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EXACT ASYMPTOTICS FOR DUARTE AND SUPERCRITICAL ROOTED KINETICALLY CONSTRAINED MODELS

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Kinetically constrained models (KCM) are a class of interacting particle systems which represent a natural stochastic (and nonmonotone) counterpart of the family of cellular automata known as $\mathcal{U}$-bootstrap percolation. A key issue for KCM is to identify the divergence of the characteristic time scales when the equilibrium density of empty sites, q, goes to zero. In (*Ann. Probab.* **47** (2019) 324–361; *Comm. Math. Phys.* **369** (2019) 761–809), a general scheme was devised to determine a sharp upper bound for these time scales. Our paper is devoted to developing a (very different) technique which allows to prove matching lower bounds. We analyse the class of two-dimensional *supercritical rooted KCM* and the *Duarte KCM*. We prove that the relaxation time and the mean infection time diverge for supercritical rooted KCM as $e^{\Theta((\log q)^2)}$ and for Duarte KCM as $e^{\Theta((\log q)^4/q^2)}$ when $q \downarrow 0$. These results prove the conjectures put forward in (*European J. Combin.* **66** (2017) 250–263; *Comm. Math. Phys.* **369** (2019) 761–809) for these models, and establish that the time scales for these KCM diverge much faster than for the corresponding $\mathcal{U}$-bootstrap processes, the main reason being the occurrence of energy barriers which determine the dominant behaviour for KCM, but which do not matter for the bootstrap dynamics.

1. Introduction. Kinetically constrained models (KCM) are interacting particle systems on the integer lattice $\mathbb{Z}^d$, which were introduced in the physics literature in the 1980s in order to model the liquid-glass transition (see, e.g., [14, 24] for reviews), a major and still largely open problem in condensed matter physics [4]. A generic KCM is a continuous time Markov process of Glauber type characterised by a finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\mathcal{U} = \{X_1, \dots, X_m\}$, its *update family*. A configuration ω is defined by assigning to each site $x \in \mathbb{Z}^d$ an occupation variable $\omega_x \in \{0, 1\}$, corresponding to an empty or occupied site, respectively. Each site $x \in \mathbb{Z}^d$ waits an independent, mean one, exponential time and then, iff there exists $X \in \mathcal{U}$ such that $\omega_y = 0$ for all $y \in X + x$, site x is updated to occupied with probability p and to empty with probability $q = 1 - p$. Since each *update set* X_i belongs to $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, the constraints never depend on the state of the to-be-updated site. As a consequence, the dynamics satisfies detailed balance w.r.t. the product Bernoulli(p) measure, μ, which is therefore a reversible invariant measure. Hence, the process started at μ is stationary.

Both from a physical and from a mathematical point of view, a central issue for KCM is to determine the speed of divergence of the characteristic time scales when $q \downarrow 0$. Two key quantities are: (i) the *relaxation time* T_{rel}, that is, the inverse of the spectral gap of the Markov generator and (ii) the *mean infection time* $\mathbb{E}_\mu(\tau_0)$, that is, the mean over the stationary process of the first time at which the origin becomes empty. The study of the infection time has been largely addressed for the $\mathcal{U}$-bootstrap percolation [3, 5, 7], a class of discrete cellular

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automata that can be viewed as the monotone deterministic counterpart of KCM. For the $\mathcal{U}$-bootstrap, given a set of "infected" sites $A_t \subset \mathbb{Z}^d$ at time t, infected sites remain infected and a site x becomes infected at time $t+1$ if the translate by x of one of the update sets in $\mathcal{U}$ belongs to A_t. Thus, if infected (noninfected) sites are regarded as empty (resp., occupied) sites, the constraint that has to be satisfied to infect a site for the $\mathcal{U}$-bootstrap is the same that is required to update the occupation variable for the KCM.

In [19], two of the authors together with R. Morris addressed the problem of identifying the divergence of time scales for two-dimensional KCM. The first goal of [19] was to identify the correct universality classes, which turn out to be different from those of $\mathcal{U}$-bootstrap percolation. Then, building on a strategy developed in [20] by two of the authors, universal upper bounds on the relaxation and mean infection time within each class were proven and were conjectured to be sharp up to logarithmic corrections [19]. On the other hand, concerning lower bounds, so far the best general result is

$$T_{\mathrm{rel}} \geq q\mathbb{E}_\mu(\tau_0) = \Omega(T) \tag{1.1}$$

where T denotes the median infection time for the $\mathcal{U}$-bootstrap process started with distribution μ (i.e., sites are initially infected independently with probability q); see [20], Lemma 4.3. However, this lower bound is in general far from optimal. Consider, for example, the *one-dimensional East model* [15] (and [13] for a review) for which a site can be updated iff its left neighbour is empty, namely $\mathcal{U} = \{\{-\vec{e}_1\}\}$. As $q \downarrow 0$, it holds

$$\mathbb{E}_\mu^{\mathrm{East}}(\tau_0) = e^{\Theta((\log q)^2)} \tag{1.2}$$

and the scaling holds for T_{rel}; see [1, 8, 9] where the sharp value of the constant has been determined. This divergence is much faster than for the corresponding $\mathcal{U}$-bootstrap model, for which it holds $T = \Theta(1/q)$. To understand this difference, it is necessary to recall a key combinatorial result [25], [11], Fact 1: in order to empty the origin, the East process has to go through a configuration with $\lceil \log_2(\ell + 1)\rceil$ simultaneous empty sites in $(-\ell, 0]$, where $-\ell$ is the position of the rightmost empty site on $(-\infty, 0]$. This logarithmic "energy barrier" (to employ the physics jargon) and the fact that at equilibrium typically $\ell \sim 1/q$ yield a divergence of the time scale as $q^{\Theta(\log q)} = e^{\Theta((\log q)^2)}$. In turn, this peculiar scaling is the reason why the East model has been extensively studied by physicists (see [16] and references therein). Indeed, if we set $q := e^{-\beta}$ with β the inverse temperature, we get the so-called *super-Arrhenius* divergence $e^{(\Theta(\beta^2))}$ which provides a very good fit of the experimental curves for fragile supercooled liquids near the glass transition [4].

In [21], together with R. Morris, we conjectured that one of the universality classes of two-dimensional KCM, that we call *supercritical rooted models*, features time scales diverging as for the East model. Our first main result (Theorem 4.2) is to establish a lower bound which allows together with the upper bound in [19], Theorem 1 to prove this conjecture,[1] namely we prove

$$\mathbb{E}_\mu^{\mathcal{U}}(\tau_0) = e^{\Theta((\log q)^2)} \quad \forall \mathcal{U} \text{ in the supercritical rooted class}$$

and the same result for T_{rel}. As for the East model, this divergence is much faster than for the corresponding $\mathcal{U}$-bootstrap process which scales as $T = 1/q^{\Theta(1)}$ [7]. A key input for our Theorem 4.2 is a combinatorial result proved by one of the authors in [18] (see also Lemma 4.5

[1]Actually, the conjecture in [21] states that $\tau_0 = e^{\Theta((\log q)^2)}$ w.h.p. when $q \to 0$. As explained in Remark 4.6, we can also prove this stronger result.

in this paper) which considerably generalises to a higher dimensional and nonoriented setting the above recalled combinatorial result for East.[2]

The $\mathcal{U}$-bootstrap results identify another universality class, the so-called *critical update families*, which display a much faster divergence. In particular, in [5] it was proven that for this class it holds $T = e^{(\Theta(\log)^c/q^\alpha)}$ with α a model dependent positive integer and $c = 0$ or $c = 2$. In [19], together with R. Morris, we analysed KCM with critical update families and we put forward the conjecture that both T_{rel} and $\mathbb{E}_\mu(\tau_0)$ diverge as $e^{\Theta((\log)^{c'}/q^\nu)}$ with ν model dependent and in general different from the exponent α of the corresponding $\mathcal{U}$-bootstrap process. In [19], we develop a technique to establish sharp upper bounds for these time scales. A matching lower bound exists only for the special class of models for which the general lower bound (1.1) is sharp, which include, for example, the 2-neighbour model. Here, we focus on the most studied update family which does not belong to this special case, the *Duarte update family*, which consists of all the 2-subsets of the North, South and West neighbours of the origin [12]. Our second main result is a sharp lower bound on the infection and relaxation time for the Duarte KCM (Theorem 5.1) that, together with the upper bound in [19], Theorem 2, establishes the scaling

$$\mathbb{E}_\mu^{\mathrm{Duarte}}(\tau_0) = e^{\Theta((\log q)^4/q^2)}$$

as $q \downarrow 0$, and the same result holds for T_{rel}. Notice that we identify also the exact power in the logarithmic correction. Finally, notice that the divergence is again much faster than for the corresponding $\mathcal{U}$-bootstrap model. Indeed, the median of the infection time for the $\mathcal{U}$-bootstrap Duarte model diverges as $T = e^{\Theta((\log q)^2/q)}$ when $q \downarrow 0$ [22].

Both for Duarte and for supercritical rooted models, the sharper divergence of time scales for KCM is due to the fact that the infection time is not well approximated by the minimal number of updates needed to infect the origin (as it is for bootstrap percolation), but it is instead the result of a much more complex infection/healing mechanism. In particular, visiting regions of the configuration space with an anomalous amount of infection is heavily penalised and requires a very long time to actually take place.[3] The basic underlying idea is that the dominant relaxation mechanism is an East-like dynamics for large *droplets* of empty sites. For supercritical rooted models, these droplets have a finite (model dependent) size, hence an equilibrium density $q_{\mathrm{eff}} = q^{\Theta(1)}$. For the Duarte model droplets have a size that diverges as $\ell = \frac{|\log q|}{q}$, and thus an equilibrium density $q_{\mathrm{eff}} = q^\ell = e^{-(\log q)^2/q}$. Then a (very) rough understanding of our results is obtained by replacing q with q_{eff} in the result for the East model (1.2). One of the key technical difficulties to translate this intuition into a lower bound is that the droplets cannot be identified with a rigid structure, at variance with the East model where the droplets are single empty sites.

2. Models and notation.

2.1. *Notation.* For the reader's convenience, we gather here some of the notation that we use throughout the paper. We will work on the probability space (Ω, μ), where $\Omega = \{0, 1\}^{\mathbb{Z}^2}$ and μ is the product Bernoulli(p) measure, and we will be interested in the asymptotic regime $q \downarrow 0$, where $q = 1 - p$. Given $\omega \in \Omega$ and $\Lambda \subset \mathbb{Z}^2$, we will often write ω_Λ or $\omega \restriction_\Lambda$ for the collection $\{\omega_x\}_{x\in\Lambda}$ and we shall write $\omega_\Lambda \equiv 0$ to indicate that $\omega_x = 0$ $\forall x \in \Lambda$. In this case,

[2]The result in [18] holds also in $d > 2$ on a properly defined class, that is, all models which are not supercritical *unrooted* (see [18] for the precise definition). Our argument immediately extends to this higher dimensional setting yielding the same lower bound as in Theorem 4.2 for T_{rel} and $\mathbb{E}_\mu(\tau_0)$.

[3]Borrowing again from physics jargon, we could say that “crossing the energy barriers” is heavily penalised.

we shall also say that Λ is empty or infected. Similarly, for $\omega_\Lambda \equiv 1$ and in this case Λ will be said to be occupied or healthy. We shall write $Y(\omega)$ for the set $\{x \in \mathbb{Z}^2 : \omega_x = 0\}$ and we shall say that $f : \Omega \mapsto \mathbb{R}$ is a *local function* if it depends on finitely many variables $\{\omega_x\}_{x\in\mathbb{Z}^2}$. Given a site $x \in \mathbb{Z}^2$ of the form $x = (a, b)$ with $a, b \in \mathbb{Z}$, we shall sometimes refer to b as the height of x. We shall also refer to a set $I \subset \mathbb{Z}^2$ of the form $I = \{x, x + \vec{e}_i, \dots, x + (n-1)\vec{e}_i\}$, $x \in \mathbb{Z}^2$, as a (horizontal or vertical) interval of length $n \in \mathbb{N}^*$. Here, $\vec{e}_1, \vec{e}_2$ denote as usual the basis vectors in $\mathbb{R}^2$. Finally, we will use the standard notation $[n]$ for the set $\{1, \dots, n\}$.

Throughout this paper, we will often make use of standard asymptotic notation. If f and g are positive real-valued functions of $q \in (0, 1)$, then we will write $f = O(g)$ if there exists a constant $C > 0$ such that $f(q) \le Cg(q)$ for every sufficiently small $q > 0$. We will also write $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$. All constants, including those implied by the notation $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$, will be such w.r.t. the parameter q.

2.2. *Models.* Fix an *update family* $\mathcal{U} = \{X_1, \dots, X_m\}$, that is, a finite collection of finite subsets of $\mathbb{Z}^2 \setminus \{\mathbf{0}\}$. Then the KCM with update family $\mathcal{U}$ is the Markov process on Ω associated to the Markov generator

$$(\mathcal{L}f)(\omega) = \sum_{x\in\mathbb{Z}^2} c_x(\omega)\big(\mu_x(f) - f\big)(\omega), \tag{2.1}$$

where $f : \Omega \mapsto \mathbb{R}$ is a local function, $\mu_x(f)$ denotes the average of f w.r.t. the variable ω_x, and c_x is the indicator function of the event that there exists $X \in \mathcal{U}$ such that $X + x$ is infected, that is, $\omega_{X+x} \equiv 0$. In the sequel, we will sometimes say that ω satisfies the update rule at x if $c_x(\omega) = 1$.

Informally, this process can be described as follows. Each vertex $x \in \mathbb{Z}^2$, with rate one and independently across $\mathbb{Z}^2$, is resampled from $(\{0, 1\}, \mathrm{Ber}(p))$ iff the update rule at x was satisfied by the current configuration. In what follows, we will sometimes call such resampling a *legal update* or *legal spin flip*. The general theory of interacting particle systems (see [17]) proves that $\mathcal{L}$ becomes the generator of a reversible Markov process $\{\omega(t)\}_{t\ge0}$ on Ω, with reversible measure μ. The corresponding Dirichlet form is

$$\mathcal{D}(f) = \sum_{x\in\mathbb{Z}^2} \mu\big(c_x \operatorname{Var}_x(f)\big),$$

where $\operatorname{Var}_x(f)$ denotes the variance of the local function f w.r.t. the variable ω_x conditionally on $\{\omega_y\}_{y\ne x}$. If ν is a probability measure on Ω, the law of the process with initial distribution ν will be denoted by $\mathbb{P}_\nu(\cdot)$ and the corresponding expectation by $\mathbb{E}_\nu(\cdot)$. If ν is concentrated on a single configuration ω, we will simply write $\mathbb{P}_\omega(\cdot)$ and $\mathbb{E}_\omega(\cdot)$.

Given a KCM and, therefore, an update family $\mathcal{U}$, the corresponding *$\mathcal{U}$-bootstrap process* on $\mathbb{Z}^2$ is defined as follows: given a set $Y \subset \mathbb{Z}^2$ of initially *infected* sites, set $Y(0) = Y$, and define for each $t \ge 0$,

$$Y(t+1) = Y(t) \cup \big\{x \in \mathbb{Z}^2 : X + x \subseteq Y(t) \text{ for some } X \in \mathcal{U}\big\}. \tag{2.2}$$

The set $Y(t)$ will represent the set of infected sites at time t and we write $[Y] = \bigcup_{t\ge0} Y(t)$ for the *closure* of Y under the $\mathcal{U}$-bootstrap process. We will also call T the median of the first infection time of the origin when the process is started with sites independently infected (healthy) with probability q (resp., $p = 1 - q$).

3. A variational lower bound for $\mathbb{E}_\mu(\tau_0)$. As mentioned in the Introduction, our main goal is to prove sharp lower bounds for the characteristic time scales of supercritical rooted KCM and of the Duarte KCM. Let us start by defining precisely these time scales, namely the relaxation time T_{rel} (or inverse of the spectral gap) and the mean infection time $\mathbb{E}_\mu(\tau_0)$.

DEFINITION 3.1 (Relaxation time, T_{rel}). Given an update family $\mathcal{U}$ and $q \in [0, 1]$, we say that $C > 0$ is a Poincaré constant for the corresponding KCM if, for all local functions f, we have

$$\mathrm{Var}_\mu(f) \le C\mathcal{D}(f). \tag{3.1}$$

If there exists a finite Poincaré constant, we then define

$$T_{\mathrm{rel}}(q,\mathcal{U}) := \inf\{C > 0 : C \text{ is a Poincaré constant}\}.$$

Otherwise, we say that the relaxation time is infinite. We will drop the $(q,\mathcal{U})$ notation setting $T_{\mathrm{rel}} := T_{\mathrm{rel}}(q,\mathcal{U})$ when confusion does not arise.

A finite relaxation time implies that the reversible measure μ is mixing for the semigroup $P_t = e^{t\mathcal{L}}$ with exponentially decaying time auto-correlations [17].

DEFINITION 3.2 (Mean infection time, $\mathbb{E}_\mu(\tau_0)$). Let $A = \{\omega \in \Omega : \omega_0 = 0\}$. Then

$$\tau_0 = \inf\{t \ge 0 : \omega(t) \in A\}.$$

Given an update family $\mathcal{U}$ and $q \in [0, 1]$, we let $\mathbb{E}_\mu^{q,\mathcal{U}}(\tau_0)$ be the mean of the infection time of the origin under the corresponding stationary KCM (i.e., when the initial configuration is distributed with Bernoulli$(1 - q)$). We will drop the $(q,\mathcal{U})$ notation setting $\mathbb{E}_\mu(\tau_0) := \mathbb{E}_\mu^{q,\mathcal{U}}(\tau_0)$ when confusion does not arise.

In the physics literature, the hitting time τ_0 is closely related to the *persistence time*, that is, the first time that there is a legal update at the origin. All our lower bounds can be easily extended to the persistence time.

It is known that the following inequality holds (see [19], Section 2.2):

$$\mathbb{E}_\mu(\tau_0) \le \frac{T_{\mathrm{rel}}(q,\mathcal{U})}{q} \quad \forall q \in (0, 1). \tag{3.2}$$

Therefore, we will focus on obtaining lower bounds on $\mathbb{E}_\mu(\tau_0)$ and then use (3.2) to derive the results for T_{rel} (indeed the correction q in the above inequality is largely subdominant w.r.t. the lower bounds we will obtain). To this aim, we establish a variational lower bound on $\mathbb{E}_\mu(\tau_0)$ (Lemma 3.3), which will be our first tool. Recall that $A = \{\omega \in \Omega : \omega_0 = 0\}$ and let H_A be the Hilbert space $\{f \in L^2(\Omega, \mu) : f \restriction_A = 0\}$ with scalar product inherited from the standard one in $L^2(\Omega, \mu)$. Let also $\mathcal{L}_A$ be the negative self-adjoint operator on H_A, whose action on local functions is given by

$$\mathcal{L}_A f(\omega) = \mathbb{1}_{A^c}(\omega)\mathcal{L} f(\omega).$$

It turns out (see, e.g., [2], Section 3) that, for any local function $f \in H_A$ and any $\omega \in A^c$,

$$\mathbb{E}_\omega\big(f(\omega(t))\mathbb{1}_{\{\tau_0>t\}}\big) = e^{t\mathcal{L}_A} f(\omega).$$

In particular, by choosing $f = \mathbb{1}_{A^c}(\cdot)$, one gets

$$\mathbb{P}_\mu(\tau_0 > t) = \int d\mu(\omega)\mathbb{1}_{A^c}(\omega)e^{t\mathcal{L}_A}\mathbb{1}_{A^c}(\omega) = \langle\mathbb{1}_{A^c}, e^{t\mathcal{L}_A}\mathbb{1}_{A^c}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\Omega, \mu)$. Thus

$$\mathbb{E}_\mu(\tau_0) = \int_0^\infty dt\langle\mathbb{1}_{A^c}, e^{t\mathcal{L}_A}\mathbb{1}_{A^c}\rangle \ge \int_0^T dt\langle\mathbb{1}_{A^c}, e^{t\mathcal{L}_A}\mathbb{1}_{A^c}\rangle \quad \forall T > 0. \tag{3.3}$$

LEMMA 3.3. *Let $\phi \in H_A$ be a local function such that $\mu(\phi^2) = 1$. Then*

$$\mathbb{E}_\mu(\tau_0) \geq T|\mu(\phi)|\big(|\mu(\phi)|e^{-T\mathcal{D}(\phi)} - \big(T\mathcal{D}(\phi)\big)^{1/2}\big) \quad \forall T > 0.$$

PROOF. Let $\phi \in H_A$ be as in the statement and write

$$\mathbb{1}_{A^c} = \alpha\phi + \psi,$$

where $\alpha = \langle \mathbb{1}_{A^c}, \phi\rangle = \mu(\phi)$ and $\langle \phi, \psi\rangle = 0$. Clearly, $\langle \psi, \psi\rangle = \mu(A^c) - \alpha^2$. We claim that, for any $T > 0$ and any $t \in [0, T]$,

$$\langle \mathbb{1}_{A^c}, e^{t\mathcal{L}_A}\mathbb{1}_{A^c}\rangle \geq \alpha^2 e^{-T\mathcal{D}(\phi)} - 2|\alpha|\big(T\mathcal{D}(\phi)\big)^{1/2}, \tag{3.4}$$

which, combined with (3.3), proves the lemma. To prove the claim, we write

$$\begin{aligned}
\langle \mathbb{1}_{A^c}, e^{t\mathcal{L}_A}\mathbb{1}_{A^c}\rangle &\geq \alpha^2\langle \phi, e^{t\mathcal{L}_A}\phi\rangle - 2|\alpha|\big|\langle \psi, e^{t\mathcal{L}_A}\phi\rangle\big| \\
&= \alpha^2\langle \phi, e^{t\mathcal{L}_A}\phi\rangle - 2|\alpha|\big|\langle \psi, (\mathbb{I} - e^{t\mathcal{L}_A})\phi\rangle\big| \\
&\geq \alpha^2\langle \phi, e^{t\mathcal{L}_A}\phi\rangle - 2|\alpha|\langle \phi, (\mathbb{I} - e^{t\mathcal{L}_A})^2\phi\rangle^{1/2}.
\end{aligned} \tag{3.5}$$

Above we discarded the nonnegative term $\langle \psi, e^{t\mathcal{L}_A}\psi\rangle$ in the first line, we used $\langle \phi, \psi\rangle = 0$ in the second line and appealed to the Cauchy–Schwarz inequality together with $\langle \psi, \psi\rangle \leq 1$ in the third line. Let now $\pi(d\lambda)$ be the spectral measure of $-\mathcal{L}_A$ associated to ϕ (see, e.g., [23], Chapter VII). Since $\mu(\phi^2) = 1$, $\pi(d\lambda)$ is a probability measure on $[0, +\infty)$. The functional calculus theorem, together with the Jensen inequality and $(1 - e^{-t\lambda})^2 \leq t\lambda$, implies that for any $t \in [0, T]$,

$$\begin{aligned}
\text{r.h.s. (3.5)} &= \alpha^2\int_0^\infty d\pi(\lambda)e^{-t\lambda} - 2|\alpha|\bigg(\int_0^\infty d\pi(\lambda)\big(1 - e^{-t\lambda}\big)^2\bigg)^{1/2} \\
&\geq \alpha^2 e^{-t\mathcal{D}_A(\phi)} - 2|\alpha|\big(t\mathcal{D}_A(\phi)\big)^{1/2} \\
&\geq \alpha^2 e^{-T\mathcal{D}(\phi)} - 2|\alpha|\big(T\mathcal{D}(\phi)\big)^{1/2},
\end{aligned}$$

where $\mathcal{D}_A(\phi) = \langle \phi, -\mathcal{L}_A\phi\rangle = \langle \phi, -\mathcal{L}\phi\rangle = \mathcal{D}(\phi)$ because ϕ is a local function in H_A. The claim is proved. $\square$

The main strategy to take advantage of Lemma 3.3 for q very small is to look for a family of local functions $\{\phi_q\}$ in H_A, normalised in such a way that $\mu(\phi_q^2) = 1$, determining a sharp lower bound when inserted in the inequality of Lemma 3.3 with a proper choice of T. More precisely, we will use the following easy corollary of Lemma 3.3.

COROLLARY 3.4 (Proxy functions). *For any family of local functions $\{\phi_q\}$ in H_A with $\mu(\phi_q^2) = 1$, it holds*

$$\mathbb{E}_\mu(\tau_0) = \Omega\big(\mu(\phi_q)^4/\mathcal{D}(\phi_q)\big). \tag{3.6}$$

PROOF. The result follows immediately using Lemma 3.3 and choosing $T \equiv T(q) = |\mu(\phi_q)|^2/(16\mathcal{D}(\phi_q))$. $\square$

Any function $\phi = \phi_q$ with the above properties will be called a *test* or *proxy* function and, in the rest of the paper, we will focus on constructing an efficient test function for the so called *supercritical rooted KCM* and for the *Duarte KCM*.

4. Supercritical rooted KCM. In order to define the class of *supercritical rooted* update families, we should begin by recalling the key geometrical notion of *stable directions* introduced in [7]. Given a unit vector $u \in S^1$, let $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u\rangle < 0\}$ denote the discrete half-plane whose boundary is perpendicular to u. Then, for a given update family $\mathcal{U}$, the set of stable directions is

$$\mathcal{S} = \mathcal{S}(\mathcal{U}) = \{u \in S^1 : [\mathbb{H}_u] = \mathbb{H}_u\}.$$

The update family $\mathcal{U}$ is *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$. In [7], it was proven that for each supercritical update family the median of the infection time of the $\mathcal{U}$-bootstrap processes diverges as $1/q^{\Theta(1)}$. In [21], the author R. Morris together with two of us, conjectured that not all supercritical update families give rise to the same scaling for KCM and that the supercritical class should be refined into two subclasses to capture the KCM scaling as follows.

DEFINITION 4.1. A supercritical two-dimensional update family $\mathcal{U}$ is said to be *supercritical rooted* if there exist two nonopposite stable directions in S^1. Otherwise it is called *supercritical unrooted*.

An example of supercritical rooted family is the two-dimensional East model, with update family $\mathcal{U} = \{\{-\vec{e}_1\}, \{-\vec{e}_2\}\}$.[4] In [19], it was proved that $\mathbb{E}_\mu(\tau_0)$ and T_{rel} diverge as an inverse power of q as $q \to 0$ in the supercritical *unrooted* case, while in the *rooted* case it satisfies (see [19], Theorem 1(b))

$$T_{\mathrm{rel}} \le e^{O((\log q)^2)}$$

and, thanks to (3.2), the same bound holds for $\mathbb{E}_\mu(\tau_0)$. Here, we prove a matching lower bound in the rooted case.

THEOREM 4.2. *Let $\mathcal{U}$ be a two-dimensional supercritical rooted update family. Then*

$$\mathbb{E}_\mu(\tau_0) \ge e^{\Omega((\log q)^2)} \quad \textit{as } q \to 0.$$

Thus we prove the following.

COROLLARY 4.3. *Let $\mathcal{U}$ be a two-dimensional supercritical rooted update family. Then*

$$T_{\mathrm{rel}}(q, \mathcal{U}) = e^{\Theta((\log q)^2)} \quad \textit{as } q \to 0.$$

and the same result holds for $\mathbb{E}_\mu(\tau_0)$.

PROOF. The lower bound follows at once from (3.2) and Theorem 4.2. The upper bound was proved in [19], Theorem 1(b). □

In order to prove Theorem 4.2, we will use the variational lower bound of Section 3 and more precisely look for a proxy function $\phi \equiv \phi_q$ satisfying the hypotheses of Corollary 3.4. We first need to introduce the notion of a legal path in Ω.

[4]We stress that the supercritical rooted class contains also update families which do not share the special "orientation" property of the East model, namely the fact that all X_i belong to a half-plane. For example, it is easy to verify that the nonoriented update family $\mathcal{U} = \{\{-\vec{e}_1\}, \{-\vec{e}_2\}, \{(\vec{e}_1, \vec{e}_2)\}\}$ has exactly two stable directions, $-\vec{e}_1$ and $-\vec{e}_2$ and, according to our Definition 4.1, it is supercritical rooted.

DEFINITION 4.4 (Legal path). Fix an update family $\mathcal{U}$, then a legal path γ in Ω is a finite sequence $\gamma = (\omega^{(0)}, \dots, \omega^{(n)})$ such that, for each $i \in [n]$, the configurations $\omega^{(i-1)}, \omega^{(i)}$ differ by a legal (with respect to the choice of $\mathcal{U}$) spin flip at some vertex $v \equiv v(\omega^{(i-1)}, \omega^{(i)})$. A generic ordered (along γ) pair of consecutive configurations in γ will be called an *edge*. Given a set $\hat{\Omega} \subset \Omega$ and a configuration ω, we say that there exists a legal path connecting $\hat{\Omega}$ to ω if there exists a legal path $\gamma = (\omega^{(0)}, \dots, \omega^{(n)})$ such that $\omega^{(0)} \in \hat{\Omega}$ and $\omega^{(n)} = \omega$.

Let $\mathcal{U}$ be a supercritical rooted update family and, for $n \geq 1$ and $\kappa \in \mathbb{N}^*$, let $\Lambda_n := \Lambda_n(\kappa) \subset \mathbb{Z}^2$ be the square centred at the origin, of cardinality $(\kappa n 2^n + 1)^2$. Let also

$$\begin{aligned} \mathcal{A}_n = \big\{\omega \in \Omega \colon\ & (\omega_{\Lambda_n}, \tilde{\omega}_{\Lambda_n^c} \equiv 0) \text{ can be reached from} \\ & (\hat{\omega}_{\Lambda_n} \equiv 1, \hat{\omega}_{\Lambda_n^c} \equiv 0) \text{ by a legal path } \gamma \text{ such that any} \\ & \omega' \in \gamma \text{ has at most } n-1 \text{ empty vertices in } \Lambda_n\big\}. \end{aligned} \tag{4.1}$$

Recall that $A = \{\omega \in \Omega \colon \omega_0 = 0\}$. In [18], one of the authors established the following key combinatorial result concerning the structure of the set $\mathcal{A}_n$.

LEMMA 4.5 ([18], Theorem 1). *There exists $\kappa_0 = \kappa_0(\mathcal{U}) > 0$ such that, for any $\kappa \geq \kappa_0$ and any $n \in \mathbb{N}^*$,*

$$\mathcal{A}_n \cap A = \varnothing.$$

Lemma 4.5 implies that if $\kappa \geq \kappa_0$, the KCM process started from any configuration with no infection inside the region Λ_n, in order to infect the origin has to leave the set $\mathcal{A}_n$ by going through its boundary set $\partial \mathcal{A}_n$ (see the proof below for a precise definition of this set). In turn, the latter is a subset of

$$\{\omega \in \Omega : \exists \text{ at least } n-1 \text{ infected vertices in } \Lambda_n\}.$$

We will therefore chose a scale n such that $2^n \simeq 1/q^{\varepsilon}$, namely w.h.p. w.r.t. the reversible measure μ there are initially no infected vertices inside Λ_n. Thus, starting from the (likely) event of no infection inside the region Λ_n, in order to infect the origin the process has to go through $\partial \mathcal{A}_n$ which has an anomalous amount, $\Theta(\log q)$, of empty sites. This mechanism, which in the physics jargon would correspond to "crossing an energy barrier" which grows logarithmically in q, is at the root of the scaling $e^{\Theta(\log q)^2}$. Let us proceed to a proof of this result, namely to the proof of Theorem 4.2.

PROOF OF THEOREM 4.2. Fix $\varepsilon < 1/2$ and choose $n := n(\varepsilon, q) = \lfloor \varepsilon \log_2(1/q) \rfloor$. Then let

$$\phi(\cdot) := \phi_q(\cdot) = \mathbb{1}_{\mathcal{A}_{\varepsilon,q}}(\cdot)/\mu(\mathcal{A}_{\varepsilon,q})^{1/2},$$

where $\mathcal{A}_{\varepsilon,q} := \mathcal{A}_{n(\varepsilon,q)}$ with $\mathcal{A}_n$ defined in (4.1) and the constant κ that enters in this definition chosen larger than the value κ_0 of Lemma 4.5. Then Lemma 4.5 implies immediately that $\phi \in H_A$. Moreover, using $\varepsilon < 1/2$ we get

$$\mu(\phi) = \mu(\mathcal{A}_{\varepsilon,q})^{1/2} \geq (1-q)^{|\Lambda_n|/2} = 1 - o(1),$$

because any configuration identically equal to one in Λ_n belongs to $\mathcal{A}_{\varepsilon,q}$ and $2^{2n} = O(1/q^{2\varepsilon})$. Finally, if

$$\partial \mathcal{A}_{\varepsilon,q} := \big\{\omega \in \mathcal{A}_{\varepsilon,q} : \exists x \in \Lambda_n \text{ with } c_x(\omega) = 1 \text{ and } \omega^x \notin \mathcal{A}_{\varepsilon,q}\big\},$$

one easily checks (see, e.g., [10], Section 3.5) that

$$\mathcal{D}(\phi) \le |\Lambda_n|\mu(\partial\mathcal{A}_{\varepsilon,q})/\mu(\mathcal{A}_{\varepsilon,q}) \le |\Lambda_n|\mu(\exists n-1 \text{ zeros in } \Lambda_n)/\mu(\mathcal{A}_{\varepsilon,q})$$
$$\le O(|\Lambda_n|^n)q^{n-1} = e^{-\Omega((\log q)^2)}.$$

Thus ϕ satisfies all the hypotheses of Corollary 3.4 and the result follows. □

REMARK 4.6. In [21], Conjecture 2.7, it was conjectured that $\tau_0 = e^{\Theta((\log q)^2)}$ w.h.p. as $q \to 0$ holds. Actually, we can also prove this stronger result. One bound immediately follows using Markov inequality and our result for the mean, Corollary 4.3. The other bound follows by using the fact that (i) the set $\mathcal{A}_{\varepsilon,q}$ has μ-probability $1 - o(1)$ (see the above proof of Theorem 4.2) and (ii) the probability of infecting the origin before $e^{\Theta((\log q)^2)}$ starting in $\mathcal{A}_{\varepsilon,q}$ goes to zero as $q \downarrow 0$. The latter result is easily obtained by a union bound on times which yields that the probability to leave $\mathcal{A}_{\varepsilon,q}$ before $e^{\Theta((\log q)^2)}$ (and, therefore, to infect the origin, thanks to Lemma 4.5) goes to zero.

5. The Duarte KCM. In this section, we analyse the mean infection time for the Duarte KCM. For this model, the update family $\mathcal{U}$ consists of the 2-subsets of the North, South and West neighbours of the origin [12]. The infection time for the Duarte bootstrap process is known to scale as $e^{\Theta((\log q)^2/q)}$ [22] (see also [6] for sharp for sharp results on the critical probability). We refer the reader to Figure 1 for a simulation of the growth of an initial anomalous pocket of infection under the Duarte bootstrap process. Concerning the Duarte KCM, in [19], Theorem 2 it was proved that

$$T_{\mathrm{rel}}(q, \mathcal{U}) \le e^{O((\log q)^4/q^2)} \quad \text{as } q \to 0$$

and, thanks to (3.2), the same result holds for $\mathbb{E}_\mu(\tau_0)$. Here, we establish a matching lower bound.

THEOREM 5.1. *Consider the Duarte KCM. Then*

$$\mathbb{E}_\mu(\tau_0) \ge e^{\Omega((\log q)^4/q^2)} \quad \textit{as } q \to 0.$$

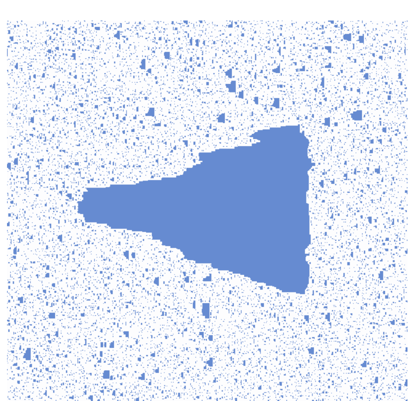

FIG. 1. *The typical spread of infection under the Duarte bootstrap process* (*courtesy of P. Smith*). *The initial pocket of infection at the left end of the cone of infection is large enough to be able to grow while all the other infected sites are not.*

Using (3.2), Theorem 5.1 and [19], Theorem 2, we get immediately the following corollary.

COROLLARY 5.2. *For the Duarte KCM, it holds*

$$T_{\mathrm{rel}}(q,\mathcal{U}) = e^{\Theta((\log q)^4/q^2)} \quad \textit{as } q \to 0.$$

and the same result for $\mathbb{E}_\mu(\tau_0)$.

Our result provides the first example of critical α-rooted KCM for which the conjecture for the divergence of time scales that we put forward in [19], Conjecture 3(a) together with R. Morris can be proven. Indeed, as explained in [19], the Duarte model is a 1-rooted model and the exponent 2 that we obtain is in agreement with [19], Conjecture 3(a). In order to prove Theorem 5.1, we will start by the variational lower bound of Section 3, as for the supercritical rooted class. However, defining the analog of the set $\mathcal{A}_n$ together with the test function ϕ satisfying the hypotheses of Corollary 3.4 is much more involved and it requires a subtle algorithmic construction. Before explaining our construction, it is useful to make some simple observations on how infection propagates in the Duarte bootstrap process.

5.1. *Preliminary tools*: *The Duarte bootstrap process*. Given $\Lambda \subset \mathbb{Z}^2$, we write $\partial\Lambda := \partial_{\|}\Lambda \cup \partial_{\perp}\Lambda$, where

$$\partial_{\|}\Lambda = \{y \in \Lambda^c : y + \vec{e}_1 \in \Lambda\},$$
$$\partial_{\perp}\Lambda = \{y \in \Lambda^c : \{y + \vec{e}_2, y - \vec{e}_2\} \cap \Lambda \neq \varnothing\}.$$

A configuration $\tau \in \{0,1\}^{\partial\Lambda}$ will be referred to as a *boundary condition* and we shall write it as $\tau = (\tau_{\|}, \tau_{\perp})$, where $\tau_{\|} := \tau \restriction_{\partial_{\|}\Lambda}$ and similarly for $\tau_{\perp}$.

DEFINITION 5.3. Given a boundary condition τ and $Y \subseteq \Lambda$, let

$$Y^\tau(t+1) = Y^\tau(t) \cup \{x \in \Lambda : X + x \subseteq Y^\tau(t) \text{ for some } X \in \mathcal{U}\}, \quad t \geq 0,$$

where $Y^\tau(0) = Y \cup \{x \in \partial\Lambda : \tau_x = 0\}$. We call the process $Y^\tau(t)$, $t \in \mathbb{N}$, the Duarte bootstrap process in Λ with τ boundary condition (for shortness, the DB^τ_Λ-process), and we shall write $[Y]^\tau_\Lambda$ for $(\bigcup_{t\geq 0} Y^\tau(t)) \cap \Lambda$. Recall also (see Section 2.2) that $[Y]$ is the analogous quantity for the bootstrap process evolving on $\mathbb{Z}^2$.

REMARK 5.4. Notice that for the DB^τ_Λ-process the boundary condition τ does not change in time.

Notation warning. If $\tau \equiv 0$ or $\tau \equiv 1$, we shall simply replace it by a 0 or a 1 in our notation. If instead τ is such that $\tau_{\|} \equiv 1$ and $\tau_{\perp} \equiv 0$, then it will be replaced by a $1,0$ in the notation.

LEMMA 5.5 (Screening property). *Consider a sequence of sites* $S := \{(i, b_i)\}_{i=1}^n$ *in* $\mathbb{Z}^2$ *with* $b_{i+1} \leq b_i$ *for all* $i \in [n-1]$, *and let*

$$S_+ = \{(i,j) \in \mathbb{Z}^2 : i \in [n], j > b_i\}, \quad S_- = \{(i,j) \in \mathbb{Z}^2 : i \in [n], j < b_i\}.$$

Let Y, Y' *be two arbitrary subsets of* $\mathbb{Z}^2$ *such that* $Y \supseteq S$ *and* $Y \cap S_+^c = Y' \cap S_+^c$. *Then* $[Y] \cap S_- = [Y'] \cap S_-$. *Similarly, if we assume that* $b_{i+1} \geq b_i$ *for all* $i \in [n-1]$ *and we exchange the role of* S_+ *and* S_-.

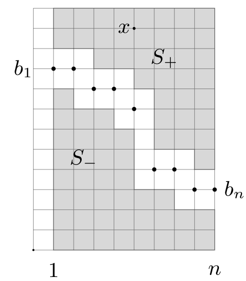

FIG. 2. *The set S (black dots) and the sets $S_\pm$ (shaded regions). If the two initial sets Y, Y' of infection contain S and differ at exactly the vertex x, it is clear that the initial discrepancy cannot influence the final infection in S_-.*

PROOF. We refer to Figure 2 for a visualisation of the geometric setting. Let Y, Y' be as in the statement and observe that $Y(s)$ and $Y'(s)$ coincide in $\{v \in \mathbb{Z}^2 : v = (a, b), a \le 0\}$ for all $s \in \mathbb{N}^*$. Let $t \in \mathbb{N}^*$ be the first time at which there exists $y \in S_-$ such that either $y \in Y'(t)$ and $y \notin Y(t)$ or vice versa. W.l.o.g we assume the first case. By construction, there exists $z \in \{y \pm \vec{e}_2, y - \vec{e}_1\}$ such that $z \in Y'(t-1)$ and $z \notin Y(t-1)$. Clearly, z cannot be of the form $z = (0, b)$ and, therefore, $z \in S_- \cup S$ because $y \in S_-$. Because of the definition of t, $z \notin S_-$ and $z \notin S$ because $S \subseteq Y(s)$ and $S \subseteq Y'(s)$ for all $s \in \mathbb{N}^*$. □

LEMMA 5.6 (Monotonicity). *Let $\Lambda \subseteq \Lambda'$ be subsets of $\mathbb{Z}^2$.*

(A) *Let $\tau, \tau' \in \{0,1\}^{\partial\Lambda}$. If $\tau_x \le \tau'_x$ for all $x \in \partial\Lambda$, then*

$$[Y]_\Lambda^{\tau'} \subseteq [Y]_\Lambda^{\tau} \quad \forall Y \subseteq \Lambda.$$

(B) *For all $Y' \subseteq \Lambda'$,*

$$[Y']_{\Lambda'}^0 \cap \Lambda \subseteq [Y' \cap \Lambda]_\Lambda^0 \quad \textit{and} \quad [Y']_{\Lambda'}^1 \cap \Lambda \supseteq [Y' \cap \Lambda]_\Lambda^1.$$

(C) *Suppose that Λ and Λ' are such that $\partial_\perp \Lambda \subseteq \partial_\perp \Lambda'$. Then for all $Y' \subseteq \Lambda'$*

$$[Y' \cap \Lambda]_\Lambda^{1,0} \subseteq [Y']_{\Lambda'}^{1,0} \cap \Lambda.$$

PROOF.

(A) It follows immediately from the fact that the DB_Λ^τ-process runs with more initial infection than the $DB_\Lambda^{\tau'}$-process.

(B) To prove the first inclusion, let $Z = (Y' \cap \Lambda) \cup (\Lambda' \setminus \Lambda)$. Clearly, $[Y']_{\Lambda'}^0 \subseteq [Z]_{\Lambda'}^0$ because $Y' \subseteq Z$. It is now sufficient to observe that, by definition,

$$[Z]_{\Lambda'}^0 \cap \Lambda = [Y' \cap \Lambda]_\Lambda^0.$$

Similarly, one proceeds for the second inclusion with $Z = Y' \cap \Lambda$.

(C) Clearly, $[Y' \cap \Lambda]_{\Lambda'}^{1,0} \subseteq [Y']_{\Lambda'}^{1,0}$. We claim that

$$[Y' \cap \Lambda]_{\Lambda'}^{1,0} \cap \Lambda \supseteq [Y' \cap \Lambda]_\Lambda^{1,0}.$$

That follows immediately from the assumption that $\partial_\perp \Lambda' \supseteq \partial_\perp \Lambda$ and the fact that the vertices of $\partial_\parallel \Lambda \cap \Lambda'$ (if any) are constrained to be healthy for all times under the $DB_\Lambda^{1,0}$-process while they are unconstrained for the $DB_{\Lambda'}^{1,0}$-process. □

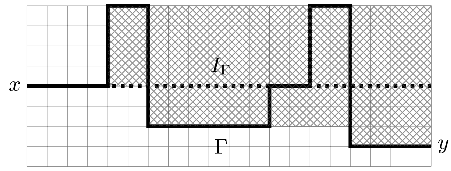

FIG. 3. *A Duarte path* Γ (*thick polygonal line*) *and the corresponding horizontal interval* I_Γ (*dotted line*). *Clearly*, $\Gamma \subseteq [Y]$ *implies that* $[Y]$ *contains the shaded region. In particular,* $I_\Gamma \subseteq [Y]$.

LEMMA 5.7 (Propagation of infection). *Let* I *be a vertical interval*, *that is*, $I = \{a, a + \vec{e}_2, \dots, a + n\vec{e}_2\}$, $a \in \mathbb{Z}^2$, *and let* $v = x + \vec{e}_1$ *for some* $x \in I$. *Suppose that* $I \cup \{v\} \subseteq [Y]$ *where* Y *is the initial set of infection. Then* $I + \vec{e}_1 \subseteq [Y]$. *In particular, if* $[Y]$ *contains* $[n] \times \{1\}$ *and* $\{1\} \times [m]$ *then* $[n] \times [m] \subseteq [Y]$.

As a corollary of the above simple property, let $x, y \in \mathbb{Z}^2$ and suppose that there exists a *Duarte path* Γ between x and y, that is, $\Gamma := (x^{(1)}, \dots, x^{(n)}) \subseteq \mathbb{Z}^2$ with $x^{(1)} = x$, $x^{(n)} = y$ and $x^{(i+1)} - x^{(i)} \in \{\vec{e}_1, \pm\vec{e}_2\}$ $\forall i \in [n-1]$. Let also I_Γ be the horizontal interval starting at x and reaching the vertical line through y (see Figure 3).

COROLLARY 5.8. *Suppose that* $\Gamma \subseteq [Y]$. *Then* $I_\Gamma \subseteq [Y]$.

5.2. *Algorithmic construction of the test function and proof of Theorem* 5.1. Fix ε a small positive constant that will be chosen later on and let

$$\ell = \left\lfloor \frac{1}{\varepsilon q} \log(1/q) \right\rfloor. \tag{5.1}$$

Suppose that a vertical interval I of length ℓ is completely infected. Notice that, with μ-probability going to 1 as $q \downarrow 0$, there is an infected site on the vertical interval sitting on the right, $I + \vec{e}_1$. Therefore, thanks to Lemma 5.7, with high probability the infection can propagate to infect $I + \vec{e}_1$. Notice that instead the infection on I does not help infecting the interval on its left, $I - \vec{e}_1$. At this point, recalling the explanation given in the Introduction, one might think that the droplets that undergo an East-like dynamics[5] are the *empty vertical intervals of length at least* ℓ. However, this is far from true, since these empty intervals might also appear (or disappear) without being facilitated by the presence of an empty interval on their left. For example, if there is an empty interval of length $\ell - 1$ and the site just above has the constraint satisfied, a single legal move may turn it into an empty interval of height ℓ. We have therefore to find a more flexible definition of the droplets respecting three key properties: (i) East-like dynamics; (ii) disjoint occurrence under the equilibrium measure μ and (iii) the density of droplets should scale as $q_{\text{eff}} = q^\ell$.[6] Our solution to the problem is the construction of an algorithm that sequentially searches for properly defined droplets on a finite volume, V, containing the origin. We let

$$N = \lfloor e^{\varepsilon(\log q)^2/q} \rfloor \quad \text{and} \quad V := V_N = \bigcup_{i=1}^{N} \mathcal{C}_i, \tag{5.2}$$

[5]Namely, a dynamics in which droplets appear/disappear only if there is a droplet on their left, as it occurs for the single empty sites in the one-dimensional East model.

[6]Indeed, since the density of droplets will play the role of the density of empty sites for East, it is natural to expect that the lower bound obtained using the droplets will be of the form (1.2) with q_{eff} replacing q. This in turn yields the result of Theorem 5.1 if $q_{\text{eff}} = q^\ell$.

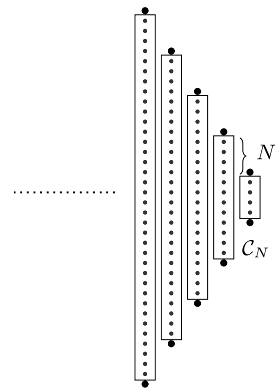

FIG. 4. *A sketchy drawing of the last few columns of the set V. The black dots represent sites belonging to $\partial_\perp V$.*

where

$$\mathcal{C}_i = \{(i, j) \in \mathbb{Z}^2 : |j| < N^2 - (i-1)N\} - N\vec{e}_1$$

as in Figure 4. In the sequel, we shall write $\bar{V}$ for set $V \cup \partial_\perp V$ and we shall refer to $\bar{\mathcal{C}}_i := \mathcal{C}_i \cup \partial_\perp \mathcal{C}_i$ as the ith column of $\bar{V}$. By construction, the origin coincides with the midpoint of the last column (see Figure 4). The core of our algorithmic construction (see Definition 5.10) consists in associating to each $\omega \in \Omega$ an element $\Phi(\omega) \in \{\downarrow, \uparrow\}^N$ via an iterative procedure based on the DB_Λ^τ-process. These arrow variables are those that satisfy the three key properties announced above, with $\Phi(\omega)_i = \uparrow$ corresponding to the occurrence of a droplet in column i, and we will use them to construct an efficient test function.

DEFINITION 5.9. Given a boundary condition τ and $\omega \in \Omega$, we shall say that $I \subseteq V$ is (ω, τ)-infectable if $I \subseteq [Y(\omega) \cap V]_V^\tau$, where we recall that $Y(\omega)$ is the set of empty vertices of ω.

Before defining the algorithm leading to the construction of an effective test function for the Duarte KCM process, it is useful to notice two simple properties of the DB_V^τ- process:

(i) Let $I \subseteq \bigcup_{i=1}^k \mathcal{C}_i$, $k \le N$. Then the property of being (ω, τ)-infectable for I depends only on the infection of the pair (ω, τ) in $\bigcup_{i=1}^k \bar{\mathcal{C}}_i$ and on $\tau_\parallel$.

(ii) If $\bar{\mathcal{C}}_i$ is healthy at time $t = 0$ (including the contribution of τ at its top and bottom boundary sites), then it will remain healthy at any later time.

The idea behind the algorithm is the following. It is tempting to decide that there is a droplet/arrow in column i when column i contains an infectable vertical interval of length at least ℓ; indeed, this has probability close to the probability that the interval is infected, which is q^ℓ. However, this brings on the following problem: as explained at the beginning of Section 5.2, once such an interval I is completely infected by the bootstrap process, with high probability the infection can propagate to $I + \vec{e}_1$, so column $i+1$ would also contain an infectable vertical interval of length at least ℓ, hence we would detect a second droplet in column $i+1$ even though the configuration on column $i+1$ is ordinary. In order to avoid that, before moving on to column $i+1$, we heal all the infections that allowed to infect I.

DEFINITION 5.10 (The algorithm). Given $\omega \in \Omega$ and $\tau \in \{0,1\}^{\partial V}$ such that $\tau_\perp \equiv 0$ and $\tau_\parallel \equiv 1$, the algorithm outputs recursively a sequence $\psi^{(k)} := (\omega^{(k)}, \tau^{(k)})$, $k \in \{0, \dots, N\}$, where $\omega^{(k)} \in \Omega$ and $\tau^{(k)} \in \{0,1\}^{\partial V}$ is such that $\tau_\parallel^{(k)} \equiv 1$. The pair $\psi^{(0)}$ coincides with

(ω, τ) and $\psi^{(k)}$ is obtained from $\psi^{(k-1)}$ by healing suitably chosen infected vertices. The iterative step goes as follows. Fix $\ell \in [N]$ and assume that $\psi^{(j)}$ has been defined for all $j = 0, \dots, k-1$, $k \in [N]$. Then:

(i) if $\bar{\mathcal{C}}_k$ contains an interval I of length at least ℓ which is $\psi^{(k-1)}$-infectable, we let $\xi_k := \xi_k(\omega) \le k$ be the largest integer such that, by removing all the empty vertices of the pair $\psi^{(k-1)}$ contained in $\bigcup_{i=1}^{\xi_k - 1} \bar{\mathcal{C}}_i$, the above property still holds. We then set both $\omega^{(k)}$ and $\tau^{(k)}$ identically equal to one (i.e., with no infection) on $\bar{\mathcal{C}}_{\xi_k}, \dots, \bar{\mathcal{C}}_k$ and equal to $\omega^{(k-1)}$ and $\tau^{(k-1)}$ elsewhere;
(ii) if not, we set $\psi^{(k)} = \psi^{(k-1)}$.

REMARK 5.11. Clearly, the above construction depends on the initial ω and we shall sometimes write $\psi^{(k)}(\omega)$ to outline this dependence.

DEFINITION 5.12 (Droplets and their range). Given k such that $\psi^{(k)}(\omega) \neq \psi^{(k-1)}(\omega)$, we define the *droplet* $D_k(\omega)$ and the *range* $r_k(\omega)$ of the kth column in ω as the set $\bigcup_{i=\xi_k}^{k} \bar{\mathcal{C}}_i$ and the integer $k - \xi_k(\omega)$, respectively. If instead $\psi^{(k)}(\omega) = \psi^{(k-1)}(\omega)$, we let $D_k(\omega) = \varnothing$ and $r_k(\omega) = 0$.

Observe that, by construction,

$$\psi^{(j)}(\omega) \restriction_{\bar{V} \setminus \bigcup_{i=1}^{j} D_i(\omega)} = \psi^{(0)}(\omega) \restriction_{\bar{V} \setminus \bigcup_{i=1}^{j} D_i(\omega)} . \tag{5.3}$$

DEFINITION 5.13 (The mapping Φ). Having defined the sequence $\{\psi^{(k)}\}_{k=1}^{N}$, we set

$$\Phi(\omega)_k = \begin{cases} \uparrow & \text{if } \psi^{(k)}(\omega) \neq \psi^{(k-1)}(\omega), \\ \downarrow & \text{otherwise,} \end{cases}$$

and $N_{\uparrow}(\omega) = \#\{i \in [N] \colon \Phi(\omega)_i = \uparrow\}$.

REMARK 5.14. Suppose that ω, ω' are such that they coincide over the first i columns. Then $\Phi(\omega)_k = \Phi(\omega')_k$ for all $k \in [i]$.

In the sequel, two events will play an important role. The first one, $\mathcal{B}_1(n)$, collects all the ω's whose image $\Phi(\omega)$ has more than n up-arrows, with $n \in [N]$:

$$\mathcal{B}_1(n) = \{\omega \in \Omega \colon N_{\uparrow}(\omega) \ge n\}. \tag{5.4}$$

The event $\mathcal{B}_2(n)$, again with $n \in [N]$, collects instead all the $\omega \in \Omega$ such that there exists n consecutive $\downarrow$-columns which are traversed by an infectable Duarte path. More precisely, for $1 \le i < j \le N$, let

$$V_{i,j} = \bigcup_{k=i}^{j} \mathcal{C}_k \tag{5.5}$$

and let

$$\mathcal{B}_2(n) = \bigcup_{j-i \ge n-1} \left(\bigcap_{k=i}^{j} \{\omega \in \Omega \colon \Phi(\omega)_k = \downarrow\} \cap \mathcal{G}_{i,j} \right), \tag{5.6}$$

where

$$\begin{aligned} \mathcal{G}_{i,j} = \{\omega \in \Omega \colon & \exists \text{ a Duarte path } \Gamma \text{ from } \mathcal{C}_i \text{ to } \mathcal{C}_j \text{ such that} \\ & \Gamma \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}\}. \end{aligned} \tag{5.7}$$

We are now ready to define our test function.

DEFINITION 5.15 (The test function). Let $I_0 = \{(0,k)\colon |k| \le \ell\}$ and

$$n_1 = \varepsilon(\log q)^2/2q, \qquad n_2 = 1/q^6 \tag{5.8}$$

where ε is the same as in the definition of N (5.2). Let also

$$\begin{aligned}
&\Omega_\downarrow = \{\omega \in \Omega : \Phi(\omega) = (\downarrow, \dots, \downarrow)\},\\
&\Omega_g = \Omega_\downarrow \cap \{\omega \in \Omega\colon \omega_{I_0} = 1\},\\
&\mathcal{A}_{\varepsilon,q} := \mathcal{A}_{N,\ell,n_1,n_2}\\
&\quad = \{\omega \in \Omega\colon \exists \text{ a legal path } \gamma \text{ connecting } \Omega_g \text{ to } (\omega_V, \tilde{\omega}_{V^c} \equiv 0)\\
&\qquad \text{s.t. } \gamma \cap \mathcal{B}_1(n_1 - 1) = \varnothing \text{ and } \gamma \cap \mathcal{B}_2(n_2 - 1) = \varnothing\},
\end{aligned} \tag{5.9}$$

where legal paths have been defined in Definition 4.4 and, for any $\mathcal{B} \subset \Omega$, we set $\gamma \cap \mathcal{B} = \varnothing$ iff none of the configurations of the path γ belongs to $\mathcal{B}$. Then we choose as a test function

$$\phi(\cdot) := \phi_q(\cdot) = \mathbb{1}_{\mathcal{A}_{\varepsilon,q}}(\cdot)/\mu(\mathcal{A}_{\varepsilon,q})^{1/2}.$$

The rest of the paper is devoted to prove that (i) ϕ satisfies the key hypothesis of Corollary 3.4, namely $\phi \in H_A$ and (ii) ϕ is an efficient proxy function, namely the bound (3.6) prove the sharp lower bound of Theorem 5.1. More precisely, we need to prove the following key propositions.

PROPOSITION 5.16. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ there exists q_ε small enough such that, for all $q \in (0, q_\varepsilon)$,*

$$\mathcal{A}_{\varepsilon,q} \cap A = \varnothing.$$

In particular, $\phi \in H_A$.

PROPOSITION 5.17. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\mu(\phi) \ge q^{O(1)} \quad \text{and} \quad \mathcal{D}(\phi) \le e^{-\Omega(\log(q)^4/q^2)} \quad \text{as } q \to 0.$$

Once the above propositions are proven, the main result of this section easily follows.

PROOF OF THEOREM 5.1. The result follows at once using Propositions 5.16 and 5.17, together with the general lower bound on $\mathbb{E}_\mu(\tau_0)$ given in (3.6). $\square$

Let us start with an easy result which will be used in the proof of both propositions.

LEMMA 5.18 (Disjoint occurrence of the droplets). *For any $\omega \in \Omega$ and any $k \ne j$, $D_k(\omega) \cap D_j(\omega) = \varnothing$.*

PROOF. Let $k_1, \dots, k_\nu$ be the labels of the columns which are of type $\uparrow$ in $\Phi(\omega)$ (for all the other columns the droplets are the empty set). Using property (ii) of the DB_V^τ-process, $D_{k_\nu}(\omega)$ cannot contain a column which is healthy for the pair $\psi^{(k_\nu - 1)}$ because any infection to the left of a healthy column cannot cross the healthy column itself. On the other hand, all the columns of the droplets $D_{k_1}, \dots, D_{k_{\nu-1}}$ are healthy for $\psi^{(k_\nu-1)}$. Thus $D_{k_\nu} \cap D_{k_j} = \varnothing$ for all $j \in [\nu - 1]$. The same reasoning applies to all the other droplets. $\square$

5.3. *East-like motion of the arrows and proof of Proposition* 5.16. Let

$$A_\ell = \{\omega \in \Omega \colon \omega_{I_0^+} \equiv 0\} \cup \{\omega \in \Omega \colon \omega_{I_0^-} \equiv 0\},$$

where $I_0^\pm = \{(0, \pm 1), \dots, (0, \pm\ell)\}$. Then it holds

LEMMA 5.19. *If $\mathcal{A}_{\varepsilon,q} \cap A \neq \varnothing$, then there exist $\omega \in A_\ell$ and a legal path γ connecting Ω_g to ω such that $\gamma \cap \mathcal{B}_i(n_i) = \varnothing$, $i = 1, 2$.*

PROOF. Fix $\omega \in \mathcal{A}_{\varepsilon,q} \cap A$, recall Definition 5.15 and let $\tilde{\gamma}$ be a legal path connecting Ω_g to $(\omega_V, \tilde{\omega}_{V^c} \equiv 0)$ such that $\tilde{\gamma} \cap \mathcal{B}_1(n_1 - 1) = \varnothing$ and $\tilde{\gamma} \cap \mathcal{B}_2(n_2 - 1) = \varnothing$. W.l.o.g., we can assume that $\tilde{\gamma}$ ends as soon as the origin is infected. It is easy to verify that $\tilde{\gamma}$ must be able to sequentially infect (and possibly heal later on) the ordered vertices of either I_0^+ starting from $(0, \ell)$ or those of I_0^- starting from $(0, -\ell)$. For simplicity, we assume that the first option holds and we let γ be the path obtained from $\tilde{\gamma}$ by deleting all the transitions in which a vertex of I_0^+ is healed.

By construction, the final configuration of γ belongs to A_ℓ. Moreover, γ is a legal path because at each step the infection in the last column of V is larger than or equal to the infection of the corresponding step of $\tilde{\gamma}$. Finally, the restriction to $\mathcal{C}_1, \dots, \mathcal{C}_{N-1}$ of any step of γ coincides with the same restriction of the appropriate step of $\tilde{\gamma}$. Using that $\tilde{\gamma} \cap \mathcal{B}_1(n_1 - 1) = \varnothing$ and $\tilde{\gamma} \cap \mathcal{B}_2(n_2 - 1) = \varnothing$, we deduce that $\gamma \cap \mathcal{B}_1(n_1) = \varnothing$ and $\gamma \cap \mathcal{B}_2(n_2) = \varnothing$. $\square$

The above lemma says that, if there exists a configuration in Ω_g for which we can infect the origin performing a legal path never crossing either $\mathcal{B}_1(n_1 - 1)$ or $\mathcal{B}_2(n_2 - 1)$, then necessarily there exists a legal path never crossing either $\mathcal{B}_1(n_1)$ or $\mathcal{B}_2(n_2)$ and connecting a configuration ω with all columns being $\downarrow$ to a configuration ω with a $\uparrow$ in the Nth column. In order to conclude that $\mathcal{A}_{\varepsilon,q} \cap A = \varnothing$, and thus prove our Proposition 5.16, we will now show that *the existence of a legal path with the above properties is impossible*. It is here that the East-like motion of the droplets emerges and plays a key role. Recall the definitions (5.2), (5.8) and let $m = 4n_1n_2$ and, for simplicity, let us suppose that m divides N. We partition $[N]$ into $M = N/m$ disjoint consecutive blocks $\{B_i\}_{i=1}^M$ of equal cardinality and, with a slight abuse of notation, we identify the columns $\bigcup_{k \in B_i} \mathcal{C}_k$ with the block B_i itself. Given $\omega \in \Omega$, we write

$$\eta_i(\omega) := \mathbb{1}_{\{\forall j \in B_i \colon \Phi(\omega)_j = \downarrow\}},$$

and we denote by $\eta(\omega)$ the collection $\{\eta_i(\omega)\}_{i=1}^M$.

CLAIM 5.20. *Given a legal path γ with the properties stated in Lemma 5.19, it is possible to construct a path $\varphi(\gamma) := (\eta^{(0)}, \dots, \eta^{(k)})$ in the space $\{0, 1\}^M$ with the following properties*:

(1) $\eta_i^{(0)} = 1$ *for all* $i \in [M]$ *and* $\eta_M^{(k)} = 0$,
(2) $\#\{i \in [M] \colon \eta_i = 0\} \leq n_1$ *for all* $\eta \in \varphi(\gamma)$,
(3) *for any edge (η, η') of $\varphi(\gamma)$, the configuration η' differs from η in exactly one coordinate. Moreover, if the discrepancy between η and η' occurs at the ith coordinate and $i \neq 1$, then $\eta_{i-1} = 0$.*

REMARK 5.21. The path $\varphi(\gamma)$ for the coarse-grained variables $\{\eta_i\}_{i=1}^M$ can be viewed as a legal path for the one-dimensional East model on $[M]$ (see, e.g., [13]).

The proof of our Proposition 5.16 then follows by using this connection with the East model, our choices (5.2), (5.8) of the parameters N, n_1, n_2 and the combinatorial result for the East model [11, 25] that we explained in the Introduction. More precisely, we have the following.

PROOF OF PROPOSITION 5.16. In [11], it was proved that a path like $\varphi(\gamma)$ above exists iff $n_1 \geq \log_2(M+1)$. With our choice (5.8) of the scaling as $q \to 0$ of n_1, n_2, N, the latter condition becomes

$$n_1 \geq \frac{1}{\log 2}(1+o(1))\varepsilon(\log q)^2/q, \quad \text{as } q \to 0,$$

violating our choice $n_1 = \varepsilon(\log q)^2/2q$. Thus $\varphi(\gamma)$ cannot exist as well as the path γ. □

We are therefore left with proving Claim 5.20. To this aim, we start by stating two preparatory results, Lemma 5.22 and Lemma 5.23, which will be the key ingredients for the proof of Claim 5.20.

LEMMA 5.22. *For any $\omega \in \mathcal{B}_2^c(n_2)$, the maximum range of a droplet of ω is $n_2 - 1$.*

PROOF. Let $\omega \in \Omega$ such that there exists $j \in [N]$ with $r_j(\omega) \geq n_2$. Denote $i = \xi_j(\omega)$. By the definition of $\xi_j(\omega) = i$, $\bar{\mathcal{C}}_j$ contains an interval I of length at least ℓ which is $\psi^{(j-1)}$-infectable by the empty sites in $\bigcup_{k=i}^{j} \bar{\mathcal{C}}_k$, but not by the empty sites in $\bigcup_{k=i+1}^{j} \bar{\mathcal{C}}_k$. Definition 5.9 implies that any $\psi^{(j-1)}$-infectable site is in V, hence $I \subseteq \mathcal{C}_j$. Furthermore, for all $k \in \{i, \dots, j-1\}$, $\Phi(\omega)_k = \downarrow$ (since thanks to Lemma 5.18 the droplets are disjoint), so by (5.3) $\psi^{(j-1)}$ and $\psi^{(0)}$ coincide on $\bigcup_{k=i}^{j} \bar{\mathcal{C}}_k$. Therefore, I is $\psi^{(0)}$-infectable by the empty sites in $\bigcup_{k=i}^{j} \bar{\mathcal{C}}_k$, but not by the empty sites in $\bigcup_{k=i+1}^{j} \bar{\mathcal{C}}_k$. We deduce that $I \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}$, but $I \nsubseteq [Y(\omega) \cap V_{i+1,j}]_{V_{i+1,j}}^{1,0}$, see (5.5) for the definition of $V_{i,j}$. Thus, there exists $z \in \mathcal{C}_j$ such that $z \in [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0} \setminus [Y(\omega) \cap V_{i+1,j}]_{V_{i+1,j}}^{1,0}$. Hence z cannot be initially empty for the Duarte bootstrap process in $V_{i,j}$; otherwise, it would also be empty for the process in $V_{i+1,j}$, hence the process in $V_{i,j}$ infects z with an update rule, so there exist $z' \in \{z - \vec{e}_1, z \pm \vec{e}_2\}$ in $[Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0} \setminus [Y(\omega) \cap V_{i+1,j}]_{V_{i+1,j}}^{1,0}$. We can iterate, creating a Duarte path in $[Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0} \setminus [Y(\omega) \cap V_{i+1,j}]_{V_{i+1,j}}^{1,0}$. There can be only a finite number of iterations because there is a finite number of sites in $V_{i,j}$, so we will stop, and the site at which we stop has to be initially empty for the process in $V_{i,j}$, but not for the process in $V_{i+1,j}$, therefore, it is in $\bar{\mathcal{C}}_i$. This implies the Duarte path can reach $\mathcal{C}_i$. Consequently, there is a Duarte path in $[Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0} \setminus [Y(\omega) \cap V_{i+1,j}]_{V_{i+1,j}}^{1,0}$ going from $\mathcal{C}_i$ to $\mathcal{C}_j$. We deduce that there exists a Duarte path in $[Y(\omega) \cap V_{i,j-1}]_{V_{i,j-1}}^{1,0}$ from $\mathcal{C}_i$ to $\mathcal{C}_{j-1}$, which is $\mathcal{G}_{i,j-1}$. Since $(j-1) - i \geq n_2 - 1$, $\omega \in \mathcal{B}_2(n_2)$. □

The next lemma is the basic technical step connecting the evolution of the coarse-grained variables $\{\Phi(\omega)_i\}_{i=1}^N$ under the Duarte KCM process to an East-like process. Given $\omega \in \Omega$ and $x \in V$, let ω^x denote the configuration ω flipped at x. We say that x is $\psi^{(k)}(\omega)$-unconstrained (or infectable in one step) if $\exists X \in \mathcal{U}$ such that $X + x$ is infected for the pair $(\omega^{(k)}, \tau^{(k)})$.

LEMMA 5.23 (East-like motion of the arrows). *Fix $\omega \in \Omega$ and let $x \in \mathcal{C}_j$. Then*:

(a) *Suppose that x is $\psi^{(0)}(\omega)$-unconstrained. Then $\Phi(\omega^x) \neq \Phi(\omega)$ implies that $j > 1$ and $\Phi(\omega)_{j-1} = \uparrow$*;

(b) *For $i > j$ suppose that $\Phi(\omega)_i =\uparrow$, $\Phi(\omega^x)_i =\downarrow$ and that $D_i(\omega) \not\ni x$. Then there exists k such that $\bar{\mathcal{C}}_k \subseteq D_i(\omega) \setminus \bar{\mathcal{C}}_i$ and $\Phi(\omega^x)_k =\uparrow$, $\Phi(\omega)_k =\downarrow$.*

PROOF. (a) If $j = 1$, then clearly $\Phi(\omega^x) = \Phi(\omega)$ because x is $\psi^{(0)}(\omega)$-unconstrained. Consider now the case $j \neq 1$ and assume that $\Phi(\omega)_{j-1} =\downarrow$. We want to prove that in this case $\Phi(\omega^x) = \Phi(\omega)$ if x is $\psi^{(0)}(\omega)$-unconstrained.

By construction, the restriction to the first $j-1$ columns of $\psi^{(k)}(\omega^x)$ and $\psi^{(k)}(\omega)$ coincide for all $k \in [j-1]$ and, as a consequence, $\Phi(\omega)_k = \Phi(\omega^x)_k$ $\forall k \in [j-1]$. Let $k_*(\omega) = \min\{k \geq j : \Phi(\omega)_k =\uparrow\}$ and similarly for ω^x. Using (5.3) together with $\Phi(\omega)_{j-1} =\downarrow$, for all $i = j-1, \dots, k_*(\omega) - 1$ the restriction of $\psi^{(i)}(\omega)$ to the columns $\bar{\mathcal{C}}_{j-1}, \dots, \bar{\mathcal{C}}_N$ coincides with the same restriction of the original pair $\psi^{(0)}(\omega)$. In particular, the fact that x is $\psi^{(0)}(\omega)$-unconstrained implies that x is also $\psi^{(k_*(\omega)-1)}(\omega)$-unconstrained, analogously for the configuration ω^x. Clearly, $k_*(\omega^x) \geq k_*(\omega)$. If not, starting from the infection of $\psi^{(j-1)}(\omega)$, we can first make a transition to $\psi^{(j-1)}(\omega^x)$ by legally flipping ω_x and from there infect an interval of length at least ℓ of $\bar{\mathcal{C}}_{k_*(\omega^x)}$ to make it of type $\uparrow$, a contradiction with the definition of $k_*(\omega)$. By exchanging the role of ω, ω^x, we conclude that $k_*(\omega^x) = k_*(\omega)$. Thus $\Phi(\omega)_k = \Phi(\omega^x)_k$ for all $k = 1 \dots, k_*(\omega)$ and, a fortiori, for all $k > k_*(\omega)$.

(b) By assumption, the restriction of ω, ω^x to $D_i(\omega)$ coincide. If $\Phi(\omega^x)_k =\downarrow$ for all the columns in $D_i(\omega)$, then $\psi^{(i-1)}(\omega) = \psi^{(i-1)}(\omega^x)$ on the set $D_i(\omega)$ implying that $\Phi(\omega^x)_i = \Phi(\omega)_i$. Thus there exists a column $\bar{\mathcal{C}}_k \subseteq D_i(\omega) \setminus \bar{\mathcal{C}}_i$ such that $\Phi(\omega^x)_k =\uparrow$ and (by the definition of $D_i(\omega)$) $\Phi(\omega)_k =\downarrow$. □

COROLLARY 5.24. *Fix $\omega \in \Omega$ and let $x \in \mathcal{C}_j$. Let also $r_\infty^x = \max_i \max(r_i(\omega), r_i(\omega^x))$ and suppose that $\Phi(\omega)_i =\uparrow$, $\Phi(\omega^x)_i =\downarrow$, with $i - j \geq m(r_\infty^x + 1)$, $m \in \mathbb{N}^*$. Then*

$$\#\{k \in \{j, \dots, i\} : \Phi(\omega)_k =\uparrow\} + \#\{k \in \{j, \dots, i\} : \Phi(\omega^x)_k =\uparrow\} \geq m.$$

PROOF. By construction, $D_i(\omega) \not\ni x$. Lemma 5.23 part (b) guarantees that there exists a column $\bar{\mathcal{C}}_k \subseteq D_i(\omega) \setminus \bar{\mathcal{C}}_i$ such that $\Phi(\omega)_k =\downarrow$ and $\Phi(\omega^x)_k =\uparrow$. We can then iterate by exchanging the role of ω, ω^x and replacing i with, for example, the largest of the labels k above. In conclusion, every $r_\infty^x + 1$ steps we are guaranteed to find a discrepancy between $\Phi(\omega)$ and $\Phi(\omega^x)$ and the result follows. □

We are now ready to conclude the proof of Claim 5.20.

PROOF OF CLAIM 5.20. To prove the claim, let $\gamma = (\omega^{(0)}, \dots, \omega^{(n)})$ and let us consider the sequence $\{\eta(\omega^{(j)})\}_{j=0}^n$. The path $\varphi(\gamma) = (\eta^{(0)}, \dots, \eta^{(k)})$ is then defined recursively by setting $\eta^{(0)} := \eta(\omega^{(0)})$ and $\eta^{(j)} := \eta(\omega^{(i_j)})$, where $i_j = \min\{i > i_{j-1} : \eta(\omega^{(i)}) \neq \eta^{(j-1)}\}$ with $i_0 = 0$, and by stopping the procedure as soon as the set $\{\eta \in \{0,1\}^M : \eta_M = 0\}$ is reached ($\phi(\gamma)$ is then a function of γ). In other words, we only keep the elements of the sequence $\eta(\omega^{(j)})$, $j = 0, \dots, n$, which change w.r.t. the previous element. Property (1) of $\varphi(\gamma)$ follows immediately from the fact that γ starts in $\Omega_\downarrow$ and ends in A_ℓ. Property (2) follows from the fact that $\gamma \cap \mathcal{B}_1(n_1) = \varnothing$. We now verify the key property (3).

Let (η, η') be an edge of $\varphi(\gamma)$ and let (ω, ω') be the edge of γ such that $\eta(\omega) = \eta$ and $\eta(\omega') = \eta'$. By construction, $\Phi(\omega) \neq \Phi(\omega')$. Let also $x \in \mathcal{C}_a$ be such that $\omega' = \omega^x$ and say that a belongs to jth block. Clearly, $\eta_i = \eta'_i$ for all $i < j$. Moreover, Lemma 5.22 and Corollary 5.24 imply that $\Phi(\omega)_v = \Phi(\omega')_v$ for all $v \in \bigcup_{i \geq j+2} B_i$ (if $j + 2 \leq N$), since otherwise either ω or ω' would have at least $\lfloor m/2(r_\infty^x + 1)\rfloor \geq \lfloor m/2n_2 \rfloor = 2n_1$ up-arrows, contradicting the assumption $\gamma \cap \mathcal{B}_1(n_1) = \varnothing$. In particular, $\eta_i = \eta'_i$ for all $i \geq j + 2$. To complete our analysis, we distinguish between two cases.

(1) $a > 1$. In this case, x must be $\psi^{(0)}(\omega)$-unconstrained and part (a) of Lemma 5.23 together with $\Phi(\omega) \neq \Phi(\omega')$ implies that $\Phi(\omega)_{a-1} = \Phi(\omega^x)_{a-1} = \uparrow$. If a is not the beginning of the block B_j then, by definition, $\eta_j = \eta'_j = 0$. Thus η, η' must differ exactly in the $(j+1)$th-block and they are both equal to zero in the previous one as required. If a is the beginning of the jth block, then necessarily $j > 1$. Moreover, $\Phi(\omega)_{a-1} = \Phi(\omega^x)_{a-1} = \uparrow$ implies that $\eta_{j-1} = \eta'_{j-1} = 0$. By the same reasoning as before, using Corollary 5.24 and Lemma 5.22 (recall that $\omega \in \mathcal{B}^c(n_2)$), we get that $\Phi(\omega)_v = \Phi(\omega')_v$ for all $v \in \bigcup_{i>j} B_i$. Thus $\eta_i = \eta'_i$ for all $i \neq j$ and $\eta_{j-1} = \eta'_{j-1} = 0$ as required.

(2) $a = 1$. Again Corollary 5.24 guarantees that $\Phi(\omega)_i = \Phi(\omega^x)_i$ for all $i \in \bigcup_{j=2}^N B_j$ so that $\eta_b = \eta'_b$ for all $b \geq 2$. $\square$

5.4. *Density of droplets and proof of Proposition* 5.17. The core of the proof of Proposition 5.17 consists in bounding from above the probabilities of the events $\mathcal{B}_1, \mathcal{B}_2$ defined in (5.4), (5.6). The first key bound is Lemma 5.25 that says that the probability that the $DB_V^{1,0}$-process restricted to an arbitrary number of consecutive columns of V is able to infect any given interval of the last column of length ℓ is $e^{-\Omega((\log q)^2/q)}$. The second key ingredient is Lemma 5.27 that bounds from above the probability of the event $\mathcal{B}_2(n_2 - 1)$. Before stating the lemmas, we need some additional notation.

Given $1 \leq i \leq j \leq N$, let $\Lambda = \bigcup_{k=i}^j \mathcal{L}_k$, where, for each $k = i, \dots, j$, $\mathcal{L}_k \supseteq \mathcal{C}_k$ is a (finite) interval of $\{(k - N, j) \colon j \in \mathbb{Z}\}$. Let also $I \subseteq \mathcal{C}_j$ be an interval of length ℓ and $\tau \in \{0,1\}^{\partial\Lambda}$ a boundary condition. The basic event that we will consider is

$$\mathcal{O}_\Lambda^\tau(I) = \{\omega \in \Omega \colon I \subseteq [Y(\omega) \cap \Lambda]_\Lambda^\tau\},$$

where we recall $Y(\omega)$ is the set of infected vertices of ω. Notice that $\mathcal{O}_\Lambda^\tau(I)$ is an increasing event (i.e., its indicator function is an increasing function) w.r.t. the partial order: $\omega \prec \omega'$ iff $\omega'_x \leq \omega_x$ $\forall x$. Our first main lemma reads as follows.

LEMMA 5.25 (Density of up-arrows). *Choose the basic scales N, ℓ, n_1, n_2 as in* (5.1), (5.2) *and* (5.8). *Then there exists $c > 0$ such that, for any $\varepsilon > 0$ sufficiently small and any $1 \leq i \leq j \leq N$,*

$$\max_I \mu(\mathcal{O}_{V_{i,j}}^{1,0}(I)) \leq e^{-c(\log q)^2/q} \quad \text{as } q \to 0,$$

where $V_{i,j} = \bigcup_{k=i}^j \mathcal{C}_k$.

PROOF. Fix $1 \leq i \leq j \leq N$ together with an interval $I \subset \mathcal{C}_j$ of length ℓ and let

$$\Lambda_{1,j} = \bigcup_{i=1}^j \{(i,k) \colon |k| < N^2\} - N\vec{e}_1.$$

We first claim that

$$\mu(\mathcal{O}_{V_{i,j}}^{1,0}(I)) \leq \mu(\mathcal{O}_{V_{1,j}}^{1,0}(I)) \leq O(1/q^2)\mu(\mathcal{O}_{\Lambda_{1,j}}^{1}(I)) \quad \text{as } q \to 0. \tag{5.10}$$

The first inequality follows from (C) in Lemma 5.6. To prove the second one, let $G = \bigcap_{k=1}^{j-1} G_k$, where G_k denotes the event that there is an empty site within the first $\lfloor N/3 \rfloor$ sites and within the last $\lfloor N/3 \rfloor$ sites of $\mathcal{C}_k$. Then, for any choice of the constant ε appearing in (5.1), (5.2) and (5.8),

$$\mu(G^c) \leq 2N(1-q)^{\frac{N}{3}-1} = o(1) \quad \text{as } q \to 0. \tag{5.11}$$

For any $\omega \in G$ and any boundary condition τ for $V_{1,j}$ such that $\tau \equiv 0$ on $\partial_\perp \mathcal{C}_j$ and $\tau_\| \equiv 1$, the screening property and translation invariance imply that $[Y(\omega) \cap V_{1,j}]^\tau_{V_{1,j}} \cap \mathcal{C}_j$ does not depend on τ. Hence,

$$\mathcal{O}^{1,0}_{V_{1,j}}(I) \cap G = \mathcal{O}^\tau_{V_{1,j}}(I) \cap G. \tag{5.12}$$

Choose τ equal to one everywhere except for $\partial_\perp \mathcal{C}_j$ where it is equal to zero. Using the FKG inequality and (5.12),

$$\begin{aligned}\mu\big(\mathcal{O}^{1,0}_{V_{1,j}}(I)\big) &\le \mu\big(\mathcal{O}^{1,0}_{V_{1,j}}(I) \mid G\big) = \mu\big(\mathcal{O}^\tau_{V_{1,j}}(I) \mid G\big) \\ &\le \big(1 + o(1)\big)\mu\big(\mathcal{O}^\tau_{V_{1,j}}(I)\big).\end{aligned}$$

We now observe that, starting from $Y(\omega)$, we can construct the set $[Y(\omega) \cap V_{1,j}]^\tau_{V_{1,j}} \cap \mathcal{C}_j$ as follows. We first output the set $[Y(\omega) \cap V_{1,j-1}]^1_{V_{1,j-1}}$ and we let $\bar\tau \in \{0, 1\}^{\partial \mathcal{C}_j}$ be such that $\bar\tau_\perp \equiv 0$ and $\{x \in \partial_\| \mathcal{C}_j : \bar\tau_x = 0\} = [Y(\omega) \cap V_{1,j-1}]^1_{V_{1,j-1}} \cap \partial_\| \mathcal{C}_j$. Then we output the set $[Y(\omega) \cap \mathcal{C}_j]^{\bar\tau}_{\mathcal{C}_j}$ which clearly coincides with $[Y(\omega) \cap V_{1,j}]^\tau_{V_{1,j}} \cap \mathcal{C}_j$.

Monotonicity and a moment of thought imply that if we repeat the above construction with $V_{1,j-1}, \mathcal{C}_j$ replaced by $\Lambda_{1,j-1}, \{(j - N, k) : |k| < N^2\}$ and $Y(\omega)$ replaced by $Y(\omega) \cup \partial_\perp \mathcal{C}_j$, then the final infection in $\mathcal{C}_j$ cannot decrease. Hence

$$\mu\big(\mathcal{O}^\tau_{V_{1,j}}(I)\big) \le \mu\big(\mathcal{O}^1_{\Lambda_{1,j}}(I) \mid \omega_{\partial_\perp \mathcal{C}_j} \equiv 0\big) \le \mu\big(\mathcal{O}^1_{\Lambda_{1,j}}(I)\big)/q^2,$$

and (5.10) follows.

Let now $T(\mathcal{U})$ be the median of the infection time of the origin (or of any other vertex of $\mathbb{Z}^2$ because of translation invariance) for the Duarte bootstrap process in $\mathbb{Z}^2$ started from $Y(\omega)$ where ω has law μ, and write

$$p(N, \ell) := \max_{j \le N} \max_I \mu\big(\mathcal{O}^1_{\Lambda_{1,j}}(I)\big), \tag{5.13}$$

where $\max_I$ is taken over all intervals $I \subset \mathcal{C}_j$ of length ℓ.

CLAIM 5.26. *If $\varepsilon < 1/4$ then, for all q small enough,*

$$p(N, \ell) \ge e^{-\frac{1}{16q} \log(q)^2}, \tag{5.14}$$

implies

$$T(\mathcal{U}) \le O\big(N^3\big) e^{\frac{1}{16q} \log(q)^2}.$$

Before proving the claim, we conclude the proof of Lemma 5.25. It follows from the main result of [6] together with a standard (and straightforward) argument that

$$T(\mathcal{U}) \ge e^{(1-o(1)) \log(q)^2/8q} \quad \text{as } q \to 0,$$

implying that for all q small enough

$$p(N, \ell) \le e^{-\frac{1}{16q} \log(q)^2},$$

if $\varepsilon < 1/48$. $\square$

PROOF OF THE CLAIM. In the sequel, it will help to refer to Figure 5 as a visual guide for the various definitions. Fix q arbitrarily small and let j be such that there exists an interval $I \subset \mathcal{C}_j$ of length ℓ such that

$$\mu\big(\mathcal{O}^1_{\Lambda_{1,j}}(I)\big) \ge e^{-\frac{1}{16q} \log(q)^2}. \tag{5.15}$$

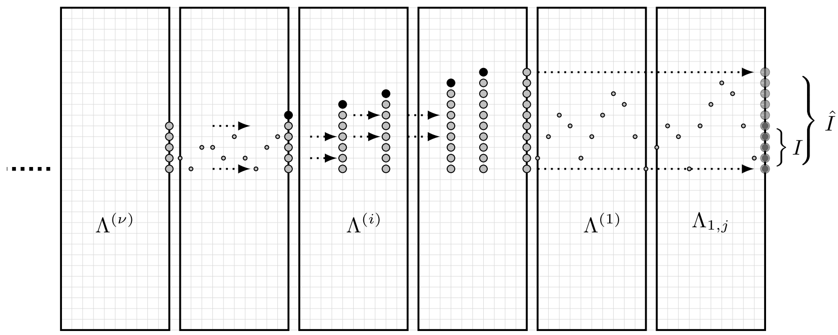

FIG. 5. *A subset of the collection of boxes $\Lambda^{(i)}$ forming $\mathcal{M}_t$. On the last column of $\Lambda_{1,j}$, the two intervals $\hat{I} \supset I$. The little gray dots denote suitable sparse single infected sites, one for each relevant column, and they have been drawn only for the initial and final stage of the infection process. The large gray dots on the right boundary of $\Lambda^{(\nu)}$ represent a shifted copy of I which is infected by the $DB^1_{\Lambda^{(\nu)}}$-process. This infected interval propagates to the right until reaching the first site of the empty upward stair (black dots). At this stage, the interval grows vertically by one unit. This process continues until the interval has become a shifted copy of the interval $\hat{I}$. The latter interval is able to continue moving to the right until infecting the interval $\hat{I}$.*

Using the symmetry w.r.t., the horizontal axis we can assume that x_I, the lowest site of I, has nonpositive height. Write $\Lambda^{(i)} := \Lambda_{1,j} - ij\vec{e}_1$ and let $\mathcal{M}_t = \bigcup_{i=0}^t \Lambda^{(i)}$, where $t = 10\lceil \max(p(N,\ell)^{-1}, 8/q^4)\rceil$. We shall define two increasing events $\mathcal{G}_1, \mathcal{G}_2 \subset \Omega$, depending only on $\omega \restriction_{\mathcal{M}_t}$, such that:

(a) if $\omega \in \mathcal{G}_1 \cap \mathcal{G}_2$ then the Duarte bootstrap process in $\mathbb{Z}^2$ is able to infect x_I within time $(2t+1)j(2N^2-1)$.

(b) $\mu(\mathcal{G}_k) > 3/4$, $k = 1, 2$.

Using the FKG inequality, $\mu(\mathcal{G}_1 \cap \mathcal{G}_2) \ge \mu(\mathcal{G}_1)\mu(\mathcal{G}_2) > 1/2$. Hence

$$T(\mathcal{U}) \le (2t+1)j\bigl(2N^2-1\bigr) \le 60N^3 e^{\frac{1}{16q}\log(q)^2}.$$

In order to define $\mathcal{G}_1, \mathcal{G}_2$, let $\hat{I} \supset I$ be the interval of $\mathcal{C}_j$ of length $\lceil 1/q^3\rceil$ and whose lowest site is x_I. Then

$$\mathcal{G}_1 = \bigl\{\forall k \in [jt], \text{the interval } \hat{I} - (k-1)\vec{e}_1 \text{ contains an empty vertex}\bigr\};$$

$$\mathcal{G}_2 = \bigl\{\exists k \in [jt]\colon \text{the } DB^1_{\mathcal{M}_t}\text{-process starting from } Y(\omega) \cap \mathcal{M}_t$$
$$\text{is able to infect } \hat{I} - k\vec{e}_1\bigr\}.$$

We now verify properties (a) and (b) above. We observe that the event $\mathcal{G}_2$ guarantees that there exists a leftmost interval of the form $\hat{I} - k\vec{e}_1$ which is infected by the Duarte bootstrap process within time $(t+1)j(2N^2-1)$.[7] The event $\mathcal{G}_1$, together with the definition of the Duarte update family $\mathcal{U}$, makes sure that the infection of $\hat{I} - k\vec{e}_1$ gets propagated forward to $\hat{I} - (k-1)\vec{e}_1, \ldots,$ until it reaches the original interval $\hat{I}$ in at most $tj(2N^2-1)$ steps. Hence, within time $(2t+1)j(2N^2-1)$ the vertex x_I becomes infected and (a) follows.

It remains to verify (b). The union bound over k gives that for any $\varepsilon > 0$,

$$\mu\bigl(\mathcal{G}_1^c\bigr) \le jt(1-q)^{\lceil 1/q^3\rceil} \le e^{-\Omega(1/q^2)} \quad \text{as } q \to 0,$$

using (5.14) and $j \le N$.

[7]The worst case is when sites are infected one by one.

In order to bound from below $\mu(\mathcal{G}_2)$, write

$$\nu := \min\{\max\{k \in [t/2, t]\colon \text{ the event } \mathcal{O}^1_{\Lambda^{(k)}}(I - kj\vec{e}_1) \text{ occurs}\}, \infty\},$$

and let $\mathcal{F} = \bigcap_{i=1}^3 \mathcal{F}_i$ where, on the event $\{\nu < +\infty\}$:

- $\mathcal{F}_1 = \{\nu \leq t\}$;
- $\mathcal{F}_2 = \{\forall k \in [\lceil 2/q^4 \rceil]$ *the interval* $I - \nu j\vec{e}_1 + k\vec{e}_1$ *contains an empty vertex*$\}$;
- $\mathcal{F}_3 = \{\exists$ *an* upward empty stair *of* $n = \lceil 1/q^3 \rceil$ *sites belonging to the first* $\lceil 2/q^4 \rceil$ *columns of* $\mathcal{M}_t$ *immediately to the right of* $\Lambda^{(\nu)}$, *that is, a sequence* $(x_1, \dots, x_n)$ *of empty sites of the form* $x_m = (j_m, h_I + m)$, *where* h_I *is the height of the uppermost site of* I *and* $\{j_m\}_{m=1}^n$ *is a strictly increasing sequence*$\}$.

We begin by observing that $\mathcal{F} \subseteq \mathcal{G}_2$. In fact, $\mathcal{F}_1$ guarantees the right amount of infection of the last column of $\Lambda^{(\nu)}$ under healthier boundary condition than those required by $\mathcal{G}_2$. $\mathcal{F}_2$ ensures that such an infection propagates over to the first $\lceil 2/q^4 \rceil$ columns to the right of $\Lambda^{(\nu)}$ while $\mathcal{F}_3$ guarantees that each time the infection meets an empty site of the upward stair it grows vertically by one unit (see Figure 5). Since the stair contains $\lceil 1/q^3 \rceil$ sites, the $\lceil 2/q^4 \rceil$th-column of $\mathcal{M}_t$ to the right of $\Lambda^{(\nu)}$ contains an infected interval which is the appropriate horizontal translation of the interval $\hat{I}$ and the inclusion $\mathcal{F} \subseteq \mathcal{G}_2$ follows.

Conditionally on $\{\nu = k\}$, the events $\mathcal{F}_2, \mathcal{F}_3$ coincide with two increasing events depending only on sites to the right of $\Lambda^{(k)}$. Hence, using the FKG inequality,

$$\begin{aligned}\mu(\mathcal{G}_2) \geq \mu(\mathcal{F}) &= \sum_{k \in [t/2, t]} \mu(\nu = k)\mu(\mathcal{F}_2 \cap \mathcal{F}_3 \mid \nu = k) \\ &\geq \sum_{k \in [t/2, t]} \mu(\nu = k)\mu(\mathcal{F}_2 \mid \nu = k)\mu(\mathcal{F}_3 \mid \nu = k).\end{aligned}$$

A union bound gives that, uniformly in $k \in [t/2, t]$,

$$\mu(\mathcal{F}_2^c \mid \nu = k) \leq \lceil 2/q^4 \rceil (1-q)^\ell \leq \lceil 2/q^4 \rceil q^{1/\varepsilon}(1 + o(1)) = o(1),$$

if $\varepsilon < 1/4$. Using the fact that $X(\omega) := \min\{i \geq 1\colon \omega_{(i,+1)} = 0\}$ is a geometric random variable of parameter q, it is easy to check that

$$\mu(\mathcal{F}_3^c \mid \nu = k) \leq \mathbb{P}\left(\sum_{i=1}^n X_i > \lceil 2/q^4 \rceil\right),$$

where $\{X_i\}_{i=1}^n$ are i.i.d. copies of X. A standard exponential Markov inequality with $\lambda = \alpha q$, $\alpha \in (0, 1)$, gives

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n X_i > \lceil 2/q^4 \rceil\right) &\leq e^{-\lambda \lceil 2/q^4 \rceil}\left(\mathbb{E}(e^{\lambda X})\right)^n \\ &\leq \left(\frac{e^{-2\alpha}}{(1-\alpha)(1+o(1))}\right)^{1/q^3} < (1-\alpha/2)^{1/q^3},\end{aligned} \tag{5.16}$$

for α small enough. In conclusion, if $\varepsilon < 1/4$,

$$\begin{aligned}\mu(\mathcal{G}_2) &\geq (1 - o(1))\mu(\mathcal{F}_1) \\ &\geq (1 - o(1))\left(1 - \left(1 - \mu(\mathcal{O}^1_{\Lambda_{1,j}}(I))\right)^{t/2}\right) \geq (1 - o(1))(1 - e^{-4})\end{aligned}$$

because of (5.15) and our choice of t. That concludes the proof of property (b). □

We now turn to the second basic lemma. Recall the definition (5.6) of the event $\mathcal{B}_2$.

LEMMA 5.27. *Choose the basic scales N, ℓ, n_1, n_2 as in* (5.1), (5.2) *and* (5.8). *Then, for ε small enough,*

$$\mu(\mathcal{B}_2(n_2 - 1)) \le e^{-\Omega(1/q^5)} \quad \text{as } q \to 0. \tag{5.17}$$

PROOF. Call $\mathcal{H}_{i,j}$ the event $\bigcap_{k=i}^{j}\{\omega \in \Omega\colon \Phi(\omega)_k = \downarrow\} \cap \mathcal{G}_{i,j}$, where $\mathcal{G}_{i,j}$ has been defined in (5.7). Clearly,

$$\mu(\mathcal{B}_2(n_2 - 1)) \le \sum_{\substack{i,j \\ j-i \ge n_2-2}} \mu(\mathcal{H}_{i,j}) \le N^2 \max_{\substack{i,j\in[N] \\ j-i\ge n_2-2}} \mu(\mathcal{H}_{i,j}),$$

and it is enough to prove that

$$\max_{\substack{i,j\in[N] \\ j-i\ge n_2-2}} \mu(\mathcal{H}_{i,j}) \le e^{-\Omega(1/q^5)}. \tag{5.18}$$

For this purpose, we first describe one important implication of the event $\mathcal{H}_{i,j}$.

CLAIM 5.28. *For any $\omega \in \mathcal{H}_{i,j}$, there exists $h \in \mathbb{Z}$ satisfying $|h| \le N^2 - (j-1)N + (j-i)\ell$, such that*

$$C_h := \left(\bigcup_{k=i}^{j}\{(k - N, h)\}\right) \cap V_{i,j} \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}.$$

Moreover, C_h has length at least $(j - i)(1 - o(1)) \ge n_2(1 - o(1))$ as $q \to 0$.

PROOF OF THE CLAIM. Given $\omega \in \mathcal{H}_{i,j}$ let $\Gamma = (x^{(1)}, \dots, x^{(n)}) \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}$ be a Duarte path from $\mathcal{C}_i$ to $\mathcal{C}_j$. Since $\Phi(\omega)_k = \downarrow$ for all $k \in \{i, \dots, j\}$ necessarily the cardinality of $\Gamma \cap \mathcal{C}_k$ is at most ℓ for all $k \in \{i, \dots, j\}$. Therefore, the height h of $x^{(1)}$ satisfies

$$|h| \le N^2 - (j - 1)N + (j - i)\ell,$$

which, in turn, implies that the corresponding interval C_h has length greater than the largest integer m such that

$$N^2 - (i - 1)N - mN \ge N^2 - (j - 1)N + (j - i)\ell.$$

Using that $m + 1$ violates the above inequality, we get

$$m \ge (j - i)(1 - \ell/N) - 1 \ge (1 - o(1))n_2.$$

The fact that $C_h \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}$ follows from Corollary 5.8. □

It is now easy to complete the proof of the lemma. As in the proof of Claim 5.26 and using a union bound over the possible value of the variable h of the claim, with probability larger than

$$1 - 2N^2 e^{-\Omega(qn_2)} \ge 1 - e^{-\Omega(1/q^5)},$$

every interval C_h as above with $|h| \le N^2 - (j-1)N + (j-i)\ell$ meets an empty upward stair, that is, a sequence $(x_1, \dots, x_\ell)$ of empty sites belonging to the first $n_2/2$ columns crossed by C_h and such that $x_m = (j_m, h + m)$ with $j_m < j_{m+1}$ for all $m \in [\ell]\}$. If C_h is also infected, then the presence of the above empty stair implies that there exists $i \le k \le i + \frac{2}{3}n_2$ and a vertical interval $I \subseteq \mathcal{C}_k$ of length at least ℓ such that $I \subseteq [Y(\omega) \cap V_{i,j}]_{V_{i,j}}^{1,0}$. The latter property implies that $\Phi(\omega)_k = \uparrow$. Hence $\mu(\mathcal{H}_{i,j})$ satisfies (5.18) uniformly in $j - i \ge n_2 - 2$. □

5.5. *Completing the proof of Proposition* 5.17. Recall Definition 5.15 of the test function ϕ and of the events Ω_g, $\Omega_\downarrow$ and $\mathcal{A}_{\varepsilon,q}$. Notice that $\Omega_g \cap \mathcal{B}_2(n_2-1)^c \subseteq \mathcal{A}_{\varepsilon,q}$ and that $\Omega_\downarrow$ is a decreasing event. Using Lemma 5.27, we get

$$\begin{aligned}\mu(\phi) &\geq \mu(\mathcal{A}_{\varepsilon,q}) \geq \mu\big(\Omega_g \cap \mathcal{B}_2(n_2-1)^c\big)\\ &\geq \mu(\Omega_\downarrow)\mu\Big(\prod_{|k|\leq \ell}\omega_{(0,k)}=1\Big) - \mu\big(\mathcal{B}_2(n_2-1)\big)\\ &\geq \mu(\Omega_\downarrow)(1-q)^{2\ell+1} - e^{-\Omega(1/q^5)} \geq q^{O(1)}\mu(\Omega_\downarrow) - e^{-\Omega(1/q^5)},\end{aligned}$$

where in the third inequality we used the FKG inequality. Using Lemma 5.25 and a union bound,

$$\begin{aligned}\mu(\Omega_\downarrow) &\geq 1-\mu\Bigg(\bigcup_{j=1}^{N}\bigcup_{I\in\mathcal{I}_j(\ell)}\mathcal{O}^{1,0}_{V_{1,j}}(I)\Bigg)\\ &\geq 1-4e^{-(c-5\varepsilon)(\log q)^2/q} = 1-o(1)\end{aligned}$$

if ε is small enough, where we let $\mathcal{I}_j(\ell)$ be the family of intervals of the jth column whose length is at least ℓ. In conclusion, $\mu(\phi) \geq q^{O(1)}$ for ε small enough.

We now turn to bound from above the Dirichlet form $\mathcal{D}(\phi)$. By definition, writing $\mathcal{A} \equiv \mathcal{A}_{\varepsilon,q}$ for notation convenience,

$$\begin{aligned}\mathcal{D}(\phi) &= \sum_{x\in\mathbb{Z}^2}\mu\big(c_x \operatorname{Var}_x(\phi)\big) = \sum_{x\in V}\mu\big(c_x \operatorname{Var}_x(\phi)\big)\\ &= \mu(\mathcal{A})^{-1}q(1-q)\sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\notin\mathcal{A}\}} + c_x(\omega)\mathbb{1}_{\{\omega\notin\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}\}}\big)\\ &\leq \mu(\mathcal{A})^{-1}\sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\notin\mathcal{A}\}}\big),\end{aligned}$$

where we used the fact that ϕ depends only on $\{\omega_x\}_{x\in V}$ in the second equality and made the change of variable $\omega\to\omega^x$ in the term $c_x(\omega)\mathbb{1}_{\{\omega\notin\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}\}}$ in the inequality. Next, we observe that

$$\begin{aligned}&\sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\notin\mathcal{A}\}}\big)\\ &\quad\leq \sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}^c,\omega^x\in\mathcal{B}_2(n_2-1)^c\}}\big) + \sum_{x\in V}\mu\big(\mathbb{1}_{\{\omega^x\in\mathcal{B}_2(n_2-1)\}}\big)\\ &\quad\leq \sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}^c,\omega^x\in\mathcal{B}_2(n_2-1)^c\}}\big)\\ &\qquad + |V|\big((1-q)/q\big)\mu\big(\mathcal{B}_2(n_2-1)\big)\\ &\quad\leq \sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}^c,\omega^x\in\mathcal{B}_2(n_2-1)^c\}}\big) + e^{-\Omega(1/q^5)},\end{aligned} \tag{5.19}$$

where in the last inequality we used Lemma 5.27 and the bound $|V| \leq 2N^3 \leq e^{O((\log q)^2/q)}$.

Given $x\in V$, let $\omega\in\mathcal{A}$ be such that $c_x(\omega)=1$ and $\omega^x\in\mathcal{A}^c\cap\mathcal{B}_2(n_2-1)^c$ and recall that $N_\uparrow(\omega)$ counts the number of up-arrows in $\Phi(\omega)$. We claim that $N_\uparrow(\omega^x)\geq n_1-1$. To prove the claim, let γ be a legal path connecting Ω_g to $(\omega_V,\tilde\omega_{V^c}\equiv 0)$ such that $\gamma\cap\mathcal{B}_i(n_i-1)=\varnothing$, $i=1,2$ and let γ^x be the path connecting Ω_g to $(\omega^x_V,\tilde\omega_{V^c}\equiv 0)$ obtained by adding to γ the transition $(\omega_V,\tilde\omega_{V^c}\equiv 0)\to(\omega^x_V,\tilde\omega_{V^c}\equiv 0)$. The path γ^x is legal because γ is legal and $c_x(\omega)=1$. Moreover, $\gamma^x\cap\mathcal{B}_2(n_2-1)=\varnothing$ because $\omega^x\notin\mathcal{B}_2(n_2-1)$. The assumption $\omega^x\in\mathcal{A}^c$ implies that $\gamma^x\cap\mathcal{B}_1(n_1-1)\neq\varnothing$. Using $\gamma\cap\mathcal{B}_1(n_1-1)=\varnothing$ the latter requirement becomes $N_\uparrow(\omega^x)\ge n_1-1$ and the claim follows.

In conclusion,

$$\begin{aligned}\sum_{x\in V}\mu\big(c_x(\omega)\mathbb{1}_{\{\omega\in\mathcal{A}\}}\mathbb{1}_{\{\omega^x\in\mathcal{A}^c,\omega^x\in\mathcal{B}_2(n_2-1)^c\}}\big)&\le\sum_{x\in V}\mu\big(N_\uparrow(\omega^x)\ge n_1-1\big)\\&\le|V|\big((1-q)/q\big)\mu\big(N_\uparrow(\omega)\ge n_1-1\big).\end{aligned}$$

We finally bound from above $\mu(N_\uparrow(\omega)\ge n_1-1)$ using Lemma 5.25. Given $n\ge n_1-1$ and $E=\{j_1<\cdots<j_n\}$, $j_i\in[N]$, let $\mathcal{N}_E$ be the event that $\Phi(\omega)_j=\uparrow$ if $j\in E$ and $\Phi(\omega)_j=\downarrow$ otherwise. By construction,

$$\mu(\mathcal{N}_E)\le\mu\Bigg(\bigcap_{k=1}^n\mathcal{Q}^{1,0}_{V_{j_{k-1}+1,j_k}}\Bigg)\le\Big(\max_{i\le j}\mu\big(\mathcal{Q}^{1,0}_{V_{i,j}}\big)\Big)^n,$$

where $j_0:=0$ and

$$\mathcal{Q}^{1,0}_{V_{i,j}}=\big\{\exists I\in\mathcal{I}_j(\ell)\text{ such that }I\subseteq\big[Y(\omega)\cap V_{i,j}\big]^{1,0}_{V_{i,j}}\big\},$$

where we recall that $\mathcal{I}_j(\ell)$ is the family of intervals of the jth column whose length is at least ℓ. Lemma 5.25 together with a union bound over $I\in\mathcal{I}_j(\ell)$ give

$$\begin{aligned}\max_{i\le j}\mu\big(\mathcal{Q}^{1,0}_{V_{i,j}}\big)&\le\max_{i\le j}\sum_{I\in\mathcal{I}_j(\ell)}\mu\big(I\subseteq\big[Y(\omega)\cap V_{i,j}\big]^{1,0}_{V_{i,j}}\big)\\&\le4N^4\max_{i\le j}\max_{I\in\mathcal{I}_j(\ell)}\mu\big(I\subseteq\big[Y(\omega)\cap V_{i,j}\big]^{1,0}_{V_{i,j}}\big)\le e^{-(c-4\varepsilon)(\log q)^2/2q}.\end{aligned}$$

In conclusion, for any ε small enough,

$$\begin{aligned}\mu\big(N_\uparrow(\omega)\ge n_1-1\big)&\le\sum_{n=n_1-1}^N\binom{N}{n}e^{-(c-4\varepsilon)n(\log q)^2/2q}\\&\le\sum_{n=n_1-1}^N\big(Ne^{-(c-4\varepsilon)(\log q)^2/2q}\big)^n\\&\le e^{-\varepsilon\Omega((\log q)^4/q^2)},\end{aligned}$$

because of the choice of $n_1=\varepsilon(\log q)^2/2q$. In conclusion, the right-hand side of (5.19) is smaller than $e^{-\varepsilon\Omega((\log q)^4/q^2)}$ and the proof of Proposition 5.17 is complete.

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