is given in Section 7.2.

7.1. *A convincing but incomplete argument for Theorem 1(i).* Let  $\lambda \in \mathbb{R}_+^V$  be a nonnegative vector on  $V$ . As

$$\exp\left(-\langle \eta, H_\beta^{-1} \lambda \rangle - \frac{1}{2} \langle \lambda, H_\beta^{-1} \lambda \rangle\right) \nu_V^{W, \theta, \eta} = \exp(-\langle \lambda, \theta \rangle) \nu_V^{W, \theta, \eta + \lambda},$$

we have

$$(7.1) \quad \int \exp\left(-\langle \eta, H_\beta^{-1} \lambda \rangle - \frac{1}{2} \langle \lambda, H_\beta^{-1} \lambda \rangle\right) \nu_V^{W, \theta, \eta}(d\beta) = \exp(-\langle \lambda, \theta \rangle).$$

On the other hand, consider  $Y(t)$ , solution of  $\mathcal{E}_V^{W, \theta, \eta}(Y)$ , and the associated processes  $(X(t))$ ,  $(\psi(t))$ . By Lemma 1 and [15], Proposition 3.4, page 148, we know that

$$\exp\left(-\langle \lambda, \psi(t) \rangle - \frac{1}{2} \langle \lambda, H_{\frac{1}{2(t \wedge T)}}^{-1} \lambda \rangle\right),$$

is a continuous martingale, dominated by 1. Moreover, we have that  $X(t) \rightarrow 0$ , a.s., when  $t \rightarrow \infty$ , hence, a.s.,

$$\lim_{t \rightarrow \infty} \psi(t) = K_T^{-1}(T\eta) = H_{\frac{1}{2T}}^{-1} \eta.$$

By the dominated convergence theorem, it implies that

$$\mathbb{E}\left(\exp\left(-\langle \lambda, H_{\frac{1}{2T}}^{-1} \eta \rangle - \frac{1}{2} \langle \lambda, H_{\frac{1}{2T}}^{-1} \lambda \rangle\right)\right) = \exp(-\langle \lambda, \psi(0) \rangle) = \exp(-\langle \lambda, \theta \rangle).$$

Hence, it implies that both  $\beta$  under  $\nu_V^{W, \theta, \eta}$  and  $\frac{1}{2T}$  obtained from  $\mathcal{E}_V^{W, \theta, \eta}$  satisfy the same functional identity (7.1). Note that the dimension of the space of variables  $(\lambda_i)_{i \in V}$  and of the random variables  $(\beta_i)_{i \in V}$  are the same. Nevertheless, it is not clear whether the functional identity (7.1) characterizes the distribution  $\nu_V^{W, \theta, \eta}$ ; at least we have no proof of this fact. If such an argument were available, it would imply Theorem 1(i) also: indeed, using the stationarity of the equation, Proposition 1, it would be possible to deduce Theorem 1(ii) by enlargement of filtration (see [8]). We do not give the detail of the argument here since the first part of the proof is missing.

7.2. *Proof.* Even if it is not obvious at first sight since the context is very different, the strategy of the proof of Theorem 1 is quite in the spirit of the proof of Theorem 2(ii) of [17]: we start from the mixture of Bessel processes and we prove that this mixture has the same law as the solutions of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$ . We use in a crucial way the fact that the law  $\nu_V^{W,\theta,\eta}$  is a probability density with explicit normalizing constant.

7.2.1. *The classical statement for  $N = 1$ .* We denote by  $\mathbf{W} = C(\mathbb{R}_+, \mathbb{R})$  the Wiener space. For  $\theta > 0$ , we denote by  $\mathbb{P}_\theta$  the law of  $X_{t \wedge T}$  where  $X_t = \theta + B_t$  and  $B_t$  is a standard Brownian motion and  $T = \inf\{t \geq 0, X_t = 0\}$  is the first hitting time of 0. For a positive real<sup>2</sup>  $T$ , we denote by  $\mathbb{B}_{\theta,0}^{3,T}$  the law of the three-dimensional Bessel Bridge from  $\theta > 0$  to 0 on time interval  $[0, T]$ , as defined in [15], Section XI-3. We always consider that the Bessel bridge is extended to time interval  $\mathbb{R}_+$ , with constant value equal to 0 after time  $T$ , and thus  $\mathbb{B}_{\theta,0}^{3,T}$  is a probability on  $\mathbf{W}$ . As mentioned in the Introduction, it is known (see [24], or p. 317 of [15]), that under  $\mathbb{P}_\theta$ ,  $\frac{1}{2T}$  has the law  $\text{Gamma}(\frac{1}{2}, \theta^2)$  and that, conditionally on  $T$ ,  $(X_t)_{t \geq 0}$  has law  $\mathbb{B}_{\theta,0}^{3,T}$ . Otherwise stated, it means that the following equality of probabilities holds on the Wiener space  $\mathbf{W}$ :

$$(7.2) \quad \mathbb{P}_\theta(\cdot) = \int_0^\infty \mathbb{B}_{\theta,0}^{3, \frac{1}{2\beta}}(\cdot) \frac{1}{\sqrt{\pi}} \frac{\theta}{\sqrt{\beta}} e^{-\theta^2 \beta} d\beta.$$

7.2.2. *Proof of Theorem 1(i) and (ii).* We use the formulation of Lemma 1(ii), and we will prove that if  $(X_i(t))_{i \in V}$  satisfies  $\mathcal{E}_V^{W,\theta,\eta}(X)$ , then  $\beta := \frac{1}{2T}$  is distributed as  $\nu_V^{W,\theta,\eta}$  and conditionally on  $T$ , the coordinates  $(X_i(t))_{t \geq 0}$  are independent 3-dimensional Bessel bridges from  $\theta_i$  to 0 on time interval  $[0, T_i]$ .

Recall that  $V = \{1, \dots, N\}$ , and denote by  $\mathbf{W}_V = C(\mathbb{R}_+, \mathbb{R}^V)$  the  $N$ -dimensional Wiener space and  $(X(t))_{t \geq 0}$  the canonical process. As usual, on this canonical space, we denote by  $T = (T_i)_{i \in V}$  the hitting times of 0,  $T_i = \inf\{t \geq 0, X_i(t) = 0\}$ . For  $\theta = (\theta_i)_{i \in V} \in \mathbb{R}_+^V$ , we set  $\mathbb{P}_{V,\theta} := \otimes_{i \in V} \mathbb{P}_{\theta_i}$  and by (7.2) we have

$$(7.3) \quad \mathbb{P}_{V,\theta} = \int_{\mathbb{R}_+^V} \left( \otimes_{i \in V} \mathbb{B}_{\theta_i,0}^{3, \frac{1}{2\beta_i}}(\cdot) \right) \left( \prod_{i \in V} \sqrt{\frac{2}{\pi}} \frac{\theta_i}{\sqrt{2\beta_i}} e^{-\theta_i^2 \beta_i} d\beta_i \right),$$

the probability on  $\mathbf{W}_V$  such that  $(X_i(t))_{i \in V}$  are  $N$  independent Brownian motions starting at positions  $(\theta_i)$  and stopped at their first hitting times of 0. The assertions of Theorem 1(i) and (ii) are equivalent to the fact that the law of the solution of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$  is a mixture of independent Bessel bridges  $\mathbb{B}_{\theta_i,0}^{3, \frac{1}{2\beta_i}}$  where  $\beta$  is a random vector with distribution  $\nu_V^{W,\theta,\eta}$ . Otherwise stated, it means that the probability distribution  $\overline{\mathbb{P}}_V^{W,\theta,\eta}$  defined by

$$(7.4) \quad \overline{\mathbb{P}}_V^{W,\theta,\eta}(\cdot) := \int \left( \otimes_{i \in V} \mathbb{B}_{\theta_i,0}^{3, \frac{1}{2\beta_i}}(\cdot) \right) (\cdot) \nu_V^{W,\theta,\eta}(d\beta),$$

is the law of the solution of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$ . The strategy is now to write the Radon–Nikodym derivative of  $\overline{\mathbb{P}}_V^{W,\theta,\eta}$  with respect to  $\mathbb{P}_{V,\theta}$  as an exponential martingale, and then to apply Girsanov’s theorem.

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<sup>2</sup>Note the different typeface embodied T to distinguish real variable representing a time from the random hitting time  $T$ .



Comparing the representations (7.3) and (7.4), together with the explicit expression for  $\nu_V^{W,\theta,\eta}$ , we get

$$\begin{aligned}
 d \frac{\overline{\mathbb{P}}_V^{W,\theta,\eta}}{\mathbb{P}_{V,\theta}} &= \mathbb{1}_{H_{\frac{1}{2T}} > 0} \cdot e^{-\frac{1}{2}\langle \theta, H_{\frac{1}{2T}} \theta \rangle + \frac{1}{2}\langle \theta, \frac{1}{T} \theta \rangle - \frac{1}{2}\langle \eta, (H_{\frac{1}{2T}})^{-1} \eta \rangle + \langle \eta, \theta \rangle} \frac{\prod_{i \in V} T_i^{-1/2}}{\sqrt{|H_{\frac{1}{2T}}|}} \\
 (7.5) \qquad &= \mathbb{1}_{H_{\frac{1}{2T}} > 0} \cdot \exp\left(\frac{1}{2}\langle \theta, W \theta \rangle - \frac{1}{2}\langle \eta, K_T^{-1} T \eta \rangle + \langle \eta, \theta \rangle\right) \frac{1}{\sqrt{|K_T|}}.
 \end{aligned}$$

(Indeed, on the right-hand side of the representation (7.3) and (7.4), the hitting times of 0  $T_i$  corresponds to the variable  $\frac{1}{2\beta_i}$ .)

Let  $t > 0$ , and define

$$\begin{cases}
 V(t) := \{i \in V, T_i > t\}, \\
 \beta^{(t)} := \frac{1}{2(t \wedge T)}, \\
 \widetilde{W}^{(t)} := W K_{t \wedge T}^{-1} = W + W K_{t \wedge T}^{-1} (t \wedge T) W, \\
 \widetilde{\eta}^{(t)} := \eta + \widetilde{W}^{(t)} (t \wedge T) \eta,
 \end{cases}$$

where the third equality comes from the fact that  $K_{t \wedge T}^{-1} = \text{Id} + (t \wedge T) W K_{t \wedge T}^{-1}$ . Note that  $\widetilde{W}^{(t)}$  is symmetric since  $K_{t \wedge T}^{-1} (t \wedge T) = H_{\frac{1}{2(t \wedge T)}}^{-1}$ . We also set

$$\begin{cases}
 \widetilde{T}^{(t)} := T - t \wedge T, \\
 \widetilde{\beta}^{(t)} := \frac{1}{2\widetilde{T}^{(t)}}, \\
 \widetilde{K}_{\widetilde{T}}^{(t)} := \text{Id} - \widetilde{T}^{(t)} \widetilde{W}^{(t)}, \\
 \widetilde{H}_{\widetilde{\beta}}^{(t)} := 2\widetilde{\beta}^{(t)} - \widetilde{W}^{(t)} = \frac{1}{\widetilde{T}^{(t)}} \widetilde{K}_{\widetilde{T}}^{(t)}.
 \end{cases}$$

Note that  $(\widetilde{H}_{\widetilde{\beta}}^{(t)})^{-1}$  is well defined for all  $t$  using  $(\widetilde{H}_{\widetilde{\beta}}^{(t)})^{-1} = (\widetilde{K}_{\widetilde{T}}^{(t)})^{-1} \widetilde{T}^{(t)}$ ; see the beginning of Section 4. By equation (4.3) applied with  $t^0 = t \wedge T$ , we get that

$$(7.6) \qquad \widetilde{\eta}^{(t)} = (t \wedge T)^{-1} H_{\beta^{(t)}}^{-1} \eta.$$

We first prove the following lemma.

LEMMA 4. *Let*

$$M_t = \exp\left(-\frac{1}{2}\langle X(t), \widetilde{W}^{(t)} X(t) \rangle + \frac{1}{2}\langle \widetilde{\eta}^{(t)}, (\widetilde{H}_{\widetilde{\beta}}^{(t)})^{-1} \widetilde{\eta}^{(t)} \rangle - \langle \widetilde{\eta}^{(t)}, X(t) \rangle\right) \sqrt{|\widetilde{K}_{\widetilde{T}}^{(t)}|}.$$

Under  $\mathbb{P}_{V,\theta}$ , we have

$$\begin{aligned}
 (7.7) \qquad \frac{M_t}{M_0} &= \exp\left(-\int_0^t \langle W \psi(s) + \eta, dX_s \rangle \right. \\
 &\quad \left. - \frac{1}{2} \int_0^t \langle W \psi(s) + \eta, \mathbb{1}_{s < T} (W \psi(s) + \eta) \rangle ds\right)
 \end{aligned}$$

with

$$\psi(t) = K_{t \wedge T}^{-1} (X(t) + (t \wedge T) \eta).$$

PROOF. We will compute the Itô derivative of  $\ln M_t$ ; the following formulae will be used several times:

$$(7.8) \quad \frac{\partial}{\partial t} K_{t \wedge T} = -\mathbb{1}_{t < T} W, \quad \frac{\partial}{\partial t} K_{t \wedge T}^{-1} = K_{t \wedge T}^{-1} \mathbb{1}_{t < T} W K_{t \wedge T}^{-1},$$

$$\frac{\partial}{\partial t} \widetilde{W}^{(t)} = \widetilde{W}^{(t)} \mathbb{1}_{t < T} \widetilde{W}^{(t)},$$

$$(7.9) \quad \frac{\partial}{\partial t} H_{\beta^{(t)}}^{-1} = H_{\beta^{(t)}}^{-1} \mathbb{1}_{t < T} \left( \frac{1}{t \wedge T} \right)^2 H_{\beta^{(t)}}^{-1}.$$

By (7.8) and the Itô formula, we have

$$(7.10) \quad \begin{aligned} d\langle X(t), \widetilde{W}^{(t)} X(t) \rangle &= 2\langle dX(t), \widetilde{W}^{(t)} X(t) \rangle + \langle \widetilde{W}^{(t)} X(t), \mathbb{1}_{t < T} \widetilde{W}^{(t)} X(t) \rangle dt \\ &\quad + \text{Trace}(\widetilde{W}^{(t)} \mathbb{1}_{t < T}) dt, \end{aligned}$$

where in the second term we used that the operator  $\widetilde{W}^{(t)}$  is symmetric.

By (4.4) of Lemma 2 applied to  $t^0 = t \wedge T$  and  $t^1 = \widetilde{T}^{(t)}$ , we get

$$\langle \widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)} \rangle = \langle \eta, (H_{\beta})^{-1} \eta \rangle - \langle \eta, (H_{\beta^{(t)}})^{-1} \eta \rangle.$$

Using (7.9) and (7.6), it implies

$$(7.11) \quad d\langle \widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)} \rangle = -\langle \widetilde{\eta}^{(t)}, \mathbb{1}_{t < T} \widetilde{\eta}^{(t)} \rangle dt.$$

We have also

$$\frac{\partial}{\partial t} \widetilde{\eta}^{(t)} = \widetilde{W}^{(t)} \mathbb{1}_{t < T} \eta + \widetilde{W}^{(t)} \mathbb{1}_{t < T} \widetilde{W}^{(t)} (t \wedge T) \eta = \widetilde{W}^{(t)} \mathbb{1}_{t < T} \widetilde{\eta}^{(t)}.$$

Hence,

$$(7.12) \quad d\langle \widetilde{\eta}^{(t)}, X(t) \rangle = \langle \widetilde{\eta}^{(t)}, dX(t) \rangle + \langle \widetilde{\eta}^{(t)}, \mathbb{1}_{t < T} \widetilde{W}^{(t)} X_t \rangle dt.$$

Finally, using (4.1) of Lemma 2 applied to  $t^0 = t \wedge T$  and  $t^1 = \widetilde{T}^{(t)}$ , we get

$$(7.13) \quad K_T^{-1} = K_{t \wedge T}^{-1} (\widetilde{K}_{\widetilde{T}}^{(t)})^{-1},$$

which implies by (7.8),

$$(7.14) \quad \begin{aligned} \frac{\partial}{\partial t} \ln |\widetilde{K}_{\widetilde{T}}^{(t)}| &= -\frac{\partial}{\partial t} \ln |K_{t \wedge T}| = -\text{Trace}(\mathbb{1}_{t < T} W K_{t \wedge T}^{-1}) \\ &= -\text{Trace}(\mathbb{1}_{t < T} \widetilde{W}^{(t)}). \end{aligned}$$

Combining (7.10), (7.11), (7.12) and (7.14), we get using that  $W\psi(t) + \eta = \widetilde{W}^{(t)} X(t) + \widetilde{\eta}^{(t)}$ ,

$$\begin{aligned} d \ln M_t &= -\langle dX(t), \widetilde{W}^{(t)} X(t) + \widetilde{\eta}^{(t)} \rangle - \frac{1}{2} \langle \widetilde{W}^{(t)} X(t), \mathbb{1}_{t < T} \widetilde{W}^{(t)} X(t) \rangle dt \\ &\quad - \frac{1}{2} \langle \widetilde{\eta}^{(t)}, \mathbb{1}_{t < T} \widetilde{\eta}^{(t)} \rangle dt - \langle \widetilde{\eta}^{(t)}, \mathbb{1}_{t < T} \widetilde{W}^{(t)} X_t \rangle dt \\ &= -\langle W\psi(t) + \eta, dX_t \rangle - \frac{1}{2} \langle W\psi(t) + \eta, \mathbb{1}_{t < T} (W\psi(t) + \eta) \rangle dt. \end{aligned}$$

Consider now a positive measurable test function  $\phi((X_s)_{s \leq t})$ . Denote by  $\overline{\mathbb{E}}_V^{W,\theta,\eta}$  (resp.,  $\mathbb{E}_{V,\theta}$ ), the expectation with respect to  $\overline{\mathbb{P}}_V^{W,\theta,\eta}$  (resp.,  $\mathbb{P}_{V,\theta}$ ). We have, by (7.5),

$$\begin{aligned}
 & \overline{\mathbb{E}}_V^{W,\theta,\eta}(\phi((X_s)_{s \leq t})) \\
 &= \mathbb{E}_{V,\theta} \left( \phi((X_s)_{s \leq t}) \mathbb{1}_{H_{\frac{1}{2T}} > 0} \cdot e^{\frac{1}{2}(\theta, W\theta) - \frac{1}{2}(\eta, (K_T)^{-1}T\eta) + (\eta, \theta)} \frac{1}{\sqrt{|K_T|}} \right) \\
 (7.15) \quad &= \mathbb{E}_{V,\theta} \left( \frac{M_t}{M_0} \phi((X_s)_{s \leq t}) \mathbb{1}_{H_{\frac{1}{2T}} > 0} \right. \\
 & \quad \left. \times e^{\frac{1}{2}(X(t), \widetilde{W}^{(t)} X(t)) - \frac{1}{2}(\widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)}) + (\widetilde{\eta}^{(t)}, X(t))} \frac{1}{\sqrt{|\widetilde{K}_{\widetilde{T}}^{(t)}|}} \right).
 \end{aligned}$$

Let us denote by  $\langle \cdot, \cdot \rangle_{V(t)}$  the usual scalar product on  $\mathbb{R}^{V(t)}$  (we keep denoting by  $\langle \cdot, \cdot \rangle$  the usual scalar product on  $\mathbb{R}^V$ ). As  $X(t)$  vanishes on  $V \setminus V(t)$ , we have

$$\langle X(t), \widetilde{W}^{(t)} X(t) \rangle = \langle X(t), \widetilde{W}^{(t)} X(t) \rangle_{V(t)}, \quad \langle \widetilde{\eta}^{(t)}, X(t) \rangle = \langle \widetilde{\eta}^{(t)}, X(t) \rangle_{V(t)}.$$

By (4.5), since  $(\widetilde{H}_{\beta}^{(t)})^{-1} = (\widetilde{K}_{\widetilde{T}}^{(t)})^{-1} \widetilde{T}^{(t)}$  and since  $\widetilde{T}^{(t)}$  vanishes on the subset  $V \setminus V(t)$  and  $\widetilde{H}_{\beta}^{(t)}$  is symmetric, we get

$$\langle \widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)} \rangle = \langle \widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)} \rangle_{V(t)}.$$

Moreover,

$$|\widetilde{K}_{\widetilde{T}}^{(t)}| = |\text{Id} - \widetilde{T}^{(t)} \widetilde{W}^{(t)}| = |(\text{Id} - \widetilde{T}^{(t)} \widetilde{W}^{(t)})_{V(t), V(t)}|$$

and

$$\mathbb{1}_{H_{\frac{1}{2T}} > 0} = \mathbb{1}_{H_{\beta(t)} > 0} \mathbb{1}_{\widetilde{H}_{\beta}^{(t)} > 0}$$

thus

$$\begin{aligned}
 & \mathbb{1}_{H_{\frac{1}{2T}} > 0} e^{\frac{1}{2}(X(t), \widetilde{W}^{(t)} X(t)) - \frac{1}{2}(\widetilde{\eta}^{(t)}, (\widetilde{H}_{\beta}^{(t)})^{-1} \widetilde{\eta}^{(t)}) + (\widetilde{\eta}^{(t)}, X(t))} \frac{1}{\sqrt{|\widetilde{K}_{\widetilde{T}}^{(t)}|}} \\
 (7.16) \quad &= \mathbb{1}_{H_{\beta(t)} > 0} \frac{d\overline{\mathbb{P}}_{V(t)}^{\widetilde{W}^{(t)}, X(t), \widetilde{\eta}^{(t)}}}{d\mathbb{P}_{V(t), X(t)}}.
 \end{aligned}$$

Therefore, using (7.15) and (7.16),

$$\begin{aligned}
 & \overline{\mathbb{E}}_V^{W,\theta,\eta}(\phi((X_s)_{s \leq t})) \\
 &= \mathbb{E}_{V,\theta} \left( \mathbb{1}_{H_{\beta(t)} > 0} \frac{M_t}{M_0} \phi((X_s)_{s \leq t}) \overline{\mathbb{E}}_{V(t)}^{\widetilde{W}^{(t)}, X(t), \widetilde{\eta}^{(t)}} [1] \right) \\
 &= \mathbb{E}_{V,\theta} (\mathbb{1}_{H_{\beta(t)} > 0} \phi((X_s)_{s \leq t}) e^{\int_0^t \langle W\psi(s) + \eta, dX_s \rangle - \frac{1}{2} \int_0^t \langle W\psi(s) + \eta, \mathbb{1}_{s < T} (W\psi(s) + \eta) \rangle ds}),
 \end{aligned}$$

where we used Lemma 4 in the second equality. It implies that

$$\begin{aligned}
 \overline{\mathbb{P}}_V^{W,\theta,\eta} &= \mathbb{1}_{H_{\beta(t)} > 0} \exp \left( \int_0^t \langle W\psi(s) + \eta, dX_s \rangle \right. \\
 & \quad \left. - \frac{1}{2} \int_0^t \langle W\psi(s) + \eta, \mathbb{1}_{s < T} (W\psi(s) + \eta) \rangle ds \right) \mathbb{P}_{V,\theta}.
 \end{aligned}$$

Finally, by Girsanov’s theorem, we know that under the law

$$(7.17) \quad \exp\left(\int_0^t \langle W\psi(s) + \eta, dX_s \rangle - \frac{1}{2} \int_0^t \langle W\psi(s) + \eta, \mathbb{1}_{s < T}(W\psi(s) + \eta) \rangle ds\right) \mathbb{P}_{V,\theta}$$

the process

$$(\tilde{B}(t))_{t \geq 0} := \left( X_t + \int_0^t \mathbb{1}_{s < T}(W\psi(s) + \eta) ds \right)_{t \geq 0}$$

is a Brownian motion stopped at time  $T$ , the first hitting time of 0 by  $(X(t))$ . (Indeed, recall that  $\mathbb{P}_{V,\theta}$  is the law of independent Brownian motions starting at  $\theta$  and stopped at their first hitting time of 0). Hence,

$$dX(t) = \mathbb{1}_{t < T} d\tilde{B}(t) + \mathbb{1}_{t < T}(W\psi(t) + \eta) dt,$$

and under the law (7.17),  $X$  is solution of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$  with driving Brownian motion  $\tilde{B}$ . By Lemma 1, we know that a.s. under the law (7.17), we have  $H_{\beta(t)} > 0$ , thus  $\bar{\mathbb{P}}_V^{W,\theta,\eta}$  and (7.17) are equal. Hence, under  $\bar{\mathbb{P}}_V^{W,\theta,\eta}$ ,  $(X(t))$  has the law of the solutions of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$ .  $\square$

**8. Proof of the Abelian properties: Theorem 2.**

PROOF OF THEOREM 2(I), (II). Consider first the restriction property (i). By Theorem 1, conditionally on  $T$ ,  $(X_i(t))_{i \in V}$  are independent Bessel bridges from  $\theta_i$  to 0 in time  $T_i$ . By Theorem 1 and Lemma C,  $\frac{1}{2T_U}$  is  $\nu_U^{W_{U,U},\theta_U,\tilde{\eta}}$  distributed. By Theorem 1 applied to the set  $U$  and parameters  $W_{U,U}, \theta_U, \tilde{\eta}$ , it implies that  $X_U$  has the law of the solutions of  $\mathcal{E}_U^{W,\theta,\tilde{\eta}}(X)$ .

For (ii), the same argument applies, using that  $\beta_{U^c}$ , conditionally on  $\beta_U$ , is  $\nu_{U^c}^{W,\theta_{U^c},\tilde{\eta}}$  distributed.  $\square$

PROOF OF THEOREM 2(III). We adopt the same notation as in the proof of Theorem 1: we denote by  $\mathbf{W} = C(\mathbb{R}_+, \mathbb{R})$  (resp.,  $\mathbf{W}_V = C(\mathbb{R}_+, \mathbb{R}^V)$ ) the Wiener space and  $X(t)$  (resp.,  $(X_i(t))_{i \in V}$ ) the canonical process of the Wiener space; we denote by  $T$  (resp.,  $T = (T_i)_{i \in V}$ ) the hitting time of 0 by  $(X(t))$  (resp., by the processes  $(X_i(t))$ ). For a real  $T > 0$ , recall that  $\mathbb{B}_{\theta,0}^{3,T}$  and  $\mathbb{E}_{\theta,0}^{3,T}$  denotes the law (resp., the expectation) on  $\mathbf{W}$  of a Bessel bridge from  $\theta$  to 0 on time interval  $[0, T]$  (and extended by 0 for  $t \geq T$ ).<sup>3</sup> Recall also that  $\mathbb{E}_V^{W,\theta,\eta}(\cdot)$  denotes the expectation with respect to the law on  $\mathbf{W}_V$  of the solution of the SDE  $\mathcal{E}_V^{W,\theta,\eta}(X)$ .

Following [15] page 463, under  $\mathbb{B}_{\theta,0}^{3,T}$ , the law of  $X(t)$  for some  $0 < t < T$  is given by  $p_{\theta,0}^{3,t,T}(y) dy$  on  $\mathbb{R}_+$ , with

$$(8.1) \quad p_{\theta,0}^{3,t,T}(y) = \frac{1}{\sqrt{2\pi t} \theta} \left(\frac{T}{T-t}\right)^{3/2} e^{-\frac{y^2}{2(T-t)} + \frac{\theta^2}{2T}} \left( e^{-\frac{(y-\theta)^2}{2t}} - e^{-\frac{(y+\theta)^2}{2t}} \right) \quad \forall y \geq 0.$$

Moreover, the Markov property of the Bessel bridge implies that under  $\mathbb{B}_{\theta,0}^{3,T}$  and conditionally on  $X(t) = x$ ,  $0 < t < T$ , the law of  $((X(u))_{0 \leq u \leq t}, (X(t+u))_{0 \leq u \leq T-t})$  is given by

$$(8.2) \quad \mathbb{B}_{\theta,x}^{3,t} \otimes \mathbb{B}_{x,0}^{3,T-t}.$$

<sup>3</sup>As before, we use a different typeface embodied T to distinguish real variables representing a time from the random hitting time  $T$ .

Let us denote by  $\bar{\nu}_V^{W,\theta,\eta}$  the distribution of  $\mathbb{T} = (\mathbb{T}_i)_{i \in V} := \frac{1}{2\beta}$  under the distribution  $\nu_V^{W,\theta,\eta}(d\beta)$ , so that

$$\begin{aligned} \bar{\nu}_V^{W,\theta,\eta}(d\mathbb{T}) &= \mathbb{1}_{H_{\frac{1}{2\mathbb{T}}} > 0} \left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac{1}{2}\langle \theta, \frac{1}{\mathbb{T}} \theta \rangle + \frac{1}{2}\langle \theta, W\theta \rangle - \frac{1}{2}\langle \eta, (H_{\frac{1}{2\mathbb{T}}})^{-1} \eta \rangle + \langle \eta, \theta \rangle} \\ &\quad \times \frac{\prod_{i \in V} \theta_i}{\sqrt{|H_{\frac{1}{2\mathbb{T}}}|}} \prod_{i \in V} \frac{1}{2\mathbb{T}_i^2} d\mathbb{T}_i. \end{aligned}$$

Let  $t^0 = (t_i^0)_{i \in V} \in \mathbb{R}_+^V$  be as in the statement of the theorem. For  $\mathbb{T} = (\mathbb{T}_i)_{i \in V} \in \mathbb{R}_+^V$ , set

$$(8.3) \quad V(t^0, \mathbb{T}) = \{i \in V, \mathbb{T}_i > t_i^0\}.$$

Fix  $U \subset V$ . Let  $h, g$  be bounded measurable test functions. By Theorem 1, we have

$$\begin{aligned} &\mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{V(t^0, \mathbb{T})=U} h((X_i[0, t_i^0])_{i \in V}) g((X_i([t_i^0, \mathbb{T}_i]))_{i \in U})] \\ &= \int \mathbb{1}_{V(t^0, \mathbb{T})=U} \bigotimes_{i \in V} \mathbb{E}_{\theta_i, 0}^{3, \mathbb{T}_i} [h((X_i[0, t_i^0])_{i \in V}) g((X_i([t_i^0, \mathbb{T}_i]))_{i \in U})] d\bar{\nu}_V^{W,\theta,\eta}(\mathbb{T}). \end{aligned}$$

By the Markov property (8.2), when  $V(t^0, \mathbb{T}) = U$ , we have that

$$\begin{aligned} &\bigotimes_{i \in V} \mathbb{E}_{\theta_i, 0}^{3, \mathbb{T}_i} [h((X_i[0, t_i^0])_{i \in V}) g((X_i([t_i^0, \mathbb{T}_i]))_{i \in U})] \\ &= \int_{\mathbb{R}_+^U} K(x_U, t_U^0, \mathbb{T}_{U^c}) \bigotimes_{i \in U} \mathbb{E}_{x_i, 0}^{3, \mathbb{T}_i - t_i^0} [g((X_i([0, \mathbb{T}_i - t_i^0]))_{i \in U})] \\ &\quad \times \left( \prod_{i \in U} p_{\theta_i, 0}^{3, t_i^0, \mathbb{T}_i}(x_i) dx_i \right), \end{aligned}$$

where

$$(8.4) \quad K(x_U, t_U^0, \mathbb{T}_{U^c}) := \left( \bigotimes_{i \in U} \mathbb{E}_{\theta_i, x_i}^{3, t_i^0} \bigotimes_{i \in U^c} \mathbb{E}_{\theta_i, 0}^{3, \mathbb{T}_i} \right) [h((X_i[0, t_i^0])_{i \in V})]$$

is a function that only depends on  $(x_i, t_i^0)_{i \in U}, (\mathbb{T}_i)_{i \in U^c}$ . We thus get

$$\begin{aligned} &\mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{V(t^0, \mathbb{T})=U} h((X_i[0, t_i^0])_{i \in V}) g((X_i([t_i^0, \mathbb{T}_i]))_{i \in U})] \\ &= \int \mathbb{1}_{V(t^0, \mathbb{T})=U} K(x_U, t_U^0, \mathbb{T}_{U^c}) \bigotimes_{i \in U} \mathbb{E}_{x_i, 0}^{3, \mathbb{T}_i - t_i^0} [g((X_i([0, \mathbb{T}_i - t_i^0]))_{i \in U})] \\ &\quad \times \left( \prod_{i \in U} p_{\theta_i, 0}^{3, t_i^0, \mathbb{T}_i}(x_i) dx_i \right) d\bar{\nu}_V^{W,\theta,\eta}(\mathbb{T}). \end{aligned}$$

In the sequel, on the event  $\{V(t^0, \mathbb{T}) = U\}$ , we set

$$(\tilde{\mathbb{T}}_i)_{i \in U} = (\mathbb{T}_i - t_i^0)_{i \in U}.$$

The strategy is now to show that we can combine the terms  $\prod_{i \in U} p_{\theta_i, 0}^{3, t_i^0, \mathbb{T}_i}(x_i)$  and the measure  $d\bar{\nu}_V^{W,\theta,\eta}(\mathbb{T})$  in such a way that on the event  $\{V(t^0, \mathbb{T}) = U\}$ , changing from variables  $(\mathbb{T}_i)_{i \in U}$  to variables  $(\tilde{\mathbb{T}}_i)_{i \in U}$ , we end up with a function of  $(x_U, t_U^0, \mathbb{T}_{U^c})$  and the measure  $\nu_U^{\tilde{W}^{(t^0)}, x, \tilde{\eta}^{(t^0)}}(d\tilde{\mathbb{T}})$ ; see the forthcoming formula (8.5).

Let us denote by  $\langle \cdot, \cdot \rangle_U$  the usual scalar product on  $\mathbb{R}^U$  (recall that we keep denoting by  $\langle \cdot, \cdot \rangle$  the usual scalar product on  $\mathbb{R}^V$ ). Note that  $\tilde{\eta}^{(t^0)}$  and  $\tilde{W}^{(t^0)}$  defined in Theorem 2(iii)

correspond to  $\tilde{\eta}$  and  $\tilde{W}$  of Lemma 2 for  $t^0 \wedge \mathbb{T}$  and  $\tilde{\mathbb{T}}$ . Hence, by (4.4) of Lemma 2, we get with  $\tilde{H}_{\frac{1}{2\tilde{\mathbb{T}}}}^{(t^0)} = \frac{1}{2\tilde{\mathbb{T}}} - \tilde{W}^{(t^0)}$  that

$$\langle \tilde{\eta}^{(t^0)}, (\tilde{H}_{\frac{1}{2\tilde{\mathbb{T}}}}^{(t^0)})^{-1} \tilde{\eta}^{(t^0)} \rangle_U - \langle \eta, (H_{\frac{1}{2\mathbb{T}}})^{-1} \eta \rangle = -\langle \eta, (H_{\frac{1}{2t^0 \wedge \mathbb{T}}})^{-1} \eta \rangle$$

and by (4.2) of Lemma 2,

$$\frac{|(\tilde{H}_{\frac{1}{2\tilde{\mathbb{T}}}}^{(t^0)})_{U,U}|}{|H_{\frac{1}{2\mathbb{T}}}|} = |K_{t^0 \wedge \mathbb{T}}| \prod_{i \in U} \left( \frac{\mathbb{T}_i}{\tilde{\mathbb{T}}_i} \right).$$

Note that we have

$$\begin{aligned} \prod_{i \in U} p_{\theta_i, 0}^{3, t_i^0, \mathbb{T}_i}(x_i) &= e^{-\frac{1}{2} \langle x, \frac{1}{\mathbb{T}} x \rangle_U + \frac{1}{2} \langle \theta, \frac{1}{\mathbb{T}} \theta \rangle_U} \prod_{i \in U} \left( e^{-\frac{(x_i - \theta_i)^2}{2t_i^0}} - e^{-\frac{(x_i + \theta_i)^2}{2t_i^0}} \right) \\ &\quad \times \frac{1}{\sqrt{2\pi t_i^0}} \frac{x_i}{\theta_i} \left( \frac{\mathbb{T}_i}{\tilde{\mathbb{T}}_i} \right)^{3/2}. \end{aligned}$$

Changing from variables  $(\mathbb{T}_i)_{i \in U}$  to  $(\tilde{\mathbb{T}}_i)_{i \in U}$ , we get

$$\begin{aligned} (8.5) \quad &\mathbb{1}_{V(t^0, \mathbb{T})=U} \left( \prod_{i \in U} p_{\theta_i, 0}^{3, \mathbb{T}_i}(x_i) \right) \bar{\nu}_V^{W, \theta, \eta}(d\mathbb{T}) \\ &= \mathbb{1}_{\mathbb{T}_i < t_i^0, i \in U^c} \Xi(x_U, t_U^0, \mathbb{T}_{U^c}) \nu_U^{\tilde{W}^{(t^0)}, x, \tilde{\eta}^{(t^0)}}(d\tilde{\mathbb{T}}) \prod_{i \in U^c} d\mathbb{T}_i \end{aligned}$$

for some explicit function  $\Xi(x_U, t_U^0, \mathbb{T}_{U^c})$  that only depends on  $(x_i, t_i^0)_{i \in U}, (\mathbb{T}_i)_{i \in U^c}$ .

Continuing our computation, we have

$$\begin{aligned} (8.6) \quad &\mathbb{E}_V^{W, \theta, \eta} [\mathbb{1}_{V(t^0, \mathbb{T})=U} h((X_i[0, t_i^0])_{i \in V}) g((X_i([t_i^0, T_i]))_{i \in U})] \\ &= \int \mathbb{1}_{\mathbb{T}_i < t_i^0, i \in U^c} K(x_U, t_U^0, \mathbb{T}_{U^c}) \Xi(x_U, t_U^0, \mathbb{T}_{U^c}) \\ &\quad \times \mathbb{E}_U^{\tilde{W}^{(t^0)}, x, \tilde{\eta}^{(t^0)}} [g((X_i([0, T_i]))_{i \in U})] \prod_{i \in U} dx_i \prod_{i \in U^c} d\mathbb{T}_i. \end{aligned}$$

Let us apply the last equality to the case where  $h$  and  $g$  are replaced by

$$\begin{aligned} \tilde{h}((X_i[0, t_i^0])_{i \in V}) &:= h((X_i[0, t_i^0])_{i \in V}) \mathbb{E}_U^{\tilde{W}^{(t^0)}, X_U(t^0), \tilde{\eta}^{(t^0)}} (g((X_i([0, T_i]))_{i \in U})), \\ \tilde{g} &:= 1. \end{aligned}$$

The identity (8.6) gives in this case

$$\begin{aligned} (8.7) \quad &\mathbb{E}_V^{W, \theta, \eta} [\mathbb{1}_{V(t^0, \mathbb{T})=U} h((X_i[0, t_i^0])_{i \in V}) \mathbb{E}_U^{\tilde{W}^{(t^0)}, X_U(t^0), \tilde{\eta}^{(t^0)}} (g((X_i([0, T_i]))_{i \in U}))] \\ &= \int \mathbb{1}_{\mathbb{T}_i < t_i^0, i \in U^c} \tilde{K}(x_U, t_U^0, \mathbb{T}_{U^c}) \Xi(x_U, t_U^0, \mathbb{T}_{U^c}) \prod_{i \in U} dx_i \prod_{i \in U^c} d\mathbb{T}_i, \end{aligned}$$

where, using (8.4) applied to  $\tilde{h}$  instead,

$$\tilde{K}(x_U, t_U^0, \mathbb{T}_{U^c}) = K(x_U, t_U^0, \mathbb{T}_{U^c}) \mathbb{E}_U^{\tilde{W}^{(t^0)}, x, \tilde{\eta}^{(t^0)}} [g((X([0, T]))_{i \in U})].$$

Remark that the right-hand sides of (8.6) and (8.8) are thus the same. Hence, we conclude that

$$\begin{aligned} & \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{V(t^0,T)=U}h((X_i[0,t_i^0])_{i \in V})g((X_i([t_i^0,T_i]))_{i \in U})] \\ &= \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{V(t^0,T)=U}h((X_i[0,t_i^0])_{i \in V})\mathbb{E}_U^{\tilde{W}^{(t^0)},X_U(t^0),\tilde{\eta}^{(t^0)}}(g((X_i([0,T_i]))_{i \in U}))]. \end{aligned}$$

Summing on all possible choices of  $U$ , we exactly get that the law of  $(X_i([t_i^0,T_i]))$ , conditionally on  $\mathcal{F}^X(t^0)$ , is the law of the solutions of the SDE  $\mathcal{E}_V^{\tilde{W}^{(t^0)},X(t^0),\tilde{\eta}^{(t^0)}}(X)$ .  $\square$

PROOF OF THEOREM 2(iv). Fix as before  $U \subset V$ . With the notation of the theorem and (8.3),

$$V(T^0, T) = \{i \in V, T_i > T_i^0\},$$

and for  $U \subset V$  fixed, we define the event

$$\mathcal{A}(T^0, T) = \{V(T^0, T) = U\} = \{T_i > t_i^0, i \in U\} \cap \{T_i \leq t_i^0, i \in U^c\}.$$

We simply write  $\{T_0 < \infty\}$  for the event  $\{T_i^0 < \infty, \forall i \in V\}$ . In order to prove the strong Markov property (iv), it is enough to prove that, for any bounded test function  $h, g$ , depending continuously on finitely many marginals of  $X$ , we have

$$\begin{aligned} & \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{T^0 < \infty}\mathbb{1}_{\mathcal{A}(T^0,T)}h((X_i[0,T_i^0])_{i \in V})g((X_i[T_i^0,T_i])_{i \in U})] \\ (8.8) \quad &= \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{T^0 < \infty}\mathbb{1}_{\mathcal{A}(T^0,T)}h((X_i[0,T_i^0])_{i \in V}) \\ & \quad \times \mathbb{E}_U^{\tilde{W}^{(T_0)},X_U(T^0),\tilde{\eta}^{(T^0)}}(g((X_i[0,T_i])_{i \in U}))]. \end{aligned}$$

We define the sequence of stopping times, for all  $i \in V$ , by

$$[T_i^0]_n = \frac{k}{2^n} \quad \text{when } \frac{k-1}{2^n} \leq T_i^0 < \frac{k}{2^n}, k \in \mathbb{N},$$

and  $[T_i^0]_n = \infty$  when  $T_i^0 = \infty$ . We can check that  $[T^0] := ([T_i^0]_n)_{i \in V}$  is a multistopping time in the sense of Theorem 2(iv), since for  $(k_i)_{i \in V} \in \mathbb{N}^V$ ,

$$\bigcap_{i \in V} \left\{ \frac{k_i - 1}{2^n} \leq T_i^0 < \frac{k_i}{2^n} \right\} \in \sigma\left(X_i(s), s \leq \frac{k_i}{2^n}, i \in V\right).$$

Moreover,  $[T_i^0]_n$  decreases a.s. to  $T_i^0$  and for  $n$  large enough  $V([T^0]_n) = V(T^0)$  a.s. This implies that a.s.,

$$\begin{aligned} & \mathbb{1}_{T^0 < \infty}\mathbb{1}_{\mathcal{A}(T^0,T)}g((X_i[T_i^0,T_i])_{i \in U}) \\ &= \lim_{n \rightarrow \infty} \mathbb{1}_{[T^0]_n < \infty}\mathbb{1}_{\mathcal{A}([T^0]_n,T)}g((X_i[[T_i^0]_n,T_i])_{i \in U}). \end{aligned}$$

Therefore, by dominated convergence theorem,

$$\begin{aligned} & \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{T^0 < \infty}\mathbb{1}_{\mathcal{A}(T^0,T)}h((X_i[0,T_i^0])_{i \in V})g((X_i[T_i^0,T_i])_{i \in U})] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_V^{W,\theta,\eta}[\mathbb{1}_{[T^0]_n < \infty}\mathbb{1}_{\mathcal{A}([T^0]_n,T)}h((X_i[0,T_i^0])_{i \in V})g((X_i[[T_i^0]_n,T_i])_{i \in U})] \\ &= \lim_{n \rightarrow \infty} \sum_{k=(k_i)_{i \in V} \in \mathbb{N}^V} \mathbb{E}_V^{W,\theta,\eta} \left[ \left( \prod_{i \in V} \mathbb{1}_{\frac{k_i-1}{2^n} \leq T_i^0 < \frac{k_i}{2^n}} \right) \mathbb{1}_{\mathcal{A}(\frac{k}{2^n},T)}h((X_i[0,T_i^0])_{i \in V}) \right. \\ & \quad \left. \times g\left(\left(X_i\left[\frac{k_i}{2^n}, T_i\right]\right)_{i \in U}\right) \right], \end{aligned}$$

where in the last equality we sum on the possible values of each  $[T_i^0]_n, i \in V$ . Note that

$$\left( \prod_{i \in V} \mathbb{1}_{\frac{k_i-1}{2^n} \leq T_i^0 < \frac{k_i}{2^n}} \right) \mathbb{1}_{\mathcal{A}(\frac{k}{2^n}, T)} h((X_i[0, T_i^0])_{i \in V})$$

is  $\mathcal{F}^X(\frac{k}{2^n})$  measurable, so we can apply the Markov property (iii), and we get

$$\begin{aligned} & \mathbb{E}_V^{W, \theta, \eta} \left[ \mathbb{1}_{\mathcal{A}(\frac{k}{2^n}, T)} \left( \prod_{i \in V} \mathbb{1}_{\frac{k_i-1}{2^n} \leq T_i^0 < \frac{k_i}{2^n}} \right) h((X_i[0, T_i^0])_{i \in V}) g \left( \left( X_i \left[ \frac{k_i}{2^n}, T_i \right] \right)_{i \in V(\frac{k}{2^n})} \right) \right] \\ &= \mathbb{E}_V^{W, \theta, \eta} \left[ \mathbb{1}_{\mathcal{A}(\frac{k}{2^n}, T)} \left( \prod_{i \in V} \mathbb{1}_{\frac{k_i-1}{2^n} \leq T_i^0 < \frac{k_i}{2^n}} \right) h((X_i[0, T_i^0])_{i \in V}) \right. \\ & \quad \left. \times \mathbb{E}_U^{\tilde{W}(\frac{k}{2^n}), X_U(\frac{k}{2^n}), \tilde{\eta}(\frac{k}{2^n})} (g((X_i[0, T_i])_{i \in U})) \right]. \end{aligned}$$

Summing on possible values of  $(k_i)$ , we get

$$\begin{aligned} & \mathbb{E}_V^{W, \theta, \eta} [\mathbb{1}_{T^0 < \infty} \mathbb{1}_{\mathcal{A}(T^0, T)} h((X_i[0, T_i^0])_{i \in V}) g((X_i[T_i^0, T_i])_{i \in U})] \\ (8.9) \quad &= \lim_{n \rightarrow \infty} \mathbb{E}_V^{W, \theta, \eta} [\mathbb{1}_{T^0 < \infty} \mathbb{1}_{\mathcal{A}([T^0]_n, T)} h((X_i[0, T_i^0])_{i \in V}) \\ & \quad \times \mathbb{E}_U^{\tilde{W}([T^0]_n), X_U([T^0]_n), \tilde{\eta}([T^0]_n)} (g((X_i[0, T_i])_{i \in U}))]. \end{aligned}$$

We conclude the proof thanks to the Feller property (see, e.g., Section 18.6 of [21]) proved in the lemma below.  $\square$

LEMMA 5. *The function  $(W, \theta, \eta) \rightarrow \mathbb{E}_V^{W, \theta, \eta}(g((X_i[0, T_i])_{i \in V}))$  is continuous on  $(\mathbb{R}_+^*)^E \times (\mathbb{R}_+^*)^V \times \mathbb{R}_+^V$  for any bounded measurable function  $g$  depending only on a finite number of marginals.*

PROOF OF LEMMA 5. It is enough to consider the case  $\eta = 0$ , since the case  $\eta \neq 0$  is a marginal of the case  $\eta = 0$  by Lemma C. Without loss of generality, we assume  $W_{i,i} = 0, \forall i$ . The proof follows from the representation Theorem 1 and the two ingredients below.

Under the three-dimensional Bessel bridge law, the expectation  $\mathbb{E}_{\theta, 0}^{3, T}(g((X_i[0, T_i])_{i \in V}))$  is continuous in  $(\theta, T)$ . Indeed, the three-dimensional Bessel bridge is the norm of a three-dimensional Brownian bridge from  $x$  to 0 if  $\|x\| = \theta$ , and the three-dimensional Brownian bridge from  $x$  to 0 can be represented as  $x + B_t^{(3)} - \frac{t}{T} B_T^{(3)} - \frac{t}{T} x$  where  $(B_t^{(3)})$  is a three-dimensional standard Brownian motion.

On the other hand, the measure  $\nu_V^W(d\beta)$  can be dominated locally on the parameters  $W, \theta$  after some change of coordinates, following [18]. (Note that the density  $\nu_V^{W, \theta}$  in the present paper correspond to  $\nu^{W, \theta^2}$  in [18].) For convenience, write  $V = \{1, \dots, N\}$ . By the change of variables,  $(\beta_i)_{i \in V} \rightarrow (x_i)_{i \in V}$  from  $\{\beta, H_\beta > 0\}$  to  $(\mathbb{R}_+^*)^V$  described in the proof of Theorem 1 of [18] (see p. 3977), we have

$$\begin{aligned} & \mathbb{1}_{H_\beta > 0} \exp\left(-\frac{1}{2}(\theta, H_\beta \theta) - \frac{1}{2} \sum_{i,j} W_{i,j} \theta_i \theta_j\right) \frac{1}{\sqrt{\det H_\beta}} d\beta \\ (8.10) \quad &= \frac{1}{2^N} \mathbb{1}_{x \in \mathbb{R}_+^N} \exp\left(-\sum_{l=1}^N \left(\frac{\theta_l^2 x_l}{2} + \frac{1}{2x_l} \left(\sum_{k=l+1}^N \theta_k^2 H_{l,k}^2\right)\right)\right) \frac{1}{\sqrt{x_1 \cdots x_N}} dx, \end{aligned}$$

following the notation there, in particular the definition of  $\{x_i, H_{i,j} : 1 \leq i, j \leq N\}$ . By definition, for any  $l \geq 1, H_{l,k} \geq W_{l,k}$ .



Now fix  $W^0, \theta^0$ , let  $\Omega$  be a neighborhood of  $(W, \theta)$ , denote

$$\underline{W}_{l,k} = \inf_{\Omega} W_{l,k}, \quad \underline{\theta}_l = \inf_{\Omega} \theta_l.$$

For any  $W, \theta \in \Omega$ , we have  $H_{l,k} \geq W_{l,k} \geq \underline{W}_{l,k}$  and  $\theta_l \geq \underline{\theta}_l$  for all  $1 \leq l, k \leq N$ , so the density in (8.10) is locally uniformly bounded (in the variables  $x$ s) by

$$\mathbb{1}_{x \geq 0} \exp\left(-\sum_{l=1}^N \left(\frac{\theta_l^2 x_l}{2} + \frac{1}{2x_l} \left(\sum_{k=l+1}^N \theta_k^2 W_{l,k}^2\right)\right)\right) \frac{1}{\sqrt{x_1 \cdots x_N}},$$

which is an integrable function, as  $x_1, \dots, x_{N-1}$  are distributed as inverse of IG distribution, and  $x_N$  is a Gamma distributed random variable.  $\square$

**9. Relation with the martingales associated with the VRJP.** Consider in this section that  $V$  is infinite and that  $W$  is such that the associated graph  $\mathcal{G}$  has finite degree at each vertex and is connected. Following [17], we extend the definition of the distribution  $\nu_V^{W,\theta}$  to the case of this infinite graph. In order to be coherent with [17], we assume that  $W$  is zero on the diagonal. Note that we slightly generalize the definition of [17] since we consider a general vector  $(\theta_i)_{i \in V} \in (\mathbb{R}_+)^V$ , which is equal to 1 in [17]. (But as noted at the beginning of Section 3 it is in fact not more general since we can always take  $\theta$  to 1 by a change of variables on  $\beta$  and  $W$ .)

Let us recall the construction of the distribution  $\nu_V^{W,\theta}$  obtained by Kolmogorov’s extension theorem. The approach is slightly different from that of [17] and make use of Lemma C(i). Let  $V_n$  be an increasing sequence of subsets such that  $\bigcup_{n \geq 1} V_n = V$ . Consider the vector  $\eta^{(n)} \in (\mathbb{R}_+)^{V_n}$  defined by

$$(9.1) \quad \eta^{(n)} = W_{V_n, V_n^c}(\theta_{V_n^c}).$$

By Lemma C(i), the sequence of distribution  $\nu_{V_n}^{W, \eta^{(n)}}$  is compatible, hence by the Kolmogorov theorem it can be extended to a measure  $\nu_V^{W,\theta}$  on  $(\mathbb{R}_+)^V$ . We define the Schrödinger operator

$$H_\beta := 2\beta - W,$$

on  $\mathbb{R}^V$  associated with the potential  $\beta \sim \nu_V^{W,\theta}$ . Note that  $H_\beta \geq 0$  as the limit of  $(H_\beta)_{V_n, V_n}$  which is positive definite since  $\beta_{V_n}$  has law  $\nu_{V_n}^{W, \eta^{(n)}}$ .

In [19], we considered the sequence of functions  $(\psi_j^{(n)})_{j \in V} \in (\mathbb{R}_+)^V$  defined by

$$(9.2) \quad \begin{cases} (H_\beta \psi^{(n)})_{V_n} = 0, \\ \psi_{V_n^c}^{(n)} = \theta_{V_n^c}, \end{cases}$$

and the operators  $(\widehat{G}^{(n)}(i, j))_{i, j \in V_n}$  by

$$\begin{cases} \widehat{G}_{V_n, V_n}^{(n)} = ((H_\beta)_{V_n, V_n})^{-1}, \\ \widehat{G}^{(n)}(i, j) = 0 \quad \text{if } i \text{ or } j \text{ in not in } V_n. \end{cases}$$

Let  $\mathcal{F}_n = \sigma(\beta_i, i \in V_n)$ , the sigma field generated by  $\beta_{V_n}$ . In [19], Proposition 9, it was proved that  $\psi^{(n)}$  is a vectorial  $\mathcal{F}_n$ -martingale, with quadratic variation given by  $\widehat{G}^{(n)}(i, j)$ , that is, that for all  $i, j$  in  $V$  and all  $n$ ,

$$\mathbb{E}(\psi^{(n+1)}(i)\psi^{(n+1)}(j) - \widehat{G}^{(n+1)}(i, j) | \mathcal{F}_n) = \psi^{(n)}(i)\psi^{(n)}(j) - \widehat{G}^{(n)}(i, j).$$

It was extended in [4] to an exponential martingale property, namely it was proved that for any compactly supported function  $\lambda \in (\mathbb{R}_+)^V$ ,

$$(9.3) \quad e^{-(\lambda, \psi^{(n)}) - \frac{1}{2}(\lambda, \widehat{G}^{(n)}\lambda)},$$

is a  $\mathcal{F}_n$ -martingale.

We can interpret the functions  $\psi^{(n)}$  that appear above in terms of the SDEs. Consider  $X^{(n)}$  the solution of the SDE  $\mathcal{E}_{V_n}^{W, \theta, \eta^{(n)}}$ , where  $\eta^{(n)}$  is defined in (9.1). Denote by  $T^{(n)}$  the associated stopping times and  $\beta^{(n)} = \frac{1}{2T^{(n)}}$  and

$$K_{t \wedge T^{(n)}}^{(n)} = \text{Id}_{V_n, V_n} - (t \wedge T^{(n)})W_{V_n, V_n}, \quad \psi^{(n)}(t) = (K_{t \wedge T^{(n)}}^{(n)})^{-1}X^{(n)}(t),$$

the associated operator and martingale that appear in Lemma 1. We always consider that  $\psi^{(n)}$  is extended to the full set  $V$  by  $\psi_{V_n^c}^{(n)}(t) = \theta_{V_n^c}$ . Considering (9.2), we have that

$$\lim_{t \rightarrow \infty} \psi^{(n)}(t) = \psi^{(n)}.$$

Hence the function  $\psi^{(n)}$  appears as the limit of the continuous martingale  $\psi^{(n)}(t)$ .

It is possible to interpret the exponential martingale property (9.3) in terms of the Abelian properties; see Theorem 2. More precisely, conditionally on  $\sigma(\beta_{V_n})$ , it is possible to construct a continuous martingale that interpolates between  $\psi^{(n)}$  and  $\psi^{(n+1)}$  and with total quadratic variation given by  $\widehat{G}^{(n+1)} - \widehat{G}^{(n)}$ , which explains the exponential martingale property as a consequence the standard exponential martingale property for continuous martingales. We do not give details of this computation which require heavy notation (but the authors will provide details under request).

**10. Case of the two points graph.** We illustrate some of the results in the case of two points. The discussion in this section will be rather informal since it is only meant to be an illustration of the results. Assume  $V = \{1, 2\}$  and  $W_{1,1} = W_{2,2} = 0$ ,  $W_{1,2} = W > 0$ , that is, the graph Laplacian is

$$\begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix}.$$

We denote  $(t \wedge T_1, t \wedge T_2) = (t^{T_1}, t^{T_2})$ . It follows that

$$K_{t \wedge T} = 1 - (t \wedge T)W = \begin{pmatrix} 1 & -t^{T_1}W \\ -t^{T_2}W & 1 \end{pmatrix},$$

$$K_{t \wedge T}^{-1} = \frac{1}{1 - t^{T_1}t^{T_2}W^2} \begin{pmatrix} 1 & t^{T_1}W \\ t^{T_2}W & 1 \end{pmatrix}.$$

And

$$\psi(t) = K_{t \wedge T}^{-1}(X(t) + t^T \eta).$$

The SDEs are now

$$X_i(t) = \theta_i + \int_0^t \mathbb{1}_{s < T_i} dB_i(s) - \int_0^t \mathbb{1}_{s < T_i} \left[ \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right]_i ds,$$

$$i = 1, 2;$$

where

$$T_i = \inf\{t \geq 0, X_i(t) = 0\}.$$

Because of the interactive drifts, it is nontrivial that this equation is well defined for all  $t \geq 0$  (i.e.,  $1 - t^{T_1}t^{T_2}W^2 > 0$ ) as shown in Lemma 1, even though intuitively it is rather clear: if  $1 - t^{T_1}t^{T_2}W^2$  approaches 0 before the hitting times of 0, then it gives a strong negative drift in the equation which pushes the process toward 0.

Theorem 1 asserts that the law of  $(\frac{1}{2T_1}, \frac{1}{2T_2})$  is  $h^W(\theta, \eta, \beta) d\beta$ , where

$$h^W(\theta, \eta, \beta) = \mathbb{1}_{H_\beta > 0} \frac{2}{\pi} e^{-\frac{1}{2}(\theta H_\beta \theta + \eta H_\beta^{-1} \eta + \theta \eta)} \frac{\theta_1 \theta_2}{\sqrt{\det H_\beta}}, \quad H_\beta = \begin{pmatrix} 2\beta_1 & -W \\ -W & 2\beta_2 \end{pmatrix}.$$

Theorem 2(i) asserts that  $X_1(t)$  has the law of a drifted Brownian motion with drift  $\eta_1 - W\theta_2$  (with corresponding formula for  $X_2$  and stopped at first hitting time of 0). Theorem 2(ii) asserts that conditionally on  $(Y_2(t))_{t \geq 0}$ ,  $(X_1(t))$  has the law of a Bessel bridge from  $\theta_1$  to 0 on time interval  $[0, \frac{2T_2}{W^2}]$ , with a constant drift  $-(\eta_1 + W \frac{\eta_2}{2T_2})$ .

Let us show that, if we assume Theorem 1, that is, that we know the law of the hitting times, then it is possible to check by direct computation that conditionally on  $(T_1, T_2)$ ,  $X_1(t)$  and  $X_2(t)$  are 3D-Bessel bridges. Let  $t > 0$ , denote  $\tilde{T} = T - t \wedge T$ . Using Proposition 1, which is a simple property of stationarity of the equation, the shifted process  $\tilde{X}(s) = X(t + s)$ ,  $s \geq 0$ , conditionally on  $\mathcal{F}^X(t)$ , is solution of the same class of SDE. More precisely, if we denote

$$\begin{aligned} \tilde{W}^{(t)} &= \frac{1}{1 - t^{T_1} t^{T_2} W^2} \begin{pmatrix} t^{T_2} W^2 & W \\ W & t^{T_1} W^2 \end{pmatrix}, \\ \begin{pmatrix} \tilde{\eta}_1^{(t)} \\ \tilde{\eta}_2^{(t)} \end{pmatrix} &= \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \frac{W}{1 - t^{T_1} t^{T_2} W^2} \begin{pmatrix} t^{T_1} t^{T_2} W \eta_1 + t^{T_2} \eta_2 \\ t^{T_1} \eta_1 + t^{T_1} t^{T_2} W \eta_2 \end{pmatrix}, \end{aligned}$$

then,  $\tilde{\psi}(s) := \psi(t + s)$ ,  $s \geq 0$ , is equal to

$$\tilde{\psi}(s) = [\text{Id} - (s \wedge \tilde{T}) \tilde{W}^{(t)}]^{-1} [\tilde{X}(s) + (s \wedge \tilde{T}) \tilde{\eta}^{(t)}]$$

and  $\tilde{X}(s)$  is solution of

$$\tilde{X}(s) = X(t) + \int_0^s \mathbb{1}_{u < \tilde{T}} dB(u) - \int_0^s \mathbb{1}_{u < \tilde{T}} [\tilde{W}^{(t)} \tilde{\psi}(u) + \tilde{\eta}^{(t)}] du.$$

In particular, Theorem 2(iii) asserts that, conditionally on  $\mathcal{F}^X(t)$ , the vector  $(\frac{1}{2T_1}, \frac{1}{2T_2})$  has distribution

$$h^{W^{(t)}}(X(t), \tilde{\eta}^{(t)}, \beta) d\beta.$$

(Note that this density can be degenerated, depending on the cardinal of  $V(t) = \{i \in \{1, 2\}, T_i > t\}$ . To be more rigorous, one would need to separate the case where  $V(t)$  is two, or one point.)

Therefore, conditionally on  $(\frac{1}{2T_1}, \frac{1}{2T_2})$ , the law of  $(X(t))$  is a Doob's  $h$ -process of the initial law of  $X(t)$  with Doob's exponential martingale given by

$$M_t = \frac{h(X(t), \tilde{\eta}^{(t)}, \beta^{(t)})}{h(\theta, \eta, \beta)}.$$

By explicit (but rather long and cumbersome) Itô differentiation, we have

$$\begin{aligned} d \log M_t &= \sum_{(i,j) \in \{(1,2), (2,1)\}} \left( \frac{1}{X_i(t)} - \frac{X_i(t)}{T_i - t} \right) dX_i(t) \\ &+ \frac{\mathbb{1}_{t < T_i}}{1 - t^{T_1} t^{T_2} W^2} (X_i(s) + t^{T_1} W X_j(t)) dB_i(t) \\ &+ \text{drift terms.} \end{aligned}$$

Therefore, by applying twice Girsanov's theorem, and using Doob's  $h$ -transform, conditionally on  $(\frac{1}{2T_1}, \frac{1}{2T_2})$ ,  $X(t) = (X_1(t), X_2(t))$  is solution of

$$X_i(t) = \theta_i + \int_0^t \mathbb{1}_{t < T_i} \left( d\tilde{B}_i(s) + \frac{1}{X_i(s)} - \frac{X_i(s)}{T_i - s} ds \right), \quad i = 1, 2,$$

where  $\tilde{B}_1(t)$ ,  $\tilde{B}_2(t)$  are independent Brownian motion under the conditional law  $\mathbb{P}(\cdot|T_1, T_2)$ . That is, conditionally on  $(T_1, T_2)$ ,  $X_1(t)$ ,  $X_2(t)$  are independent 3D-Bessel bridges.

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