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DIRECTED POLYMERS IN HEAVY-TAIL RANDOM ENVIRONMENT[1]

BY QUENTIN BERGER[2] AND NICCOLÒ TORRI[3]

Sorbonne Université

We study the directed polymer model in dimension $1+1$ when the environment is heavy-tailed, with a decay exponent $\alpha \in (0,2)$. We give all possible scaling limits of the model in the *weak-coupling* regime, that is, when the inverse temperature temperature $\beta = \beta_n$ vanishes as the size of the system n goes to infinity. When $\alpha \in (1/2, 2)$, we show that all possible transversal fluctuations $\sqrt{n} \le h_n \le n$ can be achieved by tuning properly β_n, allowing to interpolate between all superdiffusive scales. Moreover, we determine the scaling limit of the model, answering a conjecture by Dey and Zygouras [*Ann. Probab.* **44** (2016) 4006–4048]—we actually identify five different regimes. On the other hand, when $\alpha < 1/2$, we show that there are only two regimes: the transversal fluctuations are either $\sqrt{n}$ or n. As a key ingredient, we use the *Entropy-controlled Last-Passage Percolation* (E-LPP), introduced in a companion paper [*Ann. Appl. Probab.* **29** (2019) 1878–1903].

CONTENTS

1. Introduction: Directed polymers in random environment.

1.1. *General setting.* We consider the directed polymer model: it has been introduced by Huse and Henley [16] as an effective model for an interface in the

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Ising model with random interactions, and is now used to describe a stretched polymer interacting with an inhomogeneous solvent.

Let S be a nearest-neighbor simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 1$, whose law is denoted by $\mathbf{P}$, and let $(\omega_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ be a field of i.i.d. random variables (the *environment*) with law $\mathbb{P}$ (ω will denote a random variable which has the common distribution of the $\omega_{i,x}$). The *directed* random walk $(i, S_i)_{i\in\mathbb{N}_0}$ represents a polymer trajectory and interacts with its environment via a coupling parameter $\beta > 0$ (the inverse temperature). The model is defined through a Gibbs measure,

$$\frac{d\mathbf{P}^{\omega}_{n,\beta}}{d\mathbf{P}}(s) := \frac{1}{\mathbf{Z}^{\omega}_{n\beta}} \exp\left(\beta \sum_{i=1}^{n} \omega_{i,s_i}\right), \tag{1.1}$$

where $\mathbf{Z}^{\omega}_{n\beta}$ is the *partition function* of the model.

One of the main questions about this model is that of the localization and super-diffusivity of paths trajectories drawn from the measure $\mathbf{P}^{\omega}_{n,\beta}$. The transversal exponent ξ describes the fluctuation of the endpoint, that is, $\mathbb{E}\mathbf{E}^{\omega}_{n,\beta}|S_n| \approx n^{\xi}$ as $n \to \infty$. Another quantity of interest is the fluctuation exponent χ, that describes the fluctuations of $\log \mathbf{Z}^{\omega}_{n,\beta}$, that is, $|\log \mathbf{Z}^{\omega}_{n,\beta} - \mathbb{E}\log \mathbf{Z}^{\omega}_{n,\beta}| \approx n^{\chi}$ as $n \to \infty$.

This model has been widely studied in the physical and mathematical literature (we refer to [11, 12] for a general overview), in particular when $\omega_{n,x}$ have an exponential moment. The case of the dimension $d = 1$ has attracted much attention in recent years, in particular because the model is in the *KPZ universality class* ($\log \mathbf{Z}^{\omega}_{n,\beta}$ is seen as a discretization of the Hopf–Cole solution of the KPZ equation). It is conjectured that the transversal and fluctuation exponents are $\xi = 2/3$ and $\chi = 1/3$, respectively. Moreover, it is expected that the point-to-point partition function, when properly centered and renormalized, converges in distribution to the GUE distribution. Such scalings have been proved so far only for some special models; cf. [6, 21, 22].

A recent and fruitful approach to proving universality results for this model has been to consider *weak-coupling limits*, that is, when the coupling parameter β is close to criticality. This means that we allow $\beta = \beta_n$ to depend on n, with $\beta_n \to 0$ as $n \to \infty$. In [2, 3] and [10] (in dimension $d = 1$), the authors let $\beta_n = \widehat{\beta} n^{-\gamma}$, $\gamma = 1/4$ for some fixed $\widehat{\beta} > 0$, and they prove that the model (one may focus on its partition function $\mathbf{Z}^{\omega}_{n,\beta_n}$) converges to a *nontrivial* (*i.e.*, *disordered*) continuous version of the model. This is called the *intermediate disorder regime*, since it somehow interpolates between weak disorder and strong disorder behaviors. More precisely, they showed that

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\lambda(\beta_n) \xrightarrow{(\mathrm{d})} \log \mathcal{Z}_{\sqrt{2}\widehat{\beta}} \quad \text{as } n \to \infty,$$

where $\lambda(s) := \log \mathbb{E}[e^{s\omega}]$. The process $\widehat{\beta} \mapsto \log \mathcal{Z}_{\sqrt{2}\widehat{\beta}}$ is the so-called *cross-over process*, and is conjectured to interpolate between Gaussian and GUE scalings as $\widehat{\beta}$ goes from 0 to ∞ (see [4]). These results were obtained under the assumption that

ω has exponential moments, but the *universality of the limit* was conjectured to hold under the assumption of six moments [3]. In [13], Dey and Zygouras proved this conjecture, and they suggested that this result is a part of a bigger picture (when $\lambda(s)$ is not defined a different centering is necessary). We mention that in [19], the transversal fluctuations are determined in the intermediate disorder regime, in the special case of a semi-discrete directed polymer (the O'Connell–Yor model, known to be exactly solvable).

1.2. *The case of a heavy-tail environment.* In the rest of the paper, we will focus on the dimension $d = 1$ for simplicity. We consider the case where the environment distribution ω is nonnegative (for simplicity, nothing deep is hidden in that assumption) and has some heavy tail distribution: there is some $\alpha > 0$ and some slowly varying function $L(\cdot)$ such that

$$\mathbb{P}(\omega > x) = L(x)x^{-\alpha}. \tag{1.2}$$

In the case where $\beta > 0$ does not depend on n, the $\xi = 2/3$, $\chi = 1/3$ picture is expected to be modified, depending on the value of α. According to the heuristics (and terminology) of [9, 14], three regimes should occur, with different paths behaviors:

(a) if $\alpha > 5$, there should be a *collective* optimization and we should have $\xi = 2/3$, KPZ universality class, as in the finite exponential moment case;
(b) if $\alpha \in (2, 5)$, the optimization strategy should be *elitist*: most of the total energy collected should be via a small fraction of the points visited by the path, and we should have $\xi = \frac{\alpha+1}{2\alpha-1}$;
(c) if $\alpha \in (0, 2)$, the strategy is *individual*: the polymer targets few exceptional points, and we have $\xi = 1$. This case is treated in [5, 15].

As suggested by [13], this is part of a larger picture, when the inverse temperature β is allowed to depend on n. Setting $\beta_n = \widehat{\beta} n^{-\gamma}$ for some $\widehat{\beta} > 0$ and some $\gamma \in \mathbb{R}$, then we have three different classes of coupling. When $\gamma = 0$, we recover the standard directed polymer model; when $\gamma > 0$, we have a weak-coupling limit. In [13], the authors suggest that the fluctuation exponent depends on α, γ in the following manner (in [1] the case $\alpha = \infty$ is considered):

$$\xi = \begin{cases} \dfrac{2(1-\gamma)}{3} & \text{for } \alpha \geq \dfrac{5-2\gamma}{1-\gamma}, -\dfrac{1}{2} \leq \gamma \leq \dfrac{1}{4}, \\ \dfrac{1+\alpha(1-\gamma)}{2\alpha-1} & \text{for } \alpha \leq \dfrac{5-2\gamma}{1-\gamma}, \dfrac{2}{\alpha} - 1 \leq \gamma \leq \dfrac{3}{2\alpha}. \end{cases} \tag{1.3}$$

The first part is derived in [1], based on Airy process considerations, and the second part is derived in [13], based on a Flory argument inspired by [9]. Moreover, in the two regions of the (α, γ) plane defined by (1.3), the KPZ scaling relation $\chi = 2\xi - 1$ should hold. Outside of these regions, one should have $\xi = 1/2$ (γ large)

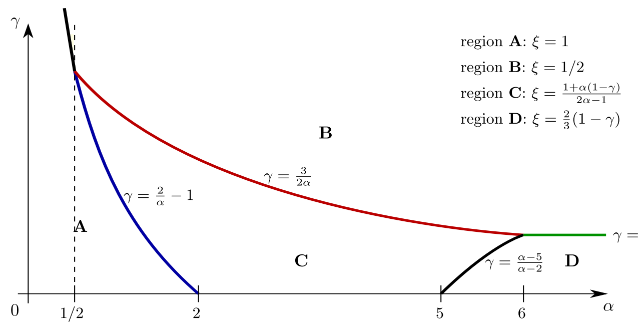

FIG. 1. *We identify four regions in the* (α, γ) *plane. Region* **A** *with* $\alpha < 2$ *is treated in* [5] *and Region* **B** *with* $\alpha > 1/2$ *in* [13]. *Regions* **C** *and* **D** *are still open, and the KPZ scaling relation* $\chi = 2\xi - 1$ *should hold in these two regions. Our main result is to settle the picture in the case* $\alpha \in (0, 2)$.

or $\xi = 1$ (γ small). This is summarized in Figure 1 below, which is the analogy of [13], Figure 1.

This picture is far from being settled, and so far only the border cases where $\xi = 1$ or $\xi = 1/2$ have been studied: Dey and Zygouras [13] proved that $\xi = 1/2$ in the cases $\alpha > 6$, $\gamma = 1/4$ and $\alpha \in (1/2, 6)$, $\gamma = 3/2\alpha$; Auffinger and Louidor [5] proved that $\xi = 1$ for $\alpha \in (0, 2)$ and $\gamma = \frac{2}{\alpha} - 1$. Here we complete the picture in the case $\alpha \in (0, 2)$. For $\alpha \in (1/2, 2)$, we go beyond the cases $\xi = 1/2$ or $\xi = 1$: we identify the correct order for the transversal fluctuations (they interpolate between $\xi = 1/2$ and $\xi = 1$), and we prove the convergence of $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ in all possible intermediate disorder regimes—this proves Conjecture 1.7 in [13]. For $\alpha < 1/2$, we show that a sharp transition occurs on the line $\gamma = \frac{2}{\alpha} - 1$, between a regime where $\xi = 1$ and a regime where $\xi = 1/2$.

2. Main results: Weak-coupling limits in the case $\alpha \in (0, 2)$.

From now on, we consider the case of a (nonnegative) environment ω verifying (1.2) with $\alpha \in (0, 2)$. For the inverse temperature, we will consider arbitrary sequences $(\beta_n)_{n\geq 1}$, but a reference example is $\beta_n = n^{-\gamma}$ for some $\gamma \in \mathbb{R}$.

For two sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$, we use the notation $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$, $a_n \ll b_n$ if $\lim_{n\to\infty} a_n/b_n = 0$, and $a_n \asymp b_n$ if $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$.

2.1. *First definitions and heuristics.*

First of all, let us present a brief energy/entropy argument to justify what the correct transversal fluctuations of the polymer should be. Let $F(x) = \mathbb{P}(\omega \leq x)$ be the disorder distribution, and define the function $m(x)$ by

$$m(x) := F^{-1}\Big(1 - \frac{1}{x}\Big) \quad \text{so we have } \mathbb{P}\big(\omega > m(x)\big) = \frac{1}{x}. \tag{2.1}$$

Note that the second identity characterizes $m(x)$ up to asymptotic equivalence: we have that $m(\cdot)$ is a regularly varying function with exponent $1/\alpha$.

Assuming that the transversal fluctuations are of order h_n (we necessarily have $\sqrt{n} \le h_n \le n$), then the amount of weight collected by a path should be of order $m(nh_n)$ (it should be dominated by the maximal value of ω in $[0,n] \times [-h_n, h_n]$). On the other hand, thanks to moderate deviations estimates for the simple random walk, the entropic cost of having fluctuations of order h_n is roughly h_n^2/n at the exponential level—at least when $h_n \gg \sqrt{n \log n}$; see (2.14) below. It therefore leads us to define h_n (seen as a function of β_n) up to asymptotic equivalence by the relation

$$\beta_n m(nh_n) \sim h_n^2/n. \tag{2.2}$$

In the case $\beta_n = n^{-\gamma}$ and $\alpha \in (1/2, 2)$, we recover (1.3), that is, we get that $h_n = n^{\xi + o(1)}$ with $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha - 1}$, which is in $(1/2, 1)$ for $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$. When $\alpha \in (0, 1/2)$, there is no h_n verifying (2.2) with $\sqrt{n} \ll h_n \ll n$, leading to believe that intermediate transversal fluctuations (i.e., $\xi \in (1/2, 1)$) cannot occur. In the following, we separate the cases $\alpha \in (1/2, 2)$ and $\alpha \in (0, 1/2)$.

2.2. *A natural candidate for the scaling limit.* Once we have identified in (2.2) the scale h_n for the transversal fluctuations, we are able to rescale both path trajectories and the field $(\omega_{i,x})$, so that we can define the rescaled "entropy" and "energy" of a path, and the corresponding continuous quantities. The rescaled paths will be in the following set:

$$\mathscr{D} := \{s : [0,1] \to \mathbb{R};\, s \text{ continuous and a.e. differentiable}\}, \tag{2.3}$$

and the (continuum) *entropy* of a path $s \in \mathscr{D}$ will derive from the rate function of the moderate deviation of the simple random walk (see [23] or (2.14) below), that is,

$$\mathrm{Ent}(s) = \frac{1}{2} \int_0^1 \big(s'(t)\big)^2 \, dt \quad \text{for } s \in \mathscr{D}. \tag{2.4}$$

As far as the disorder field is concerned, we let $\mathcal{P} := \{(w_i, t_i, x_i)\}_{i \ge 1}$ be a Poisson point process on $[0, \infty) \times [0,1] \times \mathbb{R}$ of intensity $\mu(dw\, dt\, dx) = \frac{\alpha}{2} w^{-\alpha-1} \mathbb{1}_{\{w>0\}}\, dw\, dt\, dx$. For a quenched realization of $\mathcal{P}$, the energy of a continuous path $s \in \mathscr{D}$ is then defined by

$$\pi(s) = \pi_{\mathcal{P}}(s) := \sum_{(w,t,x) \in \mathcal{P}} w \mathbb{1}_{\{(t,x) \in s\}}, \tag{2.5}$$

where the notation $(t, x) \in s$ means that $s_t = x$.

Then a natural guess for the continuous scaling limit of the partition function is to consider an energy–entropy competition variational problem. For any $\beta \in (0, +\infty]$, we let

$$\mathcal{T}_\beta := \sup_{s \in \mathscr{D}, \mathrm{Ent}(s) < +\infty} \big\{\beta \pi(s) - \mathrm{Ent}(s)\big\}. \tag{2.6}$$

This variational problem was originally introduced by Dey and Zygouras [13], Conjecture 1.7, conjecturing that it was well defined as long as $\alpha \in (1/2, 2)$ and that it was the good candidate for the scaling limit. In [7], Theorem 2.4, we show that the variational problem (2.6) is indeed well defined as long as $\alpha \in (1/2, 2)$. In Theorem 2.5 below, we prove the second part of [13], Conjecture 1.7.

THEOREM 2.1 ([7], Theorem 2.4). *For $\alpha \in (1/2, 2)$, we have that $\mathcal{T}_\beta \in (0, +\infty)$ for all $\beta > 0$ a.s. On the other hand, for $\alpha \in (0, 1/2]$ we have $\mathcal{T}_\beta = +\infty$ for all $\beta > 0$ a.s.*

Let us mention here that in [5], the authors consider the case of transversal fluctuations of order n. The natural candidate for the limit is $\widehat{\mathcal{T}}_\beta$, defined analogously to (2.6) by $\widehat{\mathcal{T}}_\beta = 0$ for $\beta = 0$, and for $\beta > 0$,

$$\widehat{\mathcal{T}}_\beta = \sup_{s \in \mathrm{Lip}_1} \left\{ \pi(s) - \frac{1}{\beta} \widehat{\mathrm{Ent}}(s) \right\}. \tag{2.7}$$

Here the supremum is taken over the set Lip_1 of 1-Lipschitz functions, and the entropy $\widehat{\mathrm{Ent}}(s)$ derives from the rate function of the large deviations for the simple random walk, that is,

$$\widehat{\mathrm{Ent}}(s) = \int_0^1 e\big(s'(t)\big)\, dt \quad \text{with } e(x) = \frac{1}{2}(1+x)\log(1+x) + \frac{1}{2}(1-x)\log(1-x).$$

2.3. *Main results I: The case $\alpha \in (1/2, 2)$.* Our first result deals with the transversal fluctuations of the polymer: we prove that h_n defined in (2.2) indeed gives the correct order for the transversal fluctuations.

THEOREM 2.2. *Assume that $\alpha \in (1/2, 2)$, that $\beta_n m(n^2) \to 0$ and that $\beta_n m(n^{3/2}) \to +\infty$, and define h_n as in (2.2): then $\sqrt{n} \ll h_n \ll n$. Then there are constants c_1, c_2 and $\nu > 0$ such that for any sequences $A_n \geq 1$ we have for all $n \geq 1$*

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n} \Big(\max_{i \leq n} |S_i| \geq A_n h_n \Big) \geq n e^{-c_1 A_n^2 h_n^2 / n} \Big) \leq c_2 A_n^{-\nu}. \tag{2.8}$$

In particular, this proves that if h_n defined in (2.2) is larger than a constant times $\sqrt{n \log n}$, then $n e^{-c_1 A h_n^2/n}$ goes to 0 as $n \to \infty$ provided that A is large enough: the transversal fluctuations are at most $A h_n$, with high $\mathbb{P}$-probability. On the other hand, if h_n is much smaller than $\sqrt{n \log n}$, then this theorem does not give sharp information: we still find that the transversal fluctuations must be smaller than $A\sqrt{n \log n}$, with high $\mathbb{P}$-probability. Anyway, in the course of the demonstration of our results, it will be clear that the main contribution to the partition function comes from trajectories with transversal fluctuations of order exactly h_n.

REMARK 2.3. In the case $L(x) \equiv 1$, Theorem 2.2 tells that if $\beta_n = n^{-\gamma}$ with $\frac{2}{\alpha} - 1 < \gamma < \frac{3}{2\alpha}$, then $h_n = n^{\xi}$ with $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$ (see (1.3)) and

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i \le n} |S_i| \ge A n^{\xi}\Big) \ge e^{-c_1' A^2 n^{2\xi-1}}\Big) \le c_2 A^{-\nu}.$$

We stress that the cases $\beta_n m(n^2) \to \beta \in (0, +\infty]$ and $\beta_n m(n^{3/2}) \to \beta \in [0, \infty)$ have already been considered by Auffinger and Louidor [5] and Dey and Zygouras [13], respectively: they find that the transversal fluctuations are of order n, respectively, $\sqrt{n}$. We state their results below, see Theorem 2.4 and Theorem 2.10, respectively. Our first series of results consists in identifying three new regimes for the transversal fluctuations ($\sqrt{n \log n} \ll h_n \ll n$, $h_n \asymp \sqrt{n \log n}$, and $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$), that interpolate between the Auffinger Louidor regime ($h_n \asymp n$) and the Dey Zygouras regime ($h_n \asymp \sqrt{n}$). We now describe more precisely these five different regimes.

Regime 1: Transversal fluctuations of order n. Consider the case where

$$\beta_n n^{-1} m(n^2) \to \beta \in (0, \infty], \tag{R1}$$

which corresponds to having transversal fluctuations of order n. If $L(x) \equiv 1$, it occurs when $\beta_n = \beta n^{-\gamma}$ with $\gamma \le \frac{2}{\alpha} - 1$. Auffinger and Louidor showed that, properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to $\widehat{\mathcal{T}}_\beta$ defined in (2.7).

THEOREM 2.4 (Regime 1, [5]). *Assume $\alpha \in (0, 2)$, and consider β_n such that (R1) holds. Then we have the following convergence:*

$$\frac{1}{\beta_n m(n^2)} \log \mathbf{Z}^{\omega}_{n,\beta_n} \xrightarrow{(\mathrm{d})} \widehat{\mathcal{T}}_\beta \quad \text{as } n \to \infty,$$

with $\widehat{\mathcal{T}}_\beta$ defined in (2.7). For $\alpha \in [1/2, 2)$, we have $\widehat{\mathcal{T}}_\beta > 0$ a.s. for all $\beta > 0$.

Regime 2: $\sqrt{n \log n} \ll h_n \ll n$. Consider the case when

$$\beta_n n^{-1} m(n^2) \to 0 \quad \text{and} \quad \beta_n (\log n)^{-1} m\big(n^{3/2}\sqrt{\log n}\big) \to \infty, \tag{R2}$$

which corresponds to having transversal fluctuations $\sqrt{n \log n} \ll h_n \ll n$, see (2.2). If $L(x) \equiv 1$, it occurs when $\beta_n = \beta n^{-\gamma}$ ($\beta \in (0, +\infty)$) with $\frac{2}{\alpha} - 1 < \gamma < \frac{3}{2\alpha}$, and we then have $h_n \sim \beta^{\frac{2\alpha}{2\alpha-1}} n^{\xi}$ with $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$. We find that, properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to $\mathcal{T}_1$ defined in (2.6)—this proves Conjecture 1.7 in [13].

THEOREM 2.5 (Regime 2). *Assume that $\alpha \in (1/2, 2)$, and consider β_n such that (R2) holds. Defining h_n as in (2.2), then $\sqrt{n \log n} \ll h_n \ll n$, and we have*

$$\frac{1}{\beta_n m(n h_n)} \big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n \beta_n \mathbb{E}[\omega] \mathbb{1}_{\{\alpha \ge 3/2\}}\big) \xrightarrow{(\mathrm{d})} \mathcal{T}_1 \quad \text{as } n \to \infty, \tag{2.9}$$

with $\mathcal{T}_1$ defined in (2.6).

We stress here that we need to recenter $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ by $n\beta_n\mathbb{E}[\omega]$ only when necessary, that is when $n/m(nh_n)$ does not go to 0: in terms of the picture described in Figure 1, this can happen only when $\gamma \geq 4-2\alpha$, and in particular when $\alpha \geq 3/2$ (this is stressed in the statement of the theorem).

REMARK 2.6. We stress that the renormalization in Theorem 2.5 verifies $\beta_n m(nh_n) \sim h_n^2/n$, so that the energy and the entropy are exactly of the same order. Roughly speaking, we have

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} \approx \sup_s \left\{ \beta_n m(nh_n)\pi(s) - \frac{h_n^2}{n}\operatorname{Ent}(s) \right\}.$$

This explains why there is no β-dependence in the limiting variational problem (it is "hidden" in the renormalization, which is the correct one both for the energy term and the entropy term).

Regime 3: $h_n \asymp \sqrt{n\log n}$. Consider the case

$$\beta_n(\log n)^{-1} m\big(n^{3/2}\sqrt{\log n}\big) \to \beta \in (0,\infty), \tag{R3}$$

which from (2.2) corresponds to transversal fluctuations $h_n \sim \beta^{1/2}\sqrt{n\log n}$; see (2.2). If $L(x)\equiv 1$, it occurs if $\beta_n = \beta(\log n)^{\zeta} n^{-\gamma}$ with $\gamma = \frac{3}{2\alpha}$ and $\zeta = \frac{2\alpha-1}{2\alpha}$. We find the correct scaling of $\log \mathbf{Z}^{\omega}_{n,\beta_n}$, which can be of two different natures (and go to $+\infty$ or 0); see Theorems 2.7–2.8 below.

We first need to introduce a few more notation. For a quenched continuum energy field $\mathcal{P}$ (as defined in Section 2.2), we define for a path s the number of weights w it collects:

$$N(s) := \sum_{(w,t,x)\in\mathcal{P}} \mathbb{1}_{\{(t,x)\in s\}}. \tag{2.10}$$

Then we define a new (continuum) energy–entropy variational problem: for a fixed realization of $\mathcal{P}$, define for any $k \geq 1$

$$\begin{aligned} \widetilde{\mathcal{T}}_{\beta}^{[k]} = \widetilde{\mathcal{T}}_{\beta}^{[k]}(\mathcal{P}) &:= \sup_{s\in\mathscr{D}, N(s)=k} \left\{ \pi(s) - \operatorname{Ent}(s) - \frac{k}{2\beta} \right\} \quad \text{and} \\ \widetilde{\mathcal{T}}_{\beta}^{[\geq r]} &:= \sup_{k\geq r} \widetilde{\mathcal{T}}_{\beta}^{[k]}. \end{aligned} \tag{2.11}$$

When $r=0$, we denote by $\widetilde{\mathcal{T}}_{\beta}$ the quantity $\widetilde{\mathcal{T}}_{\beta}^{[\geq 0]}$. In Proposition 5.5, we prove that these quantities are well defined, and that there exists $\beta_c = \beta_c(\mathcal{P}) \in (0,\infty)$ such that $\widetilde{\mathcal{T}}_{\beta} \in (0,\infty)$ if $\beta > \beta_c$ and $\widetilde{\mathcal{T}}_{\beta} = 0$ if $\beta < \beta_c$.

THEOREM 2.7 (Regime 3-a). *Assume that* $\alpha \in (1/2, 2)$, *and consider* β_n *such that* (R3) *holds. Then from* (2.2) *we have* $h_n \asymp \sqrt{n\log n}$, *and*

$$\frac{1}{\beta_n m(nh_n)}\big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n\mathbb{E}[\omega]\mathbb{1}_{\{\alpha\geq 3/2\}}\big) \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta} \quad \text{as } n\to\infty. \tag{2.12}$$

(*Recall that* $\beta_n m(nh_n) \sim h_n^2/n \sim \beta\log n$.)

Analogously to Remark 2.6, we roughly have

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} \approx \sup_{k\geq 0} \sup_{s, N(s)=k} \left\{ \beta_n m(nh_n)\pi(s) - \frac{h_n^2}{n} \operatorname{Ent}(s) - \frac{k}{2}\log n \right\}.$$

Here the extra term $\frac{k}{2}\log n$ comes from the *local* moderate deviation of the simple random walk (see (2.14) below): for each visited site, there is an extra cost $1/\sqrt{n} = e^{-\frac{1}{2}\log n}$. This explains why, renormalizing by $\beta_n m(nh_n) \sim h_n^2/n \sim \beta \log n$, one ends up with the variational problem (2.11).

If $\widetilde{\mathcal{T}}_\beta > 0$ ($\beta > \beta_c$), the scaling limit is therefore well identified, and $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ (when recentered) grows like $\beta \widetilde{\mathcal{T}}_\beta \log n$ with $\beta \widetilde{\mathcal{T}}_\beta > 0$. On the other hand, if $\widetilde{\mathcal{T}}_\beta = 0$, then the above theorem gives only a trivial limit. By an extended version of Skorokhod representation theorem [17], Corollary 5.12, one can couple the discrete environment and the continuum field $\mathcal{P}$ in order to obtain an almost sure convergence in Theorem 2.7 above. Hence, it makes sense to work conditionally on $\widetilde{\mathcal{T}}_\beta^{(\geq 1)} < 0$ (it is equivalent to $\beta < \beta_c$; see Proposition 5.5), even at the discrete level. Our next theorem says that for $\beta < \beta_c$, $\log \mathbf{Z}_{n,\beta_n}$ decays polynomially, with a random exponent $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \in (-1/2, 0)$.

THEOREM 2.8 (Regime 3-b, $\widetilde{\mathcal{T}}_\beta = 0$, $\beta < \beta_c$). *Assume that* $\alpha \in (1/2, 2)$ *and that* (R3) *holds. Then, conditionally on* $\{\widetilde{\mathcal{T}}_\beta^{[\geq 1]} < 0\}$ (*i.e.*, $\beta < \beta_c$),

$$\frac{1}{\beta_n m(nh_n)} \log\bigl(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega \leq 1/\beta_n\}}]\mathbf{1}_{\{\alpha\geq 1\}}\bigr) \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \quad \textit{as } n \to \infty.$$

Recalling that $\beta_n m(nh_n) \sim h_n^2/n \sim \beta \log n$, we note that $\exp(\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \log n)$ goes to 0 as a (random) power $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]}$ of n, with $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \in (-1/2, 0)$.

Regime 4: $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$. Consider the case

$$\text{(R4)} \qquad \beta_n m\bigl(n^{3/2}\bigr) \to \infty \quad \text{and} \quad \beta_n (\log n)^{-1} m\bigl(n^{3/2}\sqrt{\log n}\bigr) \to 0;$$

which corresponds to having transversal fluctuations $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$; see (2.2). If $L(x) \equiv 1$, it occurs if $\beta_n = \beta (\log n)^{\zeta} n^{-\gamma}$ with $\gamma = \frac{3}{2\alpha}$ and $0 < \zeta < \frac{2\alpha - 1}{2\alpha}$, in which case we have $h_n \sim \beta^{\frac{\alpha}{2\alpha-1}} (\log n)^{\frac{\alpha\zeta}{2\alpha-1}} \sqrt{n}$. Let us define

$$W_\beta := \sup_{(w,x,t)\in\mathcal{P}} \left\{ w - \frac{x^2}{2\beta t} \right\}, \tag{2.13}$$

which is a.s. positive and finite if $\alpha \in (1/2, 2)$; see Proposition 6.4 below.

THEOREM 2.9 (Regime 4). *Assume that* $\alpha \in (1/2, 2)$, *and consider* β_n *such that* (R4) *holds. Defining* h_n *as in* (2.2), *then* $\sqrt{n} \ll h_n \ll \sqrt{n \log n}$, *and we have*

$$\frac{1}{\beta_n m(nh_n)} \log\big(\sqrt{n}\big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega \mathbb{1}_{\{\omega \le 1/\beta_n\}}]\mathbb{1}_{\{\alpha \ge 1\}}\big)\big) \xrightarrow{(\mathrm{d})} W_1,$$

as $n \to +\infty$.

Recalling that $\beta_n m(nh_n) \sim h_n^2/n \ll \log n$, we note that $\exp(W_1 h_n^2/n)$ goes to infinity (at some random rate), but slower than any power of n.

Roughly speaking, in Section 6 we show that

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} \approx n^{-1/2} \exp\bigg(\sup_{(w,x,t)\in\mathcal{P}} \bigg\{\beta_n m(nh_n) w - \frac{h_n^2}{n}\frac{x^2}{2t}\bigg\}\bigg).$$

Analogously to Remark 2.6, there is no β-dependence in the limiting variational problem, since the renormalization is $\beta_n m(nh_n) \sim h_n^2/n$ (and is the correct one both for the energy term and the entropy term).

Regime 5: *Transversal fluctuations of order* $\sqrt{n}$. Consider the case

$$\beta_n m\big(n^{3/2}\big) \to \beta \in [0, \infty); \tag{R5}$$

this corresponds to having transversal fluctuations h_n of order $\sqrt{n}$. In the case $L(x) \equiv 1$, it occurs if $\beta_n = \beta n^{-\gamma}$ with $\gamma = \frac{3}{2\alpha}$. Here we state one of the results obtained by Dey and Zygouras, [13], Theorem 1.4.

THEOREM 2.10 (Regime 5, [13]). *Assume that* $\alpha \in (1/2, 2)$, *and consider* β_n *such that* (R5) *holds*, *that is*, $\beta_n m(n^{3/2}) \to \beta \in [0, \infty)$. *Then*

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\big(\log \mathbf{Z}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega \mathbb{1}_{\{\omega \le m(n^{3/2})\}}]\mathbb{1}_{\alpha \ge 1}\big) \xrightarrow{(\mathrm{d})} 2\mathcal{W}_\beta^{(\alpha)} \quad \textit{as } n \to \infty.$$

Here $\mathcal{W}_\beta^{(\alpha)}$ *is some specific* α*-stable random variable* (*defined in* [13], *p*. 4011).

Some comments about the different regimes. The regimes 2–3–4 have different behavior due to the different behaviors for the *local moderate deviation*; see [23], Theorem 3. We indeed have that, for $\sqrt{n} \ll h_n \ll n$,

$$p_n(h_n) := \mathbf{P}(S_n = h_n) = \frac{c}{\sqrt{n}} \exp\bigg(-\big(1 + o(1)\big)\frac{h_n^2}{2n}\bigg), \tag{2.14}$$

so that we identify three main possibilities: if $h_n \ll \sqrt{n \log n}$, then $p_n(h_n) = n^{-1/2+o(1)}$; if $h_n \sim c\sqrt{n \log n}$, then $p_n(h_n) = n^{-(c^2+1)/2+o(1)}$; if $h_n \gg \sqrt{n \log n}$, then $p_n(h_n) = e^{-(1+o(1))h_n^2/n}$ which decays faster than any power of n.

This is actually reflected in the behavior of the partition function. Let us denote $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = e^{-n\beta_n C_\alpha} \times \mathbf{Z}^{\omega}_{n,\beta_n}$ the renormalized (when necessary) partition function. We recall that C_α is equal either to $\mathbb{E}[\omega]\mathbb{1}_{\{\alpha \ge 3/2\}}$ (regimes 2 and 3-a) or to $\mathbb{E}[\omega \mathbb{1}_{\{\omega \le 1/\beta_n\}}]\mathbb{1}_{\alpha \ge 1}$ (regimes 3-b and 4).

- In regimes 1 and 2, transversal fluctuations are $h_n \gg \sqrt{n \log n}$, and $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$ grows faster than any power of n: roughly, it is of order $e^{\beta \widehat{\mathcal{T}}_\beta n}$ in regime 1 (for $\beta < \infty$), and of order $e^{\mathcal{T}_1 h_n^2/n}$ in regime 2.
- In regime 3, transversal fluctuations are $h_n \asymp \sqrt{n \log n}$, and $\bar{\mathbf{Z}}_{n,\beta_n}$ goes to infinity polynomially in regime 3-a, and it goes to 1 with a polynomial correction in regime 3-b. This could be summarized as $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} \approx 1 + n^{\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]}}$, with $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} > -1/2$: the transition between regimes 3-a and 3-b occurs as $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]}$ changes sign, at $\beta = \beta_c$ (note that $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]}$ keeps a mark of the local limit theorem; see (2.11) and (2.14)).
- In regime 4, $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$ goes to 1 with a correction of order $n^{-1/2} e^{W_1 h_n^2/n}$, with $e^{W_1 h_n^2/n}$ going to infinity slower than any power of n: this corresponds to the cost for a trajectory to visit a single site, at which the supremum in W_1 is attained. In regime 5, $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$ goes to 1 with a correction of order $n^{-1/2}$.

2.4. *Main results II*: *The case* $\alpha \in (0, 1/2)$. In this case, since we have $n^{-1} m(n^2)/m(n^{3/2}) \to \infty$, there is no sequence β_n such that $\beta_n n^{-1} m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$. First of all, Theorem 2.4 already gives a result, but a phase transition has been identified in [5, 24] when $\alpha \in (0, 1/2)$.

THEOREM 2.11 ([5, 24]). *When* $\alpha \in (0, 1/2)$, $\widehat{\mathcal{T}}_\beta$ *defined in* (2.7) *undergoes a phase transition*: *there exists some* $\widehat{\beta}_c = \widehat{\beta}_c(\mathcal{P})$ *with* $\widehat{\beta}_c \in (0, \infty)$ $\mathbb{P}$-*a.s.*, *such that* $\widehat{\mathcal{T}}_\beta = 0$ *if* $\beta \leq \widehat{\beta}_c$ *and* $\widehat{\mathcal{T}}_\beta > 0$ *if* $\beta > \widehat{\beta}_c$.

The fact that $\widehat{\mathcal{T}}_{\widehat{\beta}_c} = 0$ was not noted in [5, 24], but simply comes from the (left) continuity of $\beta \mapsto \widehat{\mathcal{T}}_\beta$ (the proof is identical to that for $\beta \mapsto \mathcal{T}_\beta$; see [7], Section 4.5).

In view of Theorem 2.4, the scaling limit of $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ is identified when $\widehat{\mathcal{T}}_\beta > 0$, and it is trivial when $\widehat{\mathcal{T}}_\beta = 0$. Again, by an extended version of Skorokhod representation theorem ([17], Corollary 5.12), we can obtain an almost sure convergence in Theorem 2.4. Hence, it makes sense to work conditionally on $\widehat{\mathcal{T}}_\beta > 0$ or $\widehat{\mathcal{T}}_\beta = 0$, even at the discrete level. We show here that only two regimes can hold: if $\widehat{\mathcal{T}}_\beta > 0$, then fluctuations are of order n, and properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges to $\widehat{\mathcal{T}}_\beta$ (this is Theorem 2.4); if $\widehat{\mathcal{T}}_\beta = 0$, then fluctuations are of order $\sqrt{n}$, and properly rescaled, $\log \mathbf{Z}^{\omega}_{n,\beta_n}$ converges in distribution (conditionally on $\widehat{\mathcal{T}}_\beta = 0$).

THEOREM 2.12. *Assume* $\alpha \in (0, 1/2)$, *and suppose* $\beta_n n^{-1} m(n^2) \to \beta \in [0, +\infty)$. *Then, on the event* $\{\widehat{\mathcal{T}}_\beta = 0\}$ ($\beta \leq \widehat{\beta}_c < \infty$), *transversal fluctuations are of order* $\sqrt{n}$. *More precisely, for any* $\varepsilon > 0$, *there exists some* $c_0, \nu > 0$ *such that, for any sequence* $C_n > 1$ *we have*

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i \leq n} |S_i| \geq C_n \sqrt{n}\Big) \geq e^{-c_0 C_n^2 \wedge n^{1/2}} \big| \widehat{\mathcal{T}}_\beta = 0\Big) \leq \varepsilon. \tag{2.15}$$

Moreover, conditionally on $\{\widehat{\mathcal{T}}_\beta = 0\}$, *we have that*

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})} \log \mathbf{Z}^{\omega}_{n,\beta} \xrightarrow{(\mathrm{d})} 2\mathcal{W}_0^{(\alpha)} \quad \text{as } n \to +\infty, \tag{2.16}$$

where $\mathcal{W}_0^{(\alpha)} := \int_{\mathbb{R}_+ \times \mathbb{R} \times [0,1]} w\rho(t,x)\mathcal{P}(dw,dx,dt)$ *with* $\mathcal{P}$ *a realization of the Poisson point process defined in Section* 2.2, *and* $\rho(t,x) = (2\pi t)^{-1/2} e^{-x^2/2t}$ *is the Gaussian heat kernel.*

Note that $\mathcal{W}_0^{(\alpha)}$ is well defined and has an α-stable distribution, with explicit characteristic function; see Lemma 1.3 in [13]. Theorem 2.12 therefore shows that, when $\alpha < 1/2$, a very sharp phase transition occurs on the line $\beta_n \sim \beta n/m(n^2)$: for $\beta \le \widehat{\beta}_c$, transversal fluctuations are of order $\sqrt{n}$ whereas for $\beta > \widehat{\beta}_c$ they are of order n.

2.5. *Some comments and perspectives.* We now present some possible generalizations, and we discuss some open questions.

About the case $\alpha = 1/2$. We excluded above the case $\alpha = 1/2$. In that case, both $n^{-1}m(n^2)$ and $m(n^{3/2})$ are regularly varying with index 3, and there are mostly two possibilities.

(1) If $\frac{n^{-1}m(n^2)}{m(n^{3/2})} \to 0$ (for instance if $L(x) = e^{-(\log x)^b}$ for some $b \in (0,1)$), there are sequences $(\beta_n)_{n\ge1}$ with $\beta_n n^{-1} m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$. The situation should be similar to that of Section 2.3: there should be five regimes, with transversal fluctuations h_n interpolating between $\sqrt{n}$ and n.

(2) If $\frac{n^{-1}m(n^2)}{m(n^{3/2})} \to c \in (0,\infty]$ (for instance if $L(n) = (\log x)^b$ for some b), there is no sequence $(\beta_n)_{n\ge1}$ with $\beta_n n^{-1} m(n^2) \to 0$ and $\beta_n m(n^{3/2}) \to +\infty$. Then the situation should be similar to that of Section 2.4: there should be only two regimes, with transversal fluctuations either $\sqrt{n}$ or n.

Toward the case $\alpha \in (2,5)$. When $\alpha \in (2,5)$ (more generally in region **C** in Figure 1), an important difficulty is to find the correct centering term for $\log \mathbf{Z}^{\omega}_{n,\beta_n}$. Another problem is that the variational problem $\mathcal{T}_\beta$ defined in (2.6) is $\mathcal{T}_\beta = +\infty$ a.s., since paths that collect many small weights bring an important contribution to $\mathcal{T}_\beta$. The main objective is therefore to prove a result of the type: there exists a function $f(\cdot)$ such that, for $\alpha \in (2,6)$ and any β_n in region **C** of Figure 1,

$$\frac{1}{\beta_n m(nh_n)} \big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - f(\beta_n)\big) \xrightarrow{(\mathrm{d})} \check{\mathcal{T}}_1,$$

with h_n defined as in (2.2) and where $\check{\mathcal{T}}_1$ is somehow a "recentered" version of the variational problem (2.6) (that is in which the contribution of the small weights has been canceled out). The difficulties are however serious: one needs (i) to identify the centering term $f(\beta_n)$, (ii) to make sense of the variational problem $\check{\mathcal{T}}_1$.

Path localization. We mention that in [5], Auffinger and Louidor show some path localization: they prove that, under $\mathbf{P}^{\omega}_{n,\beta_n}$, path trajectories concentrate around the (unique) maximizer γ^*_{n,β_n} of the discrete analogue of the variational problem (2.7); see Theorem 2.1 in [5]; moreover, this maximizer γ^*_{n,β_n} converges in distribution to the (unique) maximizer $\widehat{\gamma}^*_{\beta}$ of the variational problem (2.7). This could theoretically be done in our setting: in [7], Section 4.6, we prove the existence and uniqueness of the maximizer of the continuous variational problem (2.6). Then similar techniques to those of [5] could potentially be used, and one would obtain a result analogous to [5], Theorem 2.1.

Higher dimensions. Similar to [5], our methods should work in any dimension $1+d$ (one temporal dimension, d transversal dimensions). The relation (2.2) is replaced by $\beta_n m(nh_n^d) \sim h_n^2/n$: for paths with transversal scale h_n, the energy collected should be of order $\beta_n m(nh_n^d)$ while the entropy cost should remain of order h_n^2/n, at the exponential level. For $\alpha \in (0, 1+d)$, and choosing $\beta_n = n^{-\gamma}$, we should therefore find that in dimension d a similar picture to Figure 1 holds:

Case $\alpha \in (0, d/2)$	
$\gamma < \frac{1+d}{\alpha} - 1$	$\gamma > \frac{1+d}{\alpha} - 1$
$\xi = 1$	$\xi = 1/2$

Case $\alpha \in (d/2, 1+d)$		
$\gamma \le \frac{1+d}{\alpha} - 1$	$\frac{1+d}{\alpha} - 1 < \gamma < \frac{2+d}{2\alpha}$	$\gamma \ge \frac{2+d}{2\alpha}$
$\xi = 1$	$\xi = \frac{1+(1-\gamma)\alpha}{2\alpha-d} \in (\frac{1}{2}, 1)$	$\xi = \frac{1}{2}$

2.6. *Organization of the rest of the paper.* We present an overview of the main ideas used in the paper, and describe how the proofs are organized.

$*$ In Section 3, we recall some of the notation and results of the Entropy-controlled Last-Passage Percolation (E-LPP) developed in [7], which will be a central tool for the rest of the paper. In particular, we introduce a discrete energy/entropy variational problem (3.3) (which is the discrete counterpart of (2.6)), and state its convergence toward (2.6) in Proposition 3.1.

$*$ In Section 4, we prove Theorem 2.2, identifying the correct transversal fluctuations. In order to make our ideas appear clearer, we first treat the case when no centering is needed (*i.e.*, $\alpha < 3/2$) in Section 4.1. In Section 4.2, we adapt the proof to the case where it is needed. In the first case, we use a rough bound $\mathbf{P}^{\omega}_{n,\beta_n}(\max_{i\le n}|S_i| \ge A_n h_n) \le \mathbf{Z}^{\omega}_{n,\beta_n}(\max_{i\le n}|S_i| \ge A_n h_n)$, the second term being the partition function restricted to trajectories with $\max_{i\le n}|S_i| \ge A_n h_n$. The key idea is to decompose this quantity into sub-parts where trajectories have a "fixed" transversal fluctuation, as done in [13], page 4021,

$$\mathbf{Z}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \ge A_n h_n\Big) = \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} \mathbf{Z}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in \big[2^{k-1}h_n, 2^k h_n\big)\Big).$$

Then we control each term separately. Forcing the random walk to reach the scale $2^{k-1}h_n$ has an entropy cost $\exp(-c2^{2k}h_n^2/n)$ so we need to understand if the partition function, when restricted to trajectories with $\max_{i\le n}|S_i| \le 2^k h_n$, compensates this cost (cf. (4.3)): we need to estimate the probability of having $\mathbf{Z}^{\omega}_{n,\beta_n}(\max_{i\le n}|S_i|\le 2^k h_n)\ge e^{c2^{2k}h_n^2/n}$. This is the purpose of Lemma 4.1, which is the central estimate of this section, and which tediously uses estimates derived in [7] (in particular Proposition 2.6).

∗ In Section 5, we consider regimes 2 and 3-a, and we prove Theorems 2.5–2.7. The proof is decomposed into three steps. In the first step (Section 5.1), we use Theorem 2.2 in order to restrict the partition function to path trajectories that have transversal fluctuations smaller than Ah_n (for some large A fixed). In a second step (Section 5.2), we show that we can keep only the largest weights in the box of height Ah_n (more precisely a finite number of them), the small-weights contribution being negligible. Finally, the third step (Section 5.3) consists in proving the convergence of the large-weights partition function, and relies on the convergence of the discrete variational problem of Section 3.

∗ In Section 6, we treat regime 3-b and regime 4, and we prove Theorems 2.8–2.9. We proceed in four steps. In the first step (Section 6.1), we again use Theorem 2.2 to restrict the partition function to trajectories with transversal fluctuations smaller than $A\sqrt{n\log n}$ (for some large A fixed). The second step (Section 6.2) consists in showing that one can restrict to large weights. In the third step (Section 6.3), we observe that since we consider a regime $\log \mathbf{Z}^{\omega}_{n,\beta_n}\to 0$, it is equivalent to studying the convergence of $\mathbf{Z}^{\omega}_{n,\beta_n}-1$: we reduce to showing the convergence of a finite number of terms of the polynomial chaos expansion of $\mathbf{Z}^{\omega}_{n,\beta_n}-1$; see Lemmas 6.2–6.3. We prove this convergence in a last step: in Section 6.4, we show the convergence in regime 3-b (Lemma 6.2), relying on the convergence of a discrete variational problem. In Section 6.5, we show the convergence in regime 4 (Lemma 6.3), which is slightly more technical since we first need to reduce to trajectories with transversal fluctuations of order $h_n\ll\sqrt{n\log n}$.

∗ In Section 7, we consider the case $\alpha\in(0,1/2)$, and we prove Theorem 2.12. First, in Section 7.1, we prove (2.15), that is, there cannot be intermediate transversal fluctuations between $\sqrt{n}$ and n. We use mostly the same ideas as in Section 4, decomposing the contribution to the partition function according to the scale of the path, and controlling the entropic cost versus energy reward for each term. Here some simplifications occur: one can bound the maximal energy collected by a path at a given scale by the sum of all weights in a box containing the path, this sum being roughly dominated by the maximal weight in the box (this is true for $\alpha<1$). We then turn to the convergence of the partition function in Section 7.2. The idea is similar to that of [13], Section 5, and consists of several steps: first, we reduce the partition function to trajectories that stay at scale $\sqrt{n\log n}$; then we perform a polynomial chaos expansion of $\mathbf{Z}^{\omega}_{n,\beta_n}-1$ and we show that only the first term contributes; finally, we prove the convergence of the main term (see Lemma 7.2), showing in particular that the main contribution comes from trajectories that stay at scale $\sqrt{n}$.

3. Discrete energy–entropy variational problem. We introduce here some necessary notation, and state some useful results from [7]. Let us consider a box $\Lambda_{n,h} = [\![1,n]\!] \times [\![-h,h]\!]$. For any set $\Delta \subset \Lambda_{n,h}$, we define the (discrete) energy collected by Δ by

$$\Omega_{n,h}(\Delta) := \sum_{(i,x)\in\Delta} \omega_{i,x}. \tag{3.1}$$

We can also define the (discrete) entropy of a finite set $\Delta = \{(t_i, x_i); 1 \le i \le j\} \subset \mathbb{R}^2$ with $|\Delta| = j \in \mathbb{N}$ and with $0 \le t_1 \le t_2 \le \cdots \le t_j$ (with $t_0 = 0, x_0 = 0$)

$$\mathrm{Ent}(\Delta) := \frac{1}{2}\sum_{i=1}^{j} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}. \tag{3.2}$$

By convention, if $t_i = t_{i-1}$ for some i, then $\mathrm{Ent}(\Delta) = +\infty$. The set Δ is seen as a set of points a (continuous or discrete) path has to go through: if $\Delta \subset \mathbb{N}\times\mathbb{Z}$ a standard calculation gives that $\mathbf{P}(\Delta \subset S) \le e^{-\mathrm{Ent}(\Delta)}$ ($\Delta \subset S$ means that $S_{t_i} = x_i$ for all $i \le |\Delta|$), where we use that $\mathbf{P}(S_i = x) \le e^{-x^2/2i}$ by a standard Chernoff bound argument.

We are interested in the (discrete) variational problem, analogous to (2.6)

$$T_{n,h}^{\beta_{n,h}} := \max_{\Delta\subset\Lambda_{n,h}} \bigl\{\beta_{n,h}\Omega_{n,h}(\Delta) - \mathrm{Ent}(\Delta)\bigr\}, \tag{3.3}$$

with $\beta_{n,h}$ some function of n, h (soon to be specified).

We may rewrite the disorder in the region $\Lambda_{n,h}$, using the *order statistics*: we let $M_r^{(n,h)}$ be the rth largest value of $(\omega_{i,x})_{(i,x)\in\Lambda_{n,h}}$ and $Y_r^{(n,h)} \in \Lambda_{n,h}$ its position. In such a way,

$$(\omega_{i,j})_{(i,j)\in\Lambda_{n,h}} = \bigl(M_r^{(n,h)}, Y_r^{(n,h)}\bigr)_{r=1}^{|\Lambda_{n,h}|}. \tag{3.4}$$

In the following, we refer to $(M_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ as the *weight* sequence. Note also that $(Y_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ is simply a random permutation of the points of $\Lambda_{n,h}$.

The order statistics allows us to redefine the energy collected by a set $\Delta \subset \Lambda_{n,h}$, and its contribution by the first ℓ weights (with $1 \le \ell \le |\Lambda_{n,h}|$) by

$$\Omega_{n,h}^{(\ell)}(\Delta) := \sum_{r=1}^{\ell} M_r^{(n,h)} \mathbf{1}_{\{Y_r^{(n,h)}\in\Delta\}}, \qquad \Omega_{n,h}(\Delta) := \Omega_{n,h}^{(|\Lambda_{n,h}|)}(\Delta). \tag{3.5}$$

We also set $\Omega_{n,h}^{(>\ell)}(\Delta) := \Omega_{n,h}(\Delta) - \Omega_{n,h}^{(\ell)}(\Delta)$. We then define analogues of (3.3) with a restriction to the ℓ largest weights, or beyond the ℓth weight

$$\begin{aligned} T_{n,h}^{\beta_{n,h},(\ell)} &:= \max_{\Delta\subset\Lambda_{n,h}} \bigl\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\bigr\}, \\ T_{n,h}^{\beta_{n,h},(>\ell)} &:= \max_{\Delta\subset\Lambda_{n,h}} \bigl\{\beta_{n,h}\Omega_{n,h}^{(>\ell)}(\Delta) - \mathrm{Ent}(\Delta)\bigr\}. \end{aligned} \tag{3.6}$$

Estimates on these quantities are given in [7], Proposition 2.6 (most useful in Section 4). The following convergence in distribution is given in [7], Theorem 2.7, and plays a crucial role for the convergence in Theorems 2.5–2.9.

PROPOSITION 3.1. *Suppose that* $\frac{n}{h^2}\beta_{n,h}m(nh) \to \nu \in [0,\infty)$ *as* $n,h \to \infty$. *For every* $\alpha \in (1/2,2)$ *and for any* $q>0$, *we have*

$$\frac{n}{h^2}T_{n,qh}^{\beta_{n,h}} \xrightarrow{\text{(d)}} \mathcal{T}_{\nu,q} := \sup_{s\in\mathscr{M}_q}\{\nu\pi(s)-\operatorname{Ent}(s)\} \quad \text{as } n\to\infty, \tag{3.7}$$

with $\mathscr{M}_q := \{s\in\mathscr{D}, \operatorname{Ent}(s)<\infty, \max_{t\in[0,1]}|s(t)|\le q\}$. *We also have*

$$\frac{n}{h^2}T_{n,qh}^{\beta_{n,h},(\ell)} \xrightarrow{\text{(d)}} \mathcal{T}_{\nu,q}^{(\ell)} := \sup_{s\in\mathscr{M}_q}\{\nu\pi^{(\ell)}(s)-\operatorname{Ent}(s)\} \quad \text{as } n\to\infty, \tag{3.8}$$

where $\pi^{(\ell)} := \sum_{r=1}^{\ell} M_r \mathbb{1}_{\{Y_r\in s\}}$ *with* $\{(M_r,Y_r)\}_{r\ge1}$ *the order statistics of* $\mathcal{P}$ *restricted to* $[0,1]\times[-q,q]$; *see* [7], *Section* 5.1, *for details.*

Finally, we have $\mathcal{T}_{\nu,q}^{(\ell)} \to \mathcal{T}_{\nu,q}$ *as* $\ell\to\infty$, *and* $\mathcal{T}_{\nu,q}\to\mathcal{T}_\nu$ *as* $q\to\infty$, *a.s.*

4. Transversal fluctuations: Proof of Theorem 2.2.

In this section, we have $\alpha\in(1/2,2)$.

First, we partition the interval $[A_nh_n, n]$ into blocks

$$B_{k,n} := [2^{k-1}h_n, 2^k h_n), \quad k=\log_2 A_n+1,\dots,\log_2(n/h_n)+1. \tag{4.1}$$

In such a way,

$$\mathbf{P}_{n,\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\ge A_nh_n\Big) = \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} \mathbf{P}_{n,\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\in B_{k,n}\Big). \tag{4.2}$$

We first deal with the case where $n/m(nh_n) \overset{n\to\infty}{\to} 0$ for the sake of clarity of the exposition: in that case, $\log \mathbf{Z}_{n,\beta_n}^{\omega}$ does not need to be recentered. We treat the remaining case (in particular we have $\alpha\ge 3/2$) in a second step.

4.1. *Case* $n/m(nh_n)\to 0$ *as* $n\to+\infty$.

We observe that the assumption $\omega\ge 0$ implies that the partition function $\mathbf{Z}_{n,\beta_n}^{\omega}$ is larger than one. Therefore,

$$\mathbf{P}_{n,\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\in B_{k,n}\Big) \le \mathbf{Z}_{n,\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\in B_{k,n}\Big).$$

By using the Cauchy–Schwarz inequality, we get that

$$\begin{aligned}&\mathbf{Z}_{n,\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\in B_{k,n}\Big)^2\\ &\quad\le \mathbf{P}\Big(\max_{i\le n}|S_i|\ge 2^{k-1}h_n\Big)\times \mathbf{Z}_{n,2\beta_n}^{\omega}\Big(\max_{i\le n}|S_i|\le 2^k h_n\Big).\end{aligned} \tag{4.3}$$

The first probability is bounded by $2\mathbf{P}(|S_n| \geq 2^{k-1}h_n) \leq 4\exp(-2^{2k}h_n^2/2n)$ (by Lévy's inequality and a standard Chernov's bound). In the following lemma, which is the central tool for the proof of Theorem 2.2, we show that the random walk cannot "collect an energy" of order $q^2h_n^2/n$ under the condition $\max_{i\leq n}|S_i| \leq qh_n$.

LEMMA 4.1. *There exist some constant $q_0 > 0$ and some $\nu > 0$, such that for all $q \geq q_0$ we have*

$$\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,2\beta_n}\Big(\max_{i\leq n}|S_i| \leq qh_n\Big) \geq e^{\frac{1}{4}q^2\frac{h_n^2}{n}}\Big) \leq q^{-\nu}\Big(1 + 1 \wedge \frac{n}{m(nh_n)}\Big). \tag{4.4}$$

Therefore, if $n/m(nh_n) \overset{n\to\infty}{\to} 0$, this lemma gives that for $c_0 = 1/8$ and for k large enough (i.e., A_n large enough), using (4.3),

$$\begin{aligned}
&\mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,\beta_n}\Big(\max_{i\leq n}|S_i| \in B_{k,n}\Big) \geq 4e^{-c_0 2^{2k}h_n^2/n}\Big)\\
&\quad\leq \mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,2\beta_n}\Big(\max_{i\leq n}|S_i| \leq 2^k h_n\Big) \geq 4e^{(1/2-2c_0)2^{2k}h_n^2/n}\Big) \leq 2\big(2^k\big)^{-\nu}.
\end{aligned}$$

Then, using that $\sum_{k>\log_2 A_n} 4e^{-c_0 2^{2k}h_n^2/n} \leq e^{-c_1 A_n^2 h_n^2/n}$, we get that by a union bound

$$\begin{aligned}
&\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i\leq n}|S_i| \geq A_n h_n\Big) \geq e^{-c_1 A_n^2 h_n^2/n}\Big)\\
&\quad\leq \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)} \mathbb{P}\Big(\mathbf{Z}^{\omega}_{n,\beta_n}\Big(\max_{i\leq n}|S_i| \in B_{k,n}\Big) \geq 4e^{-c_0 2^{2k}h_n^2/n}\Big)\\
&\quad\leq 2\sum_{k>\log_2 A_n} 2^{-\nu k} \leq cA_n^{-\nu}.
\end{aligned} \tag{4.5}$$

We stress that in the case when $n/m(nh_n) \overset{n\to\infty}{\to} 0$, we do not need the additional n in front of $e^{-c_1 A_n^2 h_n^2/n}$ in (2.8).

PROOF OF LEMMA 4.1. For simplicity, we assume in the following that qh_n is an integer. We fix $\delta > 0$ small such that $(1+\delta)/\alpha < 2$ and $(1-\delta)/\alpha > 1/2$, and let

$$\mathtt{T} = \mathtt{T}_n(qh_n) = \frac{h_n^2}{n} q^{1/\alpha}\big(q^2h_n^2/n\big)^{-(1-\delta)^{3/2}/\alpha} \vee 1 \tag{4.6}$$

be the first level of truncation. Note that if $\alpha \leq (1-\delta)^{3/2}$ then we have $\mathtt{T} = 1$ for large n.

We decompose the partition function into three parts: thanks to Hölder's inequality, we can write that

$$\log \mathbf{Z}^{\omega}_{n,2\beta_n}\Big(\max_{i\le n}|S_i|\le qh_n\Big) \le \frac{1}{3}\log \mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} + \frac{1}{3}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n} + \frac{1}{3}\log \mathbf{Z}^{(\le 1)}_{n,6\beta_n}, \tag{4.7}$$

where the three partition functions correspond to three ranges for the weights $\beta_n\omega_{i,S_i}$:

$$\mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} := \mathbf{E}\bigg[\exp\bigg(\sum_{i=1}^n 6\beta_n\omega_{i,S_i}\mathbb{1}_{\{\beta_n\omega_{i,S_i}>\mathtt{T}\}}\bigg)\mathbb{1}_{\{\max_{i\le n}|S_i|\le qh_n\}}\bigg], \tag{4.8}$$

$$\mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n} := \mathbf{E}\bigg[\exp\bigg(\sum_{i=1}^n 6\beta_n\omega_{i,S_i}\mathbb{1}_{\{\beta_n\omega_{i,S_i}\in(1,\mathtt{T}]\}}\bigg)\mathbb{1}_{\{\max_{i\le n}|S_i|\le qh_n\}}\bigg], \tag{4.9}$$

$$\mathbf{Z}^{(\le 1)}_{n,6\beta_n} := \mathbf{E}\bigg[\exp\bigg(\sum_{i=1}^n 6\beta_n\omega_{i,S_i}\mathbb{1}_{\{\beta_n\omega_{i,S_i}\le 1\}}\bigg)\mathbb{1}_{\{\max_{i\le n}|S_i|\le qh_n\}}\bigg]. \tag{4.10}$$

We now show that with high probability, these three partition functions cannot be large. Note that when $\mathtt{T}=1$, the second term is equal to 1 and we do not have to deal with it.

Some comments on the proof. Let us stress that the main difficulty in the proof is not to deal with the contribution of the few largest weights, but rather to control the contribution of the many intermediate ones. Note that the first truncation level $\mathtt{T}$ is much smaller than $\beta_n m(qnh_n)$ (which is the order of the largest weight): for (4.8), we already deal with the contribution of many intermediate weights.

Let us now explain briefly where the choice for $\mathtt{T}$ comes from: our goal is to find $\mathtt{T}$ as small as possible (so that the first term includes already many intermediate weights) and such that $\mathbb{P}(\log \mathbf{Z}^{(>\mathtt{T})}_{n,6\beta_n} \ge c_0 q^2\frac{h_n^2}{n}) \xrightarrow[q\to+\infty]{} 0$, uniformly on n. For $\ell>0$, we let

$$\mathtt{T} := q^{1/\alpha}h_n^2/n \times \ell^{-\eta/\alpha} \sim \beta_n m(qnh_n)\times \ell^{-\eta/\alpha},$$

for some $\eta<1$. Roughly speaking, this choice of $\mathtt{T}$ allows us to safely replace $\mathbf{Z}^{(>T)}_{n,6\beta_n}$ by $\mathbf{Z}^{(\ell)}_{n,6\beta_n}$, the partition function truncated to the ℓ largest weights, cf. (4.13) below. We then compare the log-partition function with the discrete energy–entropy variational problem: we show in (4.15) that $\log \mathbf{Z}^{(\ell)}_{n,6\beta_n} \lesssim 2^\ell \exp(T^{6\beta_n,(\ell)}_{n,qh_n})$. We then choose to take $\ell = o(q^2h_n^2/n)$ as large as possible (to make $\mathtt{T}$ small), that is, $\ell=(q^2h_n^2/n)^{\eta'}$ for some $\eta'<1$: this leads to our choice of $\mathtt{T}$ in (4.6). Then the entropy-controlled last-passage percolation ([7], Proposition 2.6) allows us to control $T^{6\beta_n,(\ell)}_{n,qh_n}$.

For (4.9), we divide the partition function $\mathbf{Z}_{n,6\beta_n}^{((1,\mathtt{T}])}$ into a finite number of truncation levels $\mathtt{T}^{(j)} \le \beta_n \omega_{i,x} \le \mathtt{T}^{(j-1)}$ and we treat them separately (keeping track of the dependence of the estimates in j). The method is similar to the first term, but we chose to treat (4.9) in a second separate step to make the exposition clearer—it is the most technical part of the proof, and the core ideas are presented in the treatment of (4.8).

Term 1. For (4.8), we prove that for any $\nu < 2\alpha - 1$, for q sufficiently large, for all n large enough we have

$$\mathbb{P}\bigg(\log \mathbf{Z}_{n,6\beta_n}^{(>\mathtt{T})} \ge c_0 q^2 \frac{h_n^2}{n}\bigg) \le q^{-\nu}. \tag{4.11}$$

We compare this truncated partition function with the partition function where we keep the first ℓ weights in the order statistics $(M_i^{(n,qh_n)})_{1\le i\le 2nqh_n}$. Define

$$\ell = \ell_n(qh_n) := \big(q^2 h_n^2/n\big)^{1-\delta} \quad \text{so } \mathtt{T} = \frac{h_n^2}{n} q^{1/\alpha} \times \ell^{-(1-\delta)^{1/2}/\alpha}, \tag{4.12}$$

and set

$$\mathbf{Z}_{n,6\beta_n}^{(\ell)} := \mathbf{E}\Bigg[\exp\Bigg(\sum_{i=1}^{\ell} 6\beta_n M_i^{(n,qh_n)} \mathbb{1}_{\{Y_i^{(n,qh_n)} \in S\}}\Bigg)\Bigg]. \tag{4.13}$$

Note that the quantity in the exponent is the same writing as in the trick used in [15], page 236. Remark that, thanks to the relations (4.12) and (2.2) verified by $\mathtt{T}$ and β_n respectively, we have that for n large enough

$$\mathbb{P}\big(\beta_n M_\ell^{(n,qh_n)} > \mathtt{T}\big) \le \mathbb{P}\bigg(M_\ell^{(n,qh_n)} \ge \frac{1}{2} q^{1/\alpha} \ell^{-(1-\delta)^{1/2}/\alpha} m(nh_n)\bigg).$$

Then, since we have $q/\ell \le 1$ (see (4.12)), we can use Potter's bound (cf. [8], Theorem 1.5.6) to get that for n sufficiently large

$$m(nqh_n/\ell) \le (q/\ell)^{(1-\delta^2)/\alpha} m(nh_n),$$

and we obtain that provided that δ is small enough

$$\mathbb{P}\big(\beta_n M_\ell^{(n,qh_n)} > \mathtt{T}\big) \le \mathbb{P}\big(M_\ell^{(n,qh_n)} \ge c_0 q^{\delta^2/\alpha} \ell^{\delta^2/\alpha} m(nqh_n/\ell)\big) \le (cq\ell)^{-\delta^2 \ell/2},$$

where we used [7], Lemma 5.1, for the last inequality. We therefore get that, with probability larger than $1 - (c\ell)^{-\delta\ell/2}$ (note that $\ell^{-\delta\ell/2} \le q^{-\delta\ell/2} \le q^{-4}$ for n large enough), we have that

$$\begin{aligned} &\big\{(i,x) \in [\![1,n]\!] \times [\![-qh_n, qh_n]\!];\, \beta_n \omega_{i,x} > \mathtt{T}\big\} \subset \Upsilon_\ell \\ &\quad := \big\{Y_1^{(n,qh_n)}, \dots, Y_\ell^{(n,qh_n)}\big\}, \end{aligned} \tag{4.14}$$

and hence $\mathbf{Z}_{n,6\beta_n}^{(>T)} \le \mathbf{Z}_{n,6\beta_n}^{(\ell)}$.

We are therefore left to focus on the term $\mathbf{Z}^{(\ell)}_{n,6\beta_n}$: recalling the definitions (3.5) and (3.6), we get that

$$\begin{aligned}\mathbf{Z}^{(\ell)}_{n,6\beta_n} &= \sum_{\Delta\subset\Upsilon_\ell} e^{6\beta_n\Omega^{(\ell)}_{n,qh_n}(\Delta)}\mathbf{P}(S\cap\Upsilon_\ell=\Delta)\\ &\le \sum_{\Delta\subset\Upsilon_\ell}\exp\bigl(6\beta_n\Omega_{n,qh_n}(\Delta)-\mathrm{Ent}(\Delta)\bigr)\le 2^\ell\exp\bigl(T^{6\beta_n,(\ell)}_{n,qh_n}\bigr),\end{aligned}\tag{4.15}$$

where we used that $\mathbf{P}(\Delta\subset S)\le\exp(-\mathrm{Ent}(\Delta))$ as noted below (3.2).

Note that we have $\ell\le\frac{1}{2}c_0q^2h_n^2/n$ for n large enough (and $q\ge1$), so we get that

$$\mathbb{P}\biggl(\log\mathbf{Z}^{(\ell)}_{n,6\beta_n}\ge c_0q^2\frac{h_n^2}{n}\biggr)\le\mathbb{P}\biggl(T^{6\beta_n,(\ell)}_{n,qh_n}\ge\frac{1}{2}c_0q^2\frac{h_n^2}{n}\biggr).$$

We are going to control the last term by using the entropy-controlled last-passage percolation ([7], Proposition 2.6). For this purpose, notice that, by (2.2) and thanks to Potter's bound, for any $\eta>0$ there exists a constant c_η such that for any $q\ge1$,

$$\frac{(6\beta_n m(nqh_n))^{4/3}}{(q^2h_n^2/n)^{1/3}}\le c_\eta q^{(1+\eta)\frac{4}{3\alpha}-\frac{2}{3}}\frac{h_n^2}{n}=c_\eta\bigl(q^{4/3}\bigr)^{(1+\eta)/\alpha-2}\times q^2\frac{h_n^2}{n},$$

where we used that for any $\eta>0$, $m(nqh_n)\le c'_\eta q^{(1+\eta)/\alpha}m(nh_n)$ provided that n is large enough (Potter's bound). Therefore, provided that η is small enough so that $(1+\eta)/\alpha<2$, an application of [7], Proposition 2.6, gives that for q large enough (so that $b_q:=\frac{c_0}{2c_\eta}(q^{4/3})^{2-(1+\eta)/\alpha}$ is large),

$$\begin{aligned}\mathbb{P}\biggl(T^{6\beta_n,(\ell)}_{n,qh_n}\ge\frac{1}{2}c_0q^2\frac{h_n^2}{n}\biggr)&\le\mathbb{P}\biggl(T^{6\beta_n,(\ell)}_{n,qh_n}\ge b_q\times\frac{(6\beta_n m(nqh_n))^{4/3}}{(q^2h_n^2/n)^{1/3}}\biggr)\\ &\le cq^{-\nu},\end{aligned}\tag{4.16}$$

with $\nu=2\alpha-1-2\eta$. This gives (4.11), since η is arbitrary.

Term 2. We now turn to (4.9) We consider only the case $\mathtt{T}>1$ (and in particular we have $\alpha>(1-\delta)^{3/2}$). We show that for any $\eta>0$, there is a constant $c_\eta>0$ such that for q large enough and n large enough,

$$\mathbb{P}\bigl(\log\mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n}\ge c_0\bigl(q^2h_n^2/n\bigr)^{1-\eta}\bigr)\le\exp\bigl(-c_\eta\bigl(q^2h_n^2/n\bigr)^{1/3}\bigr).\tag{4.17}$$

Again, we need to decompose $\mathbf{Z}^{((1,\mathtt{T}])}_{n,6\beta_n}$ according to the values of the weights. We set $\theta:=(1-\delta)2/\alpha>1$, and let

$$\ell_j:=\bigl(q^2h_n^2/n\bigr)^{\theta^j(1-\delta)}=(\ell_0)^{\theta^j},\tag{4.18}$$

with $\ell_0=\ell=(q^2h_n^2/n)^{1-\delta}$ as in (4.12), and

$$\mathtt{T}^{(j)}:=\frac{h_n^2}{n}q^{1/\alpha}\times\bigl(q^2h_n^2/n\bigr)^{-\theta^j(1-\delta)^{3/2}/\alpha}=\frac{h_n^2}{n}q^{1/\alpha}(\ell_j)^{-(1-\delta)^{1/2}/\alpha}\tag{4.19}$$

for $j \in \{0, \dots, \kappa\}$ with κ the first integer such that $\theta^\kappa > \alpha/(1-\delta)^{3/2}$. We get that $\mathtt{T}^{(0)} = \mathtt{T}$, and $\mathtt{T}^{(\kappa)} < 1$. Then, thanks to Hölder inequality, we may write

$$\log \mathbf{Z}_{n,6\beta_n}^{((1,T])} \le \frac{1}{\kappa} \sum_{j=1}^{\kappa} \log \mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} \quad \text{with}$$

$$\mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} := \mathbf{E}\left[\exp\left(\sum_{i=1}^{n} 6\kappa\beta_n\omega_{i,S_i}\mathbf{1}_{\{\beta_n\omega_{i,S_i}\in(\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}]\}}\right)\mathbf{1}_{\{\max_{i\le n}|S_i|\le qh_n\}}\right].$$

To prove (4.17), it is therefore enough to prove that for any $1 \le j \le \kappa$, since $\ell_j \ge (q^2h_n^2/n)^{1-\delta}$,

$$\mathbb{P}\big(\log \mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} \ge 8\kappa\big(q^2h_n^2/n\big)\ell_j^{-\delta/10}\big) \le \exp\big(-c\big(q^2h_n^2/n\big)^{1/3}\big). \tag{4.20}$$

First of all, we notice that in view of (4.18)–(4.19), with the same computation leading to (4.14), we have that with probability larger than $1-(c\ell_j)^{-\delta\ell_j/4}$,

$$\begin{aligned} &\big\{(i,x) \in [\![1,n]\!] \times [\![-qh_n, qh_n]\!];\, \beta_n\omega_{i,x} > \mathtt{T}^{(j-1)}\big\} \\ &\quad\subset \Upsilon_{\ell_j} := \big\{Y_1^{(n,qh_n)}, \dots, Y_{\ell_j}^{(n,qh_n)}\big\}. \end{aligned} \tag{4.21}$$

On this event, and using that $\ell_j = (\ell_{j-1})^{(1-\delta)2/\alpha}$ and

$$\mathtt{T}^{(j-1)} = \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-(1-\delta)^{-1/2}/2} \le \frac{h_n^2}{n} q^{1/\alpha} \ell_j^{-1/2-\delta/5}$$

(if δ is small), we have

$$\begin{aligned} \mathbf{Z}_{n,6\kappa\beta_n}^{((\mathtt{T}^{(j)},\mathtt{T}^{(j-1)}])} &\le \mathbf{E}\left[\exp\left(6\kappa \mathtt{T}^{(j-1)} \sum_{i=1}^{\ell_j} \mathbf{1}_{\{Y_i^{(n,qh_n)} \in S\}}\right)\right] \\ &\le e^{6\kappa q^2 \frac{h_n^2}{n} \ell_j^{-\delta/10}} + \mathcal{H}_j \end{aligned} \tag{4.22}$$

with

$$\begin{aligned} \mathcal{H}_j &:= \sum_{k=q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}}^{\ell_j} \sum_{\Delta\subset\Upsilon_{\ell_j};|\Delta|=k} e^{6\kappa\frac{h_n^2}{n}q^{1/\alpha}\ell_j^{-1/2-\delta/5}k}\mathbf{P}(S\cap\Upsilon_{\ell_j}=\Delta) \\ &\le \sum_{k=q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}}^{\ell_j} \binom{\ell_j}{k} \exp\left(6\kappa\frac{h_n^2}{n}q^{1/\alpha}\ell_j^{-1/2-\delta/5}k - \inf_{\Delta\subset\Upsilon_{\ell_j},|\Delta|=k}\mathrm{Ent}(\Delta)\right). \end{aligned}$$

Then, we may bound $\binom{\ell_j}{k} \le e^{k\log\ell_j}$. We notice from the definition of κ and δ (and since $\theta \in (1,2)$) that there exists some $\nu > 0$ such that $\ell_j \le \ell_\kappa \le (q^2h_n^2/n)^{2-\nu}$ for any $1 \le j \le \kappa$: it shows in particular that $\log \ell_j \le \ell_j^{\delta^2} \le q^2\frac{h_n^2}{n}\ell_j^{-1/2-\delta/5}$, provided that n is sufficiently large and δ has been fixed sufficiently small. We end up with the following bound:

$$\mathcal{H}_j \le \sum_{k=q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}}^{\ell_j} \exp\Big(cq^2\frac{h_n^2}{n}\ell_j^{-1/2-\delta/5}k - \inf_{\Delta\subset\Upsilon_{\ell_j},|\Delta|=k}\mathrm{Ent}(\Delta)\Big).$$

Then we may use relation (2.5) of [7] (with $m=\ell_j$, $h=qh_n$) to get that, for any $k\ge q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}$,

$$\begin{aligned}\mathbb{P}\Big(&\inf_{\Delta\subset\Upsilon_{\ell_j},|\Delta|=k}\mathrm{Ent}(\Delta)\le 2cq^2\frac{h_n^2}{n}\ell_j^{-1/2-\delta/5}k\Big)\\ &\le\Big(\frac{C_0(2c\ell_j^{-1/2-\delta/5}k)^{1/2}\ell_j}{k^2}\Big)^k \\ &\le\big(cq^{\frac{3}{2\alpha}-3}\ell_j^{-\delta/4}\big)^k\le(c\ell_j)^{-\delta k/4}.\end{aligned}\tag{4.23}$$

For the last inequality, we used that $q^{\frac{3}{2\alpha}-3}\le 1$, since $\alpha>1/2$ and $q\ge 1$. Since we have that $q^2\frac{h_n^2}{n}\ell_j^{-1/2-\delta/5}\ge 1$, we get that there is a constant $c'>0$ such that

$$\sum_{k\ge q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}} e^{-cq^2\frac{h_n^2}{n}\ell_j^{-1/2-\delta/5}k}\le c'e^{-cq^2\frac{h_n^2}{n}\ell_j^{-\delta/10}}\le c'.$$

Using (4.23), we therefore obtain, via a union bound (also recalling (4.21)), that provided that n is large enough

$$\begin{aligned}\mathbb{P}\big(\mathbf{Z}_{n,6\kappa\beta_n}^{((\mathrm{T}^{(j)},\mathrm{T}^{(j-1)}])}\ge e^{8\kappa q^2\frac{h_n^2}{n}\ell_j^{-\delta/10}}\big)&\le(c\ell_j)^{-\delta\ell_j/4}+\sum_{k\ge q^{2-\frac{1}{\alpha}}\ell_j^{1/2+\delta/10}}(c\ell_j)^{-\delta k/4}\\ &\le(c\ell_j)^{-c_\delta\ell_j^{1/2}}.\end{aligned}$$

This proves (4.20) since $\ell_j\ge\ell_0=(q^2h_n^2/n)^{1-\delta}$.

Term 3. For the last part (4.10), we prove that for arbitrary $\eta>0$,

$$\mathbb{P}\Big(\log\mathbf{Z}_{n,6\beta_n}^{(\le 1)}\ge c_0q^2\frac{h_n^2}{n}\Big)\le cq^{-2}\times\begin{cases}\dfrac{n}{m(nh_n)} & \text{if } \alpha>1,\\[2mm] \dfrac{n}{m(nh_n)^{(1-\eta)\alpha}} & \text{if } \alpha\le 1.\end{cases}\tag{4.24}$$

Let us stress that in the case $\alpha\le 1$ we get that for n large $m(nh_n)^{(1-\eta)\alpha}\ge(nh_n)^{1-2\eta}$, therefore $n/(nh_n)^{(1-\eta)\alpha}$ goes to 0 provided that η is small enough, since we are considering the case when $h_n\ge\sqrt{n}$. Hence, we can replace the upper bound in (4.24) by $1\wedge(n/m(nh_n))$.

To prove (4.24), we use that $e^{6x\mathbf{1}_{\{x\le 1\}}} \le 1 + e^6 x\mathbf{1}_{\{x\le 1\}}$ for any x, and we get that

$$\mathbf{Z}_{n,6\beta_n}^{(\le 1)} \le \mathbf{E}\Bigg[\prod_{i=1}^n \big(1 + 6e^6 \beta_n \omega_{i,s_i} \mathbf{1}_{\{\beta_n \omega_{i,s_i} \le 1\}}\big)\Bigg] \quad \text{and}$$

$$\mathbb{E}\mathbf{Z}_{n,6\beta_n}^{(\le 1)} \le \mathbf{E}\Bigg[\prod_{i=1}^n \big(1 + 6e^6 \beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega \le 1/\beta_n\}}]\big)\Bigg] \le e^{6e^6 n\beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega\le 1/\beta_n\}}]}. \tag{4.25}$$

Therefore, by the Markov inequality and Jensen inequality,

$$\begin{aligned}\mathbb{P}\bigg(\log \mathbf{Z}_{n,6\beta_n}^{(\le 1)} \ge c_0 q^2 \frac{h_n^2}{n}\bigg) &\le \frac{1}{c_0 q^2} \frac{n}{h_n^2} \log \mathbb{E}\mathbf{Z}_{n,6\beta_n}^{(\le 1)} \\ &\le C q^{-2} \frac{n^2 \beta_n}{h_n^2} \mathbb{E}[\omega \mathbf{1}_{\{\omega \le 1/\beta_n\}}].\end{aligned} \tag{4.26}$$

It remains to estimate $\mathbb{E}[\omega \mathbf{1}_{\{\omega\le 1/\beta_n\}}]$. If $\alpha > 1$, then it is bounded by $\mathbb{E}[\omega] < +\infty$: this gives the first part of (4.24), using also (2.2). If $\alpha \le 1$, then for any $\eta > 0$, we have $\beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega\le 1/\beta_n\}}] \le \beta_n^{(1-\eta)\alpha}$ for n large enough: by using (2.2) together with $h_n^2/n \ge 1$, this gives the second part of (4.24).

The conclusion of Lemma 4.1 follows by collecting the estimates (4.11)–(4.17)–(4.24) of the three terms in (4.7). □

4.2. *Remaining case* ($\alpha \ge 3/2$). We now consider the remaining case, *i.e.* when we do not have that $n/m(nh_n) \overset{n\to\infty}{\to} 0$. In particular, we need to have that $\alpha \ge 3/2$, and hence $\mathbb{E}[\omega] =: \mu < +\infty$. Then, we do not simply use that $\mathbf{Z}_{n,\beta_n}^{\omega} \ge 1$ to bound $\mathbf{P}_{n,\beta_n}^{\omega}(\max_{i\le n} |S_i| \in B_{k,n})$, but instead we use a re-centered partition function $\bar{\mathbf{Z}}_{n,\beta_n}^{\omega} := e^{-n\beta_n \mu} \mathbf{Z}_{n,\beta_n}^{\omega}$, so that we can write

$$\begin{aligned}\mathbf{P}_{n,\beta_n}^{\omega}\Big(\max_{i\le n} |S_i| \in B_{k,n}\Big) &= \frac{1}{\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}} \mathbf{E}\Bigg[\exp\Bigg(\sum_{i=1}^n \beta_n(\omega_{i,s_i} - \mu)\Bigg) \mathbf{1}_{\{\max_{i\le n} |S_i| \in B_{k,n}\}}\Bigg] \\ &=: \frac{1}{\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}} \bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\Big(\max_{i\le n} |S_i| \in B_{k,n}\Big).\end{aligned} \tag{4.27}$$

First, we need to get a lower bound on $\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}$.

LEMMA 4.2. *For any $\delta > 0$, there is a constant $c > 0$ such that for any positive sequence $\varepsilon_n \le 1$ with $\varepsilon_n \ge n^{-1/2}(h_n^2/n)^{\alpha-3/2+\delta}$ (this goes to 0 for δ small enough), and any $n \ge 1$,*

$$\mathbb{P}\big(\bar{\mathbf{Z}}_{n,\beta_n}^{\omega} \ge n^{-1} e^{\varepsilon_n \frac{h_n^2}{n}}\big) \ge 1 - e^{-c/\varepsilon_n^{\alpha-1/2-\delta}} - e^{-c\varepsilon_n h_n^2/n}. \tag{4.28}$$

We postpone the proof of this lemma to the end of this subsection, and we now complete the proof of Theorem 2.2, (2.8). Lemma 4.2 gives that $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} \geq n^{-1}$ with overwhelming probability: using (4.2) combined with (4.27), we get, analogously to (4.5),

$$\begin{aligned}
&\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta}\Big(\max_{i\leq n}|S_i| \geq A_n h_n\Big) \geq n e^{-c_1 A_n^2 h_n^2/n}\Big)\\
&\quad\leq \mathbb{P}(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} \leq n^{-1}) \qquad (4.29)\\
&\qquad + \sum_{k=\log_2 A_n+1}^{\log_2(n/h_n)+1} \mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,\beta}\Big(\max_{i\leq n}|S_i| \in B_{k,n}\Big) \geq 4e^{-c_0 2^{2k} h_n^2/n}\Big).
\end{aligned}$$

To control the sum, we use the following lemma which is the analogy of Lemma 4.1 for $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$.

LEMMA 4.3. *There exist some constant $q_0 > 0$ and some $\nu > 0$, such that for all $q \geq q_0$ we have*

$$\mathbb{P}\Big(\bar{\mathbf{Z}}^{\omega}_{n,2\beta_n}\Big(\max_{i\leq n}|S_i| \leq q h_n\Big) \geq e^{\frac{1}{4}q^2\frac{h_n^2}{n}}\Big) \leq q^{-\nu}. \qquad (4.30)$$

PROOF. The proof follows the same lines as for Lemma 4.1: (4.7) still holds, with $\beta_n\omega_{i,S_i}$ replaced by $\beta_n(\omega_{i,S_i} - \mu)$ (outside of the indicator function). The bounds (4.11)–(4.17) for terms 1 and 2 still hold, since one fall back to the same estimates by using that $(\omega_{i,S_i} - \mu) \leq \omega_{i,S_i}$. It remains only to control the third term: we prove that when $\mu := \mathbb{E}[\omega] < \infty$, then for any $\delta > 0$, provided that n is large enough,

$$\mathbb{P}\Big(\log \bar{\mathbf{Z}}^{(\leq 1)}_{n,6\beta_n} \geq c_0 q^2 \frac{h_n^2}{n}\Big) \leq c q^{-2} \times n^{-1/2}\Big(\frac{h_n^2}{n}\Big)^{\alpha-\frac{3}{2}+\delta}, \qquad (4.31)$$

where we set analogously to (4.7)

$$\bar{\mathbf{Z}}^{(\leq 1)}_{n,6\beta_n} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} 6\beta_n(\omega_{i,S_i} - \mu)\mathbf{1}_{\{\beta_n\omega_{i,S_i}\leq 1\}}\Big)\Big]. \qquad (4.32)$$

Then, in the case $\alpha \geq 3/2$, using that $h_n^2/n \leq n$ we get that the upper bound in (4.31) is bounded by $cq^{-2}n^{\alpha-2+\delta}$ which is smaller than q^{-2} provided that δ had been fixed small enough.

To prove (4.31), we use that there is a constant c such that $e^x \leq 1 + x + cx^2$ as soon as $|x| \leq 6$, so that we get similar to (4.25) that

$$\begin{aligned}
&\mathbb{E}\mathbf{Z}^{(\leq 1)}_{n,6\beta_n}\\
&\quad\leq \big(1 + \beta_n\mathbb{E}[(\omega-\mu)\mathbf{1}_{\{\omega\leq 1/\beta_n\}}] + c\beta_n^2\mathbb{E}[(\omega-\mu)^2\mathbf{1}_{\{\omega\leq 1/\beta_n\}}]\big)^n \qquad (4.33)\\
&\quad\leq \exp\big(cnL(1/\beta_n)\beta_n^{\alpha}\big) \leq \exp\Big(\frac{c}{h_n}(h_n^2/n)^{\alpha+\delta}\Big).
\end{aligned}$$

For the second inequality, we used that $\mathbb{E}[(\omega-\mu)\mathbf{1}_{\{\omega\le 1/\beta_n\}}]\le 0$ and also that $\mathbb{E}[(\omega-\mu)^2\mathbf{1}_{\{\omega\le 1/\beta_n\}}]\le cL(1/\beta_n)\beta_n^{\alpha-2}$, thanks to (1.2). The last inequality holds for any fixed δ, provided that n is large enough, and comes from using Potter's bound and the relation (2.2) to get that $L(1/\beta_n)\beta_n^\alpha\le c'\mathbb{P}(\omega>1/\beta_n)\le (nh_n)^{-1}(h_n^2/n)^{\alpha+\delta}$. Then, applying Markov and Jensen inequalities as in (4.26), we get that

$$\mathbb{P}\Big(\log \bar{\mathbf{Z}}_{n,6\beta_n}^{(\le 1)}\ge c_0 q^2\frac{h_n^2}{n}\Big)\le cq^{-2}\frac{n}{h_n^3}\Big(\frac{h_n^2}{n}\Big)^{\alpha+\delta},$$

which proves (4.31). □

With Lemma 4.3 in hand, and using the Cauchy–Schwarz inequality as in (4.3), we get that

$$\mathbb{P}\Big(\bar{\mathbf{Z}}_{n,\beta}^{\omega}\Big(\max_{i\le n}|S_i|\in B_{k,n}\Big)\ge 2e^{-c_0 2^{2k}h_n^2/n}\Big)\le \big(2^k\big)^{-\nu}.$$

Plugged into (4.29), this concludes the proof of Theorem 2.2, (2.8). It therefore only remains to prove Lemma 4.2.

PROOF OF LEMMA 4.2. We need to obtain a lower bound on $\bar{\mathbf{Z}}_{n,\beta_n}$. We apply the Cauchy–Schwarz inequality to

$$\begin{aligned}\bar{\mathbf{Z}}_{n,\beta_n/2}^{(>1)} &:= \mathbf{E}\Big[\exp\Big(\sum_{i=1}^n\frac{\beta_n}{2}(\omega_{i,s_i}-\mu)\mathbf{1}_{\{\beta_n\omega_{i,s_i}>1\}}\Big)\Big]\\ &\le \big(\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\big)^{1/2}\mathbf{E}\Big[\exp\Big(\sum_{i=1}^n -\beta_n(\omega_{i,s_i}-\mu)\mathbf{1}_{\{\beta_n\omega_{i,s_i}\le 1\}}\Big)\Big]^{1/2}\\ &=: \big(\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\big)^{1/2}\big(\bar{\mathbf{Z}}_{n,-\beta_n}^{(\le 1)}\big)^{1/2},\end{aligned}$$

so that

$$\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\ge \big(\bar{\mathbf{Z}}_{n,\beta_n/2}^{(>1)}\big)^2/\bar{\mathbf{Z}}_{n,-\beta_n}^{(\le 1)}. \tag{4.34}$$

Hence, we get that

$$\mathbb{P}\big(\bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\le n^{-1}e^{\varepsilon_n\frac{h_n^2}{n}}\big)\le \mathbb{P}\big(\bar{\mathbf{Z}}_{n,-\beta_n}^{(\le 1)}\ge e^{\varepsilon_n\frac{h_n^2}{n}}\big)+\mathbb{P}\big(\bar{\mathbf{Z}}_{n,\beta_n/2}^{(>1)}\le n^{-1/2}e^{\varepsilon_n\frac{h_n^2}{n}}\big), \tag{4.35}$$

and we deal with both terms separately.

For the first term, we use that analogously to (4.33) we have

$$\begin{aligned}\mathbb{E}\bar{\mathbf{Z}}_{n,-\beta_n}^{(\le 1)} &\le \big(1-\beta_n\mathbb{E}\big[(\omega-\mu)\mathbf{1}_{\{\omega\le 1/\beta_n\}}\big]+c\beta_n^2\mathbb{E}\big[(\omega-\mu)^2\mathbf{1}_{\{\omega\le 1/\beta_n\}}\big]\big)^n\\ &\le \big(1+cL(1/\beta_n)\beta_n^\alpha\big)^n\le \exp\Big(\frac{c}{h_n}\big(h_n^2/n\big)^{\alpha+\delta/2}\Big),\end{aligned} \tag{4.36}$$

Here the difference with (4.33) is that we use for the second inequality that $-\mathbb{E}[(\omega-\mu)\mathbf{1}_{\{\omega\le 1/\beta_n\}}]=\mathbb{E}[(\omega-\mu)\mathbf{1}_{\{\omega>1/\beta_n\}}]\le cL(1/\beta_n)\beta_n^{\alpha-1}$, thanks to (1.2). Again, the second inequality holds for any fixed δ, provided that n is large enough. Using Markov's inequality, one therefore obtains that the first term in (4.35) is bounded by

$$\mathbb{P}\big(\bar{\mathbf{Z}}^{(\le 1)}_{n,-\beta_n}\ge e^{\varepsilon_n\frac{h_n^2}{n}}\big)\le \exp\biggl(\frac{c}{h_n}(h_n^2/n)^{\alpha+\delta}-\varepsilon_n\frac{h_n^2}{n}\biggr)\le \exp\biggl(-\varepsilon_n\frac{h_n^2}{2n}\biggr), \tag{4.37}$$

the second inequality holding provided that ε_n is larger than $n^{-1/2}(\frac{h_n^2}{n})^{\alpha-\frac{3}{2}+\delta}$.

As far as the second term in (4.35) is concerned, we find a lower bound on $\mathbf{Z}^{(\ge 1)}_{n,\beta_n}$ by restricting to a particular set of trajectories. Consider the set

$$\mathcal{O}_n:=\bigl\{(i,x)\in[\![n/2,n]\!]\times[\![\varepsilon_n^{1/2}h_n,2\varepsilon_n^{1/2}h_n]\!];\,\beta_n\omega_{i,x}\ge 2x^2/i\bigr\}.$$

If the set $\mathcal{O}_n$ is nonempty, then pick some $(i_0,x_0)\in\mathcal{O}_n$, and consider trajectories which visit this specific site: since all other weights are nonnegative ($(\omega-\mu)\mathbf{1}_{\{\beta_n\omega>1\}}\ge 0$ provided $\mu<1/\beta_n$), we get that

$$\begin{aligned}\bar{\mathbf{Z}}^{(\ge 1)}_{n,\beta_n}&\ge e^{\beta_n(\omega_{i_0,x_0}-\mu)}\mathbf{P}(S_{i_0}=x_0)\\ &\ge \frac{c}{\sqrt{n}}\exp\biggl(\beta_n\omega_{i_0,x_0}-\frac{x_0^2}{i_0}\biggr)\ge\frac{c}{\sqrt{n}}e^{\varepsilon_n\frac{h_n^2}{n}}.\end{aligned}\tag{4.38}$$

We used Stone's local limit theorem [23] for the second inequality (valid provided that n is large, using also that $i_0\ge n/2$). For the last inequality, we used the definition of $\mathcal{O}_n$ to bound the argument of the exponential by $x_0^2/i_0\ge\varepsilon_n h_n^2/n$. Therefore, we get that

$$\begin{aligned}\mathbb{P}\biggl(\bar{\mathbf{Z}}^{(\ge 1)}_{n,\beta_n}\le\frac{c}{\sqrt{n}}e^{\varepsilon_n\frac{h_n^2}{n}}\biggr)&\le\mathbb{P}(\mathcal{O}_n=\varnothing)=\prod_{i=n/2}^{n}\prod_{x=\varepsilon_n^{1/2}h_n}^{2\varepsilon_n^{1/2}h_n}\bigl(1-\mathbb{P}\bigl(\beta_n\omega>2x^2/i\bigr)\bigr)\\ &\le\bigl(1-\mathbb{P}\bigl(\omega>4\varepsilon_n m(nh_n)\bigr)\bigr)^{\varepsilon_n^{1/2}nh_n}.\end{aligned}$$

For the second inequality, we used that $x^2/i\ge\varepsilon_n h_n^2/n$ for the range considered, together with the relation (2.2) characterizing β_n. Then we use the definition of $m(nh_n)$ together with Potter's bound to get that for any fixed $\delta>0$, we have $\mathbb{P}(\omega>4\varepsilon_n m(nh_n))\ge c\varepsilon_n^{-\alpha+\delta}(nh_n)^{-1}$, provided that n is large enough. Therefore, we obtain that

$$\mathbb{P}\biggl(\bar{\mathbf{Z}}^{(\ge 1)}_{n,\beta_n}\le\frac{c}{\sqrt{n}}e^{\varepsilon_n\frac{h_n^2}{n}}\biggr)\le\exp\bigl(-c\varepsilon_n^{\frac{1}{2}-\alpha+\delta}\bigr),\tag{4.39}$$

which bounds the second term in (4.35). $\square$

5. Regime 2 and regime 3-a. In this section, we prove Theorem 2.5 and Theorem 2.7. We decompose the proof in three steps, Step 1 and Step 2 being the same for both theorems. For the third step, we give the details in regime 2, and adapt the reasoning to regime 3-a.

5.1. *Step* 1: *Reduction of the set of trajectories.* Recalling $\mu = \mathbb{E}[\omega]$ (which is finite for $\alpha > 1$), we define

$$\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} := \mathbf{E}\left[\exp\left(\sum_{i=1}^{n} \beta_n(\omega_{i,S_i} - \mu \mathbf{1}_{\{\alpha \geq 3/2\}})\right)\right]. \tag{5.1}$$

We show that to prove Theorem 2.5 and Theorem 2.7 we can reduce the problem to the random walk trajectories belonging to Λ_{n,Ah_n} for some $A > 0$ (large). For any $A > 0$, we define

$$\mathcal{B}_n(A) := \left\{(i, S_i)_{i=1}^{n} : \max_{i \leq n} |S_i| \leq Ah_n\right\} \tag{5.2}$$

and we let

$$\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}\big(\mathcal{B}_n(A)\big) := \mathbf{E}\left[\exp\left(\sum_{i=1}^{n} \beta_n(\omega_{i,S_i} - \mu \mathbf{1}_{\{\alpha \geq 3/2\}})\right)\mathbf{1}_{\mathcal{B}_n(A)}\right]. \tag{5.3}$$

Relation (2.8) gives that $\mathbb{P}(\mathbf{P}^{\omega}_{n,\beta_n}(\mathcal{B}_n(A)^c) \geq n e^{-c_1 A^2 h_n^2/n}) \leq c_2 A^{-\nu_1}$, uniformly on $n \in \mathbb{N}$. This implies that

$$\mathbb{P}\Big(\big|\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}\big(\mathcal{B}_n(A)\big)\big| \geq n e^{-c_1' A^2 h_n^2/n}\Big) \leq c_2 A^{-\nu_1}, \tag{5.4}$$

uniformly on $n \in \mathbb{N}$. Let us observe that in regime 2 and regime 3-a we have that $h_n^2/n \geq c_\beta \log n$, therefore $n e^{-c_1' A^2 h_n^2/n}$ goes to 0 as n gets large, provided A is sufficiently large.

In such a way, relation (5.4) implies

$$\lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = \lim_{A\to\infty} \lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}\big(\mathcal{B}_n(A)\big). \tag{5.5}$$

5.2. *Step* 2: *Restriction to large weights.* In the second step of the proof, we show that we may only consider the partition function $\mathbf{Z}^{\omega,(\mathtt{L})}_{n,\beta_n}$ truncated to a finite number $\mathtt{L}$ of large weights, with $\mathtt{L}$ independent of n. We need some intermediate truncation steps.

We start by removing the small weights. Using the notation introduced in (4.8)–(4.10) and (4.32), Hölder's inequality gives that for any $\eta \in (0, 1)$

$$\begin{aligned}&\big(\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}\big)^{\frac{1}{1-\eta}} \big(\bar{\mathbf{Z}}^{(\leq 1)}_{n,-(\eta^{-1}-1)\beta_n}\big)^{-\frac{\eta}{1-\eta}} \\ &\quad \leq \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}\big(\mathcal{B}_n(A)\big) \leq \big(\bar{\mathbf{Z}}^{(>1)}_{n,(1+\eta)\beta_n}\big)^{\frac{1}{1+\eta}} \big(\bar{\mathbf{Z}}^{(\leq 1)}_{n,(1+\eta^{-1})\beta_n}\big)^{\frac{\eta}{1+\eta}}.\end{aligned} \tag{5.6}$$

We observe that the condition $\beta_n \omega > 1$ implies (if $\mu < \infty$)

$$(5.7)\quad (1-2\eta)\beta_n\omega \le (1-\eta)\beta_n(\omega-\mu) \quad \text{and} \quad (1+\eta)\beta_n(\omega-\mu) \le (1+\eta)\beta_n\omega,$$

provided n is large enough (the first condition is equivalent to $\beta_n\omega \ge \frac{1-\eta}{\eta}\beta_n\mu$, which holds for n large enough since $\beta_n \downarrow 0$). In such a way, we can safely replace $\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n}$ and $\bar{\mathbf{Z}}^{(>1)}_{n,(1+\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1+\eta)\beta_n}$ in (5.6). The next lemma shows that the contribution given by $\log \bar{\mathbf{Z}}^{(\le 1)}_{n,\rho\beta_n}$ is negligible.

LEMMA 5.1. *Let $\rho \in \mathbb{R}$. Then*

$$(5.8)\qquad \frac{n}{h_n^2}\log \bar{\mathbf{Z}}^{(\le 1)}_{n,\rho\beta_n} \xrightarrow{\mathbb{P}} 0 \quad \textit{as } n \to \infty.$$

PROOF. The case $\rho > 0$ is a consequence of the estimate in (4.25) and (4.26), while the case $\rho < 0$ follows from the estimate in (4.36) and (4.37). □

We can further reduce the partition function $\mathbf{Z}^{(>1)}_{n,\nu\beta_n}$ to even (intermediate) larger weights (with $\nu > 0$).

We fix some $\delta > 0$ small, and define $\ell := (A^2 h_n^2/n)^{1-\delta}$ and also $\mathtt{T} = A^{1/\alpha}\frac{h_n^2}{n}\ell^{-(1-\delta)^{1/2}/\alpha}$ as in (4.12): then, Hölder's inequality gives that for any $\eta \in (0,1)$

$$\log \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \le \log \mathbf{Z}^{(>1)}_{n,\nu\beta_n} \le \frac{1}{1+\eta}\log \mathbf{Z}^{(>\mathtt{T})}_{n,(1+\eta)\nu\beta_n} + \frac{\eta}{1+\eta}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,(1+\eta^{-1})\nu\beta_n}.$$

Then (4.17) gives that for any fixed $A \ge 1$, and since $h_n^2/n \to \infty$, we have that for any $\rho > 0$,

$$(5.9)\qquad \frac{n}{h_n^2}\log \mathbf{Z}^{((1,\mathtt{T}])}_{n,\rho\beta_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Finally, we show that we can only consider a finite number of large weights. We consider $\Upsilon_\ell = \{Y_1^{(n,Ah_n)}, \dots, Y_\ell^{(n,Ah_n)}\}$ with ℓ chosen above. Using (4.14), with probability larger $1-(c\ell)^{-\delta\ell/2}$ (with $\ell \to \infty$ as $n \to \infty$) we have that

$$\Xi_{\mathtt{T}} := \{(i,x) \in [\![1,n]\!] \times [\![-Ah_n, Ah_n]\!];\, \beta_n\omega_{i,x} > \mathtt{T}\} \subset \Upsilon_\ell,$$

and thus $\mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \le \mathbf{Z}^{(\ell)}_{n,\nu\beta_n}$ with high probability. We let $\mathtt{L} \in \mathbb{N}$ be a fixed (large) constant. Since $|\Xi_{\mathtt{T}}| \to \infty$ as $n \to \infty$ in probability, we have that $\Upsilon_{\mathtt{L}} \subset \Xi_{\mathtt{T}}$ so that, $\mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n} \le \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n}$ for large n, with high probability. By using Hölder's inequality, we get

$$\mathbf{Z}^{(\mathtt{L})}_{n,\nu\beta_n} \le \mathbf{Z}^{(>\mathtt{T})}_{n,\nu\beta_n} \le \big(\mathbf{Z}^{(\mathtt{L})}_{n,\nu(1+\eta)\beta_n}\big)^{\frac{1}{1+\eta}}\big(\mathbf{Z}^{(\mathtt{L},\ell)}_{n,\nu(1+\eta^{-1})\beta_n}\big)^{\frac{\eta}{1+\eta}},$$

where

$$\mathbf{Z}_{n,\beta_n}^{(\mathrm{L},\ell)} := \mathbf{E}\left[\exp\left(\sum_{i=\mathrm{L}+1}^{\ell} \beta_n M_i^{(n,qh_n)} \mathbb{1}_{\{Y_i^{(n,qh_n)} \in S\}}\right)\right]. \tag{5.10}$$

We now show that the contribution of $\mathbf{Z}_{n,\nu(1+\eta^{-1})\beta_n}^{(\mathrm{L},\ell)}$ is negligible.

LEMMA 5.2. *For any* $\varepsilon \in (0,1)$ *and for any* $\mathrm{L} \in \mathbb{N}$ *and* $\rho > 0$, *there exists* δ_{L} *such that for all* n,

$$\mathbb{P}\left(\frac{n}{h_n^2} \log \mathbf{Z}_{n,\rho\beta_n}^{(\mathrm{L},\ell)} > \varepsilon\right) \le \delta_{\mathrm{L}}, \tag{5.11}$$

with $\delta_{\mathrm{L}} \to 0$ *as* $\mathrm{L} \to \infty$.

PROOF. We let $\rho > 0$. Recalling the definition (3.5), and using that $\mathbf{P}(\Delta \subset S) \le e^{\mathrm{Ent}(\Delta)}$, we have that

$$\begin{aligned}\mathbf{Z}_{n,\rho\beta_n}^{(\mathrm{L},\ell)} &\le \sum_{\Delta \subset \Upsilon_\ell} e^{\rho\beta_n \Omega_{n,qh_n}^{(>\mathrm{L})}(\Delta)} \mathbf{P}(S \cap \Upsilon_\ell = \Delta) \\ &\le \sum_{\Delta \subset \Upsilon_\ell} \exp\big(\rho\beta_n \Omega_{n,qh_n}^{(>\mathrm{L})}(\Delta) - \mathrm{Ent}(\Delta)\big) \le 2^\ell \exp\big(T_{n,Ah_n}^{\rho\beta_n,(>\mathrm{L})}\big).\end{aligned}$$

Using that $\ell = o(h_n^2/n)$ and the fact that

$$\mathbb{P}\left(\frac{n}{h_n^2} T_{n,qh_n}^{\beta_n,(>\mathrm{L})} \ge \varepsilon\right) \overset{\mathrm{L}\to\infty}{\longrightarrow} 0$$

(which is relation (5.5) of [7]), we conclude the proof. □

Collecting the above estimates, we can conclude that

$$\lim_{n\to\infty} \frac{n}{h_n^2} \log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega}\big(\mathcal{B}_n(A)\big) = \lim_{\nu\to 1} \lim_{\mathrm{L}\to\infty} \lim_{n\to\infty} \frac{n}{h_n^2} \log \mathbf{Z}_{n,\nu\beta_n}^{(\mathrm{L})}. \tag{5.12}$$

5.3. *Step* 3: *Regime* 2. *Convergence of the main term.* It remains to show the convergence of the partition function restricted to the large weights.

PROPOSITION 5.3. *For any* $\nu > 0$ *and* $\mathrm{L} > 0$,

$$\frac{n}{h_n^2} \log \mathbf{Z}_{n,\nu\beta_n}^{(\mathrm{L})} \overset{(\mathrm{d})}{\to} \begin{cases} \mathcal{T}_{\nu,A}^{(\mathrm{L})} & \text{in regime } 2, \\ \widetilde{\mathcal{T}}_{\beta,\nu,A}^{(\mathrm{L})} & \text{in regime } 3\text{-}a, \end{cases} \tag{5.13}$$

where $\mathcal{T}_{\beta,A}^{(\mathrm{L})}$ *was introduced in* (3.8) *and* $\widetilde{\mathcal{T}}_{\beta,\nu,A}^{(\mathrm{L})}$ *is defined in* (5.18) *below.*

One readily verifies that:

$*\ \nu \mapsto \mathcal{T}^{(\mathrm{L})}_{\nu,A}$ (resp. $\nu \mapsto \widetilde{\mathcal{T}}^{(\mathrm{L})}_{\beta,\nu,A}$) is a continuous function;

$*\ \mathcal{T}^{(\mathrm{L})}_{1,A} \to \mathcal{T}_{1,A}$ (resp., $\widetilde{\mathcal{T}}^{(\mathrm{L})}_{\beta,1,A} \to \widetilde{\mathcal{T}}_{\beta,1,A}$) as $\mathrm{L} \to \infty$ (see Proposition 3.1, resp., Proposition 5.4);

$*\ \mathcal{T}_{1,A} \to \mathcal{T}_1$ (resp., $\widetilde{\mathcal{T}}_{\beta,1,A} \to \widetilde{\mathcal{T}}_\beta$) as $A \to \infty$ (see Proposition 3.1, resp., Proposition 5.4).

Therefore, the proof of Theorem 2.5 and Theorem 2.7 is a consequence of relations (5.5), (5.12) and (5.13).

PROOF. We detail the proof for the regime 2. The regime 3-a follows similarly using the results in Section 5.4 below. To keep the notation lighter, we let $\nu = 1$.

Lower bound. For any $\mathrm{L} \in \mathbb{N}$, we consider a set $\Delta_{\mathrm{L}} \subset \Upsilon_{\mathrm{L}}$ which achieves the maximum of $T^{\beta_n,(\mathrm{L})}_{n,Ah_n}$, respectively, of $\widetilde{T}^{\beta_n,(\mathrm{L})}_{n,Ah_n}$ defined below in (5.17) for regime 3-a. We have

$$\mathbf{Z}^{(\mathrm{L})}_{n,\beta_n} \ge \exp\bigl(\beta_n \Omega_{n,Ah_n}(\Delta_{\mathrm{L}})\bigr)\mathbf{P}(S \cap \Upsilon_{\mathrm{L}} = \Delta_{\mathrm{L}}).$$

Since L is fixed, we realize that any pair of points $(i,x),(j,y) \in \Upsilon_{\mathrm{L}}$ satisfies the condition $|i-j| \ge \varepsilon n$ and $|x-y| \ge \varepsilon h_n$ with probability at least $1-c_\varepsilon$ with $c_\varepsilon \to 0$ as $\varepsilon \to 0$. In such a way, we can use the Stone local limit theorem [23] to get that $\mathbf{P}(S \cap \Upsilon_{\mathrm{L}} = \Delta_{\mathrm{L}}) = n^{-\frac{|\Delta_{\mathrm{L}}|}{2}+o(1)} e^{-\mathrm{Ent}(\Delta_{\mathrm{L}})}$. In the regime 2, in which $\mathrm{Ent}(\Delta_{\mathrm{L}}) \asymp h_n^2/n \gg \log n$, this implies that

$$\mathbf{Z}^{(\mathrm{L})}_{n,\beta_n} \ge \exp\bigl((1+o(1)) T^{\beta_n,(\mathrm{L})}_{n,Ah_n}\bigr). \tag{5.14}$$

To conclude, we use Proposition 3.1, (3.8) to obtain that $T^{\beta_n,(\mathrm{L})}_{n,Ah_n}$ converges in distribution to $\mathcal{T}^{(\mathrm{L})}_{1,A}$, concluding the lower bound.

In regime 3-a, (5.14) is replaced by

$$\mathbf{Z}^{(\mathrm{L})}_{n,\beta_n} \ge \exp\biggl(\bigl(1+o(1)\bigr)\biggl\{\beta_n \Omega_{n,Ah_n}(\Delta_{\mathrm{L}}) - \mathrm{Ent}(\Delta_{\mathrm{L}}) - \frac{|\Delta_{\mathrm{L}}|}{2}\log n\biggr\}\biggr), \tag{5.15}$$

so that $T^{\beta_n,(\mathrm{L})}_{n,Ah_n}$ is replaced by $\widetilde{T}^{\beta_n,(\mathrm{L})}_{n,Ah_n}$ defined in (5.17). Then the conclusion follows by Proposition 5.4, (5.19) below.

Upper bound. We have

$$\mathbf{Z}^{(\mathrm{L})}_{n,\beta_n} = \sum_{\Delta \subset \Upsilon_{\mathrm{L}}} e^{\beta_n \Omega^{(\mathrm{L})}_{n,qh_n}(\Delta)} \mathbf{P}(S \cap \Upsilon_{\mathrm{L}} = \Delta).$$

Using the Stone local limit theorem [23], we have that $\mathbf{P}(S \cap \Upsilon_{\mathrm{L}} = \Delta) = n^{-\frac{|\Delta|}{2}+o(1)} e^{-\mathrm{Ent}(\Delta)}$ uniformly for all $\Delta \subset \Upsilon_{\mathrm{L}}$. Since we have only a finite number of sets, we obtain that

$$\mathbf{Z}^{(\mathrm{L})}_{n,\beta_n} \le 2^{\mathrm{L}} \exp\bigl((1+o(1)) T^{\beta_n,(\mathrm{L})}_{n,Ah_n}\bigr), \tag{5.16}$$

which concludes the proof of the upper bound, again thanks to the convergence proven in Proposition 3.1, (3.8). In regime 3-a, using the Stone local limit theorem, we can safely replace $T_{n,Ah_n}^{\beta_n,(\mathrm{L})}$ by $\widetilde{T}_{n,Ah_n}^{\beta_n,(\mathrm{L})}$ defined below in (5.17), and also conclude thanks to Proposition 5.4, (5.19). $\square$

5.4. *Step* 3: *Regime* 3-*a*. *Complements for the convergence of the main term.* We complete here the proof of Theorem 2.7 by stating the results needed to complete Step 3 above in the case of regime 3-a. In analogy with (3.3), and in view of the local limit theorem (2.14), we define

$$\begin{aligned} \widetilde{T}_{n,h}^{\beta_{n,h}} &:= \max_{\Delta\subset\Lambda_{n,h}}\left\{\beta_{n,h}\Omega_{n,h}(\Delta)-\mathrm{Ent}(\Delta)-\frac{|\Delta|}{2}\log n\right\},\\ \widetilde{T}_{n,h}^{\beta_{n,h},(\ell)} &:= \max_{\Delta\subset\Lambda_{n,h}}\left\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta)-\mathrm{Ent}(\Delta)-\frac{|\Delta|}{2}\log n\right\}. \end{aligned} \tag{5.17}$$

In the next result, we state the convergence of $\frac{n}{h^2}\widetilde{T}_{n,h}^{\beta_{n,h}}$ and $\frac{n}{h^2}\widetilde{T}_{n,h}^{\beta_{n,h},(\ell)}$, analogously to Proposition 3.1.

PROPOSITION 5.4. *Suppose that* $\frac{n}{h_n^2}\beta_{n,h_n}m(nh_n)\to\nu\in(0,\infty)$ *as* $n,h_n\to\infty$ *and* $h_n\sim\beta^{1/2}\sqrt{\log n}$, *with* $\beta>0$. *Then, for every* $\alpha\in(1/2,2)$ *and for any* $q>0$, $\ell\in\mathbb{N}$ *we have the following convergence in distribution, as* $n\to\infty$:

$$\frac{n}{h_n^2}\widetilde{T}_{n,qh_n}^{\beta_{n,h_n}}\xrightarrow{(\mathrm{d})}\widetilde{\mathcal{T}}_{\beta,\nu,q}:=\sup_{s\in\mathscr{M}_q}\left\{\nu\pi(s)-\mathrm{Ent}(s)-\frac{N(s)}{2\beta}\right\}, \tag{5.18}$$

with $\mathscr{M}_q$ *as defined in Proposition* 3.1. *We also have, as* $n\to\infty$,

$$\frac{n}{h_n^2}\widetilde{T}_{n,qh_n}^{\beta_{n,h_n},(\ell)}\xrightarrow{(\mathrm{d})}\widetilde{\mathcal{T}}_{\beta,\nu,q}^{(\ell)}:=\sup_{s\in\mathscr{M}_q}\left\{\nu\pi^{(\ell)}(s)-\mathrm{Ent}(s)-\frac{N(s)}{2\beta}\right\}. \tag{5.19}$$

Moreover, we a.s. have $\widetilde{\mathcal{T}}_{\beta,\nu,q}^{(\ell)}\to\widetilde{\mathcal{T}}_{\beta,\nu,q}$ *as* $\ell\to\infty$, *and* $\widetilde{\mathcal{T}}_{\beta,\nu,q}\to\widetilde{\mathcal{T}}_{\beta,\nu}$ *as* $q\to\infty$.

The proof is identical to the proof of Proposition 3.1 (cf. proof of [7], Theorem 2.7, using also that $\frac{n}{h_n^2}\log n\to\frac{1}{\beta}$ in regime 3), for this reason it is omitted. To conclude, we show that $\widetilde{\mathcal{T}}_\beta^{[\geq r]}$ defined in (2.11) is well defined.

PROPOSITION 5.5. *For any* $r\geq 0$, *the quantities* $\widetilde{\mathcal{T}}_\beta^{[\geq r]}$ *are well defined, and for any* $\beta>0$,

$$-\frac{1}{2\beta}<\widetilde{\mathcal{T}}_\beta^{[\geq 1]}\leq\widetilde{\mathcal{T}}_\beta<\infty. \tag{5.20}$$

Moreover, $\widetilde{\mathcal{T}}_\beta\geq 0$, *and we have* $\widetilde{\mathcal{T}}_\beta>0$ *if and only if* $\widetilde{\mathcal{T}}_\beta^{[\geq 1]}>0$. *Finally, we define the critical value* $\beta_c=\inf\{\beta\colon\widetilde{\mathcal{T}}_\beta>0\}\in(0,\infty)$. *We also have that* $\beta\mapsto\widetilde{\mathcal{T}}_\beta^{[\geq 1]}$ *is continuous and* (*strictly*) *decreasing, so that* $\widetilde{\mathcal{T}}_\beta^{[\geq 1]}<0$ *if and only if* $\beta<\beta_c$.

PROOF. Since $\widetilde{\mathcal{T}}_\beta^{[0]} = 0$, we obtain that $\widetilde{\mathcal{T}}_\beta \in [0,\infty)$. As a by-product, we also have that $\widetilde{\mathcal{T}}_\beta > 0$ if and only if $\widetilde{\mathcal{T}}_\beta^{[\geq 1]} > 0$; and in that case $\widetilde{\mathcal{T}}_\beta = \widetilde{\mathcal{T}}_\beta^{[\geq 1]}$. Additionally, we have

$$\beta W_\beta - \frac{1}{2} \leq \beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \leq \beta \widetilde{\mathcal{T}}_\beta \leq \left(\beta \mathcal{T}_1 - \frac{1}{2}\right),$$

with W_β and $\mathcal{T}_1$ defined in (2.13) and (2.6), respectively. Proposition 6.4 and Theorem 2.1 ensure that for $\beta > 0$, $W_\beta \in (0,\infty)$ and $\mathcal{T}_1 < \infty$, showing (5.20).

Let us now show that $\beta_c \in (0,\infty)$. First, we have that $\beta \mapsto \beta W_\beta$ is nondecreasing and goes to $+\infty$ as $\beta \uparrow +\infty$: this implies that $\beta_c \leq \inf\{\beta : \beta W_\beta > 1/2\} < +\infty$. Second, $\beta \mapsto \beta \mathcal{T}_1$ is nondecreasing and goes to 0 as $\beta \downarrow 0$: this shows that $\beta_c \geq \sup\{\beta : \beta \mathcal{T}_1 < 1/2\} > 0$.

It remains to prove that $\beta \mapsto \widetilde{\mathcal{T}}_\beta^{[\geq 1]}$ is continuous and (strictly) decreasing. Let us note that $\beta \widetilde{\mathcal{T}}_\beta^{[\geq 1]} \geq \beta W_\beta - \frac{1}{2} \geq -\frac{1}{2}$, and also $\beta \widetilde{\mathcal{T}}_\beta^{[\geq m]} \leq \beta \mathcal{T}_1 - \frac{1}{2} m$, for any $m \geq 1$. Hence, for every fixed $\widetilde{\beta}$, there exists $\widetilde{m} = \lceil 2\widetilde{\beta} \mathcal{T}_1 \rceil + 2$ such that for all $\beta \in (0, \widetilde{\beta}]$ we have that $\beta \widetilde{\mathcal{T}}_\beta^{[\geq m]} < -\frac{1}{2}$ for all $m \geq \widetilde{m}$, so that

$$\widetilde{\mathcal{T}}_\beta^{[\geq 1]} = \widetilde{\mathcal{T}}_\beta^{[\![1,\widetilde{m}]\!]} := \sup_{1 \leq k \leq \widetilde{m}} \widetilde{\mathcal{T}}_\beta^{[k]}.$$

Since each map $\beta \mapsto \widetilde{\mathcal{T}}_\beta^{[k]}$ is continuous and (strictly) decreasing on the interval $(0, \widetilde{\beta}]$, we get that $\beta \mapsto \mathcal{T}_\beta^{[\![1,\widetilde{m}]\!]}$ is also continuous and (strictly) decreasing on $(0, \widetilde{\beta}]$. $\square$

6. Regime 3-b and regime 4. In this section, we prove Theorem 2.8 and Theorem 2.9. We decompose the proof in three steps (analogously to what is done in Section 5), Step 1 and Step 2 being the same for both regimes 3-b and 2. For the third step, we separate regime 3-b and regime 4, which have different behaviors. Note that in both regimes there is a constant $c = c_\beta > 0$ such that $h_n \leq c\sqrt{n \log n}$ (in regime 4, we have $h_n \ll \sqrt{n \log n}$).

Let us define here, analogously to (5.1), the recentered partition function

$$\bar{\mathbf{Z}}_{n,\beta_n}^\omega := \mathbf{E}\left[\exp\Big(\sum_{i=1}^n \beta_n \big(\omega_{i,S_i} - \mathbb{E}[\omega \mathbb{1}_{\omega \leq 1/\beta_n}] \mathbb{1}_{\{\alpha \geq 1\}}\big)\right]. \tag{6.1}$$

Then, roughly speaking, we show that $\log \bar{\mathbf{Z}}_{n,\beta_n}^\omega$ is of order $n^{-1/2} \exp(X h_n^2/n)$, with $X = \widetilde{\mathcal{T}}_\beta^{[\geq 1]} + \frac{1}{2\beta}$ in the regime 3-b (where $h_n^2/n \sim \beta \log n$), and with $X = W_1$ in regime 4. In all cases, we will have $\log \bar{\mathbf{Z}}_{n,\beta_n}^\omega = o(1)$ (recall that in regime 3-b, $\widetilde{\mathcal{T}}_\beta^{[\geq 1]} < 0$).

6.1. *Step* 1. *Reduction of the set of trajectories*. We proceed as for Step 1 in Section 5: for any $A > 0$ (fixed large in a moment), we define

$$\mathcal{A}_n := \Big\{(i, S_i) : \max_{i \le n} |S_i| \le A\sqrt{n \log n}\Big\}. \tag{6.2}$$

Then we let $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$ be the (normalized) partition function restricted to trajectories in $\mathcal{A}_n$. Relation (2.8) gives that, analogously to (5.4),

$$\mathbb{P}\big(\big|\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n)\big| \ge n e^{-c_1 A^2 \log n}\big) \le c_2 A^{-\nu_1}. \tag{6.3}$$

Hence, we fix A large enough so that $e^{-c_0 A^2 \log n} \le n^{-3}$. This shows that with high probability $\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = \log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n) + O(n^{-2})$. In such a way, in the following we can safely focus only on the partition function with trajectories restricted to $\mathcal{A}_n$.

6.2. *Step* 2. *Restriction to large weights*. We now fix $\eta \in (0, 1)$, small. The same Hölder inequalities as in (5.6) hold for $\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$, so that we can write, with similar notation as in (4.8)–(4.10) (the restriction to trajectories in $\mathcal{A}_n$ does not appear in the notation)

(6.4)

$$\log \bar{\mathbf{Z}}^{\omega}_{n,\beta_n}(\mathcal{A}_n) \begin{cases} \le \dfrac{1}{1+\eta} \log \mathbf{Z}^{(>1)}_{n,(1+\eta)\beta_n} + \dfrac{\eta}{1+\eta} \log \bar{\mathbf{Z}}^{(\le 1)}_{n,(1+\eta^{-1})\beta_n}, \\ \ge \dfrac{1}{1-\eta} \log \mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n} - \dfrac{\eta}{1-\eta} \log \bar{\mathbf{Z}}^{(\le 1)}_{n,-(\eta^{-1}-1)\beta_n}. \end{cases}$$

We used also (5.7) to be able to bound below $\bar{\mathbf{Z}}^{(>1)}_{n,(1-\eta)\beta_n}$ by $\mathbf{Z}^{(>1)}_{n,(1-2\eta)\beta_n}$ (using that $\beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega \le 1/\beta_n\}}] \ll 1$ when $\alpha \ge 1$). Then we need to get a more precise statement than Lemma 5.1 to deal with $\bar{\mathbf{Z}}^{(\le 1)}_{n,\rho\beta_n}$.

LEMMA 6.1. *For any* $\rho \in \mathbb{R}$,

$$\left(\frac{h_n^2}{n}\right)^{-3\alpha} \sqrt{n} \log \bar{\mathbf{Z}}^{(\le 1)}_{n,\rho\beta_n} \overset{\mathbb{P}}{\to} 0 \quad \text{as } n \to \infty.$$

PROOF. We will simply control the first moment of $\bar{\mathbf{Z}}^{(\le 1)}_{n,\rho\beta_n} - 1$. The idea is similar to that used to obtain (4.24) and (4.31). We divide the proof into two cases: when $\alpha < 1$ so that there is no renormalization necessary in (6.1), and when $\alpha \in [1, 2)$.

Let us start with the case $\alpha < 1$: using that $|\rho|\beta_n \omega_{i,S_i} \le |\rho|$ on the event $\{\beta_n \omega_{i,S_i} \le 1\}$, we get that there exists a constant c_ρ such that

$$e^{\sum_{i=1}^n \rho\beta_n \omega_{i,S_i} \mathbf{1}_{\{\beta_n \omega_{i,S_i} \le 1\}}} \le \prod_{i=1}^n \big(1 + c_\rho \beta_n \omega_{i,S_i} \mathbf{1}_{\{\beta_n \omega_{i,S_i} \le 1\}}\big). \tag{6.5}$$

By independence, and since $\mathbb{P}(\omega > t)$ is regularly varying, we get that for n sufficiently large

$$\begin{aligned}\mathbb{E}[\beta_n \omega_{i,x} \mathbf{1}_{\{\beta_n \omega_{i,x} \le 1\}}] &\le \int_0^{1/\beta_n} \beta_n \mathbb{P}(\omega > t)\, dt \le cL(1/\beta_n)\beta_n^{\alpha} \\ &\le c\mathbb{P}(\omega > 1/\beta_n) \le \frac{c'}{nh_n}\Big(\frac{h_n^2}{n}\Big)^{2\alpha}.\end{aligned} \tag{6.6}$$

For the last inequality, we used Potter's bound (see [8], Theorem 1.5.6), and the definition of β_n, that is, the fact that $\beta_n \sim \frac{h_n^2}{n} m(nh_n)$. Therefore, in view of (6.5) and using that $h_n \ge \sqrt{n}$, we get that for n sufficiently large (how large depends on ρ),

$$\mathbb{E}[\bar{\mathbf{Z}}_{n,\rho\beta_n}^{(\le 1)} - 1] \le \Big(1 + c'_\rho \frac{(h_n^2/n)^{2\alpha}}{n^{3/2}}\Big)^n - 1 \le 2c'_\rho n^{-1/2}\Big(\frac{h_n^2}{n}\Big)^{2\alpha}. \tag{6.7}$$

This concludes the proof in the case $\alpha < 1$ by using Markov's inequality, since $h_n^2/n \to +\infty$.

In the case $\alpha \in [1, 2)$, we use the expansion $e^x \le 1 + x + c_\rho x^2$ for all $|x| \le 2|\rho|$ to get, analogously to (6.5), and setting $\mu_n := \mathbb{E}[\omega \mathbf{1}_{\{\omega \le 1/\beta_n\}}] \ll 1/\beta_n$,

$$\begin{aligned}\mathbb{E}[\bar{\mathbf{Z}}_{n,\rho\beta_n}^{(\le 1)}] &\le \big(1 + \rho\beta_n \mathbb{E}[(\omega - \mu_n)\mathbf{1}_{\{\omega \le 1/\beta_n\}}] + c_\rho \beta_n^2 \mathbb{E}[(\omega - \mu_n)^2 \mathbf{1}_{\{\omega \le 1/\beta_n\}}]\big)^n \\ &\le \exp\big(cn\mathbb{P}(\omega > 1/\beta_n)\big) \le 1 + cn^{-1/2}\Big(\frac{h_n^2}{n}\Big)^{2\alpha},\end{aligned}$$

obtaining the same upper bound as in (6.7). To obtain the above inequality, we used that

$$\begin{aligned}\mathbb{E}\big[(\omega - \mu_n)\mathbf{1}_{\{\omega \le 1/\beta_n\}}\big] &= \mu_n \mathbb{P}(\omega > 1/\beta_n) \le \beta_n^{-1}\mathbb{P}(\omega > 1/\beta_n), \\ \mathbb{E}\big[(\omega - \mu_n)^2 \mathbf{1}_{\{\omega \le 1/\beta_n\}}\big] &\le \mathbb{E}\big[\omega^2 \mathbf{1}_{\{\omega \le 1/\beta_n\}}\big] \le cL(1/\beta_n)\beta_n^{\alpha-2},\end{aligned}$$

where the last inequality follows similar to (6.6). One concludes that (6.7) also holds when $\alpha \ge 1$, and the lemma follows by Markov's inequality. □

Therefore, in view of (6.4) and Lemma 6.1, we have that for both regimes 3-b and 4:

$$\lim_{n\to\infty} \frac{n}{h_n^2} \log\big(\sqrt{n} \log \bar{\mathbf{Z}}_{n,\beta_n}^{\omega}(\mathcal{A}_n)\big) = \lim_{\nu\to 1} \lim_{n\to\infty} \frac{n}{h_n^2} \log\big(\sqrt{n} \log \mathbf{Z}_{n,\nu\beta_n}^{(>1)}\big). \tag{6.8}$$

Note that in the case of regime 3-b, $h_n^2/n \sim \beta \log n$, so the limit is that of

$$\frac{1}{\beta \log n} \log\big(\log \mathbf{Z}_{n,\nu\beta_n}^{(>1)}\big) + \frac{1}{2\beta}.$$

For simplicity of notation, we will consider only the case $\nu = 1$ in the following.

6.3. *Step* 3. *Reduction of the main term.* In both regimes 3-b and 4, we show that $\log \mathbf{Z}_{n,\beta_n}^{(>1)}$ goes to 0, and we identify at which rate: to do so, it is equivalent to identify the rate at which $\mathbf{Z}_{n,\beta_n}^{(>1)} - 1$ goes to 0. The behavior for regimes 3-b and 4 are different, since the main contribution to $\mathbf{Z}_{n,\beta_n}^{(>1)} - 1$ may come from several large weights in regime 3-b, whereas it comes from a single large weight in regime 4, as it will be reflected in the proof.

Let us define $\ell = \ell(\omega)$ the number of $(i,x) \in \Lambda_{n,A_n} = [\![1,n]\!] \times [\![-A_n, A_n]\!]$ (with the notation $A_n = A\sqrt{n \log n}$ for simplicity) such that $\beta_n \omega_{i,x} \geq 1$, and let us denote

$$\{(i,x) \in \Lambda_{n,A_n}; \beta_n \omega_{i,x} \geq 1\} = \Upsilon_\ell := \{Y_1^{(n,A_n)}, \dots, Y_\ell^{(n,A_n)}\}, \tag{6.9}$$

with $Y_i^{(n,A_n)}$ the order statistics, as in Section 3. Note that

$$\mathbb{E}[\ell] = \sum_{(i,x)\in\Lambda_{n,A_n}} \mathbb{P}(\beta_n \omega_{i,x} \geq 1) \leq 2An^{3/2}\sqrt{\log n}\left(\frac{h_n^2}{n}\right)^{2\alpha} \frac{1}{nh_n}, \tag{6.10}$$

where we used that $\mathbb{P}(\omega \geq 1/\beta_n) \leq (h_n^2/n)^{2\alpha}(nh_n)^{-1}$ for n large enough, thanks to (2.2) and Potter's bound. Since $h_n^2/n \leq c \log n$, $h_n \gg \sqrt{n}$, (6.10) implies that $\ell \leq (\log n)^{3\alpha}$ with probability going to 1 (recall $\frac{1}{2} + 2\alpha < 3\alpha$).

Decomposing $\mathbf{Z}_{n,\beta_n}^{(>1)}$ according to the number of sites in Υ_ℓ visited, we can write for any fixed $k_0 > 0$,

$$\begin{aligned} &\sum_{k=1}^{k_0} \mathbf{U}_k \leq \mathbf{Z}_{n,\beta_n}^{(>1)} - 1 = \sum_{k=1}^{\ell} \mathbf{U}_k \quad \text{with} \\ &\mathbf{U}_k := \sum_{\Delta \subset \Upsilon_\ell, |\Delta| = k} e^{\beta_n \Omega_{n,A_n}(\Delta)} \mathbf{P}(S \cap \Upsilon_\ell = \Delta). \end{aligned} \tag{6.11}$$

In regime 3-b, the main contribution comes from one of the $\mathbf{U}_k$'s for some $k \geq 1$, whereas in regime 4 only the term $\mathbf{U}_1$ will contribute.

Let us now show that, with high probability, we can replace the upper bound in (6.11) by considering only a finite number of terms. For this purpose, notice that $\ell \leq (\log n)^{3\alpha}$ and $\min\{|i-j|, (i,x) \neq (j,y) \in \Upsilon_\ell\} \geq n/(\log n)^{10\alpha}$ with probability going to 1. Then we can use the Stone local limit theorem [23] to have that for any $\Delta \subset \Upsilon_\ell$

$$\mathbf{P}(S \cap \Upsilon_\ell = \Delta) \leq cn^{-(\frac{1}{2}-\eta)|\Delta|} e^{-\operatorname{Ent}(\Delta)},$$

where $\eta > 0$ is independent of Δ and can be chosen arbitrary small (by changing the value of the constant c).

As a consequence, using that $\binom{\ell}{k} \le \ell^k$ and $\ell \le (\log n)^{3\alpha}$, we have for any $1 \le k_1 \le \ell$,

$$\begin{aligned}\sum_{k=k_1}^{\ell} \mathbf{U}_k &= \sum_{k=k_1}^{\ell} \sum_{\Delta \subset \Upsilon_\ell, |\Delta|=k} e^{\beta_n \Omega_{n,A_n}(\Delta)} \mathbf{P}(S \cap \Upsilon_\ell = \Delta) \\ &\le e^{T_{n,A_n}^{\beta_n}} \sum_{k=k_1}^{\ell} \ell^k n^{-k(\frac{1}{2}-\eta)} \le c e^{T_{n,A_n}^{\beta_n}} n^{-k_1(\frac{1}{2}-\eta')}.\end{aligned} \tag{6.12}$$

Recalling Proposition 3.1 (and the fact that $h_n^2/n \le c \log n$), we have that $T_{n,A_n}^{\beta_n} \le C \log n$ with probability going to 1 as $C \to \infty$. Therefore, we obtain that (6.12) is $O(n^{-2})$ with probability close to 1, provided that k_1 is sufficiently large—this will turn out to be negligible; see Lemma 6.2. Hence, we have shown that with probability close to 1, we can keep a finite number of terms in (6.11).

This can actually be improved in regime 4, where we can keep only one term: indeed, since in that case $h_n^2/n = o(\log n)$, we get that for any fixed $\gamma > 0$, $T_{n,A_n}^{\beta_n} \le \gamma \log n$ with probability going to one. Hence, we get that in regime 4, we can take $k_1 = 2$ in (6.12) and obtain that $\sum_{k=2}^{\ell} \mathbf{U}_k = O(n^{-3/4})$ with probability close to 1, which will turn out to be negligible; see Lemma 6.3.

It remains to show the following lemmas, proving the convergence of the main term in regimes 3-b and 4.

LEMMA 6.2. *In regime* 3 (R3) (*recall* $h_n^2/n \sim \beta \log n$), *for any* $K > 0$ *we have that*

$$\frac{n}{h_n^2} \log\left(\sum_{k=1}^{K} \mathbf{U}_k\right) \xrightarrow{(\mathrm{d})} \sup_{1 \le k \le K} \widetilde{\mathcal{T}}_{\beta,A}^{[k]}, \tag{6.13}$$

where $\widetilde{\mathcal{T}}_{\beta,A}^{[k]} := \sup_{s \in \mathscr{M}_A, N(s)=k} \{\pi(s) - \mathrm{Ent}(s) - \frac{k}{2\beta}\}$, *with* $\mathscr{M}_A$ *defined below* (5.18).

Note that we have $\sup_{k \ge 1} \widetilde{\mathcal{T}}_{\beta,A}^{[k]} < 0$ in regime 3-b: this lemma proves that $\sum_{k=1}^{K} \mathbf{U}_k$ goes to 0 in probability, and hence $\mathbf{Z}_{n,\beta_n}^{(>1)} - 1$ also goes to 0 in probability. This is needed to replace the study of $\log \mathbf{Z}_{n,\beta_n}^{(>1)}$ by that of $\mathbf{Z}_{n,\beta_n}^{(>1)} - 1$, and it is actually the only place where the definition of regime 3-b is used.

LEMMA 6.3. *In regime* 4 (R4), *we have that*

$$\frac{n}{h_n^2} \log(\sqrt{n}\mathbf{U}_1) \xrightarrow{(\mathrm{d})} W_1, \tag{6.14}$$

with W_1 *defined in* (2.13).

Here also, this proves that $\mathbf{U}_1 \to 0$ in probability, and hence so does $\mathbf{Z}_{n,\beta_n}^{(>1)} - 1$.

6.4. *Regime* 3-*b*: *Convergence of the main term.* In this section, we prove Lemma 6.2.

Reduction to finitely many weights. First of all, we fix some $\mathtt{L}$ large and show that the main contribution comes from the $\mathtt{L}$ largest weights. We define

$$\mathbf{U}_k^{(\mathtt{L})} := \sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k} e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}(S\cap\Upsilon_\ell=\Delta), \tag{6.15}$$

where $\Upsilon_{\mathtt{L}}=\{Y_1^{n,A_n},\dots,Y_{\mathtt{L}}^{n,A_n}\}$ is the set of $\mathtt{L}$ largest weights in Λ_{n,A_n} (note that $\Upsilon_{\mathtt{L}}\subset\Upsilon_\ell$ for n large enough). Then we have that $\mathbf{U}_k\ge\mathbf{U}_k^{(\mathtt{L})}$, and $\sum_{k=1}^K\mathbf{U}_k$ is bounded by

$$\begin{aligned}
&\sum_{k=1}^K\sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k}\ \sum_{\Delta'\subset\Upsilon_\ell\setminus\Upsilon_{\mathtt{L}},|\Delta'|\le K} e^{\beta_n\Omega_{n,A_n}(\Delta)+\beta_n\Omega_{n,A_n}(\Delta')}\mathbf{P}\big(S\cap\Upsilon_\ell=\Delta\cup\Delta'\big)\\
&\quad\le\sum_{k=1}^K\sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k} e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}(S\cap\Upsilon_{\mathtt{L}}=\Delta)\times\exp\big(K\beta_n M_{\mathtt{L}}^{(n,A_n)}\big)\\
&\quad=\exp\big(K\beta_n M_{\mathtt{L}}^{(n,A_n)}\big)\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}.
\end{aligned}$$

In the second inequality, we simply bounded $\Omega_{n,A_n}(\Delta')$ by $KM_{\mathtt{L}}^{(n,A_n)}$ uniformly for $\Delta'\subset\Upsilon_\ell\setminus\Upsilon_{\mathtt{L}}$, with $|\Delta'|\le K$. Then, since $\beta_n\sim c_\beta(\log n)/m(nh_n)\sim c_{\beta,A}(\log n)/m(nA_n)$ as $n\to\infty$, we get that $K\beta_n M_{\mathtt{L}}^{(n,A_n)}$ is bounded above by $2c_{\beta,A}KM_{\mathtt{L}}^{(n,A_n)}/m(nA_n)\times\log n$. For any fixed $\varepsilon>0$, we can fix $\mathtt{L}$ large enough so that for large n we have $M_{\mathtt{L}}^{(n,A_n)}/m(nA_n)\le\varepsilon/(2Kc_{\beta,A})$ with probability larger than $1-\varepsilon$. We conclude that there exists some $\varepsilon_{\mathtt{L}}$ with $\varepsilon_{\mathtt{L}}\to0$ as $\mathtt{L}\to\infty$ such that

$$0\le\sum_{k=1}^K\big(\mathbf{U}_k-\mathbf{U}_k^{(\mathtt{L})}\big)\le n^{\varepsilon_{\mathtt{L}}}\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}.$$

Since $h_n^2/n\sim\beta\log n$, this proves that

$$\lim_{n\to\infty}\frac{n}{h_n^2}\log\Bigg(\sum_{k=1}^K\mathbf{U}_k\Bigg)=\lim_{\mathtt{L}\to\infty}\lim_{n\to\infty}\frac{n}{h_n^2}\log\Bigg(\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}\Bigg). \tag{6.16}$$

Convergence of the remaining term. We finally prove that

$$\frac{n}{h_n^2}\log\Bigg(\sum_{k=1}^K\mathbf{U}_k^{(\mathtt{L})}\Bigg)\xrightarrow{(\mathrm{d})}\max_{1\le k\le K}\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}, \tag{6.17}$$

where $\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}$ is the restriction of $\widetilde{\mathcal{T}}_{\beta,A}^{[k]}$ to the $\mathtt{L}$ largest weights in $[0,1]\times[-A,A]$, that is,

$$\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})} := \sup_{s\in\mathscr{M}_A, N(s)=k}\left\{\pi^{(\mathtt{L})}(s) - \mathrm{Ent}(s) - \frac{k}{2\beta}\right\}.$$

In analogy with Proposition 5.4, one shows that $\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})} \to \widetilde{\mathcal{T}}_{\beta,A}^{[k]}$ as $\mathtt{L}\to\infty$, which completes the proof.

The proof of (6.17) comes from the rewriting

$$\begin{aligned}\sum_{k=1}^{K}\mathbf{U}_k^{(\mathtt{L})} &= \sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\le K} e^{\beta_n\Omega_{n,A_n}(\Delta)}\mathbf{P}(S\cap\Upsilon_{\mathtt{L}}=\Delta)\\ &= \sum_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\le K}\exp\left(\beta_n\Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n + o(K)\right),\end{aligned}$$

where for the last inequality we used Stone local limit theorem [23] (using that any two points in $\Upsilon_{\mathtt{L}}$ have abscissa differing by at least εn with probability going to 1 as $\varepsilon\to 0$) to get that $\mathbf{P}(S\cap\Upsilon_{\mathtt{L}}=\Delta) = n^{-\frac{|\Delta|}{2}+o(1)}e^{-\mathrm{Ent}(\Delta)}$ uniformly for $\Delta\subset\Upsilon_{\mathtt{L}}$. Since there are finitely many terms in the sum, we get that analogously to (5.14)–(5.16)

$$\sum_{k=1}^{K}\mathbf{U}_k^{(\mathtt{L})} = e^{o(\log n)}\times\exp\left(\max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\le K}\left\{\beta_n\Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n\right\}\right).$$

At this stage, we write

$$\max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|\le K}\left\{\beta_n\Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{|\Delta|}{2}\log n\right\} = \max_{1\le k\le K}\widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} \quad \text{where}$$

$$\widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} := \max_{\Delta\subset\Upsilon_{\mathtt{L}},|\Delta|=k}\left\{\beta_n\Omega_{n,A_n}(\Delta) - \mathrm{Ent}(\Delta) - \frac{k}{2}\log n\right\}.$$

To complete the proof of (6.17), we only have to show that

$$\frac{n}{h_n^2}\log\left(\sum_{k=1}^{K}\mathbf{U}_k^{(\mathtt{L})}\right) = o(1) + \frac{n}{h_n^2}\max_{1\le k\le K}\widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} \xrightarrow{(\mathrm{d})} \max_{1\le k\le K}\widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}. \tag{6.18}$$

In analogy with (5.17) and Proposition 5.4, we have that for any fixed k,

$$\frac{n}{h_n^2}\widetilde{T}_{n,h}^{\beta_{n,h},(k,\mathtt{L})} \xrightarrow{(\mathrm{d})} \widetilde{\mathcal{T}}_{\beta,A}^{(k,\mathtt{L})}.$$

For the convergence of (3.8), since we have only a finite number of points, the proof is a consequence of (5.1) and (5.2) of [7] and the Skorokhod representation theorem—we use also that $\frac{n}{h_n^2}\log n\to\frac{1}{\beta}$. Since the maximum is taken over a finite number of terms, this shows (6.18) and concludes the proof.

6.5. *Regime* 4: *Convergence of the main term.* First of all, we show briefly that W_β is well defined, before we turn to the proof of Lemma 6.3. One of the difficulties here is that the reduction to trajectories operated in Section 6.1 (to trajectories with $\max_{i\le n}|S_i| \le A\sqrt{n\log n}$) is not adapted, since the transversal fluctuations are of order $h_n \ll \sqrt{n\log n}$. Therefore, we have to further reduce the set of trajectories in $\mathbf{U}_1$.

Well-posedness and properties of W_β. We prove the following.

PROPOSITION 6.4. *Assume that* $\alpha \in (1/2, 2)$. *Then for every* $\beta > 0$, $W_\beta \in (0,\infty)$ *almost surely.*

PROOF. Recalling the definition (2.13) of W_β. We fix a region $\mathcal{D}_\varepsilon := [\frac{1}{2}, 1] \times [-\varepsilon, \varepsilon]$, for $\varepsilon > 0$. In such a way, we have that

$$W_\beta \ge \sup_{(w,t.x)\in\mathcal{P};(t,x)\in\mathcal{D}_\varepsilon} \{w\} - \frac{\varepsilon^2}{\beta}. \tag{6.19}$$

We observe that

$$\max_{(w,t,x)\in\mathcal{P};(t,x)\in\mathcal{D}_\varepsilon} \{w\} \overset{(\mathrm{d})}{=} (2\varepsilon)^{1/\alpha}\,\mathrm{Exp}(1)^{-1/\alpha}.$$

Therefore, since $\frac{1}{\alpha} < 2$, the right-hand side of (6.19) is a.s. positive provided ε is sufficiently small.

For an upper bound, we simply observe that $W_\beta \le \mathcal{T}_\beta < \infty$ a.s. $\square$

Proof of Lemma 6.3. We denote $p(i,x) := \mathbf{P}(S_i = x)$ for the random walk kernel. For $A > 0$ fixed and $\delta > 0$, we split $\sqrt{n}\mathbf{U}_1$ into three parts:

$$\begin{aligned} \sqrt{n}\mathbf{U}_1 &:= \sum_{(i,x)\in\Upsilon_\ell} e^{\beta_n\omega_{i,x}}\sqrt{n}\,p(i,x) \\ &= \Bigg(\sum_{\substack{(i,x)\in\Upsilon_\ell\\ |x|>Ah_n}} + \sum_{\substack{(i,x)\in\Upsilon_\ell\\ i<\delta n,|x|\le Ah_n}} + \sum_{\substack{(i,x)\in\Upsilon_\ell\\ i\ge\delta n,|x|\le Ah_n}}\Bigg) e^{\beta_n\omega_{i,x}}\sqrt{n}\,p(i,x). \end{aligned} \tag{6.20}$$

The main term is the last one, and we now give three lemmas to control the three terms.

LEMMA 6.5. *There exist constants* c *and* $\nu > 0$ *such that for all* n *sufficiently large*, *for any* $A > 1$,

$$\mathbb{P}\Bigg(\sum_{(i,x)\in\Upsilon_\ell,|x|>Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}\,p(i,x) > A\bigg(\frac{h_n^2}{n}\bigg)^{3\alpha}\Bigg) \le cA^{-\nu}. \tag{6.21}$$

LEMMA 6.6. *There exist some $c, \nu > 0$ such that, for any $A > 1$ and $0 < \delta < A^{-1}$, we get that for n sufficiently large,*

$$\mathbb{P}\left(\frac{n}{h_n^2}\log\left(\sum_{(i,x)\in\Upsilon_\ell, i<\delta n, |x|\le Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\right) \ge (\delta A)^{\frac{1}{4\alpha}}\right) \le c(\delta A)^{1/2}. \tag{6.22}$$

And finally, for last term, we have the convergence.

LEMMA 6.7. *We have that*

$$\frac{n}{h_n^2}\log\left(\sum_{(i,x)\in\Upsilon_\ell, i\ge\delta n, |x|\le Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x)\right) \xrightarrow{(\mathrm{d})} W_1(\delta, A),$$

with $W_1(\delta, A) := \max_{(w,t,x)\in\mathcal{P}, t>\delta, |x|\le A}\{w - \frac{x^2}{2t}\}$.

Now, let us observe that taking the limit $\delta \downarrow 0$, and $A \uparrow \infty$, we readily obtain that $W_1(\delta, A) \to W_1$ (by monotonicity). Therefore, combining Lemmas 6.5–6.6–6.7, we conclude the proof of Lemma 6.3. □

PROOF OF LEMMA 6.5. Let us consider the event

$$\mathcal{G}(n, A) := \left\{\beta_n\omega_{i,x} \le \frac{x^2}{8i} \text{ for any } |x| > Ah_n, 1 \le i \le n\right\}. \tag{6.23}$$

Using this event to split the probability (and Markov's inequality), we have that, recalling the definition (6.9) of Υ_ℓ,

$$\begin{aligned}&\mathbb{P}\left(\sum_{(i,x)\in\Upsilon_\ell, |x|>Ah_n} e^{\beta_n\omega_{i,x}}\sqrt{n}p(i,x) > A\left(\frac{h_n^2}{n}\right)^{3\alpha}\right)\\ &\quad\le \frac{1}{A}\left(\frac{h_n^2}{n}\right)^{-3\alpha}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{|x|>Ah_n} e^{x^2/8i}\sqrt{n}p(i,x)\mathbb{1}_{\{\beta_n\omega_{i,x}\ge 1\}}\right]\\ &\qquad+\mathbb{P}\big(\mathcal{G}(n,A)^c\big).\end{aligned} \tag{6.24}$$

Using again that $\mathbb{P}(\omega \ge 1/\beta_n) \le (h_n^2/n)^{2\alpha}(nh_n)^{-1}$ and that $p(i,x) \le e^{-x^2/4i}$ uniformly in the range considered (provided that n is large enough), we get that the first term is bounded by

$$\frac{1}{A}\left(\frac{h_n^2}{n}\right)^{-\alpha}\frac{\sqrt{n}}{nh_n}\sum_{i=1}^{n}\sum_{|x|>Ah_n} e^{-x^2/8i} \le \left(\frac{h_n^2}{n}\right)^{-\alpha}.$$

In the last inequality, we used that the sum over x is bounded by a constant independent of i, and also that $\sqrt{n}/h_n \to 0$. The first term in (6.24) therefore goes to 0 as $n \to \infty$, and we are left to control $\mathbb{P}(\mathcal{G}(n, A)^c)$. A union bound gives

$$\begin{aligned}\mathbb{P}\big(\mathcal{G}(n, A)^c\big) &\le \sum_{i=1}^{n} \sum_{x=Ah_n}^{+\infty} \mathbb{P}\Big(\beta_n \omega_{i,x} \ge \frac{x^2}{8i}\Big) \\ &\le n \sum_{k=0}^{+\infty} \sum_{x=2^k Ah_n}^{2^{k+1}Ah_n} \mathbb{P}\Big(\beta_n \omega \ge 2^{2k} A^2 \frac{h_n^2}{8n}\Big) \\ &\le 2Anh_n \sum_{k=0}^{\infty} 2^k \mathbb{P}\Big(\omega \ge \frac{1}{10} 2^{2k} A^2 m(nh_n)\Big),\end{aligned}$$

where we used the definition (2.2) of h_n for the last inequality, with n large enough. Then, using the definition of $m(nh_n)$ and Potter's bound, we obtain that for any $\eta > 0$ (chosen such that $1 - 2\alpha + 2\eta < 0$) there is a constant $c > 0$ such that for n large enough

$$\mathbb{P}\big(\mathcal{G}(n, A)^c\big) \le cAnh_n \sum_{k \ge 1} 2^k \big(2^{2k} A^2\big)^{-\alpha+\eta} \frac{1}{nh_n} \le c' A^{1-2\alpha+2\eta},$$

where the sum over k is finite because $1 - 2\alpha + 2\eta < 0$. This concludes the proof of Lemma 6.5. □

PROOF OF LEMMA 6.6. Decomposing over the event

$$\mathcal{M}_n(\delta, A) = \Big\{ \max_{i<\delta n, |x| \le Ah_n} \beta_n \omega_{i,x} \le \frac{1}{2} (\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n} \Big\},$$

and using Markov's inequality, we get that (similar to (6.24))

$$\begin{aligned}&\mathbb{P}\Big(\sum_{(i,x) \in \Upsilon_\ell, i<\delta n, |x| \le Ah_n} e^{\beta_n \omega_{i,x}} \sqrt{n}\, p(i,x) \ge \exp\Big((\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n}\Big)\Big) \\ &\quad \le e^{-\frac{1}{2}(\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n}} \mathbb{E}\Big[\sum_{i=1}^{\delta n} \sum_{|x| \le Ah_n} \sqrt{n}\, p(i,x) \mathbb{1}_{\{\beta_n \omega_{i,x} \ge 1\}} \Big] + \mathbb{P}\big(\mathcal{M}_n(\delta, A)^c\big).\end{aligned}$$

We use again that $\mathbb{P}(\omega \ge 1/\beta_n) \le (h_n^2/n)^{2\alpha} (nh_n)^{-1}$, and the fact that $\sum_x p(i,x) = 1$ for any $i \in \mathbb{N}$, to get that the first term is bounded by

$$e^{-\frac{1}{2}(\delta A)^{\frac{1}{4\alpha}} \frac{h_n^2}{n}} \Big(\frac{h_n^2}{n}\Big)^{2\alpha} \frac{n\sqrt{n}}{nh_n} \to 0 \quad \text{as } n \to \infty.$$

For the remaining term, using that $\beta_n^{-1} h_n^2/n \sim m(nh_n)$, we have by a union bound that for n large enough

$$\mathbb{P}\big(\mathcal{M}_n(\delta, A)^c\big) \le \delta A n h_n \mathbb{P}\Big(\omega > \frac{1}{4}(\delta A)^{\frac{1}{4\alpha}} m(nh_n)\Big)$$
$$\le c\delta A n h_n \times \big((\delta A)^{\frac{1}{4\alpha}}\big)^{-2\alpha} \frac{1}{nh_n},$$

where we used Potter's bound (with $(\delta A)^{\frac{1}{4\alpha}}$ small) and the definition of $m(nh_n)$ for the last inequality (for n large). This concludes the proof of Lemma 6.6. $\square$

PROOF OF LEMMA 6.7. The Stone local limit theorem [23] (see (2.14)) gives that, for fixed $A > 0$, $\delta > 0$, there exists $c > 0$ such that uniformly for $\delta n \le i \le n$, $|x| \le A h_n$,

$$\frac{1}{c} e^{-x^2/2i} \le \sqrt{i}\, p(i, x) \le c e^{-x^2/2i}. \tag{6.25}$$

Since $\sqrt{n/i} \ge 1$ for all $i \le n$, we get the lower bound

$$\sum_{i=\delta n}^{n} \sum_{|x| \le A h_n} e^{\beta_n \omega_{i,x}} \sqrt{n}\, p(i, x) \mathbb{1}_{\{\beta_n \omega_{i,x} \ge 1\}} \ge c \exp\big(\beta_n W_n(\delta, A)\big), \tag{6.26}$$

where $W_n(\delta, A)$ is a discrete analogue of $W_1(\delta, A)$, that is,

$$W_n(\delta, A) := \max_{\substack{|x| \le A h_n, i = \delta n, \dots, n \\ \beta_n \omega_{i,x} \ge 1}} \Big\{ \omega_{i,x} - \frac{x^2}{2\beta_n i} \Big\}. \tag{6.27}$$

On the other hand, we get that $\sqrt{n/i} \le \delta^{-1/2}$ for $i \ge \delta n$, so that from (6.25) we get

$$\begin{aligned} &\sum_{i=\delta n}^{n} \sum_{|x| \le A h_n} e^{\beta_n \omega_{i,x}} \sqrt{n}\, p(i, x) \mathbb{1}_{\{\beta_n \omega_{i,x} \ge 1\}} \\ &\qquad \le \frac{c}{\sqrt{\delta}} e^{\beta_n W_n(\delta, A)} \sum_{i=1}^{n} \sum_{|x| \le A h_n} \mathbb{1}_{\{\beta_n \omega_{i,x} \ge 1\}}. \end{aligned} \tag{6.28}$$

Now, we have that $\mathbb{P}(\omega > 1/\beta_n) \le (h_n^2/n)^{2\alpha} (nh_n)^{-1}$ as already noticed, so that

$$\mathbb{E}\Bigg[\sum_{i=1}^{n} \sum_{|x| \le A h_n} \mathbb{1}_{\{\beta_n \omega_{i,x} \ge 1\}}\Bigg] \le A \Big(\frac{h_n^2}{n}\Big)^{2\alpha}. \tag{6.29}$$

Overall, combining (6.26) with (6.28)–(6.29), we get that with probability going to 1 as $n \to \infty$,

$$\Bigg|\log\Big(\sum_{(i,x) \in \Upsilon_\ell, i \ge \delta n, |x| \le A h_n} e^{\beta_n \omega_{i,x}} \sqrt{n}\, p(i, x)\Big) - \beta_n W_n(\delta, A)\Bigg| \le (2\alpha + 1) \log \frac{h_n^2}{n}.$$

To conclude the proof of Lemma 6.6, it therefore remains to show that

$$\frac{n}{h_n^2} \times \beta_n W_n(\delta, A) \xrightarrow[n\to\infty]{(\mathrm{d})} W_1(\delta, A), \tag{6.30}$$

where $W_1(\delta, A)$ is defined in Lemma 6.6.

We fix $\varepsilon > 0$ and we consider $\widetilde{W}_n(\varepsilon, \delta, A)$ the truncated version of $W_n(\delta, A)$ in which we replace the condition $\{\beta_n \omega_{i,x} \geq 1\}$ by $\{\beta_n \omega_{i,x} > \varepsilon \frac{h_n^2}{n}\}$, that is,

$$\widetilde{W}_n(\varepsilon, \delta, A) := \max_{\substack{|x|\leq Ah_n, i=\delta n,\dots,n \\ \beta_n\omega_{i,x} > \varepsilon \frac{h_n^2}{n}}} \left\{ \omega_{i,x} - \frac{x^2}{2\beta_n i} \right\}. \tag{6.31}$$

In such a way, and since $\varepsilon h_n^2/n \geq 1$ for large n, we have

$$\frac{n}{h_n^2} \beta_n \widetilde{W}_n(\varepsilon, \delta, A) \leq \frac{n}{h_n^2} \beta_n W_n(\delta, A) \leq \frac{n}{h_n^2} \beta_n \widetilde{W}_n(\varepsilon, \delta, A) + \varepsilon.$$

To prove (6.30), we need to show that

$$\frac{n}{h_n^2} \times \beta_n \widetilde{W}_n(\varepsilon, \delta, A) \xrightarrow[n\to\infty]{(\mathrm{d})} \widetilde{W}_1(\varepsilon, \delta, A) := \max_{\substack{(w,t,x)\in\mathcal{P} \\ t>\delta, |x|\leq A, w>\varepsilon}} \left\{ w - \frac{x^2}{2t} \right\}, \tag{6.32}$$

and then let $\varepsilon \downarrow 0$—notice that we have $\widetilde{W}_1(\varepsilon, \delta, A) \leq W_1(\delta, A) \leq \widetilde{W}_1(\varepsilon, \delta, A) + \varepsilon$ so that $\widetilde{W}_1(\varepsilon, \delta, A) \to W_1(\delta, A)$ as $\varepsilon \downarrow 0$.

We observe that a.s. there are only finitely many $\omega_{i,x}$ in $[\![1, n]\!] \times [\![-Ah_n, Ah_n]\!]$ that are larger than $\varepsilon m(nh_n) \sim \beta_n^{-1} \varepsilon h_n^2/n$. This is a consequence of Markov's inequality and the Borel–Cantelli lemma. Indeed, for any $K \in \mathbb{N}$ we have

$$\begin{aligned} &\mathbb{P}\big(\big|\{(i,x) \in [\![1,n]\!] \times [\![-Ah_n, Ah_n]\!] \colon \omega_{i,x} \geq \varepsilon m(nh_n)\}\big| > 2^K\big) \\ &\quad \leq 2^{-K}(2Anh_n)\mathbb{P}\big(\omega \geq \varepsilon m(nh_n)\big) \leq C_\varepsilon 2^{-K}. \end{aligned}$$

Therefore, the convergence (6.32) is a straightforward consequence of the Skorokhod representational theorem. □

7. Case $\alpha \in (0, 1/2)$.

In the first part of this section, we prove (2.15). In the second part, we prove the convergence (2.16).

7.1. *Transversal fluctuations*: *Proof of* (2.15). (a) *Paths cannot be at an intermediate scale.* We start by showing that there exists $c_0, c, \nu > 0$ such that for any sequences $C_n > 1$ and $\delta_n \in (0, 1)$ (which may go to ∞, resp., 0, as $n \to \infty$) and for any $n \geq 1$

$$\begin{aligned} &\mathbb{P}\Big(\mathbf{P}_{n,\beta_n}^\omega\Big(\max_{i\leq n} |S_i| \in [C_n\sqrt{n}, \delta_n n)\Big) \leq e^{-c_0 C_n^2} + e^{-c_0 n^{1/2}}\Big) \\ &\quad \geq 1 - c\delta_n^\nu + n^{-\frac{1-2\alpha}{4}+\varepsilon}. \end{aligned} \tag{7.1}$$

To prove it, we use a decomposition into blocks, as we did in Section 4. Here we have to partition the interval $[C_n\sqrt{n}, \delta_n n)$ into $[C_n\sqrt{n}, n^{3/4}) \cup [n^{3/4}, \delta_n n)$ (one of these intervals might be empty), obtaining

$$\begin{aligned}
&\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in [C_n\sqrt{n}, \delta n)\Big)\\
&\quad = \mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in \big[C_n\sqrt{n}, n^{3/4}\big)\Big)\\
&\quad\quad + \mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in \big(n^{3/4}, \delta_n n\big)\Big).
\end{aligned} \tag{7.2}$$

For the first term, we partition the interval $[C_n\sqrt{n}, n^{3/4})$ into smaller blocks $D_{k,n} := [2^k\sqrt{n}, 2^{k+1}\sqrt{n})$, with $k = \log_2 C_n, \dots, \log_2 n^{1/4} - 1$. Let us define

$$\Sigma(n,h) = \sum_{i=1}^{n}\sum_{x\in[\![-h,h]\!]}\omega_{i,x} \tag{7.3}$$

the sum of all weights in $[\![1,n]\!]\times[\![-h,h]\!]$. Then we write similar to (4.2) (we also use that $\mathbf{Z}^{\omega}_{n,\beta_n} \ge 1$, which is harmless here since no recentering term is needed),

$$\begin{aligned}
\mathbf{P}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in \big[C_n\sqrt{n}, n^{3/4}\big)\Big) &\le \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} \mathbf{Z}^{\omega}_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in D_{k,n}\Big)\\
&\le \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}-1} e^{\beta_n\Sigma(n,2^{k+1}\sqrt{n})}\mathbf{P}\Big(\max_{i\le n}|S_i| \in D_{k,n}\Big)\\
&\le \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} \exp\big(\beta_n\Sigma\big(n, 2^{k+1}\sqrt{n}\big) - c2^{2k}\big),
\end{aligned}$$

where for the last inequality we used a standard estimate for the deviation probability of a random walk $\mathbf{P}(\max_{i\le n}|S_i| \ge 2^k\sqrt{n}) \le e^{-c2^{2k}}$; see, for example, [18], Proposition 2.1.2(b). Therefore, on the event

$$\Big\{\forall k = \log_2 C_n, \dots, \log_2 n^{1/4}, \beta_n\Sigma\big(n, 2^{k+1}\sqrt{n}\big) \le \frac{c}{2}2^{2k}\Big\} \tag{7.4}$$

we have that

$$\mathbf{P}^{\omega}_{n,\beta}\Big(\max_{i\le n}|S_i| \in \big[C_n\sqrt{n}, n^{3/4}\big)\Big) \le \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} e^{-\frac{c}{2}2^{2k}} \le c' e^{-\frac{c}{2}C_n^2}. \tag{7.5}$$

For the second term in (7.2), we partition the interval $(n^{3/4}, \delta_n n)$ into blocks $E_{n,k} := [2^{-k-1}n, 2^{-k}n)$, $k = \log_2(1/\delta_n), \dots, \log_2 n^{1/4} - 1$. Exactly as above, we use the large deviation estimate $\mathbf{P}(\max_{i\le n}|S_i| \ge 2^{-k+1}n) \le e^{-c2^{-2k}n}$ (see, e.g., [18], Proposition 2.1.2(b)), and we obtain that on the event

$$\Big\{\forall k = \log_2(1/\delta_n), \dots, \log_2 n^{1/4}, \beta_n \Sigma\big(n, 2^{-kn}\big) \le \frac{c}{2} 2^{-2k} n\Big\} \tag{7.6}$$

we have

$$\mathbf{P}^{\omega}_{n,\beta}\Big(\max_{i\le n}|S_i| \in \big(n^{3/4}, \delta_n n\big)\Big) \le \sum_{k=\log_2(1/\delta_n)}^{\log_2 n^{1/4}} e^{-\frac{c}{2}2^{-2k}n} \le c' e^{-\frac{c}{2}n^{1/2}}. \tag{7.7}$$

It now only remains to show that the complementary events of (7.4) and (7.6) have small probability. We start with (7.6). Using that $\beta_n \le 2\beta n/m(n^2)$ for n large, we get by a union bound that

$$\begin{aligned} &\mathbb{P}\Big(\exists k \ge \log_2 1/\delta_n, \beta_n \Sigma\big(n, 2^{-k}n\big) > \frac{c}{2} 2^{-2k} n\Big) \\ &\quad \le \sum_{k\ge \log_2 1/\delta_n} \mathbb{P}\big(\Sigma\big(n, 2^{-k}n\big) > c_\beta 2^{-2k} m\big(n^2\big)\big). \end{aligned} \tag{7.8}$$

Then, by Potter's bound we have that $m(2^{-k+1}n^2) \le 2^{-2k}m(n^2)$ since $\alpha < 1/2$ (recall $m(\cdot)$ (2.1) is regularly varying with exponent $1/\alpha$). As a consequence, the last probability in (7.8) is in the so-called one-jump large deviation domain (see [20], Theorem 1.1, we are using $\alpha < 1$ here), that is,

$$\mathbb{P}\big(\Sigma\big(n, 2^{-k}n\big) > c_\beta 2^{-2k} m\big(n^2\big)\big) \sim 2^{-k+1} n^2 \mathbb{P}\big(\omega > c_\beta 2^{-2k} m\big(n^2\big)\big).$$

Therefore, using again Potter's bound, we get that for arbitrary η there is some constant c such that

$$\mathbb{P}\big(\Sigma\big(n, 2^{-k}n\big) > c_\beta 2^{-2k} m\big(n^2\big)\big) \le c\big(2^{2k}\big)^{\alpha+\eta} n^{-2},$$

where we also used that $\mathbb{P}(\omega > m(n^2)) = n^{-2}$. Therefore, taking η small enough so that $2\alpha - 1 + 2\eta < 0$, we obtain that (7.8) is bounded by a constant times

$$\sum_{k\ge \log_2 1/\delta_n} 2^{k(2\alpha-1+2\eta)} \le c\delta_n^{1-2\alpha+2\eta}.$$

Similarly, for (7.4), we have by a union bound that

$$\begin{aligned} &\mathbb{P}\Big(\exists k \in \big\{\log_2 C_n, \dots, \log_2 n^{1/4}\big\}, \beta_n \Sigma\big(n, 2^{k+1}\sqrt{n}\big) > \frac{c}{2} 2^{2k}\Big) \\ &\quad \le \sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} \mathbb{P}\big(\Sigma\big(n, 2^{k+1}\sqrt{n}\big) > c_\beta 2^{2k} n^{-1} m\big(n^2\big)\big). \end{aligned} \tag{7.9}$$

Then again, we notice that $m(2^{k+2}n^{3/2}) \le 2^{2k}n^{-1}m(n^2)$ (using Potter's bound, as $\alpha < 1/2$). Hence, the last probability in (7.9) is in the one-jump large deviation domain (see [20], Theorem 1.1), that is,

$$\mathbb{P}\big(\Sigma\big(n, 2^{k+1}\sqrt{n}\big) > c2^{2k}n^{-1}m\big(n^2\big)\big) \le c2^k n^{3/2}\mathbb{P}\big(\omega > c_\beta 2^{2k}n^{-1}m\big(n^2\big)\big).$$

Then we also get that for any $\eta > 0$ we have that there is a constant $c > 0$ such that

$$\mathbb{P}\big(\omega > c_\beta 2^{2k}n^{-1}m\big(n^2\big)\big) \le c\big(2^{2k}n^{-1}\big)^{-\alpha-\eta},$$

so that provided that $1 - 2\alpha - 2\eta > 0$, (7.9) is bounded by a constant times

$$\sum_{k=\log_2 C_n}^{\log_2 n^{1/4}} 2^{k(1-2\alpha-2\eta)}n^{\alpha-\frac{1}{2}+\eta} \le cn^{-\frac{1}{4}(1-2\alpha-2\eta)}.$$

(b) *Paths cannot be at scale n conditionally on* $\widehat{\mathcal{T}}_\beta = 0$. We have shown in (7.1) that paths cannot be on an intermediate scale: it remains to prove that on the event $\widehat{\mathcal{T}}_\beta = 0$, paths cannot be at scale n. For this purpose, we use [5], Theorem 2.1, and [24], Theorem 1.8, which ensure that for any δ and $\varepsilon > 0$ there exists $\nu > 0$ such that

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\le n}|S_i| \in (\delta n, n]\Big) \le e^{-n\nu}|\widehat{\mathcal{T}}_\beta = 0\Big) \ge 1 - \varepsilon. \tag{7.10}$$

Therefore, we get that for any $\varepsilon > 0$ and $\delta > 0$, combining (7.1) with (7.10), for any sequence $C_n > 1$, provided that n is large enough we have

$$\mathbb{P}\Big(\mathbf{P}^\omega_{n,\beta_n}\Big(\max_{i\le n}|S_i| \ge C_n\sqrt{n}\Big) \ge e^{-c_0C_n^2} + e^{-c_0 n^{1/2}} + e^{-n\nu}|\widehat{\mathcal{T}}_\beta = 0\Big) \le c\delta^\nu + 2\varepsilon,$$

which concludes the proof of (2.15).

7.2. *Convergence in distribution conditionally on* $\widehat{\mathcal{T}}_\beta = 0$, *proof of* (2.16). In the following, we consider the case where $\beta_n n^{-1}m(n^2) \to \beta$ with $\beta < \infty$. In the case $\beta = +\infty$, we would indeed have that $\widehat{\mathcal{T}}_\beta > 0$. The proof follows the same idea as that of [13], Theorem 1.4 (and similar steps as above), but with many adaptations (and simplifications) in our case. We focus on the case $\beta > 0$, in which $\frac{\sqrt{n}}{\beta_n m(n^{3/2})}$ goes to infinity as a regularly varying function with exponent $\frac{2}{\alpha} - \frac{1}{2} - \frac{3}{2\alpha} = \frac{1-\alpha}{2\alpha} > 0$ (if $\beta = 0$, it goes to infinity faster).

Step 1. Reduction of the set of trajectories. Equation (2.15) (with $C_n = A\sqrt{\log n}$) gives that, with $\mathbb{P}$ probability larger than $1-\varepsilon$ (conditionally on $\widehat{\mathcal{T}}_\beta = 0$), we have $\mathbf{P}^\omega_{n,\beta_n}(\max_{i\le n}|S_i| \le A\sqrt{n\log n}) \ge 1 - e^{-c_0 A\log n}$ provided that n is large enough. We therefore get

$$\mathbb{P}\big(\big|\log \mathbf{Z}^\omega_{n,\beta_n} - \log \mathbf{Z}^\omega_{n,\beta_n}(\mathcal{A}_n)\big| \le n^{-c_0 A}|\widehat{\mathcal{T}}_\beta = 0\big) \ge 1 - \varepsilon, \tag{7.11}$$

where $\mathcal{A}_n$ is defined in (6.2). Note that, provided A has been fixed large enough, we have that $\frac{\sqrt{n}}{\beta_n m(n^{3/2})} n^{-c_0 A} \to 0$ as $n \to \infty$: we conclude that, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{\sqrt{n}}{\beta_n m(n^{3/2})} \left|\log \mathbf{Z}^{\omega}_{n,\beta_n} - \log \mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)\right| > \varepsilon \,|\, \widehat{\mathcal{T}}_\beta = 0\right) \le \varepsilon, \tag{7.12}$$

provided that n is large enough. We will therefore focus on $\log \mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$.

As in Section 6, we use the notation $A_n = A\sqrt{n \log n} = C_n \sqrt{n}$ and $\Lambda_{n,A_n} = [\![1, n]\!] \times [\![-A_n, A_n]\!]$.

Step 2. Truncation of the weights. We let $k_n := m(n^{3/2} \log n)$ be a sequence of truncation levels, and $\widetilde{\omega}_x := \omega_x \mathbf{1}_{\{\omega_x \le k_n\}}$ be the truncated environment. Then we have that

$$\mathbb{P}(\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n) \neq \mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n)) = \mathbb{P}\Big(\max_{(i,x)\in\Lambda_{n,A_n}} \omega_{i,x} > m\big(n^{3/2}\log n\big)\Big)$$
$$\le \frac{2A}{\sqrt{\log n}} \overset{n\to\infty}{\to} 0,$$

where we used a union bound for the last inequality, together with the definition of $m(\cdot)$ (2.1). Henceforth, we can safely replace $\mathbf{Z}^{\omega}_{n,\beta_n}(\mathcal{A}_n)$ with the truncated partition function $\mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n)$.

Step 3. Expansion of the partition function. We write again $p(i,x) = \mathbf{P}(S_i = x)$ for the random walk kernel, and let $\lambda_n(t) = \log \mathbb{E}[e^{t\widetilde{\omega}_x}]$. Then expanding

$$\exp\left(\sum_{i=1}^{n} (\beta_n \omega_{i,S_i} - \lambda_n(\beta_n))\right) = \prod_{(i,x)\in\Lambda_{n,A_n}} \left(1 + e^{\beta_n \widetilde{\omega}_{i,x} - \lambda_n(\beta_n)} - 1\right)^{\mathbf{1}_{\{S_i = x\}}},$$

we obtain

$$\begin{aligned} &e^{-n\lambda_n(\beta_n)} \mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n) \\ &\quad = 1 + \sum_{(i,x)\in\Lambda_{n,A_n}} \left(e^{\beta_n \widetilde{\omega}_{i,x} - \lambda_n(\beta_n)} - 1\right) p(i,x) + \mathbf{R}_n, \end{aligned} \tag{7.13}$$

with

$$\mathbf{R}_n := \sum_{k=2}^{\infty} \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ |x_i| \le A_n, i=1,\dots,k}} \prod_{j=1}^{k} \left(e^{\beta_n \widetilde{\omega}_{j,x_j} - \lambda_n(\beta_n)} - 1\right) p_n(i_j - i_{j-1}, x_j - x_{j-1}).$$

LEMMA 7.1. *We have that for n large*

$$\mathbb{P}\left(\frac{\sqrt{n}}{\beta_n m(n^{3/2})} \mathbf{R}_n \ge n^{-1/4}\right) \le \frac{(\log n)^{4/\alpha}}{\sqrt{n}} \to 0.$$

In particular, $\mathbf{R}_n \to 0$ *in probability.*

PROOF. Note that $\mathbb{E}[\mathbf{R}_n]=0$, so it will be enough to control the second moment of $\mathbf{R}_n$. Since the $\widetilde{\omega}_{i,x}$ are independent and $\mathbb{E}[e^{\beta_n\widetilde{\omega}_{i,x}-\lambda_n(\beta_n)}-1]=0$,

$$\mathbb{E}[\mathbf{R}_n^2]=\sum_{k=2}^{\infty}\sum_{\substack{1\le i_1<\cdots<i_k\le n\\ |x_i|\le A_n, i=1,\dots,k}}(e^{\lambda_n(2\beta_n)-\lambda_n(\beta_n)}-1)^k\prod_{j=1}^{k}p_n(i_j-i_{j-1},x_j-x_{j-1})^2$$

$$\le\sum_{k=2}^{\infty}(e^{\lambda_n(2\beta_n)}-1)^k\Biggl(\sum_{i=1}^{n}\sum_{x\in\mathbb{Z}}p(i,x)^2\Biggr)^k.$$

First, we have that

$$\sum_{i=1}^{n}\sum_{x\in\mathbb{Z}}p(i,x)^2=\mathbf{E}^{\otimes 2}\Biggl[\sum_{i=1}^{n}1_{\{S_n=S_n'\}}\Biggr]\le c\sqrt{n},$$

where S and S' are two independent simple random walks. Then, since $\beta_n\widetilde{\omega}\le\beta_n k_n\to 0$, we can write $e^{2\beta_n\widetilde{\omega}}\le 1+3\beta_n\widetilde{\omega}$ for n large, so that

$$\begin{aligned} e^{\lambda_n(2\beta_n)}-1&\le 3\beta_n\mathbb{E}[\widetilde{\omega}]=3\beta_n\int_0^{k_n}\mathbb{P}(\omega>u)\,du\\ &\le c\beta_n L(k_n)k_n^{1-\alpha}\le\frac{c\beta_n k_n}{n^{3/2}\log n}.\end{aligned}\tag{7.14}$$

To estimate the integral, we used the tail behavior of $\mathbb{P}(\omega>u)$ (1.2) (see [8], Theorem 1.5.8), while for the last inequality, we used that $k_n=m(n^{3/2}\log n)$ and the definition (2.1) of $m(\cdot)$, so that $L(k_n)k_n^{-\alpha}\sim n^{-3/2}(\log n)^{-1}$. We therefore get that for n large enough

$$\mathbb{E}[\mathbf{R}_n^2]\le\sum_{k\ge 2}\biggl(\frac{\beta_n k_n}{n}\biggr)^k\le 2\biggl(\frac{\beta_n k_n}{n}\biggr)^2.$$

To conclude, by Potter's bounds we get that $k_n\le m(n^{3/2})(\log n)^{2/\alpha}$ for n large, so that

$$\mathbb{E}[\mathbf{R}_n^2]\le\biggl(\frac{\beta_n m(n^{3/2})}{\sqrt{n}}\biggr)^2\times\frac{(\log n)^{\frac{4}{\alpha}}}{n},\tag{7.15}$$

and the conclusion of the lemma follows by using Markov's inequality. □

Going back to (7.13), we get that

$$\begin{aligned}&\mathbf{Z}_{n,\beta_n}^{\widetilde{\omega}}(\mathcal{A}_n)\\ &\quad=e^{(n-1)\lambda_n(\beta_n)}\biggl(e^{\lambda_n(\beta_n)}+\sum_{(i,x)\in\Lambda_{n,A_n}}(e^{\beta_n\widetilde{\omega}_{i,x}}-e^{\lambda_n(\beta_n)})p(i,x)+e^{\lambda_n(\beta_n)}\mathbf{R}_n\biggr)\\ &\quad=e^{(n-1)\lambda_n(\beta_n)}(1+\mathbf{V}_n+\mathbf{W}_n+e^{\lambda_n(\beta_n)}\mathbf{R}_n),\end{aligned}$$

with

$$\mathbf{V}_n := \sum_{(i,x)\in\Lambda_{n,A_n}} \big(e^{\beta_n\widetilde{\omega}_{i,x}} - 1\big)p(i,x) \quad \text{and}$$

$$\mathbf{W}_n := \big(e^{\lambda_n(\beta_n)} - 1\big)\Big(1 - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x)\Big).$$

We show below that $\lim_{n\to\infty}\mathbf{W}_n = 0$ and that $\mathbf{V}_n$ converges in probability to 0, so that using also Lemma 7.1, we get

$$\begin{aligned}(7.16)\qquad &\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\log \mathbf{Z}^{\widetilde{\omega}}_{n,\beta_n}(\mathcal{A}_n)\\ &= \frac{\sqrt{n}}{\beta_n m(n^{3/2})}\mathbf{V}_n + \frac{\sqrt{n}}{\beta_n m(n^{3/2})}\big((n-1)\lambda_n(\beta_n) + \mathbf{W}_n\big) + o(1).\end{aligned}$$

Before we prove the convergence of the first term (see Lemma 7.2), we show that the second term goes to 0—note that this implies that $\mathbf{W}_n \to 0$ since $\beta_n n^{-1/2} m(n^{3/2}) \to 0$. We write that

$$\begin{aligned}(7.17)\qquad \big|(n-1)\lambda_n(\beta_n) + \mathbf{W}_n\big| \le{}& (n-1)\big|e^{\lambda_n(\beta_n)} - 1 - \lambda_n(\beta_n)\big|\\ &+ \Big|n - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x)\Big|.\end{aligned}$$

For the second term, using standard large deviation for the simple random walk (e.g., [18], Proposition 2.1.2(b)), there is a constant $c > 0$ such that

$$(7.18)\qquad n - \sum_{(i,x)\in\Lambda_{n,A_n}} p(i,x) = \sum_{i=1}^{n} \mathbf{P}\big(S_i > A\sqrt{n\log n}\big) \le n e^{-cA^2 \log n}.$$

For the first term, since we have $\lambda_n(\beta_n) \to 0$, we get that for n large enough

$$(7.19)\qquad \big|e^{\lambda_n(\beta_n)} - 1 - \lambda_n(\beta_n)\big| \le \lambda_n(\beta_n)^2 \le \Big(\frac{\beta_n m(n^{3/2})}{n^{3/2}}(\log n)^{2/\alpha}\Big)^2,$$

where for the second inequality we used (7.14) (note that $\lambda_n(\beta_n) \le e^{\lambda_n(\beta_n)} - 1$), together with the fact that $k_n \le m(n^{3/2})(\log n)^{2/\alpha}$.

Hence plugging (7.18) and (7.19) into (7.17), we get that provided that A is large enough,

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\big|(n-1)\lambda_n(\beta_n) + \mathbf{W}_n\big| \le \frac{\beta_n m(n^{3/2})}{n^{3/2}}(\log n)^{4/\alpha} + o(1) \xrightarrow{n\to\infty} 0$$

so that the second term in (7.16) goes to 0 as $n \to \infty$, proving also that $\mathbf{W}_n \to 0$ (recall also $\beta_n n^{-1/2} m(n^{3/2}) \to 0$).

Step 4. Convergence of the main term. We conclude the proof by showing the convergence in distribution of the first term in (7.16), which proves also that $\mathbf{V}_n$ goes to 0 in probability, since $\beta_n n^{-1/2} m(n^{3/2}) \to 0$.

LEMMA 7.2. *We have the following convergence in distribution*:

$$\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\mathbf{V}_n := \frac{\sqrt{n}}{\beta_n m(n^{3/2})} \sum_{(i,x)\in\Lambda_{n,A_n}} \left(e^{\beta_n \widetilde{\omega}_{i,x}} - 1\right) p(i,x) \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{W}_0^{(\alpha)},$$

with $\mathcal{W}_0^{\alpha}$ *defined in Theorem* 2.12.

PROOF. First of all, since $\beta_n \widetilde{\omega}_{i,x} \le \beta_n k_n \to 0$ as $n \to \infty$ (and using that $0 \le e^x - 1 - x \le x^2$ for x small), we have that for n large

$$0 \le \mathbf{V}_n - \beta_n \sum_{(i,x)\in\Lambda_{n,A_n}} \widetilde{\omega}_{i,x} p(i,x) \le \sum_{(i,x)\in\Lambda_{n,A_n}} (\beta_n \widetilde{\omega}_{i,x})^2 p(i,x). \tag{7.20}$$

Then we can estimate the expectation of the upper bound, using that similar to (7.14) we have $\mathbb{E}[(\widetilde{\omega})^2] \le cL(k_n)k_n^{2-\alpha} \sim ck_n^2/(n^{3/2}\log n)$. Using also that $k_n \le m(n^{3/2})(\log n)^{2/\alpha}$ for n large, we obtain that

$$\begin{aligned}\frac{\sqrt{n}}{\beta_n m(n^{3/2})}\mathbb{E}\Big[\sum_{(i,x)\in\bar{\Lambda}_n} (\beta_n \widetilde{\omega}_{i,x})^2 p(i,x)\Big] &\le c\frac{k_n}{m(n^{3/2})}\beta_n k_n n^{-1} \sum_{i=1}^{n}\sum_{x\in\mathbb{Z}} p(i,x) \\ &\le c(\log n)^{2/\alpha}\beta_n k_n \xrightarrow{n\to\infty} 0.\end{aligned}$$

The proof of the lemma is therefore reduced to showing the convergence in distribution of the following term:

$$\begin{aligned}&\frac{\sqrt{n}}{m(n^{3/2})}\sum_{(i,x)\in\bar{\Lambda}_n} \widetilde{\omega}_{i,x} p(i,x) \\ &\quad = \sum_{i=1}^{n} \sum_{|x|\le K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}\, p(i,x) \\ &\qquad + \sum_{i=1}^{n} \sum_{K\sqrt{n}<|x|\le A_n} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}\, p(i,x),\end{aligned} \tag{7.21}$$

where we fixed some level $K > 0$ (we take the limit $K \to \infty$ in the end).

First term in (7.21). First, note that the first term converges in distribution to

$$\mathcal{W}_{0,K}^{(\alpha)} := 2\int_{\mathbb{R}_+}\int_0^1\int_{-K}^{K} w\rho(t,x)\mathcal{P}(dw\,dt\,dx), \tag{7.22}$$

where $\rho(t,x) := (2\pi t)^{-1/2}e^{-x^2/2t}$ is the Gaussian kernel and $\mathcal{P}(w,t,x)$ is a PPP on $[0,\infty)\times[0,1]\times\mathbb{R}$ of intensity $\mu(dw\,dt\,dx) = \frac{\alpha}{2}w^{-\alpha-1}\mathbb{1}_{\{w>0\}}\,dw\,dt\,dx$. The proof of (7.22) is identical to that in [13], page 4036, so we omit details.

Then, since $\mathcal{W}_0^{(\alpha)} < \infty$ a.s. (see [13], Lemma 1.3), one readily gets that $\mathcal{W}_{0,K}^{(\alpha)} \to \mathcal{W}_0^{(\alpha)}$ as $K \to \infty$, by monotonicity.

Second term in (7.21). To conclude the proof, it remains to show that the second term in (7.21) goes to 0 in probability as $K \to \infty$, uniformly in n: for any K (large), we have for n sufficiently large,

$$\mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{K\sqrt{n}<|x|\le A_n} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}\,p(i,x) \ge K^{-1}\bigg) \le c e^{-\alpha K}. \tag{7.23}$$

To prove (7.23), we split the sum in parts with $|x| \in (2^{k-1}K\sqrt{n}, 2^k K\sqrt{n}]$ for $k = 1, 2\dots$. By a union bound, we have

$$\begin{aligned}
&\mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{|x|>K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}\,p(i,x) \ge K^{-1}\bigg)\\
&\quad\le \sum_{k=1}^{\infty} \mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{|x|=2^{k-1}K\sqrt{n}}^{2^k K\sqrt{n}} \frac{\omega_{i,x}}{m(n^{3/2})}\sqrt{n}\,p(i,x) \ge K^{-1}2^{-k}\bigg) \qquad (7.24)\\
&\quad\le \sum_{k\ge 1} \mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{|x|\le 2^k K\sqrt{n}} \omega_{i,x} \ge e^{c'(2^k K)^2} m\big(n^{3/2}\big)\bigg).
\end{aligned}$$

In the last inequality, we used that there is a constant c such that for any k, uniformly in $i \in \{1,\dots,n\}$ and $|x| \ge 2^{k-1}K\sqrt{n}$, we have $\sqrt{n}\,p(i,x) \le e^{-c(2^k K)^2} \le 2^{-k}K^{-1}e^{-c'(2^k K)^2}$ (since $K2^k \ge 1$).

Now, we use that $m(2^{k+1}Kn^{3/2}) \ge (2^k K)^{-2/\alpha} m(n^{3/2})$ by Potter's bound, and also that for all k, $e^{c'(2^k K)^2}(2^k K)^{-2/\alpha} \ge e^{2^k K}$ if K is large: the last probability in (7.24) is in the one-jump large deviation domain (see [20], Theorem 1.1, we use here that $\alpha < 1$): there is a $c > 0$ such that for all $k \ge 1$,

$$\begin{aligned}
&\mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{|x|\le 2^k K\sqrt{n}} \omega_{i,x} \ge e^{2^k K} m\big(2^{k+1}Kn^{3/2}\big)\bigg)\\
&\quad\le c2^k K n^{3/2}\,\mathbb{P}\big(\omega \ge e^{2^k K} m\big(2^k K n^{3/2}\big)\big) \le c e^{-\frac{\alpha}{2}2^k K}.
\end{aligned}$$

The second inequality comes from Potter's bound, provided that $e^{2^k K}$ is large enough, and also the definition (2.1) of $m(\cdot)$. Plugged in (7.24), we get

$$\mathbb{P}\bigg(\sum_{i=1}^{n} \sum_{|x|>K\sqrt{n}} \frac{\widetilde{\omega}_{i,x}}{m(n^{3/2})}\sqrt{n}\,p(i,x) \ge \varepsilon\bigg) \le c\sum_{k\ge 1} e^{-\frac{\alpha}{2}2^k K} \le c e^{-\alpha K},$$

which is (7.23). □

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REFERENCES

[1] ALBERTS, T., KHANIN, K. and QUASTEL, J. (2010). Intermediate disorder regime for directed polymers in dimension $1 + 1$. *Phys. Rev. Lett.* **105** 090603.

[2] ALBERTS, T., KHANIN, K. and QUASTEL, J. (2014). The continuum directed random polymer. *J. Stat. Phys.* **154** 305–326. MR3162542

[3] ALBERTS, T., KHANIN, K. and QUASTEL, J. (2014). The intermediate disorder regime for directed polymers in dimension $1 + 1$. *Ann. Probab.* **42** 1212–1256. MR3189070

[4] AMIR, G., CORWIN, I. and QUASTEL, J. (2011). Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.* **64** 466–537. MR2796514

[5] AUFFINGER, A. and LOUIDOR, O. (2011). Directed polymers in a random environment with heavy tails. *Comm. Pure Appl. Math.* **64** 183–204. MR2766526

[6] BALÁZS, M., QUASTEL, J. and SEPPÄLÄINEN, T. (2011). Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.* **24** 683–708. MR2784327

[7] BERGER, Q. and TORRI, N. (2019). Entropy-controlled last-passage percolation. *Ann. Appl. Probab.* **29** 1878–1903. MR3914559

[8] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1989). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR1015093

[9] BIROLI, G., BOUCHAUD, J.-P. and POTTERS, M. (2007). Extreme value problems in random matrix theory and other disordered systems. *J. Stat. Mech. Theory Exp.* **7** P07019, 15. MR2335697

[10] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2017). Polynomial chaos and scaling limits of disordered systems. *J. Eur. Math. Soc.* (*JEMS*) **19** 1–65. MR3584558

[11] COMETS, F. (2017). *Directed Polymers in Random Environments. Lecture Notes in Math.* **2175**. Springer, Cham. MR3444835

[12] COMETS, F., SHIGA, T. and YOSHIDA, N. (2004). Probabilistic analysis of directed polymers in a random environment: A review. In *Stochastic Analysis on Large Scale Interacting Systems. Adv. Stud. Pure Math.* **39** 115–142. Math. Soc. Japan, Tokyo. MR2073332

[13] DEY, P. S. and ZYGOURAS, N. (2016). High temperature limits for $(1 + 1)$-dimensional directed polymer with heavy-tailed disorder. *Ann. Probab.* **44** 4006–4048. MR3572330

[14] GUEUDRE, T., LE DOUSSAL, P., BOUCHAUD, J.-P. and ROSSO, A. (2015). Ground-state statistics of directed polymers with heavy-tailed disorder. *Phys. Rev. E* (3) **91** 062110, 10. MR3491341

[15] HAMBLY, B. and MARTIN, J. B. (2007). Heavy tails in last-passage percolation. *Probab. Theory Related Fields* **137** 227–275. MR2278457

[16] HUSE, D. A. and HENLEY, C. L. (1985). Pinning and roughening of domain walls in Ising systems due to random impurities. *Phys. Rev. Lett.* **54** 2708–2711.

[17] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169

[18] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk*: *A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157

[19] MORENO FLORES, G. R., SEPPÄLÄINEN, T. and VALKÓ, B. (2014). Fluctuation exponents for directed polymers in the intermediate disorder regime. *Electron. J. Probab.* **19** no. 89, 28. MR3263646

[20] NAGAEV, S. V. (1979). Large deviations of sums of independent random variables. *Ann. Probab.* **7** 745–789. MR0542129

[21] SEPPÄLÄINEN, T. (2012). Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.* **40** 19–73. MR2917766

[22] SEPPÄLÄINEN, T. and VALKÓ, B. (2010). Bounds for scaling exponents for a $1+1$ dimensional directed polymer in a Brownian environment. *ALEA Lat. Am. J. Probab. Math. Stat.* **7** 451–476. MR2741194

[23] STONE, C. (1967). On local and ratio limit theorems. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability* (*Berkeley, Calif.*, 1965/66), *Vol. II*: *Contributions to Probability Theory*, *Part* 2 217–224. Univ. California Press, Berkeley, CA. MR0222939

[24] TORRI, N. (2016). Pinning model with heavy tailed disorder. *Stochastic Process. Appl.* **126** 542–571. MR3434993

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75252 PARIS CEDEX 5
FRANCE
E-MAIL: quentin.berger@sorbonne-universite.fr
niccolo.torri@sorbonne-universite.fr