The Annals of Applied Statistics
2020, Vol. 14, No. 1, 160–177
https://doi.org/10.1214/19-AOAS1304

ASSESSING WAGE STATUS TRANSITION AND STAGNATION USING QUANTILE TRANSITION REGRESSION

BY CHIH-YUAN HSU[1], YI-HAU CHEN[2], RUOH-RONG YU[3,*] AND TSUNG-WEI HUNG[3]

[1]*Department of Biostatistics, Vanderbilt University Medical Center, chih-yuan.hsu@vumc.org*

[2]*Institute of Statistical Science, Academia Sinica, yhchen@stat.sinica.edu.tw*

[3]*Center for Survey Research, Research Center for Humanities and Social Sciences, Academia Sinica,*
[*]*yurr@gate.sinica.edu.tw*

Workers in Taiwan overall have been suffering from long-lasting wage stagnation since the mid-1990s. In particular, there seems to be little mobility for the wages of Taiwanese workers to transit across wage quantile groups. It is of interest to see if certain groups of workers, such as female, lower educated and younger generation workers, suffer from the problem more seriously than the others. This work tries to apply a systematic statistical approach to study this issue, based on the longitudinal data from the Panel Study of Family Dynamics (PSFD) survey conducted in Taiwan since 1999. We propose the quantile transition regression model, generalizing recent methodology for quantile association, to assess the wage status transition with respect to the marginal wage quantiles over time as well as the effects of certain demographic and job factors on the wage status transition. Estimation of the model can be based on the composite likelihoods utilizing the binary, or ordinal-data information regarding the quantile transition, with the associated asymptotic theory established. A goodness-of-fit procedure for the proposed model is developed. The performances of the estimation and the goodness-of-fit procedures for the quantile transition model are illustrated through simulations. The application of the proposed methodology to the PSFD survey data suggests that female, private-sector workers with higher age and education below postgraduate level suffer from more severe wage status stagnation than the others.

1. Introduction. Workers in Taiwan have been suffering from long-lasting wage stagnation since the mid-1990s. In particular, there seems to be little mobility for the wages of Taiwanese workers to transit across wage quantile groups. It is believed that certain groups of workers, such as the younger generation workers, are faced with this problem more seriously than the others (Huang, Liu and Yang (2014); Chen and Kuo (2014), Haepp and Hsin (2016), Li and Fang (2015); Lin, Chang and Lu (2017)). However, there still lacks systematic investigation into the degrees of severity of wage stagnation across different subgroups of workers. To examine the transition of personal earnings status across time, as well as how the transition is related to demographic and job factors, we employ individual-level panel data from the Panel Study of Family Dynamics (PSFD) survey conducted by the Center for Survey Research at Academia Sinica in Taiwan. The PSFD is a face-to-face survey project launched in 1999. The questionnaire contains general and elaborated questions on the interviewee's education, work, marriage, residence, income and expenditures, interaction with family members, attitudes toward family values, etc. The core question modules, including work, marriage, residence, income and expenditures and interaction with family members, are retained in each follow-up questionnaire conducted annually through 2012. Since 2012, follow-ups have been conducted on a biennial basis.

Received July 2019; revised October 2019.

Key words and phrases. Longitudinal data, panel study, quantile association, quantile regression, transition probability.

Owing to its skewed nature, the (marginal) wage distribution at a time point is better captured by its quantiles at several quantile levels than by just its mean. When there exist covariates that influence the marginal quantiles, quantile regression Koenker (2005), Koenker and Bassett (1978) at several representative quantiles (e.g., median or quartiles) can be applied to summarize the marginal wage distribution at each time point given the covariates. Also, given the marginal quantiles, it is informative to trace the time dynamics of the wage distribution through the transition of the wage status relative to the marginal wage quantiles over time. For example, to study the wage stagnation issue, researchers may focus on workers continuously having earnings below the 25% quantiles of the earnings distributions over time. Also, it may be meaningful to find characteristics of the workers with wages below the 25% quantile of the wage distribution at an initial time but with wages above the median of the wage distribution 10 years later; that is, the transition from the wage status below the 25% quantile to the wage status above the 50% quantile over a 10-year period is of interest.

In this work, we propose quantile transition regression analysis which focuses on the transition of outcome statuses indexed by marginal outcome quantiles over time as well as the effects of certain covariates on the outcome status transition. Mathematically, let Y_{ij} denote the outcome for the ith subject at the jth time point and $\boldsymbol{X}_{ij}$ the associated covariate vector $(j = 1, \dots, n_i \geq 2; i = 1, \dots, m)$. We use the local odds ratio (LOR)

$$\begin{aligned}
&\psi_{ijk}(\tau_1, \tau_2) \\
&= \frac{P(\tilde{Y}_{ij}(\tau_1) = 1, \tilde{Y}_{ik}(\tau_2) = 1) P(\tilde{Y}_{ij}(\tau_1) = 0, \tilde{Y}_{ik}(\tau_2) = 0)}{P(\tilde{Y}_{ij}(\tau_1) = 1, \tilde{Y}_{ik}(\tau_2) = 0) P(\tilde{Y}_{ij}(\tau_1) = 0, \tilde{Y}_{ik}(\tau_2) = 1)} \qquad (1.1) \\
&= \frac{P\{Y_{ij} \leq q_{\tau_1}(Y_{ij}), Y_{ik} \leq q_{\tau_2}(Y_{ik})\} P\{Y_{ij} > q_{\tau_1}(Y_{ij}), Y_{ik} > q_{\tau_2}(Y_{ik})\}}{P\{Y_{ij} \leq q_{\tau_1}(Y_{ij}), Y_{ik} > q_{\tau_2}(Y_{ik})\} P\{Y_{ij} > q_{\tau_1}(Y_{ij}), Y_{ik} \leq q_{\tau_2}(Y_{ik})\}}
\end{aligned}$$

to assess the tendency of transition of outcome statuses between time points j and k $(1 \leq j < k \leq n_i)$ and quantile levels τ_1 and $\tau_2 \in (0, 1)$, where, $\tilde{Y}_{ij}(\tau) = I\{Y_{ij} \leq q_\tau(Y_{ij})\}$ with $I(\cdot)$ the indicator function and, suppressing the conditioning on covariates, $q_\tau(Y_{ij}) = q_\tau(Y_{ij}|\boldsymbol{X}_{ij})$ denoting the τth quantile of Y_{ij} given the covariates $\boldsymbol{X}_{ij}$, $\tau \in (0, 1)$; $P(\cdot)$ should be interpreted as the conditional probability given the covariates. It can be seen that a larger value of $\psi_{ijk}(\tau_1, \tau_2)$ indicates a lower tendency of transition across statuses of Y_{ij} and Y_{ik}, with the outcome statuses defined in terms of the marginal outcome quantiles. We term such a tendency of transition as the "quantile transition" between Y_{ij} and Y_{ik} which reflects how the longitudinal outcomes transit across different locations in the marginal outcome distributions.

Li, Cheng and Fine (2014) first introduced the LOR similar to (1.1) for local association analysis with bivariate data. Yang, Chen and Chang (2017) employed the LOR (1.1) to model the covariance structure of longitudinal outcomes but only considered a single quantile level. Compared to these existing analyses, the proposed quantile transition analysis based on (1.1) is substantially more complicated since it involves multiple quantiles for longitudinal outcome vectors of possibly different lengths. In literature, the "quantile dynamic model" (see, e.g., Galvao (2011)), which is a quantile regression model for the outcome at a time point that includes the time lag of the outcome as a covariate in addition to other covariates, has been developed for assessing the influence of some past outcome on the quantiles of the current outcome. Unlike the quantile transition analysis proposed in this work, the quantile dynamic model cannot provide a direct assessment for the dynamics of the transition among outcome quantiles over time which is of relevance in the wage stagnation study and many other applications as mentioned above.

We propose two estimation procedures for the quantile transition regression analysis, applying the composite likelihood approach based on binary and ordinal codings for longitudinal outcome statuses over multiple quantile levels, respectively. For the proposed estimators of the parameters in the quantile transition regression model, we obtain the asymptotic normality together with the asymptotic variances that can be explicitly estimated. Also, we propose a procedure for assessing the goodness of fit of a quantile transition regression model.

This paper is organized as follows. We introduce the proposed quantile transition regression model in Section 2.1, and the estimation procedures in Sections 2.2 and 2.3. Procedures for testing modeling assumptions and goodness of fit are presented in Section 2.4. The simulation examples and the analysis of Taiwan wage data based on the PSFD survey are reported in Sections 3 and 4, respectively. The conclusions are presented in Section 5. The large sample properties of the proposed estimators are described in the Supplementary Material (Hsu et al. (2020)).

2. Quantile transition regression analysis.

2.1. *The model.* To assess covariate effects on the quantile transition (1.1), we extend the quantile association model of Yang, Chen and Chang (2017) at a single quantile level to the regression model for multiple quantile levels $[\tau_c, c \in \mathcal{L} \equiv \{1, \dots, L\}]$:

$$\log(\psi_{ijk(c_1,c_2)}) = \boldsymbol{Z}_{ijk}^{\mathrm{T}} \boldsymbol{\alpha}_{(c_1,c_2)}, \quad c_1, c_2 \in \mathcal{L}, \tag{2.1}$$

where $\psi_{ijk(c_1,c_2)} = \psi_{ijk}(\tau_{c_1}, \tau_{c_2})$, $\boldsymbol{Z}_{ijk}$ is the covariate vector that may include a subset of $(\boldsymbol{X}_{ij}, \boldsymbol{X}_{ik})$ and other covariates expected to influence the quantile transition between Y_{ij} and Y_{ik} and $\boldsymbol{\alpha}_{(c_1,c_2)} = \boldsymbol{\alpha}(\tau_{c_1}, \tau_{c_2})$ is the covariate effect on the quantile transition associated with quantile levels τ_{c_1} and τ_{c_2}. A larger positive (negative) $\boldsymbol{\alpha}_{(c_1,c_2)}$ value implies a higher tendency to stay in (escape from) the initial outcome status at a later time, with the outcome statuses at the initial and later times defined with respect to the outcomes' marginal quantiles at levels τ_{c_1} and τ_{c_2}, respectively. When suitable, we can impose certain restrictions on the parameters $\boldsymbol{\alpha}_{(c_1,c_2)}$ to have a parsimonious model. For example, it may be reasonable to assume that $\boldsymbol{\alpha}_{(c_1,c_2)}$ is constant for all $c_1, c_2 \in \mathcal{L}$. Let $\boldsymbol{\alpha}_* = (\boldsymbol{\alpha}_{(1,1)}^{\mathrm{T}}, \dots, \boldsymbol{\alpha}_{(L,L)}^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\alpha}$ the set of distinct parameters in $\boldsymbol{\alpha}_*$ and write $\boldsymbol{\alpha}_* = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\alpha}$ for a known constant matrix $\boldsymbol{C}$ formulating the restrictions. The assumption $\boldsymbol{\alpha}_* = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\alpha}$ can be confirmed by the hypothesis testing based on the asymptotic distributional theory for the estimator of $\boldsymbol{\alpha}_*$, as mentioned in Section 2.4.1 and Section S.2 of the Supplementary Material (Hsu et al. (2020)).

To implement the quantile transition analysis, we also need to estimate marginal quantiles $q_\tau(Y_{ij})$ involved in $\tilde{Y}_{ij}(\tau) = I\{Y_{ij} \le q_\tau(Y_{ij})\}$. For simplicity, we consider the linear quantile regression model with $q_\tau(Y_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\boldsymbol{\beta}(\tau)$. When estimating the marginal quantile regression parameter $\boldsymbol{\beta} = \{\boldsymbol{\beta}(\tau_c)^{\mathrm{T}}, c \in \mathcal{L}\}$ for multiple quantile levels, we solve the estimating equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} (\boldsymbol{I}_L \otimes \boldsymbol{X}_{ij}^{\mathrm{T}})^{\mathrm{T}} \boldsymbol{V}_{ij}^{-1} \boldsymbol{h}_{ij} = \boldsymbol{0}, \tag{2.2}$$

where $\boldsymbol{I}_L$ is the $L \times L$ identity matrix, $\otimes$ is the Kronecker product, $\boldsymbol{h}_{ij} = (h_{ij(1)}, \dots, h_{ij(L)})^{\mathrm{T}}$ with $h_{ij(c)} = 1 - \Phi\{\delta_{ij}(\tau_c)\} - \tau_c$ and $\boldsymbol{V}_{ij}$ is an $L \times L$ matrix whose (c_1, c_2) element $\boldsymbol{V}_{ij}[c_1, c_2] = \mathrm{cov}\{\tilde{Y}_{ij}(\tau_{c_1}), \tilde{Y}_{ij}(\tau_{c_2})\} = \tau_{c_1} \wedge \tau_{c_2} - \tau_{c_1}\tau_{c_2}$, where $a \wedge b \equiv \min(a, b)$. The estimating equation (2.2) uses the induced smoothing (Brown and Wang (2005); Yang, Chen and Chang (2017)) to smooth the indicator function $\tilde{Y}_{ij}(\tau)$ as $1 - \Phi\{\delta_{ij}(\tau)\}$, where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, $\delta_{ij}(\tau) = \{Y_{ij} - \hat{q}_\tau(Y_{ij})\}/\hat{r}_{ij}(\tau)$ and $\hat{r}_{ij}(\tau)$ is the estimated asymptotic standard error of $\hat{q}_\tau(Y_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\hat{\boldsymbol{\beta}}(\tau)$; details about how to estimate the standard errors can be found in Yang, Chen and Chang (2017). Besides, (2.2) uses working independence between outcomes at two time points but accounts for the correlation across different quantile levels within the same outcome. With the estimates $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$, $\tilde{Y}_{ij}(\tau)$ is then redefined as $\tilde{Y}_{ij}(\tau) = I\{Y_{ij} \le \boldsymbol{X}_{ij}^{\mathrm{T}}\hat{\boldsymbol{\beta}}(\tau)\}$.

The proposed quantile transition analysis allows for time-varying coefficients in both the marginal quantile and the quantile transition regression models. We consider the marginal quantile regression model with time-varying coefficients as $q_\tau(Y_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\boldsymbol{\beta}_j(\tau)$ with $\boldsymbol{\beta}_j(\tau) = \boldsymbol{A}(\tau)\boldsymbol{b}(t_j)$, where t_j is the time value for the jth time point, $\boldsymbol{b}(t) = \{b_1(t), \dots, b_K(t)\}^{\mathrm{T}}$ is a vector of basis functions of time variable t and $\boldsymbol{A}(\tau)$ is the corresponding $P \times K$ unknown coefficient matrix with column vectors $\boldsymbol{a}_k = (a_{1k}, \dots, a_{Pk})^{\mathrm{T}}$ for $k = 1, \dots, K$. That is, the pth component of $\boldsymbol{\beta}_j(\tau)$ is $\beta_{jp}(\tau) = \sum_{k=1}^{K} a_{pk} b_k(t_j)$. Both parametric and nonparametric basis functions can be used for $\boldsymbol{b}(t)$. In particular, piecewise-constant and piecewise-linear time-varying coefficients can be considered. Note that the time-varying coefficient quantile regression model given above can be rewritten as a conventional quantile regression model with time-independent coefficients: $q_\tau(Y_{ij}|\tilde{\boldsymbol{X}}_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\boldsymbol{\beta}_j(\tau) = \tilde{\boldsymbol{X}}_{ij}^{\mathrm{T}}\tilde{\boldsymbol{a}}(\tau)$, where $\tilde{\boldsymbol{X}}_{ij} = \boldsymbol{b}(t_j) \otimes \boldsymbol{X}_{ij}$ is the expanded covariate vector of length $P \times K$ that includes the "interactions" between the original covariates $\boldsymbol{X}_{ij}$ and the time basis functions $\boldsymbol{b}(t_j)$ and $\tilde{\boldsymbol{a}}(\tau)$ is the $P \times K$ vector stacking the columns of $\boldsymbol{A}(\tau)$. Using the alternative representation of the model, the proposed estimation procedure can thus be applied to the time-varying coefficient quantile regression model in the same way as to the time-independent coefficient model.

To consider the quantile transition regression model (2.1) with time-varying coefficients $\boldsymbol{\alpha}_{jk(c_1,c_2)}$, note that there are two time variables (j, k) involved, which can be equivalently represented as $(j, k - j)$, namely an initial time and a time lag. So we consider the time varying coefficient $\boldsymbol{\alpha}_{jk(c_1,c_2)} = \boldsymbol{A}^*(\tau_{c_1}, \tau_{c_2})\boldsymbol{b}^*(t_j, s_{jk})$, where t_j is the time value for the initial time point j, $s_{jk} = t_k - t_j$ is the value of time lag between the kth and jth time points, $\boldsymbol{b}^*(t, s)$ is a vector of basis functions of the initial time and the time lag variables (t, s), and $\boldsymbol{A}^*(\tau_{c_1}, \tau_{c_2})$ is the corresponding coefficient matrix. Following the same arguments as above for the marginal quantile regression model, the quantile transition regression with time varying coefficients can be reexpressed as a quantile transition regression model with time-independent coefficients but with the covariate vector $\boldsymbol{Z}_{ijk}$ expanded to include interactions between the original covariates $\boldsymbol{Z}_{ijk}$ and the time basis functions for both the initial time and the time-lag variables. The time-varying coefficient quantile transition model can be reexpressed as a time-independent coefficient model.

REMARK 1. It is of interest to see if there is a bona fide joint probability distribution for $\boldsymbol{Y}_i = (Y_{i1}, \dots, Y_{in_i})^{\mathrm{T}}$ that satisfies the association model (2.1) and, in addition, that the marginal quantiles of $\boldsymbol{Y}_i$ are given by $q_{\tau_c}(Y_{ij}|\boldsymbol{X}_{ij})$ for $1 \le j \le n_i$ and all c. For $1 \le j < k \le n_i$, write $\tilde{Y}_{ij(c)} = \tilde{Y}_{ij}(\tau_c) = I\{Y_{ij} \le q_{\tau_c}(Y_{ij})\}$ and $\mu_{ijk(c_1,c_2)} = P(\tilde{Y}_{ij(c_1)} = 1, \tilde{Y}_{ik(c_2)} = 1)$ which is a one-to-one function of $\psi_{ijk(c_1,c_2)}$. For given $0 \le w_r \le 1$ for $r \in \mathcal{L}^2 = \{1, \dots, L^2\}$ with $\sum_{r\in\mathcal{L}^2} w_r = 1$, consider $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \dots, \epsilon_{in_i})^{\mathrm{T}}$ from a mixture of multivariate normal distributions with the joint density $\sum_{r\in\mathcal{L}^2} w_r\phi(\boldsymbol{x}; \boldsymbol{\Sigma}_r)$, where $\phi(\boldsymbol{x}; \boldsymbol{\Sigma})$ denotes a multivariate normal density with zero mean and a $n_i \times n_i$ covariance matrix $\boldsymbol{\Sigma}$ whose diagonal elements are ones. For $1 \le j < k \le n_i$, the (j, k) element $\boldsymbol{\Sigma}_r[j, k]$ of $\boldsymbol{\Sigma}_r$ for $r \in \mathcal{L}^2$ are the solutions of the L^2 equations

$$\int_{-\infty}^{q_{c_1}} \int_{-\infty}^{q_{c_2}} \sum_{r\in\mathcal{L}^2} w_r \phi_2\big(x_1, x_2; \boldsymbol{\Sigma}_r[j, k]\big)\, dx_1\, dx_2 = \mu_{ijk(c_1,c_2)}, \qquad c_1, c_2 \in \mathcal{L}$$

subject to the condition that all $\boldsymbol{\Sigma}_r$'s are positive definite, where $q_c = \Phi^{-1}(\tau_c)$ and $\phi_2(x_1, x_2; \rho)$ is a standard bivariate normal density with correlation ρ. Then, the random variables $Y_{ij} = q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) + \{\epsilon_{ij} - q_\tau(\epsilon_{ij})\}$ for $1 \le j \le n_i$ would satisfy both the marginal quantile regression models $\{q_{\tau_c}(Y_{ij}|\boldsymbol{X}_{ij}), c \in \mathcal{L}\}$ and the quantile transition model (2.1). It can thus be seen that the quantile transition can address local association imbedded in a mixture distribution and hence is suitable for modeling a heterogeneous correlation structure.

2.2. *Composite binary likelihood.* We first consider the estimation procedure for the quantile transition model (2.1) based on the composite binary likelihood. The composite binary likelihood generalizes the alternating logistic regression approaches for clustered ordinal outcomes (Carey, Zeger and Diggle (1993); Heagerty and Zeger (1996)) and for quantile association (Yang, Chen and Chang (2017)). With given marginal quantiles, we estimate the parameter $\boldsymbol{\alpha}$ in model (2.1) by maximizing the composite likelihood based on the pairwise conditional distributions of $\{Y_{ij}\}$:

$$\mathcal{L}_B(\boldsymbol{\alpha}) = \prod_{i=1}^{m} \prod_{1\le j<k\le n_i} \prod_{c_1,c_2=1}^{L} P(\tilde{Y}_{ik(c_2)} | \tilde{Y}_{ij(c_1)}), \tag{2.3}$$

where $\tilde{Y}_{ij(c)} = \tilde{Y}_{ij}(\tau_c)$ for $c \in \mathcal{L}$, $P(\tilde{Y}_{ik(c_2)}|\tilde{Y}_{ij(c_1)}) = \xi_{ijk(c_1,c_2)}^{\tilde{Y}_{ik(c_2)}}(1-\xi_{ijk(c_1,c_2)})^{1-\tilde{Y}_{ik(c_2)}}$ and $\xi_{ijk(c_1,c_2)} = P(\tilde{Y}_{ik(c_2)} = 1|\tilde{Y}_{ij(c_1)})$ having the form

$$\log\left(\frac{\xi_{ijk(c_1,c_2)}}{1-\xi_{ijk(c_1,c_2)}}\right) = \log(\psi_{ijk(c_1,c_2)})\tilde{Y}_{ij(c_1)} + \nu_{ijk(c_1,c_2)} \tag{2.4}$$

with $\nu_{ijk(c_1,c_2)}$ a one-to-one function of $\psi_{ijk(c_1,c_2)}$ (Diggle et al. (2002), p. 145). The resulting estimator of $\boldsymbol{\alpha}$ is denoted by $\hat{\boldsymbol{\alpha}}_B$. Appendix A.1 provides some details for the estimation.

REMARK 2. As in Yang, Chen and Chang (2017), to pursue more stable computation in maximizing (2.3), the induced smoothing technique is applied which approximates the indicator $\tilde{Y}_{ij}(\tau)$ with the smooth function $1-\Phi\{\delta_{ij}(\tau)\}$ and approximates the pairwise products of the indicator $\tilde{Y}_{ij}(\tau_1)\tilde{Y}_{ik}(\tau_2)$ with the smooth function $\Psi\{\delta_{ij}(\tau_1), \delta_{ik}(\tau_2); \rho_{ijk}(\tau_1,\tau_2)\}$, where $\Psi(a,b;\rho)$ stands for the tail probability $P(\mathcal{Z}_1 > a, \mathcal{Z}_2 > b)$ of a standard bivariate normal $(\mathcal{Z}_1, \mathcal{Z}_2)$ with correlation ρ and $\rho_{ijk}(\tau_1,\tau_2)$ is the estimated asymptotic correlation of $\hat{q}_{\tau_1}(Y_{ij}|\boldsymbol{X}_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\hat{\boldsymbol{\beta}}(\tau_1)$ and $\hat{q}_{\tau_2}(Y_{ik}|\boldsymbol{X}_{ik}) = \boldsymbol{X}_{ik}^{\mathrm{T}}\hat{\boldsymbol{\beta}}(\tau_2)$; see Yang, Chen and Chang (2017) for details about the estimation of the asymptotic correlation.

2.3. *Composite ordinal likelihood.* Let O_{ij} be the ordinal value of Y_{ij} classified by its quantiles $\{q_{\tau_c}(Y_{ij}|\boldsymbol{X}_{ij}), c \in \mathcal{L}\}$; namely, $O_{ij} = c$ if $q_{\tau_{c-1}}(Y_{ij}|\boldsymbol{X}_{ij}) < Y_{ij} \le q_{\tau_c}(Y_{ij}|\boldsymbol{X}_{ij})$ for $c = 1,\dots,L+1$, where $q_{\tau_0}(Y_{ij}|\boldsymbol{X}_{ij}) \equiv -\infty$ and $q_{\tau_{L+1}}(Y_{ij}|\boldsymbol{X}_{ij}) \equiv \infty$. In practice, the quantiles $\{q_{\tau_c}(Y_{ij}|\boldsymbol{X}_{ij}), c \in \mathcal{L}\}$ are estimated, and the O_{ij}'s are defined as above with the quantiles replaced by their estimates $\hat{q}_{\tau}(Y_{ij}|\boldsymbol{X}_{ij}) = \boldsymbol{X}_{ij}^{\mathrm{T}}\hat{\boldsymbol{\beta}}(\tau)$. Alternatively to (2.3), we propose to estimate the quantile transition model (2.1) by maximizing the composite likelihood based on the pairwise conditional distributions of $\{O_{ij}\}$:

$$\mathcal{L}_O(\boldsymbol{\alpha}) = \prod_{i=1}^{m} \prod_{1\le j<k\le n_i} P(O_{ik}|O_{ij}), \tag{2.5}$$

where

$$P(O_{ik}|O_{ij}) = \sum_{c=1}^{L} I(O_{ik} = c)p_{ijk(c)} = \prod_{c=1}^{L}\left(\frac{p_{ijk(c)}}{p_{ijk(c+1)}}\right)^{\tilde{Y}_{ik(c)}} \cdot (1 - P_{ijk(L)}),$$

and $p_{ijk(c)} = P(O_{ik} = c|O_{ij})$, $P_{ijk(c)} = P(O_{ik} \le c|O_{ij})$. Observe that

$$P(O_{ik} \le c_2 | O_{ij} = c_1) = \frac{\tau_{c_1}}{\tau_{c_1} - \tau_{c_1-1}}\xi_{ijk(c_1,c_2)}^{(1)} - \frac{\tau_{c_1-1}}{\tau_{c_1} - \tau_{c_1-1}}\xi_{ijk(c_1-1,c_2)}^{(1)}$$

with $\xi_{ijk(c_1,c_2)}^{(1)} = P(\tilde{Y}_{ik(c_2)} = 1|\tilde{Y}_{ij(c_1)} = 1)$ and $P(\tilde{Y}_{ik(c_2)} = 1|\tilde{Y}_{ij(c_1)})$ given in (2.4). The estimator of $\boldsymbol{\alpha}$ obtained by maximizing (2.5), denoted by $\hat{\boldsymbol{\alpha}}_O$, directly utilizes the ordinal-data information induced by marginal quantiles and is expected to be be more efficient than the estimator obtained from the composite binary likelihood (2.3). The numerical studies in Section 3 confirm this expectation. Kuk (2007) reported a parallel phenomenon for clustered ordinal data.

Similar to $\hat{\boldsymbol{\alpha}}_B$, $\hat{\boldsymbol{\alpha}}_O$ can also be obtained by the induced smoothing method mentioned in Section 2.2 for stable computation; details are given in Appendix A.2. The large sample properties of the estimators $\hat{\boldsymbol{\alpha}}_B$ and $\hat{\boldsymbol{\alpha}}_O$ for the quantile transition parameter $\boldsymbol{\alpha}$ derived from the composite binary and composite ordinal likelihoods, respectively, are described in Section S.1 of the Supplementary Material (Hsu et al. (2020)). In particular, the asymptotic normality of the estimators $\hat{\boldsymbol{\alpha}}_B$ and $\hat{\boldsymbol{\alpha}}_O$ is established, together with the closed-form estimators for the asymptotic variances of the estimators.

2.4. *Testing modeling assumptions and goodness of fit.* We develop formal procedures for testing modeling assumptions and evaluating goodness of fit for the proposed quantile transition regression analysis.

2.4.1. *Testing constancy of coefficients.* As mentioned above, in practice we may impose the restriction that the marginal quantile regression coefficients are equal across all or some of the quantile levels. Similarly, the coefficients in the quantile transition model may be assumed equal across all or some of the quantile level pairs. To test the adequacy of such modeling constraints, we test the null hypothesis of the adequacy of the modeling constraints via the asymptotic chi-square statistic which is based on the asymptotic normality results of the proposed estimators obtained under the model without modeling constraints. When the null hypothesis is rejected, we conclude that the modeling constraints are inadequate and do not consider the model with such constraints. Details about the asymptotic chi-square test for the adequacy of the modeling constraints are given in Section S.2 of the Supplementary Material (Hsu et al. (2020)). For models that are not rejected by the asymptotic chi-square test, we can further evaluate and compare the goodness of fit of the candidate models by the procedure developed below.

As mentioned in Section 2.1, the time-varying coefficient quantile regression and quantile transition models can be reexpressed as the corresponding time-independent coefficient models with additional interaction terms between the covariate variables and the time basis functions. The asymptotic chi-square test in Section S.2 of the Supplementary Material (Hsu et al. (2020)) can be applied to test for the null hypothesis of zero coefficients for these interaction terms which is equivalent to the null hypothesis that the coefficients for the covariate variables are constant over time, that is, the coefficients are time independent.

2.4.2. *Goodness of fit.* In this section we detail the goodness-of-fit procedure proposed for evaluating and testing the adequacy of a quantile transition model. Write $\tilde{\boldsymbol{Y}}_i = (\tilde{\boldsymbol{Y}}_{i1(\cdot)}^{\mathrm{T}}, \ldots, \tilde{\boldsymbol{Y}}_{in_i(\cdot)}^{\mathrm{T}})^{\mathrm{T}}$ with $\tilde{\boldsymbol{Y}}_{ij(\cdot)} = (\tilde{Y}_{ij(1)}, \ldots, \tilde{Y}_{ij(L)})^{\mathrm{T}}$, and denote $\boldsymbol{V}_i = \mathrm{cov}(\tilde{\boldsymbol{Y}}_i)$ as the covariance matrix of $\tilde{\boldsymbol{Y}}_i$. In $\boldsymbol{V}_i$, the elements corresponding to $\mathrm{cov}(\tilde{Y}_{ij(c_1)}, \tilde{Y}_{ij(c_2)}) = \tau_{c_1} \wedge \tau_{c_2} - \tau_{c_1}\tau_{c_2}$ and the elements corresponding to $\mathrm{cov}(\tilde{Y}_{ij(c_1)}, \tilde{Y}_{ik(c_2)}) = \mu_{ijk(c_1,c_2)} - \tau_{c_1}\tau_{c_2}$ for $c_1, c_2 \in \mathcal{L}$, where $\mu_{ijk(c_1,c_2)} = P(\tilde{Y}_{ij(c_1)} = 1, \tilde{Y}_{ik(c_2)} = 1)$, is a function of $\psi_{ijk(c_1,c_2)}$. Let $\tilde{\boldsymbol{s}}_i = \boldsymbol{V}_i^{-1/2}\{\tilde{\boldsymbol{Y}}_i - \mathbf{1}_{n_i} \otimes (\tau_1, \ldots, \tau_L)^{\mathrm{T}}\}$ be the standardized version of $\tilde{\boldsymbol{Y}}_i$ with the involved parameters $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ given by their estimates from the specified models.

When a quantile transition model is adequately specified, the resulting $\boldsymbol{V}_i$ will capture well the actual correlation structure in $\tilde{\boldsymbol{Y}}_i$, and hence all the $n_i \times L$ elements $\tilde{s}_{il}$ of the standardized outcome $\tilde{\boldsymbol{s}}_i$ will essentially be uncorrelated with one another. For $1 \le l < l' \le n_i \times L$, let $\widehat{\mathrm{cor}}(\tilde{\boldsymbol{s}}_{\cdot l}, \tilde{\boldsymbol{s}}_{\cdot l'})$ be the Pearson correlation coefficient based on the paired data $\{(\tilde{s}_{il}, \tilde{s}_{il'}); i \in \mathcal{M}_{ll'}\}$ with $\mathcal{M}_{ll'}$ the set of subjects for whom both $\tilde{s}_{il}$ and $\tilde{s}_{il'}$ are available and $p_{ll'}$ be the corresponding p-value of the hypothesis testing for zero correlation obtained from the Student's

t distribution (Kendall and Stuart (1973)). Denote by κ the number of (l, l') pairs for which the size of $\mathcal{M}_{ll'}$ is sufficiently large (e.g., > 30) and $p_{(1)}$ the smallest one among the κ p-values $p_{ll'}$'s. For balanced longitudinal data with $n_i = n$, $\kappa = \binom{nL}{2}$ and the size of $\mathcal{M}_{ll'} = m$ for all the (l, l') pairs. The proposed goodness-of-fit measure for the quantile transition model is given by

$$T = \kappa p_{(1)}, \tag{2.6}$$

where a larger value of T would suggest a better goodness of fit for the considered model. Further, following the Bonferroni method of multiple testing, we may use T as a goodness-of-fit test statistic and reject the null hypothesis that the quantile transition model is adequate when $T < \alpha$ with α a prespecified significance level.

3. Simulations.

3.1. *Performance of the proposed estimators.* We conduct simulations to examine the performances of the estimators proposed in Sections 2.2 and 2.3 for the quantile transition model parameters. The two covariates are generated as $X_{ij1} \sim \text{Ber}(0.5)$ and $X_{ij2} \sim \text{Uniform}(0, 1)$, $1 \le j \le n_i$, $1 \le i \le m$, $m = 100$ or 200. Two scenarios are considered for generating the outcome data. In Scenario 1, $Y_{ij} = 0.5 + 0.5X_{ij1} + X_{ij2} + \epsilon_{ij}$ $(1 \le j \le n_i$, $1 \le i \le m)$, where $(\epsilon_{i1}, \dots, \epsilon_{in_i})^{\mathrm{T}} = (1 - z_i)\boldsymbol{\epsilon}_{i(1)} + z_i\boldsymbol{\epsilon}_{i(2)}$, $z_i \sim \text{Ber}(0.5)$, $n_i = 3$ or 5 for all i and $\boldsymbol{\epsilon}_{i(c)}$ follows the mixture multivariate normal distribution for $c = 1, 2$. Details of the error distribution is given in Section S.7 of the Supplementary Material (Hsu et al. (2020)).

The marginal quantile regression model for Y_{ij} is correctly specified in the analysis as $q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) = \beta_0(\tau) + \beta_1 X_{ij1} + \beta_2 X_{ij2}$, where only the intercept is allowed to vary with the quantile level τ and the other regression parameters are assumed to be constant across the quantile levels. The true parameter values are given as $\beta_0(\tau) = 0.5 + q_\tau(\epsilon_{ij})$, $\beta_1 = 0.5$ and $\beta_2 = 1$. We consider $\tau \in (0.2, 0.8)$, and the true intercept parameter values in the marginal quantile regression are $\beta_0(0.2) = -0.34$ and $\beta_0(0.8) = 1.34$. The quantile transition analysis assumes the model $\log(\psi_{ijk(c_1,c_2)}) = z_{ijk,1}\alpha_1 + z_{ijk,2}\alpha_2$ with $z_{ijk,1} = 1/|k-j|$ and $z_{ijk,2} = z_i$ which is a correctly specified model. The true parameter values are $\alpha_1 = 2$, $\alpha_2 = -0.5$ when $n_i = 3$ and are $\alpha_1 = 1.8$, $\alpha_2 = -0.5$ when $n_i = 5$.

In Scenario 2 the outcome data are generated from $Y_{ij} = q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) + \{\epsilon_{ij} - q_\tau(\epsilon_{ij})\}$ for $1 \le j \le 4$, and $q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) = \beta_0(\tau) + \beta_{j,1}X_{ij1} + \beta_{j,2}X_{ij2}$, where $\beta_0(\tau)$ is a time-independent intercept, $\beta_{j,1} = a_{11} + a_{12}I(j \ge 3)$ with $a_{11} = 0.5$ and $a_{12} = 0$, and $\beta_{j,2} = a_{21} + a_{22}I(j \ge 3)$ with $a_{21} = 1$ and $a_{22} = 0$, and $(\epsilon_{i1}, \dots, \epsilon_{i4})^{\mathrm{T}}$ follows a mixture multivariate normal distribution. Details of the error distribution is given in Section S.7 of the Supplementary Material (Hsu et al. (2020)).

In the analysis of the simulated data, the models

$$\begin{aligned} q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) &= \beta_0(\tau) + (X_{ij1}, X_{ij2})\boldsymbol{\beta}_j, \quad \text{with } \boldsymbol{\beta}_j = \boldsymbol{A}\boldsymbol{b}(j), \\ \log(\psi_{ijk(c_1,c_2)}) &= \alpha_{jk(c_1,c_2)} = \boldsymbol{A}^*(\tau_{c_1}, \tau_{c_2})\boldsymbol{b}^*(j, k-j) \end{aligned} \tag{3.1}$$

are considered with quantile levels $\tau, \tau_{c_1}, \tau_{c_2} \in (0.25, 0.50)$, the piecewise-linear basis functions $\boldsymbol{b}(j) = \{1, I(j \ge 3)\}$ and $\boldsymbol{b}^*(j, k-j) = \{1, I(j \ge 2), I(k - j \ge 2) \times (k - j)\}$, $\boldsymbol{A}$ a constant coefficient matrix independent of quantile level and $\boldsymbol{A}^*(\tau_{c_1}, \tau_{c_2})$ a coefficient matrix depending on quantile levels. As mentioned in Section 2.1, the models (3.1) can be reexpressed time-independent coefficient models as

$$\begin{aligned} q_\tau(Y_{ij}|\boldsymbol{X}_{ij}) &= \beta_0(\tau) + \beta_1 X_{ij1} + \beta_2 X_{ij2} + \beta_3 X_{ij1} I(j \ge 3) \\ &\quad + \beta_4 X_{ij2} I(j \ge 3), \\ \log(\psi_{ijk(c_1,c_2)}) &= \alpha_0(\tau_{c_1}, \tau_{c_2}) + \alpha_1(\tau_{c_1}, \tau_{c_2}) I(j \ge 2) \\ &\quad + \alpha_2(\tau_{c_1}, \tau_{c_2}) I(k - j \ge 2) \times (k - j). \end{aligned} \tag{3.2}$$

TABLE 1

Simulation results ($\times 10^2$) *for Scenario* 1 *with true parameter values* $\beta_0(0.2) = -0.34$, $\beta_0(0.8) = 1.34$, $\beta_1 = 0.5$, $\beta_2 = 1$, $\alpha_1 = 2$, $\alpha_2 = -0.5$ *and* $\alpha_1 = 1.8$, $\alpha_2 = -0.5$ *for* $n_i = 3$ *and* $n_i = 5$, *respectively*

	$m = 100$				$m = 200$			
	Bias	ESE	ASE	CP	Bias	ESE	ASE	CP
$n_i = 3$								
$\hat{\beta}_0(0.2)$	−0.37	14.82	14.16	92.0	0.35	10.37	10.25	92.8
$\hat{\beta}_0(0.8)$	−0.14	14.21	14.05	93.6	0.08	10.42	10.19	93.2
$\hat{\beta}_1$	−0.07	10.45	10.53	94.1	0.02	7.53	7.63	95.2
$\hat{\beta}_2$	0.33	18.70	18.19	92.7	−0.41	13.67	13.17	92.9
$\hat{\alpha}_{B1}$	2.69	37.82	38.79	94.7	2.18	27.36	28.02	94.8
$\hat{\alpha}_{O1}$	−1.86	35.87	37.40	95.4	−2.08	25.93	26.87	95.1
$\hat{\alpha}_{B2}$	−3.35	50.08	50.00	94.3	−2.60	34.96	35.87	94.3
$\hat{\alpha}_{O2}$	−0.02	47.08	48.42	95.4	−0.47	33.37	34.67	94.7
$n_i = 5$								
$\hat{\beta}_0(0.2)$	−0.34	11.83	11.39	92.6	−0.09	8.26	8.13	93.7
$\hat{\beta}_0(0.8)$	0.04	11.70	11.44	93.7	−0.14	8.32	8.14	94.4
$\hat{\beta}_1$	0.10	8.68	8.40	93.5	0.21	6.06	6.01	94.4
$\hat{\beta}_2$	0.61	14.29	14.40	95.1	−0.06	10.60	10.41	94.3
$\hat{\alpha}_{B1}$	3.15	30.79	30.67	93.8	4.21	22.18	21.93	95.1
$\hat{\alpha}_{O1}$	−0.41	28.78	28.70	93.5	0.90	20.88	20.54	95.2
$\hat{\alpha}_{B2}$	−2.74	30.62	32.03	94.5	−1.69	22.58	23.02	95.1
$\hat{\alpha}_{O2}$	−1.40	29.49	30.91	94.7	−0.44	21.68	22.22	94.9

ESE: empirical standard error, ASE: asymptotic standard error, CP: coverage of 95% C.I.

The models (3.2) are indeed correct models for the data generated under Scenario 2, with the true parameter values $\beta_0(0.25) = -0.17$, $\beta_0(0.50) = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 1$, $\beta_3 = 0$ and $\beta_4 = 0$, $\{\alpha_0(\tau_{c_1}, \tau_{c_2}), \tau_{c_1}, \tau_{c_2} \in (0.25, 0.50)\} = (1.09, 1.07, 1.07, 1.00)$, $\{\alpha_1(\tau_{c_1}, \tau_{c_2}), \tau_{c_1}, \tau_{c_2} \in (0.25, 0.50)\} = (1.04, 1.14, 1.14, 1.00)$ and $\{\alpha_2(\tau_{c_1}, \tau_{c_2}), \tau_{c_1}, \tau_{c_2} \in (0.25, 0.50)\} = (-0.2, -0.2, -0.2, -0.2)$. Since the true value of $\alpha_2(\tau_{c_1}, \tau_{c_2})$ is constant across the quantile level pairs, in the analysis we simplify the models (3.2) assuming $\alpha_2(\tau_{c_1}, \tau_{c_2}) = \alpha_2$ for $\tau_{c_1}, \tau_{c_2} \in (0.25, 0.50)$.

The simulation results based on 1000 replicates for $m = 100$ and 200 are shown in Tables 1 (Scenario 1) and 2 (Scenario 2). We can see that both estimators, the composite binary likelihood and the composite ordinal likelihood estimators, proposed for the quantile transition model (2.1) have negligible finite sample bias. In addition, for both estimators the averages of the standard error estimates over simulations are close to the simulation standard deviations, and the coverage probabilities of the 95% confidence intervals derived from the asymptotic normality are close to the nominal value. This confirms the adequacy of the asymptotic theory in the Supplementary Material (Section S.1; Hsu et al. (2020)). The estimator $\hat{\boldsymbol{\alpha}}_O$, based on the composite ordinal likelihood, exhibits higher efficiency in inference than the estimator $\hat{\boldsymbol{\alpha}}_B$ based on the composite binary likelihood, as expected. On the other hand, the estimator $\hat{\boldsymbol{\alpha}}_B$ has better efficiency in computation than the estimator $\hat{\boldsymbol{\alpha}}_O$. For instance, completing 1000 replicates for $\hat{\boldsymbol{\alpha}}_B$ and $\hat{\boldsymbol{\alpha}}_O$ in Scenario 1 with $n_i = 3$ and $m = 100$ takes 21 and 61 minutes, respectively, in an ordinary personal computer.

3.2. *Performance of the goodness-of-fit procedure.* To examine the performance of the goodness-of-fit procedure proposed in Section 2.4.2, we assume that the marginal quantile regression is correctly specified, and three candidate quantile transition models are to be

TABLE 2

Simulation results ($\times 10^2$) *for Scenario* 2. *The true parameter values are*: $\beta_0(0.25) = -0.17$, $\beta_0(0.5) = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 1$, $\beta_3 = 0$ *and* $\beta_4 = 0$; $\alpha_0(0.25, 0.25) = 1.09$, $\alpha_0(0.25, 0.50) = 1.07$, $\alpha_0(0.50, 0.25) = 1.07$, $\alpha_0(0.50, 0.50) = 1.00$, $\alpha_1(0.25, 0.25) = 1.04$, $\alpha_1(0.25, 0.50) = 1.41$, $\alpha_1(0.50, 0.25) = 1.41$, $\alpha_1(0.50, 0.50) = 1.00$, $\alpha_2 = -0.2$

	$m = 100$				$m = 200$			
	Bias	ESE	ASE	CP	Bias	ESE	ASE	CP
$\hat{\beta}_0(0.25)$	0.27	13.11	12.40	91.2	0.25	8.94	8.95	94.1
$\hat{\beta}_0(0.50)$	−0.33	12.55	12.12	93.5	−0.23	8.75	8.76	94.8
$\hat{\beta}_1$	0.00	12.94	12.78	93.7	−0.05	9.18	9.12	94.1
$\hat{\beta}_2$	0.05	19.82	18.92	92.7	−0.19	14.21	13.71	94.1
$\hat{\beta}_3$	−0.41	16.62	16.25	92.9	−0.07	11.79	11.73	94.4
$\hat{\beta}_4$	0.54	19.76	19.63	93.3	0.04	14.83	14.36	94.5
$\hat{\boldsymbol{\alpha}}_B$								
$\hat{\alpha}_0(0.25, 0.25)$	−5.18	37.73	40.68	96.3	−1.66	27.81	29.05	96.0
$\hat{\alpha}_0(0.25, 0.50)$	−3.87	36.07	39.04	95.8	−1.33	26.92	28.08	96.0
$\hat{\alpha}_0(0.50, 0.25)$	−3.44	36.84	38.44	95.5	−0.81	26.59	27.69	95.6
$\hat{\alpha}_0(0.50, 0.50)$	−1.23	32.17	34.28	96.3	1.43	23.41	24.79	96.2
$\hat{\alpha}_1(0.25, 0.25)$	−3.23	44.06	49.15	95.9	−3.19	32.93	35.23	94.9
$\hat{\alpha}_1(0.25, 0.50)$	−5.99	46.79	49.76	95.6	−4.65	34.46	36.14	96.8
$\hat{\alpha}_1(0.50, 0.25)$	−7.01	47.17	48.71	94.4	−6.03	34.69	35.48	95.2
$\hat{\alpha}_1(0.50, 0.50)$	−6.28	39.56	41.23	95.8	−4.24	27.22	29.89	96.2
$\hat{\alpha}_2$	1.21	9.70	10.01	95.5	0.12	6.98	7.28	95.5
$\hat{\boldsymbol{\alpha}}_O$								
$\hat{\alpha}_0(0.25, 0.25)$	−6.22	37.10	40.09	96.1	−2.89	27.41	28.57	95.7
$\hat{\alpha}_0(0.25, 0.50)$	−5.13	35.26	38.46	96.1	−2.62	26.51	27.66	95.6
$\hat{\alpha}_0(0.50, 0.25)$	−4.52	36.20	37.84	95.7	−2.15	26.00	27.21	95.8
$\hat{\alpha}_0(0.50, 0.50)$	−2.17	31.76	33.68	95.9	0.29	23.02	24.39	94.6
$\hat{\alpha}_1(0.25, 0.25)$	−2.67	43.44	48.13	96.1	−2.43	32.33	34.53	95.1
$\hat{\alpha}_1(0.25, 0.50)$	−5.43	45.87	48.62	95.8	−4.11	33.97	35.29	96.3
$\hat{\alpha}_1(0.50, 0.25)$	−6.91	46.40	47.73	94.6	−5.45	33.98	34.83	94.6
$\hat{\alpha}_1(0.50, 0.50)$	−5.84	38.91	40.36	95.9	−3.46	26.71	29.31	96.1
$\hat{\alpha}_2$	1.77	9.33	9.86	94.7	0.78	6.82	7.17	95.2

ESE: empirical standard error, ASE: asymptotic standard error, CP: coverage of 95% C.I.

compared and tested with respect to their goodness of fit to the data simulated from Scenario 1 of Section 3.1 with $n_i = 3$. The candidate models are: Model 1: $\log(\psi_{ijk(c_1,c_2)}) = \alpha_{(c_1,c_2)}$; Model 2: $\log(\psi_{ijk(c_1,c_2)}) = \alpha_0 + \alpha_1 z^*_{ijk} + \alpha_2 z_i$, where $z^*_{i12} = 0$ and $z^*_{ijk} = 1$ for $(j, k) \neq (1, 2)$; and Model 3: $\log(\psi_{ijk(c_1,c_2)}) = \alpha_0 + \alpha_1 I(k - j = 2) + \alpha_2 z_i$. Note that Model 3 is the correct model with the true parameter values $\alpha_0 = 2$, $\alpha_1 = -1$ and $\alpha_2 = -0.5$.

The goodness-of-fit measure T proposed in Section 2.4.2 is applied for evaluating the adequacy of the three candidate models; recall that a larger T value indicates a better fit. Table 3 presents the percentages (among 1000 simulations) for each candidate model whose T value is the largest among those from the three candidate models. We see that the goodness-of-fit measure T nicely picks up the adequate quantile transition model, no matter which estimation ($\hat{\boldsymbol{\alpha}}_B$ or $\hat{\boldsymbol{\alpha}}_O$) is used; the performance enhances when the sample size increases. Table 3 also presents the percentages of the proposed goodness-of-fit test rejecting the null hypothesis that the model under consideration is adequate, namely, the type I error rates (powers) of the goodness-of-fit test when the model under consideration is correct (incorrect). We see that the goodness-of-fit test, based on T using either $\hat{\boldsymbol{\alpha}}_B$ or $\hat{\boldsymbol{\alpha}}_O$, has adequate type I

TABLE 3

The percentage (%) over 1000 simulations of each model whose T value is the largest among those of the three candidate models, with the T value calculated based on the composite binary likelihood (T_B) or the composite ordinal likelihood (T_O) estimation. Also reported are the percentage (%) over 1000 simulations of each model whose goodness-of-fit test based on T rejects the null hypothesis of the model being adequate

	$m = 100$			$m = 200$		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
T_B	13.8	17.4	68.8	4.0	9.1	86.9
T_O	12.4	17.0	70.6	3.6	8.5	87.9
Test, T_B	18.9	20.0	5.9	41.6	40.3	5.6
Test, T_O	18.7	19.4	5.6	41.6	39.1	4.8

error rates close to the desired 5% level, and the powers of the goodness-of-fit test increase with the sample sizes.

In Section S.3 of the Supplementary Material (Hsu et al. (2020)), we provide simulation results for the performance of the proposed estimators under lower and higher quantiles. The results reveal that the proposed estimators still work reasonably well for lower and higher quantiles, such as 5% and 95% quantiles, although a larger sample size is required for the asymptotic normality theory to work well. In Section S.4 of the Supplementary Material (Hsu et al. (2020)), we provide more simulation results confirming the adequacy of the proposed goodness-of-fit procedure.

4. Analysis of the wage data in Taiwan.

4.1. *The wage data.* The wage data used in the analysis are from the PSFD survey whose details have been given in the Introduction section. The sample used in our analysis consists of two groups of respondents in the PSFD survey who were first interviewed in 2003 and 2009, respectively. The corresponding birth years of these two cohorts are 1964–76 and 1977–82. These two cohorts of 1152 and 2179 individuals, respectively, are referred to as the older and younger generations in our analysis.

The observed longitudinal wage data for the younger and older generations in the PSFD survey are unbalanced, where the number n_i of observations per subject ranges from two to 12. The unbalanced data for a subject is mainly of an intermittent style due to accidentally missing the interview visits. Also, there exists no significant difference in baseline characteristics such as gender, age generation and job type among subjects with different missing data patterns. The total number of observations amounts to 19,052 up to the year of 2016. Observations with no job or no earnings from work are removed from the analysis. The resulting analytical sample contains 16,222 wage observations, in which 7444 and 8778 observations belong to the older and younger generations, respectively.

4.2. *The marginal quantile and quantile transition models.* Our main goal is to examine how the wages of Taiwanese workers transit across the wage quantiles during the follow-up years as well as the factors associated with the transition. For a time point (year), we consider wage statuses classified by wage quantiles at levels 25%, 50% and 75%. The quantile transition regression analysis proposed in Section 2 is applied to the longitudinal wage data from the PSFD survey.

First, we obtain wage quantile estimates over the follow-up years conditional on demographic, job-, eduction- and heath-related covariates. We use the consumer price index (CPI) adjusted real hourly earnings, in units of thousand New Taiwan Dollars (NTD), as the outcome variable in our analysis. According to our study interests and following our preliminary analyses, the covariates considered for the marginal earnings quantile regression include respondent's gender (female vs. male), generation (younger vs. older), age (years), education (high school, vocational school, university, postgraduate vs. below high school), employment status (government/public-sector employee, private-sector employee vs. employer/self employed), job (service industry, blue collar vs. white collar), health condition (moderate/poor vs. good) and residential area (rural vs. urban); the former two are time-fixed while the others are time-varying covariates. We apply the linear quantile regression as mentioned in Section 2.1 to estimate the marginal earnings quantiles given the covariates, with the regression coefficients estimated from the estimating equation (2.2).

When applying the quantile transition regression model (2.1) to the PSFD wage data, the covariates considered in the model for the wage status transition between two time points include gender, generation, age, education, employment status and time lag between the two time points (years) where the time-varying covariates, except for the time lag in the model, take on the values at the earlier time point. The composite binary and composite ordinal likelihood approaches proposed in Sections 2.2 and 2.3 are applied to make inference on the quantile transition regression model.

The time variable in the analysis of the PSFD wage data is the calendar year minus 2003 (the starting year of the PSFD study) and takes integer values from 0 to 13; the time lag variable takes integer values from 1 to 13. Following Section 2.1, in addition to the time-independent coefficient models, we also consider time-varying coefficient models for the marginal quantile and the quantile transition models. We consider piecewise-constant time-varying coefficients in the marginal quantile regression coefficients using the basis functions $\boldsymbol{b}(t) = \{I(t \in I_k); k = 1, 2, 3\}$ with the time intervals $I_1 = [0, 5)$, $I_2 = [5, 10)$, $I_3 = [10, 13]$. Similarly, the quantile transition regression coefficients are also time-varying and given by the basis functions $\boldsymbol{b}(t, s) = \{I(t \in I_k), I(s \in U_l); k, l = 1, 2, 3\}$, where (t, s) are the initial time and the time lag for the two time points considered and the time intervals I_k are given as above and the time-lag intervals are $U_1 = [1, 5)$, $U_2 = [5, 10)$, $U_3 = [10, 13]$. The choice of the time and time-lag intervals as above, which are roughly five-year intervals, is based on the consideration of the full follow-up period of the analysis as well as the fact that the wage quantile transition over five-year periods is of our main interest. We also consider piecewise constant intervals of shorter (roughly three years) or longer (roughly seven years) lengths. In addition, we employ linear basis functions, namely, the basis function in the marginal quantile regression is $\boldsymbol{b}(t) = t$ and the basis functions in the quantile transition model are $\boldsymbol{b}(t, s) = \{t, s\}$ with (t, s) defined as above. All these time-varying coefficient models result in models with time effects (i.e., estimated coefficients for the interaction terms between the covariates and the time basis functions, see Section 2.1 for details) being virtually null, as revealed by the asymptotic chi-square test in Section 2.4.1 and Section S.2 of the Supplementary Material (Hsu et al. (2020)). We thus report only the analysis results from the time-independent coefficient models.

For the time-independent coefficient marginal quantile and quantile transition models, we apply the asymptotic chi-square test proposed in Section S.2 of the Supplementary Material (Hsu et al. (2020)) to see whether simpler models with constant coefficients across different quantile levels (or quantile level pairs in the quantile transition model) can be considered. To streamline the testing procedure, we apply the asymptotic chi-square test in a variable-wise manner, namely, we test the constancy of coefficients across quantile levels (quantile level pairs) for one covariate variable at a time. When the constancy of the coefficients for a covariate variable is tested, the coefficients for all the other covariate variables are left as unconstrained. When the null hypothesis of constant coefficients is rejected for one covariate variable, the coefficients for that variable across different quantile levels (quantile level pairs) are set to different coefficient parameters; otherwise, the coefficients for that variable across different quantile levels (quantile level pairs) are set to a single parameter. After all the covariate variables have been tested, the resulting final model is further confirmed by the goodness-of-fit procedure mentioned in Section 2.4.2.

4.3. *Results and main empirical findings.* The estimated coefficients for the marginal quantiles and quantile transition regression models, determined by the procedures in the previous section, are reported in Tables 4 and 5, respectively. It is seen from Table 4, or Figure 1, that the 25%, 50% and 75% quantiles of real hourly earnings of female workers are significantly lower than those of males. Also, after adjusting for age, the younger-generation workers have lower earnings quantiles than do the older-generation workers. The earnings quantiles are observed to increase with age but decrease with age squared, revealing a concave pattern of the relationship between age and earnings. The earnings quantiles of higher-educated workers are significantly higher. According to the test procedure mentioned in the end of the last subsection, the linear age and the university education level effects are determined to be nonconstant across quantile levels, with both effects being positive and increasing with quantile levels. Workers in moderate or poor health condition tend to have lower quantiles of earnings. Compared to those of white-collar workers, earnings quantiles are significantly lower for service industry and blue-collar workers. In addition, earnings quantiles of workers residing in rural areas are significantly lower than those of workers residing in ur-

TABLE 4

Parameter estimates and standard errors for marginal quantile regression analysis of the real hourly earnings (in thousand NTD) in Taiwan with quantile levels (0.25, 0.50, 0.75)

Marginal quantile regression	$\hat{\boldsymbol{\beta}} \times 10^2$	SE $\times 10^2$	p-value
Intercept (0.25)	11.46	0.72	0.00‡
Intercept (0.50)	14.16	0.71	0.00‡
Intercept (0.75)	17.47	0.73	0.00‡
Gender (female vs. male)	−2.61	0.24	0.00‡
Generation (younger vs. older)	−1.78	0.31	0.00‡
Age (0.25)	0.38	0.05	0.00‡
Age (0.50)	0.56	0.05	0.00‡
Age (0.75)	0.83	0.05	0.00‡
Age^2	−0.01	0.00	0.00‡
Education (vs. below high school)			
high school	1.77	0.49	0.00‡
vocational school	4.32	0.52	0.00‡
university (0.25)	6.33	0.57	0.00‡
university (0.50)	7.22	0.60	0.00‡
university (0.75)	7.95	0.69	0.00‡
postgraduate	10.86	0.64	0.00‡
Employment status (vs. emplyer/self-employed)			
government/public-sector employees	1.77	0.65	0.01‡
private-sector employees	0.10	0.51	0.84
Job (vs. white collar)			
blue collar	−2.00	0.33	0.00‡
service industry	−1.22	0.22	0.00‡
Health condition (moderate/poor vs. good)	−0.50	0.11	0.00‡
Residential area (rural vs. urban)	−1.26	0.23	0.00‡

‡ p-value < 0.01; † p-value < 0.05; * p-value < 0.1

TABLE 5

The composite binary likelihood ($\hat{\boldsymbol{\alpha}}_B$) and composite ordinal likelihood ($\hat{\boldsymbol{\alpha}}_O$) estimates and standard errors, in parentheses, for the quantile transition regression analysis of the wage data in Taiwan with quantile levels (0.25, 0.50, 0.75)

Quantile transition regression	$\hat{\boldsymbol{\alpha}}_B \times 10^2$ (SE $\times 10^2$)	$\hat{\boldsymbol{\alpha}}_O \times 10^2$ (SE $\times 10^2$)
Intercept (0.25, 0.25)	139.47‡ (27.74)	135.51‡ (22.70)
Intercept (0.25, 0.50)	122.01‡ (27.33)	120.51‡ (22.14)
Intercept (0.25, 0.75)	112.43‡ (27.63)	113.18‡ (22.58)
Intercept (0.50, 0.25)	135.70‡ (27.80)	131.55‡ (22.60)
Intercept (0.50, 0.50)	127.04‡ (27.42)	123.35‡ (22.26)
Intercept (0.50, 0.75)	124.07‡ (27.40)	122.27‡ (22.22)
Intercept (0.75, 0.25)	128.54‡ (28.29)	126.87‡ (23.57)
Intercept (0.75, 0.50)	132.48‡ (27.88)	130.37‡ (22.72)
Intercept (0.75, 0.75)	146.45‡ (27.88)	144.45‡ (22.49)
Gender (female vs. male)	44.38‡ (10.88)	47.61‡ (8.92)
Generation (younger vs. older)	−1.90 (12.08)	13.41 (9.81)
Age (year)	7.69‡ (1.25)	8.37‡ (0.98)
Time lag (year)	−11.05‡ (0.97)	−10.90‡ (0.92)
Education (vs. below high school)		
high school	1.12 (20.41)	−25.84 (16.46)
vocational school	−6.98 (21.33)	−35.80† (17.07)
university	39.32 (24.10)	2.29 (18.64)
postgraduate	−4.29 (26.10)	−55.68‡ (20.82)
Employment status (vs. employer/self employed)		
government/public-sector employees	14.06 (21.53)	36.95† (17.01)
private-sector employees	34.86† (15.71)	60.03‡ (12.68)

‡ p-value < 0.01; † p-value < 0.05; * p-value < 0.1.

ban areas. Government/public-sector employees have significantly higher earnings quantiles than workers of other types of employment status.

Table 5 displays results for regression analysis of the quantile transition of earnings statuses. Both analyses, based on the composite binary and composite ordinal likelihoods, show that, compared to male workers, female workers are more likely to stay in the same earn-

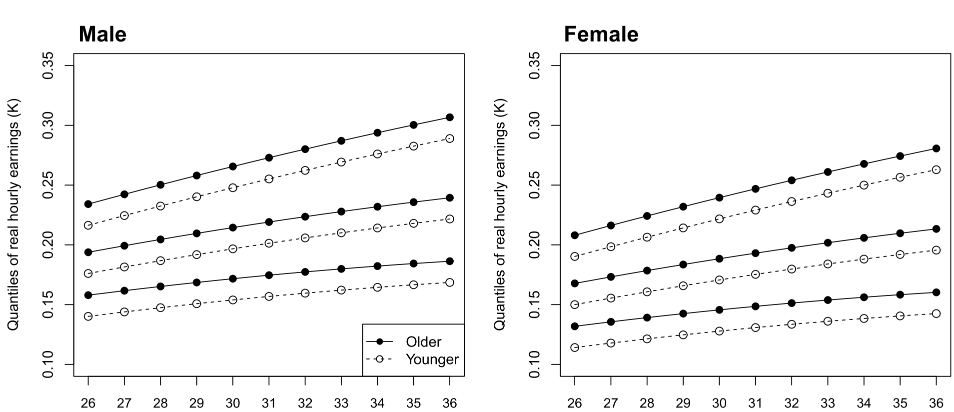

FIG. 1. *Quantiles at levels* (0.25, 0.50, 0.75) *of real hourly earnings* (*in thousand NTD*) *for urban blue-collar workers of* 26 *to* 36*-years-old*, *university education level and good health condition*, *by their gender and generation*, *based on the marginal quantile regression model shown in Table* 4.

ings status over time with respect to the covariate-adjusted marginal earnings quantiles. The tendencies of earnings status transition are not significantly different between younger- and older-generation workers. Also, as the age of a worker increases the likelihood of being stuck in the same earnings status becomes higher. The likelihood of transition into some other earnings statuses increases with time, as revealed by the estimated coefficient of the time lag. Compared to those who are employers/self employed and government/public-sector employees, private-sector employees are more likely to get stuck in the same earnings status over time. Also, workers with postgraduate education are found to be more likely to escape from the previous earnings status in the composite ordinal likelihood analysis. The other factors, such as job, health condition and residential area have no significant effects on the quantile transition and, hence, are excluded from the model. We have confirmed that the model presented in Table 5 is adequate by applying the goodness-of-fit evaluation procedure proposed in Section 2.4.2. The model has the goodness-of-fit measure $T = 0.45$, and the null hypothesis of the model, being adequate, cannot be rejected at 5% significance level.

Based on the the composite ordinal likelihood estimates, for different gender, generation and employment status groups, Figure 2 displays the tendencies of stagnation of earnings status over time, given by the conditional probabilities that the wage is below the τth quintile after some years given that the wage is initially below the τth quantile with initial age 26 years and university education level. We can see that female and younger generation workers are associated with higher degrees of earnings status stagnation than male and older generation workers, in particular, for earnings statuses corresponding to lower earnings quantile levels. Also, private-sector employees suffer from earnings status stagnation more seriously than workers of other employment types, in particular, for earnings statuses corresponding to lower earnings quantile levels. For example, the probability that female, younger generation and private-sector workers, who were initially in the lowest 25% wage group, still remain in the lowest 25% wage group 12 years later is as high as 45%, while the corresponding quantile transition probability for male, older generation and private-sector workers is 35%.

In Section S.6 of the Supplementary Material (Hsu et al. (2020)), we provide the analysis of the PSFD wage data based on the quantile dynamic regression model, which is the same as the marginal quantile regression model considered in Table 4 but further includes the real hourly earnings at the previous time point as a covariate in the model. The covariate effects obtained from the quantile dynamic model are similar to those shown in Table 4. The coefficient for the previous earnings reveals that the quantiles of the current real hourly earnings are highly associated with the previous earnings. However, the results from this model cannot reveal information about how the earnings status at a time point, defined with respect to the earnings quantiles at that time, is related to the past earnings statuses, such as the quantile transition probabilities mentioned at the end of the last paragraph.

5. Conclusions. Motivated by the desire to realize which groups of workers in Taiwan suffer more seriously from the wage stagnation problem, this study considers a regression model for the quantile transition which quantifies the degrees of transition from a quantile level to another for the outcomes between two time points. Based on the PSFD survey data, the proposed quantile transition analysis demonstrates that, compared to male workers, female workers in Taiwan have lower quantiles of real earnings and lower probabilities of transition across earnings statuses defined in terms of marginal covariate-adjusted earnings quantiles. The younger generation, who were born between 1977 and 1982, have a lower quantile of real earnings adjusting for age and other covariates than the older generation born between 1964 and 1976. Besides, private-sector employees are more likely to get stuck in the same earnings status over time compared to the government/public-sector employees and employers/self-employed workers. Higher age and education below postgraduate level are

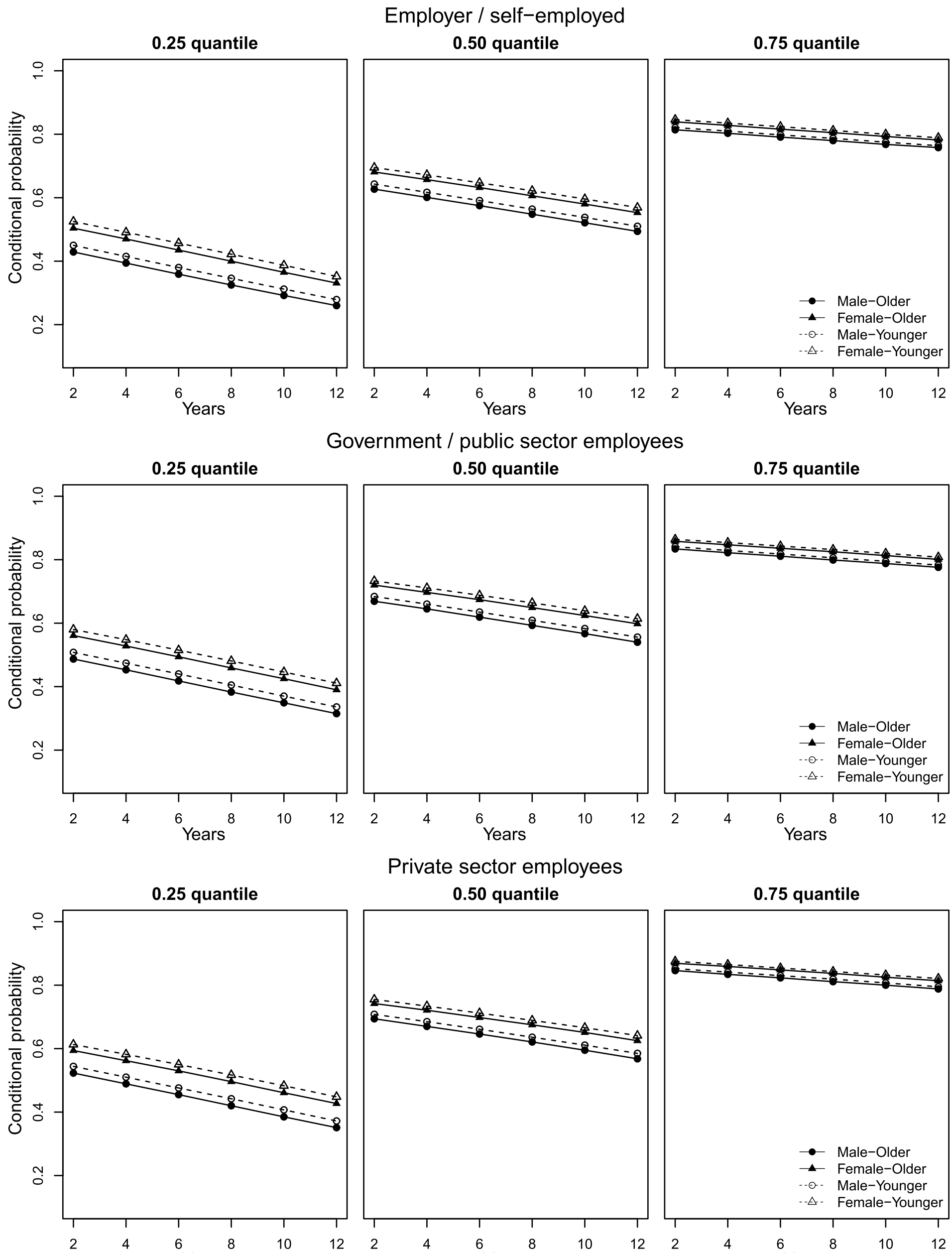

FIG. 2. *Conditional probabilities of earnings status stagnation over time and different quantiles by gender, generation, and employment status, with age fixed at* 26 *years initially and education level fixed at university.*

also associated with higher risks of wage stagnation. The proposed statistical methodology helps identify some critical demographic and job-related factors associated with long-term stagnation of earnings status.

The model employed in the current work extends the previous ones coping specifically with bivariate data (Li, Cheng and Fine (2014)) and longitudinal data at a single quantile level (Yang, Chen and Chang (2017)). We propose two estimators for the regression coefficients in the quantile transition regression model based, respectively, on the composite binary and composite ordinal likelihoods. Both estimators are consistent, provided that the quantile transition regression model is correctly specified. The estimator from the composite ordinal likelihood is more efficient in inference than that from the composite binary likelihood. Simulation results in the Supplementary Material (Section S.5; Hsu et al. (2020)) also show that, when the quantile transition regression model is subject to moderate model misspecification, the composite ordinal likelihood results in more accurate estimates of the local odds ratio (LOR) for the quantile transition. On the other hand, the composite binary likelihood estimator is more efficient in computation than the composite ordinal likelihood estimator. Asymptotic theory developed for the proposed composite binary and ordinal likelihood estimators facilitates interval estimation and hypothesis testing regarding the significance of the regression parameters in the quantile transition model. In particular, certain modeling assumptions, such as constant coefficients over quantile levels and/or time, can be tested via the asymptotic distribution theory for the proposed estimators. Also, a goodness-of-fit procedure is proposed for evaluating and testing the adequacy of a quantile transition model.

In some applications extreme quantiles are of interest. Yet, the extension of our quantile transition analysis to the latter setting would require results and methods from extreme value theory (de Haan and Ferreira (2006)), via extremal quantile regression (Chernozhukov (2005)) and is beyond the scope of the current work.

The methodology introduced in this paper is restricted to a discrete set of predetermined quantile levels. It is of interest to examine the quantile transition over a continuum of quantile levels which may require extensions of the method in Frumento and Bottai (2016) to the quantile transition. We will investigate such an extension in our future work.

APPENDIX: DETAILS FOR PARAMETER ESTIMATION

A.1. The estimating equation for $\boldsymbol{\alpha}_B$. Following Yang, Chen and Chang (2017), the maximizer of $\mathcal{L}_B(\boldsymbol{\alpha})$ can be obtained by the solution to the estimating equation

$$U_B(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \boldsymbol{C}\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{w}_i = \boldsymbol{0}, \tag{A.1}$$

where $\boldsymbol{Z}_i = (\boldsymbol{Z}_{i12}^{*\mathrm{T}}, \boldsymbol{Z}_{i13}^{*\mathrm{T}}, \dots, \boldsymbol{Z}_{i1n_i}^{*\mathrm{T}}, \boldsymbol{Z}_{i23}^{*\mathrm{T}}, \dots, \boldsymbol{Z}_{i(n_i-1)n_i}^{*\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{Z}_{ijk}^{*} = \boldsymbol{I}_{L^2} \otimes \boldsymbol{Z}_{ijk}^{\mathrm{T}}$ with $\boldsymbol{I}_{L^2}$ the $L^2 \times L^2$ identity matrix, $\boldsymbol{w}_i = (\boldsymbol{w}_{i12}^{\mathrm{T}}, \boldsymbol{w}_{i13}^{\mathrm{T}}, \dots, \boldsymbol{w}_{i1n_i}^{\mathrm{T}}, \boldsymbol{w}_{i23}^{\mathrm{T}}, \dots, \boldsymbol{w}_{i(n_i-1)n_i}^{\mathrm{T}})^{\mathrm{T}}$ with

$$\boldsymbol{w}_{ijk} = (w_{ijk(1,1)}, w_{ijk(1,2)}, \dots, w_{ijk(1,L)}, w_{ijk(2,1)}, \dots, w_{ijk(L,L-1)}, w_{ijk(L,L)})^{\mathrm{T}}$$

and

$$w_{ijk(c_1,c_2)} = \tilde{Y}_{ij(c_1)}\tilde{Y}_{ik(c_2)} - \xi_{ijk(c_1,c_2)}^{(1)}\tilde{Y}_{ij(c_1)} + d_{ijk(c_1,c_2)}(\tilde{Y}_{ik(c_2)} - \xi_{ijk(c_1,c_2)}), \tag{A.2}$$

where $\xi_{ijk(c_1,c_2)}^{(1)} = P(\tilde{Y}_{ik(c_2)} = 1 | \tilde{Y}_{ij(c_1)} = 1)$, and $d_{ijk(c_1,c_2)} = \partial \nu_{ijk(c_1,c_2)} / \partial \log(\psi_{ijk(c_1,c_2)})$.

Through the induced smoothing technique, the smoothed version of (A.1) is given by

$$\tilde{U}_B(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \boldsymbol{C}\boldsymbol{Z}_i^{\mathrm{T}}\tilde{\boldsymbol{w}}_i = \boldsymbol{0}, \tag{A.3}$$

where $\tilde{\boldsymbol{w}}_i$ is defined analogously to $\boldsymbol{w}_i$ in (A.1) with $w_{ijk(c_1,c_2)}$ replaced by

$$\tilde{w}_{ijk(c_1,c_2)} = g_{ijk(c_1,c_2)} - \xi_{ijk(c_1,c_2)}^{(1)} g_{ij(c_1)} + d_{ijk(c_1,c_2)}(g_{ik(c_2)} - \xi_{ijk(c_1,c_2)}),$$

with $g_{ij(c_1)} = 1 - \Phi\{\delta_{ij(c_1)}\}$, $g_{ijk(c_1,c_2)} = \Psi\{\delta_{ij(c_1)}, \delta_{ik(c_2)}; \rho_{ijk(c_1,c_2)}\}$ for $1 \le i \le m$, $1 \le j \le n_i$, $c_1, c_2 \in \mathcal{L}$, with $\delta_{ij(c)} = \delta_{ij}(\tau_c)$, $\rho_{ijk(c_1,c_2)} = \rho_{ijk}(\tau_{c_1}, \tau_{c_2})$. The estimator $\hat{\boldsymbol{\alpha}}_B$ can be obtained as the solution of (A.3).

A.2. The estimating equation for $\boldsymbol{\alpha}_O$. Taking logarithm of (2.5) and differentiation with respect to $\boldsymbol{\alpha}$ leads to the following estimating equation for $\boldsymbol{\alpha}$:

$$\text{(A.4)}\quad \begin{aligned} U_O(\boldsymbol{\alpha}) &= \sum_{i=1}^{m} \sum_{1 \le j < k \le n_i} \left(\frac{\partial \boldsymbol{P}_{ijk(\cdot)}}{\partial \boldsymbol{\alpha}} \right)^{\mathrm{T}} \boldsymbol{S}_{ijk}^{-1} (\tilde{\boldsymbol{Y}}_{ik(\cdot)} - \boldsymbol{P}_{ijk(\cdot)}) \\ &= \sum_{i=1}^{m} \boldsymbol{C}\boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{u}_i = \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{P}_{ijk(\cdot)} = (P_{ijk(1)}, \dots, P_{ijk(L)})^{\mathrm{T}}$, $\tilde{\boldsymbol{Y}}_{ik(\cdot)} = (\tilde{Y}_{ik(1)}, \dots, \tilde{Y}_{ik(L)})^{\mathrm{T}}$, $\boldsymbol{S}_{ijk}$ is an $L \times L$ matrix whose (c_1, c_2) element $\boldsymbol{S}_{ijk}[c_1, c_2] = P_{ijk(c_1 \wedge c_2)} - P_{ijk(c_1)} P_{ijk(c_2)}$, $\boldsymbol{u}_i = (\boldsymbol{u}_{i12}^{\mathrm{T}}, \boldsymbol{u}_{i13}^{\mathrm{T}}, \dots, \boldsymbol{u}_{i1n_i}^{\mathrm{T}}, \boldsymbol{u}_{i23}^{\mathrm{T}}, \dots, \boldsymbol{u}_{i(n_i-1)n_i}^{\mathrm{T}})^{\mathrm{T}}$ with

$$\text{(A.5)}\quad \begin{aligned} \boldsymbol{u}_{ijk} &= (\boldsymbol{u}_{ijk(1,\cdot)}^{\mathrm{T}}, \boldsymbol{u}_{ijk(2,\cdot)}^{\mathrm{T}}, \dots, \boldsymbol{u}_{ijk(L,\cdot)}^{\mathrm{T}})^{\mathrm{T}}, \\ \boldsymbol{u}_{ijk(c,\cdot)} &= \operatorname{diag}\{\boldsymbol{e}_{ijk(c,\cdot)}\} \left\{ \frac{\tau_c}{\tau_c - \tau_{c-1}} \boldsymbol{S}_{ijk(c)}^{-1} (\tilde{Y}_{ij(c)} - \tilde{Y}_{ij(c-1)})(\tilde{\boldsymbol{Y}}_{ik(\cdot)} - \boldsymbol{P}_{ijk(c,\cdot)}) \right. \\ &\quad \left. - \frac{\tau_c}{\tau_{c+1} - \tau_c} \boldsymbol{S}_{ijk(c+1)}^{-1} (\tilde{Y}_{ij(c+1)} - \tilde{Y}_{ij(c)})(\tilde{\boldsymbol{Y}}_{ik(\cdot)} - \boldsymbol{P}_{ijk(c+1,\cdot)}) \right\}, \end{aligned}$$

and $\boldsymbol{e}_{ijk(c,\cdot)} = (e_{ijk(c,1)}, \dots, e_{ijk(c,L)})^{\mathrm{T}}$ with $e_{ijk(c_1,c_2)} = \xi_{ijk(c_1,c_2)}^{(1)}(1 - \xi_{ijk(c_1,c_2)}^{(1)}) \times (1 + d_{ijk(c_1,c_2)})$ and $d_{ijk(c_1,c_2)}$ defined in (A.2). In the above, $\boldsymbol{S}_{ijk(c)}$ and $\boldsymbol{P}_{ijk(c,\cdot)}$ are defined as $\boldsymbol{S}_{ijk}$ and $\boldsymbol{P}_{ijk(\cdot)}$, respectively, with O_{ij} set to c, $\tilde{Y}_{ij(0)} = 0$ and $\tilde{Y}_{ij(L+1)} = 1$.

A smoothed version $\tilde{U}_O(\boldsymbol{\alpha}) = \sum_i \boldsymbol{C}\boldsymbol{Z}_i^{\mathrm{T}} \tilde{\boldsymbol{u}}_i$ of $U_O(\boldsymbol{\alpha})$ in (A.4), similar to $\tilde{U}_B(\boldsymbol{\alpha})$ in (A.3), can also be obtained by the induced smoothing method mentioned in Remark 2 of Section 2.2. The estimator $\hat{\boldsymbol{\alpha}}_O$ can be obtained as the solution of $\tilde{U}_O(\boldsymbol{\alpha}) = \boldsymbol{0}$.

Acknowledgments. We are grateful to the Editor, Beth Ann Griffin, the Associate Editor and two referees for their very helpful comments and suggestions. This work is supported by the Ministry of Science and Technology of Taiwan (MOST 106-2118-M-001-016-MY3).

SUPPLEMENTARY MATERIAL

Asymptotic theory, asymptotic chi-square test, and additional simulation studies and data analysis (DOI: 10.1214/19-AOAS1304SUPP; .pdf). A PDF document providing the asymptotic theory, the asymptotic chi-square test for the adequacy of the modeling constraints, and additional simulation studies and data analysis.

REFERENCES

BROWN, B. M. and WANG, Y.-G. (2005). Standard errors and covariance matrices for smoothed rank estimators. *Biometrika* **92** 149–158. MR2158616 https://doi.org/10.1093/biomet/92.1.149

CAREY, V. C., ZEGER, S. L. and DIGGLE, P. J. (1993). Modelling multivariate binary data with alternating logistic regressions. *Biometrika* **80** 517–526.

CHEN, H.-J. and KUO, C.-W. (2014). The possible causes and solutions for Taiwan's wage stagnation. In *The R.O.C. Economy Yearbook* (Economic Daily News, ed.) 32–39. Economic Daily News, Taipei.

CHERNOZHUKOV, V. (2005). Extremal quantile regression. *Ann. Statist.* **33** 806–839. MR2163160 https://doi.org/10.1214/009053604000001165

DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory: An Introduction*. *Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR2234156 https://doi.org/10.1007/0-387-34471-3

DIGGLE, P. J., HEAGERTY, P. J., LIANG, K.-Y. and ZEGER, S. L. (2002). *Analysis of Longitudinal Data*, 2nd ed. *Oxford Statistical Science Series* **25**. Oxford Univ. Press, Oxford. MR2049007

FRUMENTO, P. and BOTTAI, M. (2016). Parametric modeling of quantile regression coefficient functions. *Biometrics* **72** 74–84. MR3500576 https://doi.org/10.1111/biom.12410

GALVAO, A. F. JR. (2011). Quantile regression for dynamic panel data with fixed effects. *J. Econometrics* **164** 142–157. MR2821799 https://doi.org/10.1016/j.jeconom.2011.02.016

HAEPP, T. and HSIN, P.-L. (2016). Is Taiwan's workforce underpaid? Evidence from marginal product of labor estimates at the company level. *J. Soc. Sci. Philos.* **28** 299–331.

HEAGERTY, P. J. and ZEGER, S. L. (1996). Marginal regression models for clustered ordinal measurements. *J. Amer. Statist. Assoc.* **91** 1024–1036.

HSU, C.-Y., CHEN, Y.-H., YU, R.-R. and HUNG, T.-W. (2020). Supplement to "Assessing wage status transition and stagnation using quantile transition regression." https://doi.org/10.1214/19-AOAS1304SUPP.

HUANG, D.-S., LIU, B.-J. and YANG, T.-H. (2014). The phenomena of Taiwan's real wage stagnation: Global trend and Taiwanese characteristics. IEAS Working Paper No. 14-A012.

KENDALL, M. G. and STUART, A. (1973). *The Advanced Theory of Statistics. Vol.* 2: *Inference and Relationship*, 3rd ed. Hafner, New York. MR0474561

KOENKER, R. (2005). *Quantile Regression. Econometric Society Monographs* **38**. Cambridge Univ. Press, Cambridge. MR2268657 https://doi.org/10.1017/CBO9780511754098

KOENKER, R. and BASSETT, G. JR. (1978). Regression quantiles. *Econometrica* **46** 33–50. MR0474644 https://doi.org/10.2307/1913643

KUK, A. Y. C. (2007). A hybrid pairwise likelihood method. *Biometrika* **94** 939–952. MR2416800 https://doi.org/10.1093/biomet/asm051

LI, R., CHENG, Y. and FINE, J. P. (2014). Quantile association regression models. *J. Amer. Statist. Assoc.* **109** 230–242. MR3180559 https://doi.org/10.1080/01621459.2013.847375

LI, C.-T. and FANG, C.-T. (2015). *The Causes of the Decoupling of Taiwan's Labor Productivity and Wages. Presented at Conference of the Taiwanese Economic Associations*. National Taiwan Univ., Taipei.

LIN, C.-C., CHANG, J.-J. and LU, S.-S. (2017). Wage stagnation? Fact disclosure and cross-country comparison. *J. Soc. Sci. Philos.* **29** 87–125.

YANG, C.-C., CHEN, Y.-H. and CHANG, H.-Y. (2017). Joint regression analysis of marginal quantile and quantile association: Application to longitudinal body mass index in adolescents. *J. R. Stat. Soc. Ser. C. Appl. Stat.* **66** 1075–1090. MR3715598 https://doi.org/10.1111/rssc.12214