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STRICT MONOTONICITY OF PERCOLATION THRESHOLDS UNDER COVERING MAPS

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We answer a question of Benjamini and Schramm by proving that under reasonable conditions, quotienting a graph strictly increases the value of its percolation critical parameter p_c. More precisely, let $\mathcal{G} = (V, E)$ be a quasi-transitive graph with $p_c(\mathcal{G}) < 1$, and let G be a nontrivial group that acts freely on V by graph automorphisms. Assume that $\mathcal{H} := \mathcal{G}/G$ is quasi-transitive. Then one has $p_c(\mathcal{G}) < p_c(\mathcal{H})$.

We provide results beyond this setting: we treat the case of general covering maps and provide a similar result for the uniqueness parameter p_u, under an additional assumption of boundedness of the fibres. The proof makes use of a coupling built by lifting the exploration of the cluster, and an exploratory counterpart of Aizenman–Grimmett's essential enhancements.

Bernoulli percolation is a simple model for problems of propagation in porous media that was introduced in 1957 by Broadbent and Hammersely [4]: given a graph $\mathcal{G}$ and a parameter $p \in [0, 1]$, erase each edge independently with probability $1 - p$. Studying the connected components of this random graph (which are referred to as *clusters*) has been since then an active field of research; see the books [6, 13]. A prominent quantity in this theory is the so-called *critical parameter* $p_c(\mathcal{G})$, which is characterised by the following dichotomy: for every $p < p_c(\mathcal{G})$, there is almost surely no infinite cluster, while for every $p > p_c(\mathcal{G})$, there is almost surely at least one infinite cluster.

Originally, the main focus was on the Euclidean lattice $\mathbb{Z}^d$. In 1996, Benjamini and Schramm initiated the systematic study of Bernoulli percolation on more general graphs, namely quasi-transitive graphs [3]. A graph is *quasi-transitive* (resp., *transitive*) if the action of its automorphism group on its vertices yields finitely many orbits (resp., a single orbit). Intuitively, a graph is quasi-transitive if it has finitely many types of vertices, and transitive if all the vertices look the same. The paper [3] contains, as its title suggests, many questions and a few answers: in their Theorem 1 and Question 1, they investigate the monotonicity of p_c under quotients. Their Question 1 is precisely the topic of the present paper. It goes as follows.

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Setting of [3]. Let $\mathcal{G} = (V, E)$ be a locally finite *connected* graph. Let G be a group acting on V by graph automorphisms. A vertex of the *quotient graph* $\mathcal{G}/G$ is an orbit of $G \curvearrowright V$, and two distinct orbits are connected by an edge if and only if there is an edge of $\mathcal{G}$ intersecting both orbits.

Theorem 1 of [3] asserts that $p_c(\mathcal{G}) \le p_c(\mathcal{G}/G)$. It is proved by lifting the exploration of a spanning tree of the cluster of the origin from $\mathcal{G}/G$ to $\mathcal{G}$. They then ask the following natural question. Recall that a group action $G \curvearrowright X$ is *free* if the only element of G that has a fixed point is the identity element:

$$\forall g \in G\backslash\{1\}, \forall x \in X, \quad gx \neq x.$$

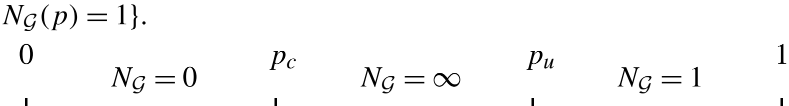
The main result of the present paper is the following theorem, which gives a positive answer to Question 1 from [3].

THEOREM 0.1. *Let G be a nontrivial group acting on a graph $\mathcal{G}$ by graph automorphisms. Assume that $p_c(\mathcal{G}) < 1$, that G acts freely on $V(\mathcal{G})$, and that both $\mathcal{G}$ and $\mathcal{H} := \mathcal{G}/G$ are quasi-transitive. Then one has $p_c(\mathcal{G}) < p_c(\mathcal{H})$.*

EXAMPLE. Let G be a group and S be a finite generating subset of G. The *Cayley graph* $\mathcal{G}$ associated with (G, S) has vertex-set G, and two *distinct* elements g and h of G are connected by an edge if and only if $g^{-1}h \in S^{\pm 1}$. Let N be a normal subgroup of G, and let it act on G by left multiplication: for every $(n, g) \in N \times G$, one sets $n \cdot g := ng$. Then N acts freely and by graph automorphisms on $G = V(\mathcal{G})$. Besides, $\mathcal{G}$ and $\mathcal{G}/N \simeq \mathsf{Cayley}(G/N, \overline{S})$ are transitive (the set $\overline{S}$ stands for the reduction of S modulo N).

REMARK. By using the techniques of [15], one can deduce from Theorem 0.1 and [10], exercise page 4, that when $\mathcal{G}$ ranges over Cayley graphs of 3-solvable groups, $p_c(\mathcal{G})$ takes uncountably many values. Actually, the set of such values contains a subset homeomorphic to $\{0, 1\}^{\mathbb{N}}$. This is optimal in the following sense: there are only countably many 2-solvable finitely generated groups (see Corollary 3 in [9]), hence only countably many Cayley graphs of such groups. The same result without the solvability condition has been obtained previous to [15] by Kozma [12], by working with graphs of the form $\mathcal{G} \star \mathcal{G}$.

We also address in Theorem 0.2 below a similar question for the *uniqueness parameter* p_u. Recall that given a *quasi-transitive graph* $\mathcal{G}$, the number of infinite connected components for Bernoulli percolation of parameter p takes an almost sure value $N_{\mathcal{G}}(p) \in \{0, 1, \infty\}$, and that the following monotonicity property holds: $\forall p < q$, $N_{\mathcal{G}}(p) = 1 \implies N_{\mathcal{G}}(q) = 1$; see [17]. One thus defines $p_u(\mathcal{G}) := \inf\{p \in [0, 1] : N_{\mathcal{G}}(p) = 1\}$.



THEOREM 0.2. *Let G be a nontrivial* finite *group acting on a graph $\mathcal{G}$ by graph automorphisms. Assume that $p_u(\mathcal{G}) < 1$, that G acts freely on $V(\mathcal{G})$ and that both $\mathcal{G}$ and $\mathcal{H} := \mathcal{G}/G$ are quasi-transitive. Then one has $p_u(\mathcal{G}) < p_u(\mathcal{H})$.*

In addition to Theorems 0.1 and 0.2, we also provide similar results for the case of general covering maps (see Section 1 for definition and statements). In particular, one does not need quasi-transitivity in order to prove strict inequalities for p_c; see Theorem 1.1.

In our proofs, we use an exploratory version of Aizenman–Grimmett's essential enhancements [1], and build a coupling between p-percolation on $\mathcal{G}$ and enhanced percolation on $\mathcal{H}$ by lifting the exploration of the cluster of the origin. The part of our work devoted to essential enhancements (Section 2.2) follows the Aizenman–Grimmett strategy, thus making crucial use of certain differential inequalities; see also [16]. Our coupling (Section 2.1) improves on that used in [3].

Let us mention that a theorem quite similar to our Theorem 0.1 has already been obtained for the connective constant for the self-avoiding walk instead of p_c; see Theorem 3.8 in [7]. However, we would like to stress that our techniques are completely different from those of [7].

Structure of the paper. Section 1 provides the relevant definitions and the statements of two general theorems, namely Theorems 1.1 and 1.4. Theorem 1.1 is proved in Section 2 and Theorem 1.4 is established in Section 3. Section 4 explains why Theorems 1.1 and 1.4 imply Theorems 0.1 and 0.2 (as well as Corollaries 1.2 and 1.5). Finally, Section 5 discusses the hypotheses of our results and raises several questions.

1. The case of general covering maps.

To avoid any ambiguity, let us review the relevant vocabulary.

Convention. Graphs are taken to be nonempty, locally finite (every vertex has finitely many neighbours) and *connected*. Subgraphs (e.g., percolation configurations) may not be connected. Unless otherwise stated, our graphs are taken to be simple (no multiple edges, no self-loops, edges are unoriented). A graph $\mathcal{G}$ may be written in the form (V, E), where $V = V(\mathcal{G})$ denotes its set of vertices and $E = E(\mathcal{G})$ its set of edges. An edge is a subset of V with precisely two elements. The *degree* of a vertex is its number of neighbours. Graphs are endowed with their respective graph distance, denoted by d. Finally, percolation is taken to mean Bernoulli *bond* percolation, but our proofs can be adapted to Bernoulli *site* percolation.

In Theorem 0.1, the graphs $\mathcal{G}$ and $\mathcal{H}$ are related via the quotient map $\pi : x \mapsto Gx$. This map is a *weak covering map*, meaning that it is 1-Lipschitz for the graph distance and that it has the *weak lifting property*: for every $x \in V(\mathcal{G})$ and every neighbour u of $\pi(x)$, there is a neighbour of x that is mapped to u. This fact does not use the freeness of the action of G or quasi-transitivity.

Weak covering maps are by definition able to lift edges, but it turns out they can also lift trees, meaning that for every subtree of the target space and every vertex in the preimage of the tree, there is a lift of the tree that contains this vertex. Recall that given a subtree[1] $\mathcal{T}$ of $\mathcal{H}$, a *lift* of $\mathcal{T}$ is a subtree $\mathcal{T}'$ of $\mathcal{G}$ such that π induces a graph isomorphism from $\mathcal{T}'$ to $\mathcal{T}$, that is, it induces well-defined bijections from $V(\mathcal{T}')$ to $V(\mathcal{T})$ and from $E(\mathcal{T}')$ to $E(\mathcal{T})$. Lifting trees enables us to lift paths: as a consequence, if $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ is a weak covering map, then π maps the ball $B_r(x)$ surjectively onto the ball $B_r(\pi(x))$ for any $x \in V(\mathcal{G})$ and $r \geq 0$.

The map $\pi : x \mapsto Gx$ satisfies a second property, namely *disjoint tree-lifting*: if $\mathcal{T}$ is a subtree of $\mathcal{H}$ and if x and y are distinct vertices of $\mathcal{G}$ such that $\pi(x) = \pi(y)$ belongs to $V(\mathcal{T})$, then one can find two *vertex-disjoint* lifts of $\mathcal{T}$ such that one of them contains x and the other y. This fact uses the freeness of G, and is established in Lemma 4.1.

Finally, the map π *has uniformly nontrivial fibres*: there is some R such that for every $x \in V(\mathcal{G})$, there is some $y \in V(\mathcal{G})$ satisfying $\pi(x) = \pi(y)$ and $0 < d(x, y) \leq R$; see Lemma 4.2.

It turns out that these three properties of π suffice to prove strict inequality, so that there is actually no need for group actions and quasi-transitivity.

THEOREM 1.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be graphs of bounded degree. Assume that there is a weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with uniformly nontrivial fibres and the disjoint tree-lifting property. If $p_c(\mathcal{G}) < 1$, then one has $p_c(\mathcal{G}) < p_c(\mathcal{H})$.*

Theorem 0.1 then follows from Theorem 1.1 and Lemmas 4.1 and 4.2. Theorem 1.1 yields a second corollary. Say that a map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ is a *strong covering map* if it is 1-Lipschitz for the graph distance and has the *strong lifting property*: for every $x \in V(\mathcal{G})$, for every neighbour u of $\pi(x)$, there is a *unique* neighbour of x that maps to u. Recall that for many authors, the definition of a "covering map" is taken to be even stricter: a *classical covering map* is a *graph homomorphism* with the strong lifting property. By Theorem 1.1 and Lemma 4.3, the following result holds.

COROLLARY 1.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be graphs of bounded degree. Assume that there is a strong covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with uniformly nontrivial fibres. If $p_c(\mathcal{G}) < 1$, then one has $p_c(\mathcal{G}) < p_c(\mathcal{H})$.*

We also study the monotonicity question for p_u. This question was already investigated in the following setting: a particular kind of weak covering map is given by taking two graphs $\mathcal{G}$ and $\mathcal{H}$ and considering the natural projection $\pi : V(\mathcal{G} \times \mathcal{H}) \to V(\mathcal{G})$. Theorem 6.12 in [14] implies that if $\mathcal{G}$ and $\mathcal{H}$ are unimodular transitive graphs, then $p_u(\mathcal{G} \times \mathcal{H}) \leq p_u(\mathcal{G})$. If $\mathcal{H}$ has at least two vertices and

[1] That is, a tree with $V(\mathcal{T}) \subset V(\mathcal{H})$ and $E(\mathcal{T}) \subset E(\mathcal{H})$.

$p_u(\mathcal{G}) < 1$, one can deduce that $p_u(\mathcal{G} \times \mathcal{H}) \le p_u(\mathcal{G} \times \{0,1\}) < p_u(\mathcal{G})$. The first inequality follows from Theorem 6.12 of [14] while the second one can be proved by hand or by using our Theorem 0.2.

In this paper, we work in a different setting, namely weak covering maps with bounded fibres. Say that a weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ has *bounded fibres* if there is some K such that

$$\forall x, y \in V(\mathcal{G}), \quad \pi(x) = \pi(y) \quad \Longrightarrow \quad d(x, y) \le K.$$

The following two theorems are, respectively, the p_u counterparts of Theorem 1 from [3] and Theorem 1.1 above.

THEOREM 1.3. *Let $\mathcal{G}$ and $\mathcal{H}$ be quasi-transitive graphs. Assume that there is a weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with bounded fibres.*
Then one has $p_u(\mathcal{G}) \le p_u(\mathcal{H})$.

THEOREM 1.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be quasi-transitive graphs. Assume that there is a noninjective weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with bounded fibres and the disjoint tree-lifting property. If $p_u(\mathcal{G}) < 1$, then one has $p_u(\mathcal{G}) < p_u(\mathcal{H})$.*

Theorem 0.2 follows directly from Theorem 1.4 and Lemmas 4.1, 4.2 and 4.4. The next corollary follows from Theorem 1.4 and Lemma 4.3.

COROLLARY 1.5. *Let $\mathcal{G}$ and $\mathcal{H}$ be quasi-transitive graphs. Assume that there is a noninjective strong covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with bounded fibres. If $p_u(\mathcal{G}) < 1$, then one has $p_u(\mathcal{G}) < p_u(\mathcal{H})$.*

Let us mention that our proofs can be made explicit in that they actually yield quantitative (but poor) lower bounds on the differences $p_c(\mathcal{H}) - p_c(\mathcal{G})$ and $p_u(\mathcal{H}) - p_u(\mathcal{G})$.

2. Proof of Theorem 1.1. Let $\mathcal{G}$, $\mathcal{H}$ and π be as in Theorem 1.1. Let r be a positive integer. Pick a root o in $\mathcal{H}$, and some $o' \in \pi^{-1}(\{o\})$.

Notation. Given a graph (V, E), the ball of centre x and radius r is $B_r(x) := \{y \in V : d(x, y) \le r\}$. It is considered as a set of vertices, but it may also be considered as a graph—with the structure the ambient graph induces on it. For $r \in \mathbb{N}$, the sphere of centre x and radius r is $S_r(x) := \{y \in V : d(x, y) = r\}$. We also set $S_{r+\frac{1}{2}}(x) := \{e \in E : e \cap S_r(x) \neq \varnothing \text{ and } e \cap S_{r+1}(x) \neq \varnothing\}$.

We are going to construct a random subset $\mathcal{C}_0$ of $V(\mathcal{H})$ which will be a "strict enhancement" of the cluster of o in a p-percolation model on $\mathcal{H}$. Given a configuration $(\omega, \alpha) \in \{0,1\}^{E(\mathcal{H})} \times \{0,1\}^{V(\mathcal{H})}$, we define inductively a sequence $(C_n)_{n \ge 0}$ of subsets of $V(\mathcal{H})$ as follows. We sometimes identify ω with the subset of edges

$\{e : \omega_e = 1\}$ or the subgraph of $\mathcal{H}$ associated with it. Set $C_0 := \{o\}$. For $n \geq 0$, let C_{2n+1} be the union of the ω-clusters of the vertices of C_{2n}. Then let C_{2n+2} be the union of C_{2n+1} and the vertices v such that there is some $u \in C_{2n+1}$ satisfying the following conditions:

1. $d(u, v) = r + 1$,
2. $\omega_e = 1$ for all edges e in $B_r(u)$,
3. $\alpha_u = 1$.

The sequence of sets (C_n) is nondecreasing, and we define $\mathcal{C}_o = \mathcal{C}_o(\omega, \alpha) := \bigcup_n C_n$. Given $p, s \in [0, 1]$, the distribution of the random variable $\mathcal{C}_o(\omega, \alpha)$ under the probability measure $\mathbb{P}_{p,s} := \mathsf{Ber}(p)^{\otimes E(\mathcal{H})} \otimes \mathsf{Ber}(s)^{\otimes V(\mathcal{H})}$ is denoted by $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$. In a similar way, we can define $\mathcal{C}_A = \mathcal{C}_A(\omega, \alpha)$—and its distribution under $\mathbb{P}_{p,s}$, denoted by $\mathcal{C}_{\mathcal{H}}^{p,s}(A)$—by considering the same process but initialising it with $C_0 = A$. We also set $\mathcal{C}_{\mathcal{G}}^{p}(A)$ to be the distribution of the cluster of A in bond percolation of parameter p on $\mathcal{G}$.

REMARK. Note that $\mathcal{C}_o(\omega, \alpha)$ does not coincide with the cluster of o for the following model: declare an edge e to be open if "e is ω-open or there is a vertex u such that $e \in S_{r+\frac{1}{2}}(u)$, all the edges in $B_r(u)$ are ω-open and $\alpha_u = 1$." This would be an instance of the *classical* enhancement introduced by Aizenman and Grimmett; see [1]. Indeed, the model we consider here is an *exploratory version* of their model. For example, in our model the assertion $v \in \mathcal{C}_u(\omega, \alpha)$ does not necessarily imply $u \in \mathcal{C}_v(\omega, \alpha)$. Also, our model is stochastically dominated by the classical one.

We will prove the following two propositions. The proof of Proposition 2.1 proceeds by lifting some exploration process from $\mathcal{H}$ to $\mathcal{G}$: in that, it is similar to the proof of Theorem 1 of [3]. The proof of Proposition 2.2 uses an exploratory variation of the techniques of Aizenman and Grimmett [1]. Even though essential enhancements are delicate in general [2], it turns out that our particular enhancement *can* be handled for general graphs, even for site percolation.

PROPOSITION 2.1. *Take $\mathcal{G}$, $\mathcal{H}$ and π to satisfy the hypotheses of Theorem 1.1* (*but not necessarily $p_c(\mathcal{G}) < 1$*). *Then there is a choice of $r \geq 1$ such that the following holds*: *for every $\varepsilon > 0$, there is some $s \in (0, 1)$ such that for every $p \in [\varepsilon, 1]$, $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is stochastically dominated*[2] *by $\pi(\mathcal{C}_{\mathcal{G}}^{p}(o'))$.*

PROPOSITION 2.2. *Let $\mathcal{H}$ be a graph of bounded degree such that $p_c(\mathcal{H}) < 1$. Then, for any choice of $r \geq 1$, the following holds*: *for every $s \in (0, 1]$, there exists $p_s < p_c(\mathcal{H})$ such that for every $p \in [p_s, 1]$, the cluster $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is infinite with positive probability.*

[2]There is a coupling such that the $(\mathcal{H}, p, s)$-cluster is a subset of the π-image of the $(\mathcal{G}, p)$-cluster.

Assuming these propositions, let us establish Theorem 1.1.

PROOF OF THEOREM 1.1. First, notice that if $p_c(\mathcal{H}) = 1$, then the conclusion holds trivially. We thus assume that $p_c(\mathcal{H}) < 1$. We pick r so that the conclusion of Proposition 2.1 holds. Since boundedness of the degree of $\mathcal{H}$ implies that $p_c(\mathcal{H}) > 0$, we can pick some ε in $(0, p_c(\mathcal{H}))$. By Proposition 2.1, we can pick $s \in (0, 1)$ such that for every $p \in [\varepsilon, 1]$, $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is stochastically dominated by $\pi(\mathcal{C}_{\mathcal{G}}^{p}(o'))$. By Proposition 2.2, there is some $p_s < p_c(\mathcal{H})$ such that for every $p \in [p_s, 1]$, the cluster $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is infinite with positive probability. Fix such a p_s, and set $p := \max(p_s, \varepsilon) < p_c(\mathcal{H})$. By definition of p_s, the cluster $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is infinite with positive probability. As $p \geq \varepsilon$, the definition of s implies that $\mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is stochastically dominated by $\pi(\mathcal{C}_{\mathcal{G}}^{p}(o'))$. As a result, $\pi(\mathcal{C}_{\mathcal{G}}^{p}(o'))$ is infinite with positive probability. In particular, $\mathcal{C}_{\mathcal{G}}^{p}(o')$ is infinite with positive probability, so that $p_c(\mathcal{G}) \leq p < p_c(\mathcal{H})$. □

2.1. *Proof of Proposition* 2.1. The choice of a suitable value of r is given by the following lemma.

LEMMA 2.3. *There is a choice of $r \geq 1$ such that for every $x \in V(\mathcal{G})$, the set $Z = Z(x, r)$ defined as the connected component*[3] *of x in $\pi^{-1}(B_r(\pi(x))) \cap B_{3r}(x)$ satisfies that for any $u \in S_{r+1}(\pi(x))$, the fibre $\pi^{-1}(\{u\})$ contains at least two vertices adjacent to Z.*

PROOF. Let R be given by the fact that π has uniformly nontrivial fibres and set $r := \lceil \frac{R}{2} \rceil$. Let x be any vertex of $\mathcal{G}$. Take some $y \in V(\mathcal{G})$ such that $\pi(x) = \pi(y)$ and $0 < d(x, y) \leq R$. Let $\mathcal{T}$ be a spanning tree of $B_{r+1}(\pi(x))$ obtained by adding first the vertices at distance 1, then at distance 2, etc. As π has the disjoint tree-lifting property, one can pick two vertex-disjoint lifts $\mathcal{T}_x$ and $\mathcal{T}_y$ of $\mathcal{T}$ such that $x \in V(\mathcal{T}_x)$ and $y \in V(\mathcal{T}_y)$.

Let γ be a geodesic path from x to y, thus staying inside $\pi^{-1}(B_r(\pi(x)))$ as $R \leq 2r$. The set Z' consisting in the union of the span of γ and $(V(\mathcal{T}_x) \cup V(\mathcal{T}_y)) \cap \pi^{-1}(B_r(\pi(x)))$ is a connected subset of $Z(x, r)$: its connectedness results from the choice of the spanning tree $\mathcal{T}$. It thus suffices to prove that for any $u \in S_{r+1}(\pi(x))$, the fibre $\pi^{-1}(\{u\})$ contains at least two vertices adjacent to Z'. But this is the case as every such u admits a lift in $\mathcal{T}_x$ and another one in $\mathcal{T}_y$. □

Take r to satisfy the conclusion of Lemma 2.3. Let $\varepsilon > 0$. Set M and s to be so that the following two conditions hold:

$$\forall e = \{x, y\} \in E(\mathcal{H}), \quad M \geq |B_r(x) \cup B_r(y)|,$$

$$\forall x \in V(\mathcal{G}), \quad s \leq \left(1 - (1 - \varepsilon)^{1/M}\right)^{|E(B_{3r+1}(x))|}.$$

[3]Here, $\pi^{-1}(B_r(\pi(x))) \cap B_{3r}(x)$ is seen as endowed with the graph structure induced by $\mathcal{G}$.

For instance, one may take $M := D^{r+2}$ and $s := (1-(1-\varepsilon)^{1/M})^{D^{3r+2}}$, where D stands for the maximal degree of a vertex of $\mathcal{G}$. Let $p \in [\varepsilon, 1]$.

We define the multigraph $\hat{\mathcal{G}}$ as follows: the vertex-set is $V(\mathcal{G})$, the edge-set is $E(\mathcal{G}) \times \{1, \dots, M\}$ and $(\{x, y\}, k)$ is interpreted as an edge connecting x and y. The multigraph $\hat{\mathcal{H}}$ is defined in the same way, with $\mathcal{H}$ instead of $\mathcal{G}$. The purpose of this multigraph is to allow multiple use of each edge for a bounded number of "s-bonus." They will play no role as far as p-exploration is concerned: concretely, for "p-exploration," each edge will be considered together with all its parallel copies.

Let ω be a Bernoulli percolation of parameter $\hat{p} := 1 - (1-p)^{1/M}$ on $\hat{\mathcal{H}}$, so that $\hat{p}$-percolation on $\hat{\mathcal{H}}$ corresponds to p-percolation on $\mathcal{H}$. Let ω' be a Bernoulli percolation of parameter $\hat{p}$ on $\hat{\mathcal{G}}$ that is independent of ω. Choose an injection from $E(\mathcal{H})$ to $\mathbb{N}$, so that $E(\mathcal{H})$ is now endowed with a well-ordering; do the same with $E(\mathcal{G})$, $V(\mathcal{G})$ and $V(\mathcal{H})$.

We now define algorithmically an exploration process. This dynamical process will construct edge after edge a Bernoulli percolation η of parameter $\hat{p}$ on $\hat{\mathcal{G}}$ and an α with distribution $\mathsf{Ber}(s)^{\otimes V(\mathcal{H})}$. The random variables η, α and ω will be coupled in a suitable way, and α will be independent of ω.

We are also going to build two random sets, namely $C_\infty \subset V(\mathcal{H})$ and $C'_\infty \subset V(\mathcal{G})$. The set C_∞ will have the same distribution as $\mathcal{C}^{p,s}_{\mathcal{H}}(o)$, while C'_∞ will be stochastically dominated by $\mathcal{C}^{p}_{\mathcal{G}}(o')$. The set C_∞ (resp., C'_∞) will be constructed step by step, as a nondecreasing union $\bigcup_\ell C_\ell$ (resp., $\bigcup_\ell C_\ell$). Likewise, C_ℓ will be built as $\bigcup_n C_{\ell,n}$ and C'_ℓ as $\bigcup_n C'_{\ell,n}$. The set $C_{\ell,n}$ (resp., $C'_{\ell,n}$) thus stands for the "currently explored portion of C_∞ (resp., C'_∞)."

Structure of the process. In the exploration, edges in $\hat{\mathcal{G}}$ may get explored in two different ways, called p-explored and s-explored. Edges in $\mathcal{H}$ may get p-explored, and vertices in $\mathcal{H}$ may get s-explored. No vertex or edge will get explored more than once. In particular, no edge of $\hat{\mathcal{G}}$ will get p- *and* s-explored.

For every $\ell > 0$, during Step ℓ, we will define inductively a sequence $(C_{\ell,n})_n$ of subsets of $V(\mathcal{H})$ and a sequence $(C'_{\ell,n})_n$ of subsets of $V(\mathcal{G})$. At the end of each iteration of the process, it will be the case that the following conditions hold:

(A) If an edge e in $\mathcal{H}$ is p-explored, then there is a lift e' of e in $\mathcal{G}$ such that the set of the p-explored lifts of e is precisely $\{e'\} \times \{1, \dots, M\}$.

(B) If an edge e in $E(\mathcal{H})$ is p-unexplored, then all of its lifts are unexplored.

(C) Every element of $C'_{\ell,n}$ is connected to o' by an η-open path.

(D) For every edge e in $\mathcal{H}$ and each lift e' of e in $\mathcal{G}$, the number of s-explored edges of the form (e', k) is at most the number of s-explored vertices u in $\mathcal{H}$ at distance at most r from some endpoint of e.

(E) The map π induces a well-defined surjection from $C'_{\ell,n}$ to $C_{\ell,n}$.

Step 0. Set $C_0 = \{o\}$ and $C'_0 = \{o'\}$. Initially, nothing is considered to be p- or s-explored.

Step $2K+1$. Set $C_{2K+1,0} := C_{2K}$ and $C'_{2K+1,0} := C'_{2K}$.

While there is an unexplored edge that intersects $C_{2K+1,n}$ in $\mathcal{H}$, do the following (otherwise finish this step):

1. take $e = \{u, v\}$ to be the smallest such edge, with $u \in C_{2K+1,n}$ say,
2. pick $e' = \{x, y\}$ some lift of e with $x \in \pi^{-1}(\{u\}) \cap C'_{2K+1,n}$,
3. declare e and all (e', k)'s to be p-explored (they were unexplored before because of Conditions (A) and (B)),
4. for every $k \leq M$, define $\eta_{(e',k)} := \omega_{(e,k)}$,
5. set $(C_{2K+1,n+1}, C'_{2K+1,n+1}) := (C_{2K+1,n}, C'_{2K+1,n})$ if all the (e, k)'s are ω-closed; else, set $(C_{2K+1,n+1}, C'_{2K+1,n+1}) := (C_{2K+1,n} \cup \{v\}, C'_{2K+1,n} \cup \{y\})$.

When this step is finished, which occurs after finitely or countably many iterations, set $C_{2K+1} := \bigcup_n C_{2K+1,n}$ and $C'_{2K+1} := \bigcup_n C'_{2K+1,n}$.

Step $2K+2$. Set $C_{2K+2,0} := C_{2K+1}$ and $C'_{2K+2,0} := C'_{2K+1}$.

Say that an r-ball is *"fully open"* if for each $\mathcal{H}$-edge lying inside it, *at least* one of its copies in $\hat{\mathcal{H}}$ is open. While there is at least one s-unexplored vertex in C_{2K+1} whose r-ball is "fully open" in ω, do the following (otherwise finish this step):

1. Take u to be the smallest such vertex.
2. Pick some $x \in C'_{2K+1} \cap \pi^{-1}(\{u\}) \neq \varnothing$.
3. *This paragraph is not an algorithmic substep, but gathers a few relevant observations. Call an edge in $\mathcal{G}$ p-explored if one (hence every by* (A)*) of its copies in $\hat{\mathcal{G}}$ is p-explored. Call a p-explored edge of $\mathcal{G}$ open if at least one of its copies is η-open. Notice that by construction and as the r-ball of u is "fully open" in ω, all the p-explored edges of $\mathcal{G}$ that lie inside $\pi^{-1}(B_r(u))$ are open. Also note that for each edge lying in $Z(x, r)$, Condition* (D) *and the value of M guarantee that at least one of its copies in $\hat{\mathcal{G}}$ has not been s-explored. As a result, for every edge in $Z(x, r)$, either all its copies have a well-defined η-status and one of them is open, or at least one of these copies has a still undefined η-status. This is what makes Substep 4 possible.*
4. For each p-unexplored edge e' in $Z(x, r)$, take its s-unexplored copy (e', k) in $\hat{\mathcal{G}}$ of smallest label k, set $\eta_{(e',k)} := \omega'_{(e',k)}$, and switch its status to s-explored.
5. If all these newly s-explored edges are open (so that Z is "fully η-open"), then perform this substep. By (A) and the definition of r, for every $\mathcal{H}$-edge $e \in S_{r+\frac{1}{2}}(u)$, there is at least one lift e' of e that is adjacent to $Z(x, r)$ and p-unexplored: pick the smallest one. By (D) and the value of M, one of its copies (e', k) is s-unexplored: pick that with minimal $k =: k_e$. Declare all these edges to be s-explored and set $\eta_{(e',k_e)} := \omega'_{(e',k_e)}$. If all these (e', k_e)'s are ω'-open, then say that this substep is successful.
6. Notice that conditionally on everything that happened strictly before the current Substep 4, the event "Substep 5 is performed and successful" has some

(random) probability $q \geq \hat{p}^{|E(Z(x,r))|} \geq \hat{p}^{|E(B_{3r}(x))|} \geq s$. If the corresponding event does not occur, set $\alpha_u := 0$. If this event occurs, then independently on (ω, ω') and everything that happened so far, set $\alpha_u := 1$ with probability $s/q \leq 1$ and $\alpha_u := 0$, otherwise. Declare u to be s-explored.

7. If $\alpha_u = 1$, then set $C_{2K+2,n+1} := C_{2K+2,n} \cup S_{r+1}(u)$ and $C'_{2K+2,n+1}$ to be the union of $C_{2K+2,n}$, $Z(x,r)$, and the e''s of Substep 5. Notice that Condition (C) continues to hold as in this case Z is "fully η-open" and η-connected to $C_{2K+2,n}$. Otherwise, set $C_{2K+2,n+1} := C_{2K+2,n}$ and $C'_{2K+2,n+1} := C'_{2K+2,n}$.

As before, when this step is finished set $C_{2K+2} := \bigcup_n C_{2K+2,n}$ and $C'_{2K+2} := \bigcup_n C'_{2K+2,n}$.

Step ∞. Set $C_\infty := \bigcup_K C_K$ and $C'_\infty := \bigcup_K C'_K$. Take η' independent of everything done so far, with distribution $\mathsf{Ber}(\hat{p})^{\otimes E(\hat{\mathcal{G}})}$. Wherever η is undefined, define it to be equal to η'. In the same way, wherever α is undefined, toss independent Bernoulli random variables of parameter s, independent of everything done so far.

By construction, C_∞ has the distribution of the cluster of the origin for the (p,s)-process on $\mathcal{H}$: it is the cluster of the origin of $((\bigvee_k \omega_{e,k})_e, \alpha)$ which has distribution $\mathsf{Ber}(p)^{\otimes E(\mathcal{H})} \otimes \mathsf{Ber}(s)^{\otimes V(\mathcal{H})}$. Recall that $\vee$ stands for the maximum operator. Besides, C'_∞ is included in the cluster of o' for $(\bigvee_k \eta_{e,k})_e$, which is a p-bond-percolation on $\mathcal{G}$. Finally, the coupling guarantees that π surjects C'_∞ onto C_∞. Proposition 2.1 follows.

REMARK. This construction adapts to site percolation. The lift is the same as in [3] while the "multiple edges" trick now consists in defining $\hat{\mathcal{G}}$ as follows: each vertex has M possible states, and it is p-open if one of its $\hat{p}$-states say so.

2.2. *Proof of Proposition* 2.2. In this proof, we follow the strategy of Aizenman and Grimmett [1, 2].

By monotonicity, we can assume without loss of generality that $s < 1$. Let $\theta_L(p,s)$ be the $\mathbb{P}_{p,s}$-probability of the event $\mathcal{E}_L := \{\mathcal{C}_o(\omega,\alpha) \cap S_L(o) \neq \varnothing\}$, and $\theta(p,s) = \lim_{L\to\infty} \theta_L(p,s)$ be the probability that $\mathcal{C}_o(\omega,\alpha) = \mathcal{C}_{\mathcal{H}}^{p,s}(o)$ is infinite. We claim that in order to prove Proposition 2.2, we only need to show that for any $\varepsilon > 0$, there exist $c = c(\varepsilon) > 0$ and $L_0(\varepsilon) \geq 1$ such that for any $p, s \in [\varepsilon, 1-\varepsilon]$ and $L \geq L_0$, we have

$$\frac{\partial}{\partial s}\theta_L(p,s) \geq c\frac{\partial}{\partial p}\theta_L(p,s). \tag{1}$$

Indeed, assume that (1) is true. It is easy to see that, since $p_c(\mathcal{H}) \in (0,1)$, for any $s \in (0,1)$, there is some $\varepsilon > 0$ such that we can find a curve—actually a line segment—$(\mathbf{p}(t), \mathbf{s}(t))_{t\in[0,s]}$ inside $[\varepsilon, 1-\varepsilon]^2$ satisfying $\frac{\mathbf{p}'(t)}{\mathbf{s}'(t)} = -c$ for all $t \in [0,s]$ and $p_0 := \mathbf{p}(0) > p_c(\mathcal{H})$, $p_s := \mathbf{p}(s) < p_c(\mathcal{H})$, $\mathbf{s}(s) = s$. But now note that (1) implies that $t \mapsto \theta_L(\mathbf{p}(t), \mathbf{s}(t))$ is a nondecreasing function for all $L \geq L_0$. In particular, we have $\theta(p_s, s) = \theta(\mathbf{p}(s), \mathbf{s}(s)) = \lim_L \theta_L(\mathbf{p}(s), \mathbf{s}(s)) \geq \lim_L \theta_L(\mathbf{p}(0), \mathbf{s}(0)) = \theta(\mathbf{p}(0), \mathbf{s}(0)) \geq \theta(p_0, 0) > 0$, where in the last inequality we use $p_0 > p_c(\mathcal{H})$. By monotonicity, we conclude that for every $p \in [p_s, 1]$, we have $\theta(p, s) > 0$ as desired.

Now note that since the event $\mathcal{E}_L$, which depends only on finitely many coordinates, is increasing in both ω and α, the Margulis–Russo formula gives us

$$\frac{\partial}{\partial p}\theta_L(p, s) = \sum_e \mathbb{P}_{p,s}(e \text{ is } p\text{-pivotal for } \mathcal{E}_L),$$
$$\frac{\partial}{\partial s}\theta_L(p, s) = \sum_x \mathbb{P}_{p,s}(x \text{ is } s\text{-pivotal for } \mathcal{E}_L).$$

Recall that an edge e is said to be *p-pivotal* for an increasing event $\mathcal{E}$ in a configuration (ω, α) if $(\omega \cup \{e\}, \alpha) \in \mathcal{E}$ but $(\omega \setminus \{e\}, \alpha) \notin \mathcal{E}$. Similarly, a vertex x is said to be *s-pivotal* for an increasing event $\mathcal{E}$ in a configuration (ω, α) if $(\omega, \alpha \cup \{x\}) \in \mathcal{E}$ but $(\omega, \alpha \setminus \{x\}) \notin \mathcal{E}$.

It follows from the above formulas that in order to derive (1), it is enough to prove that for some $R, L_0 > 0$, for every $\varepsilon > 0$, there is some $c' > 0$ such that for any edge e, any $p, s \in [\varepsilon, 1 - \varepsilon]$, and any $L \geq L_0$, one has

$$\sum_{x \in B_R(e)} \mathbb{P}_{p,s}(x \text{ is } s\text{-pivotal for } \mathcal{E}_L) \geq c' \mathbb{P}_{p,s}(e \text{ is } p\text{-pivotal for } \mathcal{E}_L), \tag{2}$$

where for $e = \{x, y\}$, we set $B_R(e) := B_R(x) \cup B_R(y)$. Indeed, since each vertex can be in $B_R(e)$ for at most $C := \max_x |E(B_{R+1}(x))|$ different e's, summing (2) over e gives

$$\sum_x C\mathbb{P}_{p,s}(x \text{ is } s\text{-pivotal for } \mathcal{E}_L) \geq c' \sum_e \mathbb{P}_{p,s}(e \text{ is } p\text{-pivotal for } \mathcal{E}_L)$$

which implies (1) for $c := c'/C$.

The following deterministic lemma directly implies (2).

LEMMA 2.4. *There are constants R and L_0 such that the following holds. If $L \geq L_0$ and an edge e is p-pivotal for $\mathcal{E}_L$ in a configuration (ω, α), then there exist a configuration (ω', α') differing from (ω, α) only inside $B_R(e)$ and a vertex z in $B_R(e)$ such that z is s-pivotal for $\mathcal{E}_L$ in (ω', α').*

PROOF. Take $R := 3r + 1$ and $L_0 := 2r + 2$. Let (ω, α) and e be as in Lemma 2.4 and assume without loss of generality that $(\omega, \alpha) \in \mathcal{E}_L$. Now, remove from α all the vertices in $B_R(e)$ one by one. If at some point, we get for the first time a configuration (ω, α') that is not in $\mathcal{E}_L$ anymore, then it means that the last vertex z that was removed is s-pivotal for that configuration (ω, α'), thus yielding the conclusion of the lemma. Therefore, we can assume that $(\omega, \alpha') \in \mathcal{E}_L$ where $\alpha' := \alpha \setminus B_R(e)$. In particular, e is still p-pivotal in (ω, α'). We now have two cases.

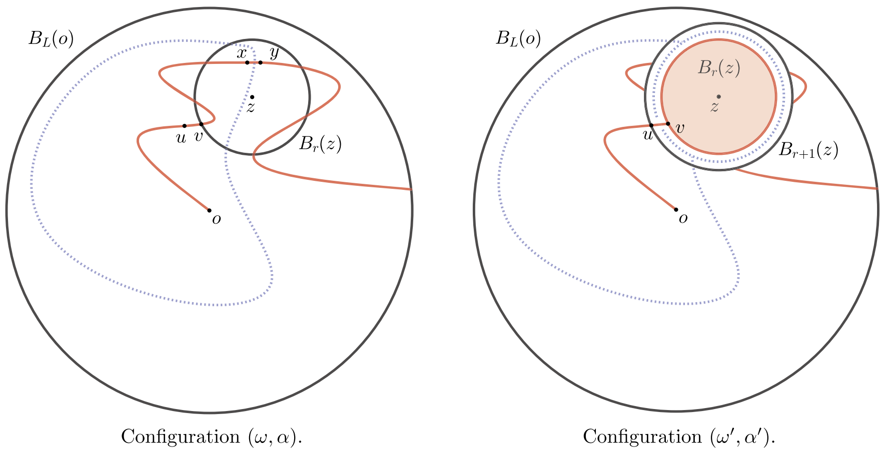


FIG. 1. *A picture of Case a in the proof of Lemma 2.4. The colour red represents open edges, either in odd or even steps. The dashed lines in blue represent closed edges preventing certain connections.*

Case a. The edge $e = \{x, y\}$ is far from the origin o, namely $d(o, e) > r$.

See Figure 1 for a representation of this case. Since e is p-pivotal for $\mathcal{E}_L$, we have $e \subset B_L(o)$ and $e \not\subset S_L(o)$. So we can assume without loss of generality that $x \in B_{L-1}(o)$. Take z to be a vertex such that $x \in B_r(z) \subset B_{L-1}(o)$ and $o \notin B_r(z)$.[4] Now, take some vertex $u \in S_{r+1}(z)$ such that $u \in \mathcal{C}_o(\tilde{\omega}, \alpha')$, where $\tilde{\omega}$ is given by closing in ω all the edges inside $B_{r+1}(z)$, that is, $\tilde{\omega} := \omega \setminus E(B_{r+1}(z))$. Such a vertex can be obtained as follows. Let n be the first step of the exploration that contains some vertex of $S_{r+1}(z)$, that is, such that $C_n(\omega, \alpha') \cap S_{r+1}(z) \neq \varnothing$. The previous step $n-1$ does not depend on the state of the edges inside $B_{r+1}(z)$. In particular, one has $C_{n-1} := C_{n-1}(\omega, \alpha') = C_{n-1}(\tilde{\omega}, \alpha')$. Notice that as $\alpha' \cap B_{2r+1}(z) = \varnothing$, the step n is actually an odd one (in which we only explore things in ω). Therefore, C_{n-1} is ω-connected to $S_{r+1}(z)$. In particular, there is some $u \in S_{r+1}(z)$ such that C_{n-1} is ω-connected to u outside $B_{r+1}(z)$, thus also $\tilde{\omega}$-connected. All of this implies that $u \in C_n(\tilde{\omega}, \alpha') \subset \mathcal{C}_o(\tilde{\omega}, \alpha')$. Let v be any neighbour of u in $B_r(z)$. Finally, define ω' by opening in $\tilde{\omega}$ the edge $\{u, v\}$ together with all the edges inside $B_r(z)$. Formally, one has

$$\omega' := [\omega \setminus E(B_{r+1}(z))] \cup [E(B_r(z)) \cup \{\{u, v\}\}].$$

Case b. The edge e is close to the origin, namely $d(o, e) \leq r$.

Without loss of generality, assume $d(o, x) \leq r$. Then simply take $z = x$ and ω' given by closing in ω all the edges inside $B_{r+1}(x)$ and then opening all the edges inside $B_r(x)$, that is, $\omega' := [\omega \setminus E(B_{r+1}(x))] \cup E(B_r(x))$.

[4]Just take a suitable vertex in some geodesic from x to o. In the case where $d(x, S_{L-1}(o)) \geq r$, one can simply take $z = x$. Here, we are using that $L \geq L_0 = 2r + 2$.

We claim that, in both cases above, z is s-pivotal for the event $\mathcal{E}_L$ in the configuration (ω', α'). We are only going to treat Case (a). We leave the slightly simpler Case (b) to the reader.

Remind that by definition of u, we have $u \in \mathcal{C}_o(\tilde{\omega}, \alpha')$. Since $\alpha' \cap B_{2r+1}(z) = \varnothing$, one can see that after opening at $\tilde{\omega}$ all the edges inside $B_r(z)$ together with $\{u, v\}$ (thus yielding ω'), we do not add any extra vertex in even steps but we add $B_r(z)$ at a certain odd step, so that $\mathcal{C}_o(\omega', \alpha') = \mathcal{C}_o(\tilde{\omega}, \alpha') \cup B_r(z)$. In particular, one has $\mathcal{C}_o(\omega', \alpha') \cap S_L(o) = \varnothing$, so that $(\omega', \alpha') \notin \mathcal{E}_L$.

Recall that $z \in \mathcal{C}_o(\omega', \alpha') \subset \mathcal{C}_o(\omega', \alpha' \cup \{z\})$ and that $B_r(z)$ is p-open. This implies that $B_{r+1}(z)$ is contained in $\mathcal{C}_o(\omega', \alpha' \cup \{z\})$. Together with $\omega \subset \omega' \cup B_{r+1}(z)$ and $B_{2r+1}(z) \cap \alpha' = \varnothing$, this implies that $\mathcal{C}_o(\omega, \alpha') \subset \mathcal{C}_{B_{r+1}(z) \cup \{o\}}(\omega, \alpha') \subset \mathcal{C}_{B_{r+1}(z) \cup \{o\}}(\omega', \alpha' \cup \{z\}) = \mathcal{C}_o(\omega', \alpha' \cup \{z\})$. As a result, $\mathcal{C}_o(\omega', \alpha' \cup \{z\}) \cap S_L(o) \neq \varnothing$, so that $(\omega', \alpha' \cup \{z\}) \in \mathcal{E}_L$. $\square$

REMARK. As in Section 2.1, the proof above can be adapted to site percolation in a straightforward way.

3. Proof of Theorems 1.3 and 1.4. As a warm-up, let us first prove Theorem 1.3.

3.1. *Proof of Theorem* 1.3. In what follows, we will denote by $\mathbb{P}_p$ the percolation measure of parameter p on both graphs $\mathcal{G}$ and $\mathcal{H}$, but this will not cause any confusion. For A and B, two subsets of the vertices of a graph, we write "$A \leftrightarrow B$" for the event that there is an open path intersecting both A and B. Similarly, "$A \leftrightarrow \infty$" will denote the event that there is an infinite (self-avoiding) open path intersecting A.

Let $\mathcal{G}$, $\mathcal{H}$ and π be as in Theorem 1.3. The coupling used in [3] to prove the monotonicity of p_c under covering maps yields straightforwardly the following fact: for any two finite subsets $A, B \subset V(\mathcal{H})$ one has

$$\mathbb{P}_p[\pi^{-1}(A) \leftrightarrow \pi^{-1}(B)] \geq \mathbb{P}_p[A \leftrightarrow B]. \tag{3}$$

Assume that $p > p_u(\mathcal{H})$. By uniqueness of the infinite cluster at p and the Harris-FKG inequality, one has

$$\begin{aligned}\mathbb{P}_p[B_\ell(u) \leftrightarrow B_\ell(v)] &\geq \mathbb{P}_p[B_\ell(u) \leftrightarrow \infty, B_\ell(v) \leftrightarrow \infty] \\ &\geq \mathbb{P}_p[B_\ell(u) \leftrightarrow \infty]\mathbb{P}_p[B_\ell(v) \leftrightarrow \infty]\end{aligned}$$

for any two vertices $u, v \in V(\mathcal{H})$. This implies, by quasi-transitivity, that

$$\lim_{\ell \to \infty} \inf_{u,v \in V(\mathcal{H})} \mathbb{P}_p[B_\ell(u) \leftrightarrow B_\ell(v)] = 1.$$

Let K be given by the boundedness of the fibres. As for any vertex $x \in V(\mathcal{G})$, one has $\pi^{-1}(B_\ell(\pi(x))) \subset B_{\ell+K}(x)$, inequality (3) and the previous equation imply that

$$\lim_{\ell \to \infty} \inf_{x,y \in V(\mathcal{G})} \mathbb{P}_p[B_\ell(x) \leftrightarrow B_\ell(y)] = 1.$$

Now simply remind that the above equation guarantees that $p \geq p_u(\mathcal{G})$; see [17].

3.2. *Proof of Theorem* 1.4. The proof of Theorem 1.4 follows quite closely that of Theorem 1.1.

Let $\mathcal{G}$, $\mathcal{H}$ and π be as in Theorem 1.4. Let r be a positive integer. We use the (p,s)-model of Section 2, except that we now initialise it at any finite set A, instead of just at a single point o. When using the (p,s)-model initialised at some finite set $A \subset V(\mathcal{H})$, if B is a subset of $V(\mathcal{H})$, we write "$A \rightsquigarrow B$" for the event "$\mathcal{C}_A \cap B \neq \varnothing$."

Here are two propositions, which are reminiscent of Propositions 2.1 and 2.2.

PROPOSITION 3.1. *Take $\mathcal{G}$, $\mathcal{H}$ and π to satisfy the hypotheses of Theorem* 1.1 (*but not necessarily $p_c(\mathcal{G}) < 1$*). *Then there is some choice of $r \geq 1$ such that the following holds*: *for every $\varepsilon > 0$, there is some $s \in (0,1)$ such that for every $p \in [\varepsilon, 1]$, for every nonempty finite subset A' of $V(\mathcal{G})$, the random set $\mathcal{C}_{\mathcal{H}}^{p,s}(\pi(A'))$ is stochastically dominated by $\pi(\mathcal{C}_{\mathcal{G}}^{p}(A'))$. In particular, for any two finite subsets $A, B \subset V(\mathcal{H})$, one has*

$$\mathbb{P}_p\big[\pi^{-1}(A) \leftrightarrow \pi^{-1}(B)\big] \geq \mathbb{P}_{p,s}[A \rightsquigarrow B].$$

Given a positive integer r, we say that a finite, nonempty subset B of $V(\mathcal{H})$ is **r**-*nice* if its complement can be written as a union of balls of radius r.

PROPOSITION 3.2. *Let $\mathcal{H}$ be a graph of bounded degree. For every $r \geq 1$ and $s, \epsilon > 0$, there exists $\delta > 0$ such that the following holds*: *for every $p \in [\epsilon, 1-\epsilon]$ and any two nonempty finite subsets $A, B \subset V(\mathcal{H})$ such that B is r-nice and $d(A,B) > 3r$,*[5] *one has*

$$\mathbb{P}_{p,s}[A \rightsquigarrow B] \geq \mathbb{P}_{p+\delta}[A \leftrightarrow B].$$

Proposition 3.1 is proved exactly as Proposition 2.1, except that the process is initialised at $(A', \pi(A'))$ instead of $(\{o'\}, \{o\})$. Recall that the assumptions of Theorem 1.4 imply that π has uniformly nontrivial fibres.

In Section 3.3, we explain how to adjust the proof of Proposition 2.1 in order to get Proposition 3.2.

PROOF OF THEOREM 1.4. If $p_u(\mathcal{H}) = 1$, then the conclusion holds trivially, so we can assume that $p_u(\mathcal{H}) < 1$. Since in addition $p_u(\mathcal{H}) \geq p_c(\mathcal{H}) > 0$, we can find some $\varepsilon > 0$ such that $p_u(\mathcal{H}) \in (\epsilon, 1-\epsilon)$. Notice that boundedness of fibres together with the disjoint tree-lifting property and the non-injectivity of π easily implies that the fibres are uniformly nontrivial, so that we can apply Proposition 3.1

[5] Recall that $d(A,B) := \min\{d(u,v) : u \in A, v \in B\}$.

above. We can thus pick $r \in \mathbb{N}$ and $s \in (0,1)$ such that for every $p \in [\varepsilon, 1]$, for any two nonempty finite subsets A, B of $V(\mathcal{H})$, one has

$$\mathbb{P}_p[\pi^{-1}(A) \leftrightarrow \pi^{-1}(B)] \geq \mathbb{P}_{p,s}[A \rightsquigarrow B].$$

By applying Proposition 3.2 to some parameter $p \in (\epsilon, 1-\epsilon)$ that satisfies $p < p_u(\mathcal{H}) < p+\delta =: q$, we get that for any two nonempty finite subsets $A, B \subset V(\mathcal{H})$ such that B is r-nice and $d(A,B) > 3r$, one has

$$\mathbb{P}_{p,s}[A \rightsquigarrow B] \geq \mathbb{P}_q[A \leftrightarrow B].$$

Let K be given by the fact that π has bounded fibres. Notice that for every $x, y \in V(\mathcal{G})$, one has $d(x,y) - K \leq d(\pi(x), \pi(y)) \leq d(x,y)$. Let ℓ be a positive integer and x, y be vertices of $\mathcal{G}$ such that $d(x,y) > L(\ell) := 2\ell + 4r + K$. Define $u := \pi(x)$, $v := \pi(y)$, $A := B_\ell(u)$ and $B := V(\mathcal{H}) \setminus \bigcup_{w : d(w,v) > r+\ell} B_r(w)$. Since B is r-nice and $d(A,B) > 3r$, we have

$$\mathbb{P}_p[\pi^{-1}(A) \leftrightarrow \pi^{-1}(B)] \geq \mathbb{P}_{p,s}[A \rightsquigarrow B] \geq \mathbb{P}_q[A \leftrightarrow B].$$

Also notice that $B_\ell(v) \subset B \subset B_{\ell+r}(v)$, $\pi^{-1}(A) \subset B_{\ell+K}(x) \subset B_L(x)$ and $\pi^{-1}(B) \subset B_{\ell+r+K}(y) \subset B_L(y)$. These inclusions combined with the previous inequality give

$$\mathbb{P}_p[B_{L(\ell)}(x) \leftrightarrow B_{L(\ell)}(y)] \geq \mathbb{P}_q[B_\ell(\pi(x)) \leftrightarrow B_\ell(\pi(y))]$$

for any two vertices $x, y \in V(\mathcal{G})$ such that $d(x,y) > L(\ell)$. Notice that this inequality is still true when $d(x,y) \leq L(\ell)$, as the left-hand side is then equal to 1. Taking the infimum over $x, y \in V(\mathcal{G})$ and then sending ℓ to infinity gives

$$\lim_{L\to\infty} \inf_{x,y \in V(\mathcal{G})} \mathbb{P}_p[B_L(x) \leftrightarrow B_L(y)] \geq \lim_{\ell\to\infty} \inf_{u,v \in V(\mathcal{H})} \mathbb{P}_q[B_\ell(u) \leftrightarrow B_\ell(v)] = 1$$

where the last equality follows, as in Section 3.1, from the fact that $q > p_u(\mathcal{H})$. It follows from the above equation (see [17]) that $p_u(\mathcal{G}) \leq p < p_u(\mathcal{H})$. $\square$

REMARK. A recent paper of Tang [18] proves that on any quasi-transitive graph, uniqueness of infinite cluster at p is equivalent to $\inf_{u,v \in V} \mathbb{P}_p[u \leftrightarrow v] > 0$. By using this theorem instead of [17], one can simplify the above proof: one does not need to connect large balls anymore, but only vertices.

3.3. *Proof of Proposition* 3.2. The proof follows the same lines as that of Proposition 2.2, so we will only highlight the necessary adaptations here.

For any two finite subsets $A, B \subset V(\mathcal{H})$, we consider the following finite dimensional approximation of the event $A \rightsquigarrow B$: for each L, define $\mathcal{E}_L^{A,B} := \{(\omega, \alpha) : \mathcal{C}_A(\omega_L, \alpha_L) \cap B \neq \varnothing\}$, where ω_L (resp., α_L) is the configuration equal to ω (resp., α) in $B_L(o)$ and equal to 0 elsewhere. By the argument presented at the beginning of Section 2.2, one can easily reduce the proof to the following deterministic lemma.

LEMMA 3.3. *There is a constant R such that the following holds. For any two nonempty finite subsets $A, B \subset V(\mathcal{H})$ such that B is r-nice and $d(A, B) > 3r$, there is some $L_0 = L_0(A, B)$ such that for all $L \geq L_0$, if an edge e is p-pivotal for $\mathcal{E}_L^{A,B}$ in a configuration (ω, α), then there exist a configuration (ω', α') differing from (ω, α) only inside $B_R(e)$ and a vertex z in $B_R(e)$ such that z is s-pivotal for $\mathcal{E}_L^{A,B}$ in (ω', α').*

PROOF. As in Lemma 2.4, it is enough to take $R = 3r + 1$. Given A and B as above, take L_0 such that $A \cup B \subset B_L(o)$ and $d(A \cup B, S_L(o)) > 3r$ for all $L \geq L_0$. Let (ω, α) and e be as in Lemma 3.3. As before, we can assume that e is p-pivotal for $\mathcal{E}_L^{A,B}$ in (ω, α'), where $\alpha' := \alpha \setminus B_R(e)$. Again, we have two cases.

Case a. The edge $e = \{x, y\}$ is far from A, namely $d(e, A) > r$.

Notice that, since e is p-pivotal, we can assume without loss of generality that $x \notin B$. In this case, one can always find a vertex z such that $B_r(z) \subset B_L \setminus (A \cup B)$ and $x \in B_r(z)$. Indeed, if $d(x, B) > r$ and $d(x, S_L(o)) \geq r$, it suffices to take $z = x$; if $d(x, B) \leq r$, we use the fact that B is r-nice to find z such that $B_r(z) \cap B = \varnothing$ and $x \in B_r(z)$, which directly implies $B_r(z) \subset B_L(o) \setminus A$ since $d(B, S_L(o)) > 3r$ and $d(A, B) > 3r$; and if $d(x, S_L(o)) < r$, we can take an appropriate z in the geodesic path from o to x in such a way that $x \in B_r(z) \subset B_L$, which directly implies $B_r(z) \cap (A \cup B) = \varnothing$ since $d(A \cup B, S_L(o)) > 3r$. As in the proof of Lemma 2.4, we can find $u \in S_{r+1}(z)$ such that $u \in \mathcal{C}_o(\tilde{\omega}, \alpha')$, where $\tilde{\omega} := \omega \setminus E(B_{r+1}(z))$. Pick $v \in B_r(z)$ some neighbour of u and define $\omega' := [\omega \setminus E(B_{r+1}(z))] \cup [E(B_r(z)) \cup \{\{u, v\}\}]$.

Case b. The edge $e = \{x, y\}$ is close to A, namely $d(e, A) \leq r$.

Without loss of generality, assume $d(x, A) \leq r$. Then simply take $z = x$ and ω' given by closing in ω all the edges inside $B_{r+1}(x)$ and then opening all the edges inside $B_r(x)$, that is, $\omega' := [\omega \setminus E(B_{r+1}(x))] \cup E(B_r(x))$.

One can check in the same way as in the proof of Lemma 2.4 that in both cases above, z is s-pivotal for the event $\mathcal{E}_L^{A,B}$ in the configuration (ω', α'). $\square$

4. Deriving Theorems 0.1 and 0.2 and Corollaries 1.2 and 1.5. Theorem 0.1 results from Theorem 1.1 and Lemmas 4.1 and 4.2, while Theorem 0.2 results from Theorem 1.4 and Lemmas 4.1, 4.2 and 4.4. Likewise, Corollaries 1.2 and 1.5 follow by combining Lemma 4.3 with Theorems 1.1 and 1.4, respectively.

LEMMA 4.1. *Let $\mathcal{G}$ be a graph, and let G be a group acting on $V(\mathcal{G})$ by graph automorphisms. Let $\mathcal{H}$ be the quotient graph $\mathcal{G}/G$ and $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ denote the quotient map $x \mapsto Gx$.*

If $G \curvearrowright V(\mathcal{G})$ is free, then π has the disjoint tree-lifting property.

PROOF. With the notation of Lemma 4.1, let x and y be two distinct vertices of $\mathcal{G}$ such that $\pi(x) = \pi(y)$. Let $\mathcal{T}$ be a subtree of $\mathcal{H}$, and let $\mathcal{T}_x$ be a lift of $\mathcal{T}$ that contains x: recall that such a lift exists, as π is a weak covering map. As $Gx = Gy$, let us take some $g \in G$ such that $gx = y$. Since x and y are distinct, g is not the identity element. Therefore, by freeness of the action, g has no fixed point.

We claim that $\mathcal{T}_y := g\mathcal{T}_x$ is a lift of $\mathcal{T}$ that is vertex-disjoint from $\mathcal{T}_x$. It is indeed a lift, as $\forall z \in V(\mathcal{G})$, $\pi(z) = \pi(gz)$. To prove vertex-disjunction, let $z \in V(\mathcal{T}_x) \cap gV(\mathcal{T}_x)$. Thus, one can pick $z_\star$ in $V(\mathcal{T}_x)$ such that $z = gz_\star$. As $\pi(z) = \pi(gz_\star) = \pi(z_\star)$, by bijectivity of $\pi : V(\mathcal{T}_x) \to V(\mathcal{T})$, one has $z = z_\star$. Therefore, $z = gz$, which contradicts the fact that g has no fixed point. $\square$

LEMMA 4.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be quasi-transitive graphs. Let $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ be a noninjective weak covering map with the disjoint tree-lifting property.*

Then π has uniformly nontrivial fibres.

PROOF. Let $(\mathcal{G}, \mathcal{H}, \pi)$ satisfy the assumptions of Lemma 4.2. First, assume additionally that there is some r such that for every $x \in V(\mathcal{G})$, one has $|B_r(x)| > |B_r(\pi(x))|$. Fix such an r. Let x be any vertex of $\mathcal{G}$. As $\pi(B_r(x)) = B_r(\pi(x))$, by the pigeonhole principle, one can pick two vertices y and z in $B_r(x)$ such that $\pi(y) = \pi(z)$. Pick a self-avoiding path of length at most r from $\pi(y)$ to $\pi(x)$ in $B_r(\pi(x))$. As π has the disjoint tree-lifting property, one can obtain two vertex-disjoint lifts of this path with one starting at y and the other at z. Each of these paths ends inside $\pi^{-1}(\{\pi(x)\}) \cap B_{2r}(x)$: therefore, this set contains at least one vertex distinct from x, thus establishing that the fibres are uniformly nontrivial with $R := 2r$.

Let us now prove that the assumptions of the lemma imply the existence of such an r. Pick one vertex in each $\mathsf{Aut}(\mathcal{G})$-orbit, thus yielding a finite set $\{x_1, \dots, x_m\} \subset V(\mathcal{G})$. Define $\{u_1, \dots, u_n\} \subset V(\mathcal{H})$ by doing the same in $\mathcal{H}$. Proceeding by contradiction and as π is a weak covering map, we may assume that for every r, there is some $x \in V(\mathcal{G})$ such that $B_r(x)$ and $B_r(\pi(x))$ are isomorphic as rooted graphs. As a result, for every r, there are some i and j such that $B_r(x_i)$ and $B_r(u_j)$ are isomorphic as rooted graphs. As i and j can take only finitely many values, there is some (i_0, j_0) such that for infinitely many values of r—hence all values of r—the rooted graphs $B_r(x_{i_0})$ and $B_r(u_{j_0})$ are isomorphic. It results from local finiteness and diagonal extraction (or equivalently from the fact that the local topology on locally finite connected rooted graphs is Hausdorff) that $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.

This is a contradiction for the following reason. There are two vertices x and y in $\mathcal{G}$ such that $\pi(x) = \pi(y)$: fix such a pair (x, y). For r_0 large enough, for all $i \leq m$, the r_0-ball centred at x_i contains x and y. Pick such an r_0 and pick i such that the cardinality of $B_{r_0}(x_i)$ is minimal: as $\pi(x) = \pi(y)$, the cardinality of $B_{r_0}(\pi(x_i))$ is strictly less than that of $B_{r_0}(x_i)$. Therefore, the minimal cardinality of a r_0-ball is not the same for $\mathcal{H}$ and $\mathcal{G}$. $\square$

REMARK. Notice that in the above proof we only needed to use that we can lift *paths* disjointly.

LEMMA 4.3. *Any strong covering map has the disjoint tree-lifting property.*

PROOF. Let $\pi : \mathcal{G} \to \mathcal{H}$ denote a strong covering map. Let x and y be two vertices of $\mathcal{G}$ such that $\pi(x) = \pi(y)$. Let $\mathcal{T}$ be a subtree of $\mathcal{H}$, and let $\mathcal{T}_x$ and $\mathcal{T}_y$ be lifts of $\mathcal{T}$ such that x belongs to $V(\mathcal{T}_x)$ and y to $V(\mathcal{T}_y)$. Assume that $V(\mathcal{T}_x) \cap V(\mathcal{T}_y) \neq \varnothing$. Let us prove that $x = y$.

As $\mathcal{T}_x$ is connected, it suffices to prove that if z_0 belongs to $V(\mathcal{T}_x) \cap V(\mathcal{T}_y)$, then all its $\mathcal{T}_x$-neighbours belong to $V(\mathcal{T}_x) \cap V(\mathcal{T}_y)$. But this is the case: indeed, any $\mathcal{T}_x$-neighbour z_1 of z_0 is, by the strong lifting property, the unique neighbour $z_\star$ of z_0 such that $\pi(\{z_0, z_\star\}) = \pi(\{z_0, z_1\})$, so that $\pi^{-1}(\{\pi(z_1)\}) \cap V(\mathcal{T}_y) = \{z_1\}$. $\square$

In the following lemma, we show that the assumption of bounded fibres in Theorems 1.4 and Corollary 1.5 can actually be relaxed to that of *fibres of bounded* cardinality, that is, the condition that $\sup_{u \in V(\mathcal{H})} |\pi^{-1}(\{u\})| < \infty$.

LEMMA 4.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be quasi-transitive graphs. Assume that there is a noninjective weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with the disjoint tree-lifting property and fibres of bounded cardinality.*

Then there is a map $\pi_\star : V(\mathcal{G}) \to V(\mathcal{H})$ satisfying all these conditions and that furthermore has bounded and uniformly nontrivial fibres.

REMARK. Concerning Corollary 1.5, the boundedness assumption can be relaxed further to the condition that $\pi^{-1}(\{o\})$ is finite. Indeed, for a *strong* covering map, the cardinality of $\pi^{-1}(\{u\})$ does not depend on u.

PROOF. First, let us prove that there is a weak covering map $\pi_\star : V(\mathcal{G}) \to V(\mathcal{H})$ with the disjoint tree-lifting property and bounded fibres. If π has bounded fibres, then we are done. Thus, assume that this is not the case. Let K denote the maximal cardinality of a fibre, that is, $K = \max_{u \in V(\mathcal{H})} |\pi^{-1}(\{u\})|$. Since π does not have bounded fibres and since $u \mapsto \mathsf{diam}(\pi^{-1}(\{u\})$ is 2-Lipschitz, for every n, there is some $x_n \in V(\mathcal{G})$ such that

$$\forall u \in V(\mathcal{H}), \quad \left|\pi^{-1}(\{u\}) \cap B_n(x_n)\right| \leq K - 1.$$

As $\mathcal{G}$ is quasi-transitive, one can pick F some finite set of vertices of $\mathcal{G}$ that intersects every $\mathsf{Aut}(\mathcal{G})$-orbit. For every n, pick some graph automorphism φ_n of $\mathcal{G}$ such that $\varphi^{-1}(x_n) \in F$, and define the equivalence relation $\mathcal{R}_n$ on $V(\mathcal{G})$ by

$$x \mathcal{R}_n y \iff \pi(\varphi_n(x)) = \pi(\varphi_n(y)).$$

By taking a pointwise limit of these relations along a converging subsequence, one can endow $V(\mathcal{G})$ with an equivalence relation $\mathcal{R}$ such that:

- $\mathcal{G}/\mathcal{R}$ is isomorphic to $\mathcal{H}$,
- the projection $\pi_1 : V(\mathcal{G}) \to V(\mathcal{G})/\mathcal{R}$ is a weak covering map with the disjoint tree-lifting property,
- every $\mathcal{R}$-class has cardinality at most $K - 1$.

If π_1 has bounded fibres, then we are done. Otherwise, iterate the process, applying the same construction to π_1 instead of π. Since the maximal cardinality of a fibre cannot decrease forever, this process stops at some suitable $\pi_\star$.

Now, we need to show that $\pi_\star$ has uniformly nontrivial fibres. Notice that the weak covering map $\pi_\star$ cannot be injective, as $\mathcal{G}$ and $\mathcal{H}$ are not isomorphic: see the last paragraph of the proof of Lemma 4.2. As $\pi_\star$ has the disjoint tree-lifting property, every $\pi_\star$-fibre $\pi_\star^{-1}(\{u\})$ has cardinality at least 2. As $\pi_\star$ has bounded fibres, this implies that $\pi_\star$ has uniformly nontrivial fibres. □

5. On the hypotheses of our results. None of the assumptions of Theorem 0.1 can be removed. This is obvious for the hypothesis that $p_c(\mathcal{G}) < 1$. As for freeness, take $\mathcal{G}$ to be $\mathbb{Z}^2$ with two extra pendant edges attached to each vertex. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{G}$ by swapping the two pendant edges at each vertex. Since $\mathbb{Z}^2$ is amenable, one has $p_u(\mathcal{G}) = p_c(\mathcal{G})$ and $p_u(\mathcal{H}) = p_c(\mathcal{H})$, so that freeness is also necessary in Theorem 0.2; see [5].

For the hypothesis that $\mathcal{G}$ is quasi-transitive, let $\mathcal{G}$ be defined by taking two disjoint copies of $\mathbb{Z}^2$ and putting an additional edge between the two copies of the origin. The group $G := \mathbb{Z}/2\mathbb{Z}$ acts by swapping copies. As for quasi-transitivity of $\mathcal{H}$, take $\mathcal{G}$ to be the square lattice $\mathbb{Z}^2$ and G to be $\mathbb{Z}/2\mathbb{Z}$ acting via the reflection $(x, y) \mapsto (x, 1 - y)$. See [6] for the classical fact that $p_c(\mathbb{N} \times \mathbb{Z}) = p_c(\mathbb{Z}^2)$.

Still, we do not know what happens if freeness is relaxed to the absence of trivial G-orbit.

QUESTION 5.1. Let G be a group acting on a graph $\mathcal{G}$ by graph automorphisms. Assume that both $\mathcal{G}$ and $\mathcal{H} := \mathcal{G}/G$ are quasi-transitive, and that for every vertex $x \in V(\mathcal{G})$ there exists $g \in G$ such that $gx \neq x$. Is it necessarily the case that $p_c(\mathcal{G}) < 1$ implies $p_c(\mathcal{G}) < p_c(\mathcal{H})$? If we assume further that G is finite, is it necessarily the case that $p_u(\mathcal{G}) < 1$ implies $p_u(\mathcal{G}) < p_u(\mathcal{H})$?

REMARK. An interesting particular case (which we also do not know how to solve) is when G is normal in a quasi-transitive subgroup of $\mathsf{Aut}(\mathcal{G})$. In that setting, $\mathcal{H}$ is automatically quasi-transitive, and the map π always has uniformly nontrivial fibres.

As for Theorem 1.1 and Corollary 1.2, notice that the assumption that fibres are uniformly nontrivial cannot be simply replaced by nontriviality of the fibres (namely $\forall u \in V(\mathcal{H}), |\pi^{-1}(\{u\})| \neq 1$), even if π is taken to be a *classical* covering map. Indeed, take $\mathcal{H}$ to be a graph with bounded degree and $p_c < 1$, and pick some edge e in $\mathcal{H}$. To define $\mathcal{G}$, start with two copies of $\mathcal{H}$, and denote by $\{x, y\}$ and $\{x', y'\}$ the two copies of e. Then replace these two edges by $\{x', y\}$ and $\{x, y'\}$, thus yielding a connected graph. Take π to be the natural projection from $\mathcal{G}$ to $\mathcal{H}$.

We do not know how to answer the following question, which investigates a generalisation of Theorem 1.1/Corollary 1.2.

QUESTION 5.2. Let $\mathcal{G}$ and $\mathcal{H}$ be graphs of bounded degree. Assume that there is a weak covering map $\pi : V(\mathcal{G}) \to V(\mathcal{H})$ with uniformly nontrivial fibres.

Is it necessarily the case that $p_c(\mathcal{G}) < 1$ implies $p_c(\mathcal{G}) < p_c(\mathcal{H})$? If we assume further that π has bounded fibres and that both $\mathcal{G}$ and $\mathcal{H}$ are quasi-transitive, is it necessarily the case that $p_u(\mathcal{G}) < 1$ implies $p_u(\mathcal{G}) < p_u(\mathcal{H})$?

Notice that one cannot remove the assumption of boundedness of the fibres (or finiteness of G, in the case of Theorem 0.2) from Theorems 1.3 and 1.4 or Corollary 1.5. Indeed, without this assumption, it is even possible to have the strict inequality in the *reverse* direction. The following example shows that this is easy to obtain if one further relaxes the assumption that $p_u(\mathcal{G}) < 1$: take $\mathcal{G}$ to be the $2d$-regular tree and $\mathcal{H}$ to be the d-dimensional hypercubic lattice, for some $d \geq 2$, then we have $p_u(\mathcal{H}) = p_c(\mathcal{H}) < 1 = p_u(\mathcal{G}) = 1$. Notice that $\mathcal{H}$ indeed is a quotient of $\mathcal{G}$: as $\mathcal{H}$ is $2d$-regular, one can realise $\mathcal{G}$ as the set of finite nonbactracking paths of $\mathcal{H}$ launched at 0, and mapping such a path to its final position yields a strong covering map from $\mathcal{G}$ to $\mathcal{H}$. If one does not want to relax the assumption that $p_u(\mathcal{G}) < 1$, one can take d to be large enough, $\mathcal{G}_d$ to be the product of the $2d$-regular tree and the biinfinite line $\mathbb{Z}$, and $\mathcal{H}_d$ to be the $(d+1)$-dimensional hypercubic lattice. Indeed, $p_u(\mathcal{G}_d) \sim \frac{1}{\sqrt{d}}$ but $p_u(\mathcal{H}_d) = p_c(\mathcal{H}_d) \sim \frac{1}{2d}$; see respectively [8] and [11].

Finally, it is natural to look for a milder condition than the finiteness of G in Theorem 0.2. A natural condition to consider is that of amenability. Recall that a group G is said to be *amenable* if there is a sequence $(F_n)_n$ of nonempty finite subsets of G such that for every $g \in G$, one has $|F_n \triangle g F_n| = o(|F_n|)$.

QUESTION 5.3. Does Theorem 0.2 still hold if G is only assumed to be amenable instead of finite?

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