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KINETICALLY CONSTRAINED MODELS WITH RANDOM CONSTRAINTS

BY ASSAF SHAPIRA

LPSM, Paris Diderot University, assafshap@gmail.com

We study two kinetically constrained models in a quenched random environment. The first model is a mixed threshold Fredrickson–Andersen model on $\mathbb{Z}^2$, where the update threshold is either 1 or 2. The second is a mixture of the Fredrickson–Andersen 1-spin facilitated constraint and the North-East constraint in $\mathbb{Z}^2$. We compare three time scales related to these models—the bootstrap percolation time for emptying the origin, the relaxation time of the kinetically constrained model, and the time for emptying the origin of the kinetically constrained model—and understand the effect of the random environment on each of them.

1. Introduction. Kinetically constrained models (KCMs) are a family of interacting particle systems introduced by physicists in order to study glassy and granular materials [15, 27]. These are reversible Markov processes on the state space $\{0, 1\}^V$, where V is the set of vertices of some graph. The equilibrium measure of these processes is a product measure of i.i.d. Bernoulli random variables, and their nontrivial behavior is due to kinetic constraints—the state of each site is resampled at rate 1, but only when a certain local constraint is satisfied. This condition expresses the fact that sites are blocked when there are not enough empty sites in their vicinity. One example of such a constraint is that of the Fredrickson–Andersen j-spin facilitated model on $\mathbb{Z}^d$, in which an update is only possible if at least j nearest neighbors are empty [14]. We will refer to this constraint as the FAjf constraint. Another example is the North-East constraint: the underlying graph is $\mathbb{Z}^2$, and an update is possible only if both the site above and the site to the right are empty [23]. These constraints result in the lengthening of the time scales describing the dynamics as the density of empty sites q tends to 0. This happens since sites belonging to large occupied regions could only change their state when empty sites penetrate from the outside.

The main difficulty in the analysis of KCMs is that they are not attractive, which prevents us from using tools such as monotone coupling and censoring often used in the study of Glauber dynamics. For this reason spectral analysis and inequalities related to the spectral gap are essential for the study of time scales in these models. See [25] for more details.

A closely related family of models are the bootstrap percolation models, which are, unlike KCMs, monotone deterministic processes in discrete time. The state space of the bootstrap percolation is the same as that of the KCM, and they share the same family of constraints; but in the bootstrap percolation occupied sites become empty (deterministically) whenever the constraint is satisfied, and empty sites can never be filled. The initial conditions of the bootstrap percolation are random i.i.d. Bernoulli random variables with parameter $1 - q$, that is, they are chosen according to the equilibrium measure of the KCM. In this paper we will refer to the bootstrap percolation that corresponds to a certain constraint by its KCM name, so, for example, the j-neighbor bootstrap percolation will be referred to as the bootstrap percolation with the FAjf constraint.

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In the examples given previously of the FAjf model and the North-East model the constraints are translation invariant. Universality results for general homogeneous models have been studied recently for the bootstrap percolation in a series of works that provide a good understanding of their behavior [5, 8, 10, 19]. Inspired by the tools developed for the bootstrap percolation, universality results on the KCMs could also be obtained for systems with general translation invariant constraints [25, 26]. Another type of models vastly studied for the bootstrap percolation are models in a random environment, such as the bootstrap percolation on the polluted lattice [16, 17], Galton–Watson trees [9], random regular graphs [7, 21] and the Erdős–Rényi graph [22]. KCMs in random environments have also been studied in the physics literature, see [28, 30].

KCMs in random environments may correspond to inhomogeneous physical systems, for example, granular materials in which grains are of different size. There are many possible choices of random environments—making the graph random, adding a random chemical potential, or taking the constraints to be random. In this paper we will consider two models on the two-dimensional lattice with random constraints. We will focus on the divergence of time scales when the equilibrium density of empty sites q is small.

The time scale that is commonly considered in KCMs is the relaxation time, that is, the inverse of the spectral gap. This time scale determines the slowest possible relaxation of correlation between observables, and for homogeneous system it often coincides with typical time scales of the system (see, e.g., [24]). However, when the system is not homogeneous the relaxation time will in general not describe actual observed time scales. We will see in this paper that very unlikely configurations of the disorder that appear far away determine the relaxation time, even though the observed local behavior is not likely to be affected by these remote regions.

Another time scale that is natural to look at is the first time at which the origin (or any arbitrary vertex) is empty. In the bootstrap percolation literature, this time is indeed the most commonly studied. It could be observed physically, and we will see that it is not significantly effected by the "bad" regions far away from the origin.

In this paper we compare the three time scales—the time it takes for the origin to be emptied with the bootstrap percolation, the relaxation time for the KCM, and the first time the origin is empty in the KCM. We will study them in two toy models that provide examples of new behavior occurring in inhomogeneous systems; and demonstrate how the tools developed in Section 4 could be applied to KCMs in random environments.

We will first analyze these time scales in a mixed threshold FA model on the two-dimensional lattice. Unlike the classical model, in which all vertices have the same threshold of empty neighbors needed for the constraint to be satisfied, in this model different vertices have a different threshold. This threshold will be equal to 1 or 2 (such that the system remains ergodic), chosen independently at random in the beginning and fixed throughout the dynamics. For this model, the bootstrap percolation time scales as $q^{-1/2}$, the relaxation time scales as $e^{c/q}$, and the emptying time of the origin as a random power of q, which depends on the realization of the quenched environment.

The second model we consider is a mixture of the FA1f and the North-East models. Similarly to the first model, the constraint at each vertex is determined before starting the dynamics. It is chosen independently at random, and equals either the FA1f constraint or the North-East constraint. We show that in the appropriate parameter regime the relaxation time is infinite, but still the distribution of the origin's emptying time decays exponentially, with a rate which is polynomial in q.

2. Mixed threshold Fredrickson–Andersen model.

2.1. *Model and notation.* In this section we will treat two models—the mixed threshold bootstrap percolation on $\mathbb{Z}^2$ and the mixed threshold FA model on $\mathbb{Z}^2$. Both models live on the same random environment, that will determine the threshold at each vertex of $\mathbb{Z}^2$. It is denoted ω, and the threshold at a vertex x equals $\omega_x \in \{1, 2\}$. This environment is chosen according to a measure ν, which depends on a parameter $\pi \in (0, 1)$. ν will be a product measure—for each vertex $x \in \mathbb{Z}^2$, ω_x equals 1 with probability π and 2 with probability $1 - \pi$, independently from the other vertices. Sites with threshold 1 will be called *easy*, and sites with threshold 2 *difficult*.

Both the bootstrap percolation and the FA dynamics are defined on the state space $\Omega = \{0, 1\}^{\mathbb{Z}^2}$. For a configuration $\eta \in \Omega$, we say that a site x is *empty* if $\eta_x = 0$ and *occupied* if $\eta_x = 1$. For $\eta \in \Omega$ and $x \in \mathbb{Z}^2$ define the constraint

$$c_x(\eta) = \begin{cases} 1 & \sum_{y \sim x} (1 - \eta_y) \geq \omega_x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We can now define the bootstrap percolation with these constraints—it is a deterministic dynamics in discrete time, where at each time step t empty vertices stay empty, and an occupied vertex x becomes empty if the constraint is satisfied, namely $c_x(\eta(t-1)) = 1$. The initial conditions for the bootstrap percolation are random, depending on a parameter $q \in (0, 1)$. They are chosen according to the measure μ, defined as a product of independent Bernoulli measures:

$$\mu = \bigotimes_{x \in \mathbb{Z}^2} \mu_x,$$

$$\mu_x \sim \mathrm{Ber}(1 - q).$$

The Fredrickson–Andersen model is a continuous time Markov process on Ω. It is reversible with respect to the equilibrium measure μ defined above, and its generator $\mathcal{L}$ is defined by

$$\mathcal{L} f = \sum_x c_x (\mu_x f - f) \tag{2.2}$$

for any local function f. We will denote by $\mathcal{D}$ the Dirichlet form associated with $\mathcal{L}$. Probabilities and expected values with respect to this process starting at η will be denoted by $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$. When starting from equilibrium we will use $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$.

Finally, for any event $A \subseteq \Omega$, we define the hitting time

$$\tau_A = \inf\{t : \eta(t) \in A\}.$$

The hitting time is defined for both the KCM and the bootstrap percolation. For the time it takes to empty the origin we will use the notation

$$\tau_0 = \tau_{\{\eta_0 = 0\}}.$$

2.2. *Results*. The first result concerns the bootstrap percolation. It will say that for small values of q, τ_0 scales as $\frac{1}{\sqrt{q}}$. To avoid confusion we stress that μ and $\mathbb{P}_\mu$ depend on q, even though this dependence is not expressed explicitly in the notation.

THEOREM 2.1. *Consider the bootstrap percolation with the mixed FA constraint. Then ν-almost surely*

$$\lim_{q \to 0} \mu\left[\tau_0 \geq \frac{a}{\sqrt{q}}\right] \xrightarrow{a \to \infty} 0, \tag{2.3}$$

$$\lim_{q \to 0} \mu\left[\tau_0 \leq \frac{a}{\sqrt{q}}\right] \xrightarrow{a \to 0} 0. \tag{2.4}$$

For the KCM we have an exponential divergence of the relaxation time, but a power law behavior of τ_0.

THEOREM 2.2. *Consider the KCM with the mixed FA constraint.*

1. *There exists a constant $c > 0$ (that does not depend on π, q) such that ν-almost surely the relaxation time of the dynamics is at least $e^{c/q}$.*

2. *ν-almost surely there exist $\underline{\alpha}$ and $\overline{\alpha}$ (which may depend on ω) such that*

$$\mathbb{P}_\mu[\tau_0 \geq q^{-\overline{\alpha}}] \xrightarrow{q\to 0} 0, \tag{2.5}$$

$$\mathbb{P}_\mu[\tau_0 \leq q^{-\underline{\alpha}}] \xrightarrow{q\to 0} 0. \tag{2.6}$$

Moreover, $\mathbb{E}_\mu[\tau_0] \geq q^{-\underline{\alpha}}$ for q small enough.

REMARK 2.3. We will see that the two exponents $\underline{\alpha}$ and $\overline{\alpha}$ cannot be deterministic—there is $\alpha_0 \in \mathbb{R}$ such that $\nu(\overline{\alpha} < \alpha_0) > 0$ but $\nu(\underline{\alpha} < \alpha_0) < 1$.

REMARK 2.4. In these two theorems we see that while τ_0 for the bootstrap percolation behaves like $q^{-1/2}$, its scaling for the KCM is random. In the proof we will see in details the reason for this difference, but we could already try to describe it heuristically. The bootstrap percolation is dominated by the sites far away from the origin, and once these sites are emptied the origin will be emptied as well. The influence of the environment far away becomes deterministic by a law of large numbers, so we do not see the randomness of ω in the exponent. To the contrary, in the FA dynamics even when sites far away are empty, one must empty many sites in a close neighborhood of the origin simultaneously before the origin could be emptied. Therefore, in order to empty the origin we must overcome a large energy barrier, which makes τ_0 bigger. This effect depends on the structure close to the origin, so it feels the randomness of the environment.

For simplicity, we have chosen to focus on the two-dimensional case. However, a more general result can also be obtained. In the next two theorems we will consider the bootstrap percolation and KCM on $\mathbb{Z}^d$. The thresholds $\{\omega_x\}_{i\in\mathbb{Z}^2}$ are i.i.d., according to a law that we denote by ν. We will also assume that the probability that the threshold is 1 is nonzero, and that the probability that the threshold is more than d is zero.

THEOREM 2.5. *For the bootstrap percolation model described above, ν-almost surely*

$$\lim_{q\to 0} \mu[\tau_0 \geq a q^{-1/d}] \xrightarrow{a\to\infty} 0, \tag{2.7}$$

$$\lim_{q\to 0} \mu[\tau_0 \leq a q^{-1/d}] \xrightarrow{a\to 0} 0. \tag{2.8}$$

THEOREM 2.6. *For the KCM described above, ν-almost surely there exist $\underline{\alpha}$ and $\overline{\alpha}$ (which may depend on ω) such that*

$$\mathbb{P}_\mu[\tau_0 \geq q^{-\overline{\alpha}}] \xrightarrow{q\to 0} 0, \tag{2.9}$$

$$\mathbb{P}_\mu[\tau_0 \leq q^{-\underline{\alpha}}] \xrightarrow{q\to 0} 0. \tag{2.10}$$

3. Mixed North-East and FA1f KCM.

3.1. *Model and notation.* In this section we will consider again a kinetically constrained dynamics in an environment with mixed constraints. This time, however, the two constraints we will have are FA1f and north-east. That is, using the same ω and ν as before, for x such that $\omega_x = 1$,

$$c_x(\eta) = \begin{cases} 1 & \sum_{y\sim x}(1-\eta_y) \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and when $\omega_x = 2$,

$$c_x(\eta) = \begin{cases} 1 & \eta_{x+e_1} = 0 \text{ and } \eta_{x+e_2} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For the same μ, we can define $\mathcal{L}$ as in (2.2). Note that c_x (and therefore $\mathcal{L}$) are not the same as those of the previous section, even though we use the same letters to describe them. Again, the hitting time of a set A will be denoted by τ_A, and $\tau_0 = \tau_{\{\eta_0=0\}}$.

We restrict ourselves to the case where π is greater than the critical probability for the Bernoulli site percolation on $\mathbb{Z}^2$, denoted by p^{SP}. The critical probability for the oriented percolation on $\mathbb{Z}^2$ will be denoted by p^{OP}.

REMARK 3.1. Our choice of regime, where easy sites percolate, guarantees that all sites are emptiable for the bootstrap percolation. The infinite cluster $\mathcal{C}$ of easy sites is emptiable since it must contain an empty site somewhere. The connected components of $\mathbb{Z}^2 \setminus \mathcal{C}$ are finite, and have an emptiable boundary, so each of them will also be emptied eventually.

This choice, however, is not the only one for which all sites are emptiable. For any fixed environment ω there is a critical value q_c such that above q_c all sites are emptiable and below q_c some sites remain occupied forever. For $\pi > p^{\mathrm{SP}}$ we already know that ν-almost surely $q_c = 0$. In fact, the same argument gives a slightly better result by allowing sites to be difficult if they are also empty. This implies that $q_c \leq 1 - \frac{1-p^{\mathrm{SP}}}{1-\pi}$. On the other hand, if there is an infinite up-right path of difficult sites that are all occupied, this path could never be emptied. This will imply that $q_c \geq 1 - \frac{p^{\mathrm{OP}}}{1-\pi}$.

3.2. *Results*. We will see for this model that it is possible to have an infinite relaxation time, and still the tail of the distribution of τ_0 decays exponentially, with a rate that scales polynomially with q.

THEOREM 3.2. *Consider the kinetically constrained model described above, with $\pi > p^{\mathrm{SP}}$ and $q \leq q^{\mathrm{OP}}$.*

1. *ν-almost surely the spectral gap is* 0, *that is*, *the relaxation time is infinite*.
2. *There exist two positive constants c, C depending on π and a ν-random variable τ such that*:
   (a) $\mathbb{P}_\mu(\tau_0 \geq t) \leq e^{-t/\tau}$ *for all $t > 0$*,
   (b) $\nu(\tau \geq t) \leq Ct^{\frac{c}{\log q}}$ *for t large enough*.

4. Some tools. In this section we will present some tools that will help us analyze the kinetically constrained models that we have introduced. We will start by considering a general state space Ω, and any Markov process on Ω that is reversible with respect to a certain measure μ. We denote its generator by $\mathcal{L}$ and the associated Dirichlet form by $\mathcal{D}$. We will consider, for some event A, its hitting time τ_A. With some abuse of notation, we use τ_A also for the μ-random variable giving for every state $\eta \in \Omega$ the expected hitting time at A starting from that state:

$$\tau_A(\eta) = \mathbb{E}_\eta(\tau_A).$$

$\tau_A(\eta)$ satisfies the following Poisson problem (see, e.g., [11], equation (7.2.45)):

$$\begin{aligned} \mathcal{L}\tau_A &= -1 \quad \text{on } A^c, \\ \tau_A &= 0 \quad \text{on } A. \end{aligned} \tag{4.1}$$

By multiplying both sides of the equation by τ_A and integrating with respect to μ, we obtain

COROLLARY 4.1. $\mu(\tau_A) = \mathcal{D}\tau_A$.

Rewriting this corollary as $\mu(\tau_A) = \frac{\mu(\tau_A)^2}{\mathcal{D}\tau_A}$, it resembles a variational principle introduced in [4] that will be useful in the following. In order to formulate it we will need to introduce some notation.

DEFINITION 4.2. For an event $A \subseteq \Omega$, V_A is the set of all functions in the domain of $\mathcal{L}$ that vanish on the event A. Note that, in particular, $\tau_A \in V_A$.

DEFINITION 4.3. For an event $A \subseteq \Omega$,

$$\overline{\tau}_A = \sup_{0 \neq f \in V_A} \frac{\mu(f^2)}{\mathcal{D}f}.$$

The following proposition is given in the first equation of the proof of Theorem 2 in [4]:

PROPOSITION 4.4. $\mathbb{P}_\mu[\tau_A > t] \leq e^{-t/\overline{\tau}_A}$.

REMARK 4.5. In particular, Proposition 4.4 implies that $\mu(\tau_A) \leq \overline{\tau}_A$. This, however, could be derived much more simply from Corollary 4.1:

$$\mu(\tau_A)^2 \leq \mu(\tau_A^2) \leq \overline{\tau}_A \mathcal{D}\tau_A = \overline{\tau}_A \mu(\tau_A).$$

Note that whenever τ_A is not constant on A^c this inequality is strict. Thus on one hand Proposition 4.4 gives an exponential decay of $\mathbb{P}_\mu[\tau_A > t]$, which is stronger than the information on the expected value we can obtain from the Poisson problem in equation (4.1). On the other hand, $\overline{\tau}_A$ could be longer than the actual expectation of τ_A.

In order to bound the hitting time from below we will formulate a variational principle that will characterize τ_A.

DEFINITION 4.6. For $f \in V_A$, let

$$\mathcal{T}f = 2\mu(f) - \mathcal{D}f.$$

PROPOSITION 4.7. *τ_A maximizes $\mathcal{T}$ in V_A. Moreover, $\mu(\tau_A) = \sup_{f \in V_A} \mathcal{T}f$.*

PROOF. Consider $f \in V_A$, and let $\delta = f - \tau_A$. Using the self-adjointness of $\mathcal{L}$, equation (4.1), and the fact that $\delta \in V_A$ we obtain

$$\begin{aligned} \mathcal{T}f &= \mathcal{T}(\tau_A + \delta) \\ &= 2\mu(\tau_A) + 2\mu(\delta) - \mathcal{D}\tau_A - \mathcal{D}\delta + 2\mu(\delta\mathcal{L}\tau) \\ &= \mathcal{T}\tau_A - \mathcal{D}\delta. \end{aligned}$$

By the positivity of the Dirichlet form, $\mathcal{T}$ is indeed maximized by τ_A. Finally, by Corollary 4.1,

$$\sup_{f\in V_A} \mathcal{T} f = \mathcal{T}\tau_A = 2\mu(\tau_A) - \mathcal{D}\tau_A = \mu(\tau_A).$$

□

As an immediate consequence we can deduce the monotonicity of the expected hitting time:

COROLLARY 4.8. *Let $\mathcal{D}$ and $\mathcal{D}'$ be the Dirichlet forms of two reversible Markov processes defined on the same space, such that both share the same equilibrium measure μ. We denote the expectations with respect to these processes starting at equilibrium by $\mathbb{E}_\mu$ and $\mathbb{E}'_\mu$. Assume that the domain of $\mathcal{D}$ is contained in the domain of $\mathcal{D}'$, and that for every $f \in \operatorname{Dom}\mathcal{D}$*

$$\mathcal{D} f \le \mathcal{D}' f.$$

Then, for an event $A \subseteq \Omega$,

$$\mathbb{E}_\mu \tau_A \le \mathbb{E}'_\mu \tau_A.$$

We will now restrict ourselves to kinetically constrained models. Fix a graph G and take $\Omega = \{0,1\}^G$. For every vertex $x \in G$ and a state $\eta \in \Omega$ we define a constraint $c_x(\eta) \in \{0,1\}$. The constraint does not depend on the value at x, and is nonincreasing in η. The equilibrium measure μ is a product measure. The generator of this process, operating on a local function f, is given by

$$\mathcal{L} f = \sum_x c_x(\mu_x f - f)$$

and its Dirichlet form by

$$\mathcal{D} f = \mu\Bigl(\sum_x c_x \operatorname{Var}_x f\Bigr).$$

Fix a subgraph H of G, and denote the complement of H in G by H^c.

We will compare the dynamics of this KCM to the dynamics restricted to H, with boundary conditions that are the most constrained ones.

DEFINITION 4.9. The restricted dynamics on H is the KCM defined by the constraints

$$c_x^H(\eta) = c_x(\eta^H),$$

where, for $\eta \in \{0,1\}^H$, η^H is the configuration given by

$$\eta^H(x) = \begin{cases} \eta_x & x \in H, \\ 1 & x \in H^c. \end{cases}$$

We will denote the corresponding generator by $\mathcal{L}_H$ and its Dirichlet form by $\mathcal{D}_H$.

CLAIM 4.10. *For any f in the domain of $\mathcal{L}$,*

$$\mathcal{D} f \ge \mu_{H^c} \mathcal{D}_H f.$$

PROOF. $c_x^H \le c_x$ and $\operatorname{Var}_x f$ is positive, therefore

$$\mathcal{D} f = \mu\Bigl(\sum_x c_x \operatorname{Var}_x f\Bigr) \ge \mu\Bigl(\sum_{x\in H} c_x^H \operatorname{Var}_x f\Bigr).$$

□

The next claim will allow us to relate the spectral gap of the restricted dynamics to the variational principles discussed earlier.

CLAIM 4.11. *Let γ_H be the spectral gap of $\mathcal{L}_H$, and fix an event A that depends only on the occupation of the vertices of H. Then, for all $f \in V_A$:*

1. $\mathcal{D}f \geq \mu(A)\gamma_H(\mu f)^2$,
2. $\mathcal{D}f \geq \frac{\mu(A)}{1+\mu(A)}\gamma_H\mu(f^2)$.

PROOF. First, note that $\mu_H(A) \leq \mu_H(f=0) \leq \mu_H(|f-\mu_H f| \geq \mu_H f)$. Therefore, by Chebyshev inequality and the fact that $\mu(A) = \mu_H(A)$,

$$\mu(A) \leq \frac{\mathrm{Var}_H f}{(\mu_H f)^2}. \tag{4.2}$$

Then, Claim 4.10 implies

$$\mathcal{D}f \geq \mu_{H^c}\mathcal{D}_H f \geq \gamma_H\mu_{H^c}\,\mathrm{Var}_H f \geq \mu(A)\gamma_H\mu_{H^c}(\mu_H f)^2 \geq \mu(A)\gamma_H(\mu f)^2$$

by Jensen's inequality. For the second part, we use inequality (4.2),

$$\mathrm{Var}_H f \geq \mu(A)\bigl(\mu_H(f^2) - \mathrm{Var}_H f\bigr),$$

which implies

$$\mathrm{Var}_H f \geq \frac{\mu(A)}{1+\mu(A)}\mu_H(f^2).$$

The result then follows by applying Claim 4.10. □

5. Proof of the results.

5.1. *Mixed threshold bootstrap percolation on $\mathbb{Z}^2$.*

5.1.1. *Proof of equation* (2.3). For the upper bound we will find a specific mechanism in which a cluster of empty sites could grow until it reaches the origin.

DEFINITION 5.1. A square (i.e., a subset of $\mathbb{Z}^2$ of the form $x + [L]^2$) is *good* if it contains at least one easy site in each line and in each column.

CLAIM 5.2. *Fix L. The probability that a square of side L is good is at least $1 - 2Le^{-\pi L}$.*

PROOF.

$$\mathbb{P}[\text{easy site in each line}] = \bigl[1 - (1-\pi)^L\bigr]^L \geq 1 - Le^{-\pi L}.$$

The same bound holds for $\mathbb{P}$[easy site in each column], and then we conclude by the union bound. □

DEFINITION 5.3. The square $[L]^2$ is *excellent* if for every $2 \leq i \leq L$ at least one of the sites in $\{i\} \times [i-1]$ is easy, and at least one of the sites in $[i-1] \times \{i\}$ is easy. For other squares of side L being excellent is defined by translation.

We will use p_L to denote the probability that a square of side L is excellent. Note that p_L depends only on π and not on q.

The next two claims will show how a cluster of empty sites could propagate. See Figure 1.

CLAIM 5.4. *Assume that $[L]^2$ is excellent, and that $(1,1)$ is initially empty. Then $[L]^2$ will be entirely emptied by time L^2.*

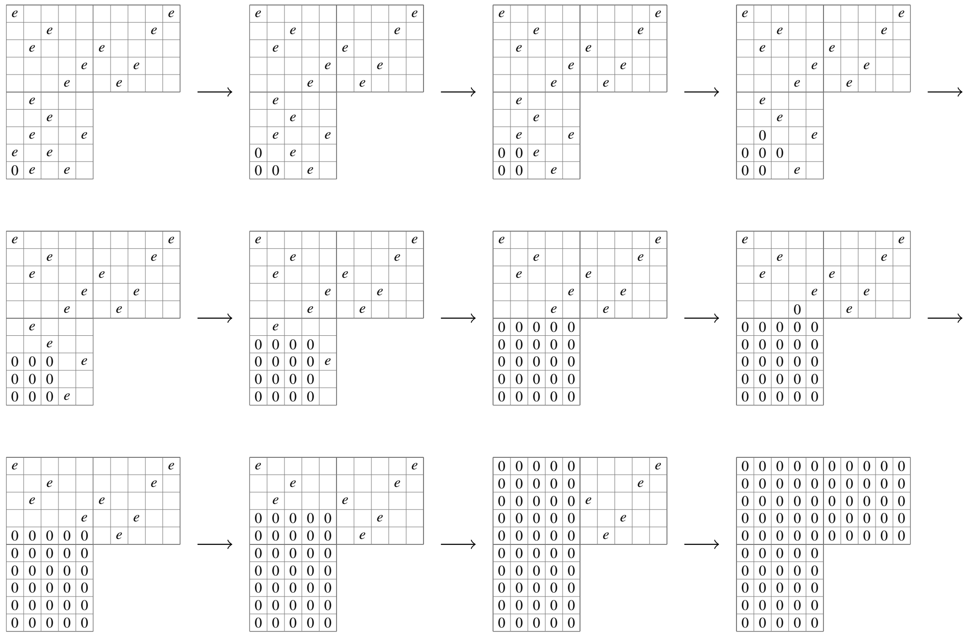

FIG. 1. *First steps of propagating the empty cluster.* 0 *represents an empty site, otherwise the state is the initial one. e stands for an easy site.*

PROOF. This could be done by induction on the size of the empty square—assume that $[l]^2$ is entirely emptied for some $l \le L$. By the definition of an excellent square, there is an easy site $x \in \{l+1\} \times [l]$. Its neighbor to the left is empty (since it is in $[l]^2$), so at the next time step this site will also be empty. Once x is empty, the two sites $x \pm e_2$ could be emptied, and then the sites $x \pm 2e_2$ and so on, as long as they stay in $\{l+1\} \times [l]$. Thus, at time l all sites in $\{l+1\} \times [l]$ will be empty, and by the same reasoning the sites of $[l] \times \{l+1\}$ will also be empty. Since $(l+1, l+1)$ has two empty neighbors it will be emptied at step $l+1$, and thus $[l+1]^2$ will be emptied. $\square$

CLAIM 5.5. *Assume that $[L]^2$ is good, and that it has a neighboring square that is entirely empty by time T. Then $[L]^2$ will be entirely empty by time $T + L^2$.*

PROOF. We can empty $[L]^2$ line by line (or column by column, depending on whether its empty neighbor is in the horizontal direction or the vertical one). For each line, we start by emptying the easy site that it contains, and then continue to propagate. $\square$

DEFINITION 5.6. Until the end of the proof of the upper bound, L will be the minimal length for which the probability to be good exceeds $p^{\mathrm{SP}} + 0.01$.

DEFINITION 5.7. $\mathcal{C}$ will denote the infinite cluster of good boxes of the form $Li + [L]^2$ for $i \in \mathbb{Z}^2$. $\mathcal{C}_0$ will denote the cluster of the origin surrounded by a path in $\mathcal{C}$, or just the origin if it is in $\mathcal{C}$. Note that $\mathcal{C}_0$ could also be seen as the connected component of the origin in $\mathbb{Z}^2 \setminus \mathcal{C}$, but not in the standard $\mathbb{Z}^2$ but rather in the matching graph (see [18], Figure 3.2). $\partial\mathcal{C}_0$ will be the outer boundary of $\mathcal{C}_0$ (namely the boxes of $\mathcal{C}$ that have a neighbor in $\mathcal{C}_0$). Note that $\mathcal{C}_0$ is finite and that $\partial\mathcal{C}_0$ is connected.

CLAIM 5.8. *Assume that at time T one of the boxes on $\partial\mathcal{C}_0$ is entirely empty. Then by time $T + T_0$ the origin will be empty, where $T_0 = (|\partial\mathcal{C}_0| + |\mathcal{C}_0|)L^2$.*

PROOF. By Claim 5.5, the boundary $\partial\mathcal{C}_0$ will be emptied by time $T_0 + L^2|\partial\mathcal{C}_0|$. Then, at each time step at least one site of $\mathcal{C}_0$ must be emptied, since no finite region could stay occupied forever. □

CLAIM 5.9. *Assume that a box $Li + [L]^2$ in $\mathcal{C}$ is empty at time T. Also, assume that the graph distance in $\mathcal{C}$ between this box and $\partial\mathcal{C}_0$ is l. Then by time $T + lL^2 + T_0$ the origin will be empty.*

PROOF. This is again a direct application of Claims 5.5 and 5.8. □

Finally, we will use the following result from percolation theory.

CLAIM 5.10. *For l large enough, the number of boxes in $\mathcal{C}$ that are at graph distance in $\mathcal{C}$ at most l from $\partial\mathcal{C}_0$ is greater than θl^2, where θ depends only on the probability that a box is good.*

PROOF. By ergodicity the cluster $\mathcal{C}$ has an almost sure positive density, so in particular

$$\liminf_{l\to\infty} \frac{|\mathcal{C} \cap [-l, l]^2|}{|[-l, l]^2|} > 0.$$

By [3], there exists a positive constant ρ such that boxes of graph distance l from the origin must be in the box $[-\frac{1}{\rho}l, \frac{1}{\rho}l]^2$ for l large enough. Combining these two facts proves the claim. □

This claim together with a large deviation estimate yields the following.

COROLLARY 5.11. *For l large enough, the number of excellent boxes in $\mathcal{C}$ that are at graph distance in $\mathcal{C}$ at most l from $\partial\mathcal{C}_0$ is greater than $\theta' l^2$, where $\theta' = 0.99\theta p_L$.*

We can now put all the ingredients together and obtain the upper bound.

Fix $c > 0$, and $l = \frac{c}{\sqrt{q}}$. By Corollary 5.11, for q small, there are at least $\frac{\theta' c^2}{q}$ excellent boxes at distance smaller than l from $\partial\mathcal{C}_0$. If one of them contains an empty site at the bottom left corner, the origin will be emptied by time $(l+1)L^2 + T_0$. For q small enough, this time is bounded by $\frac{2cL^2}{\sqrt{q}}$. The complement of this event, that is, that none of these boxes contain such an empty site, has probability $(1-q)^{\frac{\theta' c^2}{q}}$. This probability tends to 0 uniformly in q as $c \to \infty$, which concludes the proof.

5.1.2. *Proof of equation* (2.4). The lower bound results from the simple observation, that the origin could only be emptied by time t if there is an empty site at distance smaller than t. The probability of that event is $1 - (1-q)^{4t^2}$, and taking $t = \frac{a}{\sqrt{q}}$ and q small enough this probability is bounded by $1 - e^{-2a}$. This tends to 0 with a uniformly in q, which finishes the proof.

5.2. *Mixed threshold KCM on $\mathbb{Z}^2$.*

5.2.1. *Spectral gap.* The spectral gap of this model is dominated by that of the FA2f model. Fix any γ strictly greater than the gap of FA2f. Then there is a local nonconstant function f such that

$$\frac{\mathcal{D}^{\mathrm{FA2f}} f}{\operatorname{Var} f} \le \gamma,$$

where $\mathcal{D}^{\mathrm{FA2f}}$ is the Dirichlet form of the FA2f model.

Since f is local, it is supported in some square of size $L \times L$, for L big enough. ν-almost surely it is possible to find a far away square in $\mathbb{Z}^2$ of size $L \times L$ that contains only difficult sites. By translation invariance of the FA2f model we can assume that this is the square in which f is supported. In this case, $\mathcal{D} f = \mathcal{D}^{\mathrm{FA2f}} f$, and this shows that indeed the gap of the model with random threshold is smaller than that of FA2f, which by [12] is bounded by $e^{-c/q}$.

5.2.2. *Proof of equation* (2.5). In this part we will use Corollary 4.1 in order to bound τ_0 by a path argument. As in the proof of the upper bound for the bootstrap percolation, we will consider the good squares (see Definition 5.1) and their infinite cluster. In fact, by Claim 5.2, by choosing L big enough we may assume that the box $[L]^2$ is in this cluster. Let us fix this L until the end of this part. We will also choose an infinite self avoiding path of good boxes starting at the origin and denote it by $i_0, i_1, i_2, \ldots$. Note that this path depends on ω but not on η.

On this cluster empty sites will be able to propagate, and the next definition will describe the seed needed in order to start this propagation.

DEFINITION 5.12. A box in $\mathbb{Z}^2$ is *essentially empty* if it is good and contains an entire line or an entire column of empty sites. This will depend on both ω and η.

In order to guarantee the presence of an essentially empty box we will fix $l = q^{-L-1}$, and define the bad event

DEFINITION 5.13. Fix a disorder ω, and let $i_0, \ldots, i_l$ as described above. $B \subseteq \Omega$ is defined as the event, that none of the boxes $i_0, \ldots, i_l$ is essentially empty. Note that B depends on ω even though for brevity this dependence does not appear explicitly in the notation.

A simple bound shows that

$$\mu(B) \le \left(1 - q^L\right)^l \le e^{-1/q}. \tag{5.1}$$

We can use this bound in order to bound the hitting time at B:

CLAIM 5.14. *There exists $C > 0$ such that $\mathbb{P}_\mu(\tau_B \le t) \le C e^{-1/q} t$.*

PROOF. We use the graphical construction of the Markov process. In order to hit B, we must hit it at a certain clock ring taking place in one of the sites of $\bigcup_{n=1}^{l} (Li + [L]^2)$. Therefore,

$$\begin{aligned}\mathbb{P}(\tau_B \le t) &\le \mathbb{P}\big[\text{more than } 2(2L+1)^2 lt \text{ rings by time } t\big] + 2(2L+1)^2 lt \mu(B) \\ &\le e^{-(2L+1)^2 q^{-L-1} t} + 2(2L+1)^2 q^{-L-1} t e^{-1/q} \le C e^{-1/q} t.\end{aligned}$$ □

In order to bound τ_0 we will study the hitting time of $A = B \cup \{\eta(0) = 0\}$.

LEMMA 5.15. *Fix $\eta \in \Omega$. Then there exists a path $\eta_0, \ldots, \eta_N$ of configurations and a sequence of sites $x_0, \ldots, x_{N-1}$ such that*:

1. $\eta_0 = \eta$,
2. $\eta_N \in A$,

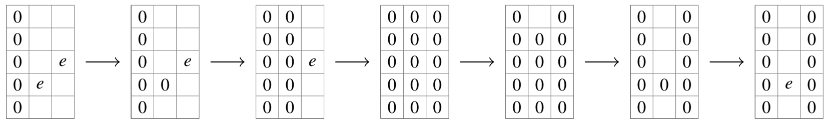

FIG. 2. *Creating an empty column and propagating it using the easy sites. 0 represents an empty site, otherwise the state is the initial one. e stands for an easy site.*

3. $\eta_{i+1} = \eta_i^{x_i}$,
4. $c_{x_i}(\eta_i) = 1$,
5. $N \le 4L^2 l$,
6. *For all $i \le N$, η_i differs from η at at most $3L$ points, contained in at most two neighboring boxes.*

PROOF. If $\eta \in A$, we take the path η with $N = 0$. Otherwise $\eta \in B^c$, so there is an essentially empty box in $i_0, \dots, i_l$, which contain an empty column (or row). We can then create an empty column (row) next to it and propagate that column (row) as in Figure 2. When the path rotates we can rotate this propagating column (row) as show in Figure 3. □

We can use this path together with Corollary 4.1 in order to bound τ_A.

LEMMA 5.16. *There exists $C_L > 0$ (that may depends on L but not on q) such that $\mu(\tau_A) \le C_L q^{-5L-2}$.*

PROOF. Since τ vanishes on A, taking the path defined in Lemma 5.15,

$$\tau_A(\eta) = \sum_{i=0}^{N-1} \bigl(\tau_A(\eta_i) - \tau_A(\eta_{i+1})\bigr).$$

In the following we use the notation

$$\nabla_x \tau_A(\eta) = \tau_A(\eta) - \tau_A(\eta^x).$$

Then, by Cauchy–Schwarz inequality,

$$\begin{aligned}
\mu(\tau_A)^2 \le \mu(\tau_A^2) &= \sum_\eta \mu(\eta) \left(\sum_{i=0}^{N-1} \nabla_{x_i} \tau_A(\eta_i) \right)^2 \\
&\le \sum_\eta \mu(\eta) N \sum_i c_{x_i}(\eta_i) \bigl(\nabla_{x_i} \tau_A(\eta_i)\bigr)^2 \\
&= \sum_\eta \mu(\eta) N \sum_i \sum_z \sum_{\eta'} c_z(\eta') \bigl(\nabla_z \tau_A(\eta')\bigr)^2 \mathbb{1}_{z=x_i} \mathbb{1}_{\eta'=\eta_i}.
\end{aligned}$$

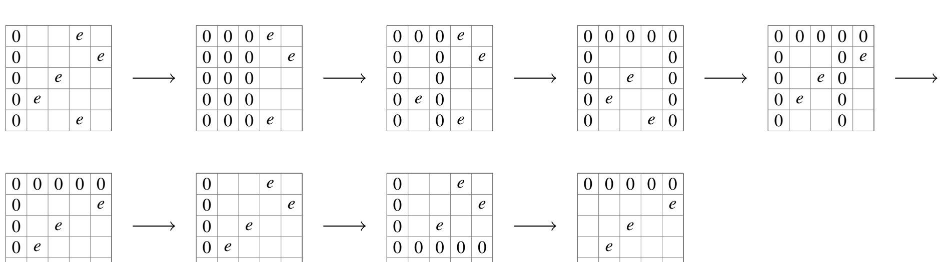

FIG. 3. *Rotating an empty column in a good box. 0 represents an empty site, otherwise the state is the initial one. e stands for an easy site.*

By property number 6 of the path, we know that $\mu(\eta) \le q^{-3L}\mu(\eta')$, so we obtain

$$\mu(\tau_A)^2 \le q^{-3L} N \sum_{\eta'} \mu(\eta') \sum_z c_z(\eta')(\nabla_z \tau_A(\eta'))^2 \sum_i \mathbb{1}_{z=x_i} \sum_\eta \mathbb{1}_{\eta'=\eta_i}.$$

Still using property 6, η differs from η' at at most $3L$ points, all of them in the box containing z or in one of the two neighboring boxes. This gives the bound $\sum_\eta \mathbb{1}_{\eta'=\eta^{(i)}} \le (3L^2)^{3L}$. Finally, bounding $\mathbb{1}_{z=x_i}$ by 1,

$$\begin{aligned}\mu(\tau_A)^2 &\le q^{-3L}(3L^2)^{3L} N^2 \sum_{\eta'} \mu(\eta') \sum_z c_z(\eta')(\nabla_z \tau_A(\eta'))^2 \\ &\le 16(3L^2)^{3L} L^4 q^{-5L-2} \mathcal{D}\tau_A.\end{aligned}$$

This concludes the proof of the lemma by Corollary 4.1. $\square$

Using this lemma and the bound on τ_B in Claim 5.14, we can finish the estimation of the upper bound. By the Markov inequality

$$\mu(\tau_A \ge C_L q^{-5L-3}) \le q.$$

On the other hand, by Claim 5.14,

$$\begin{aligned}\mu(\tau_A < C_L q^{-5L-5}) &\le \mu(\tau_0 < C_L q^{-5L-3}) + \mu(\tau_B < C_L q^{-5L-3}) \\ &\le \mu(\tau_0 < C_L q^{-5L-3}) + C'_L e^{-1/q}.\end{aligned}$$

Therefore

$$\mu(\tau_0 \ge C_L q^{-5L-3}) \le q + C'_L e^{-1/q},$$

and taking $\overline{\alpha} = 5L + 3$ will suffice.

Concerning Remark 2.3, fix L_0 such that the probability that $[L_0]^2$ is good exceeds p^{SP}. Then, for $\alpha_0 = 5L_0 + 3$, $\nu(\overline{\alpha} \le \alpha_0) > 0$. We will see later that the other inequality holds as well for the same α_0.

5.2.3. *Proof of equation* (2.6). A trivial bound could be obtained by taking any $\underline{\alpha} < 1$, since the rate at which the origin becomes empty is always at most q. We will, however, look for a bound that will better describe the effect of the disorder, and will allow us to prove Remark 2.3. The basic observation for the estimation of this lower bound is that if $[-L, L]^2$ is initially occupied, and if it contains only difficult sites, then at some point we will need to empty at least $\frac{L}{2}$ sites before the origin could be emptied. This energy barrier forces τ_A to be greater than $q^{-L/2}$.

In the following we will fix L such that $[-L, L]^2$ contains only difficult sites (so it is not the same L we have used for the upper bound).

DEFINITION 5.17. For a rectangle R, the span of R are the sites that could be emptied with the bootstrap percolation using only the 0s of R. If the span of R equals R we say that R is internally spanned.

The next fact is a consequence of the fact that a set which is stable under the bootstrap percolation must be a rectangle [2].

FACT 5.18. The span of a rectangle R is a union of internally spanned rectangles.

DEFINITION 5.19. For $x \in \mathbb{Z}^2$, let $\overline{G}_x$ be the event that the origin is in the span of $[-L, L]^2$ for η, but not for η^x. G_x is defined as the intersection of $\overline{G}_x$ with the event $\{c_x = 1\}$.

First, note that legal flips of sites in the interior of a rectangle or outside the rectangle cannot change its span. Therefore, $G_x = \varnothing$ for x which is not on the inner boundary of $[-L, L]^2$.

CLAIM 5.20. *Fix x on the inner boundary of $[-L, L]^2$, and let $\eta \in G_x$. Then x and the origin belong to the same internally spanned rectangle.*

PROOF. Recalling Fact 5.18, we consider the internally spanned rectangle containing the origin. If it didn't contain x, the origin would be in the span of $[-L, L]^2$ also for η^x, contradicting the definition of G_x. □

COROLLARY 5.21. *Fix x on the inner boundary of $[-L, L]^2$. Then $\mu(G_x) \le \binom{L^2}{L/2} q^{L/2}$.*

PROOF. Assume without loss of generality that x is on the right boundary. Then there must be an internally spanned rectangle in $[-L, L]^2$ whose width is at least L. Since all sites of $[-L, L]^2$ are difficult, it cannot contain two consequent columns that are entirely occupied, therefore it must contain at least $\frac{L}{2}$ empty sites. □

We can use the same argument as in the proof of Claim 5.14. Defining $G = \bigcup_x G_x$, this argument will tell us that the hitting time τ_G is bigger than $Cq^{-L/2+1}$ with probability that tends to 1 as $q \to 0$, for some constant C that depends on L. If we start with a configuration for which the origin is not in the span of $[-L, L]^2$, it could only be emptied after τ_G—at the first instant in which the span of $[-L, L]^2$ includes the origin, G_x must occur for the site that has just been flipped. Since the probability to start with an entirely occupied $[-L, L]^2$ tends to 1 as $q \to 0$, equation (2.6) is satisfied for $\underline{\alpha} = \frac{L}{2} - 1$.

In order to bound also the expected value of τ_0 we will use Proposition 4.7. Let us consider the function

$$f = \mathbb{1}_{0 \text{ is not in the span of } [-L,L]^2}.$$

We can bound its Dirichlet form using Corollary 5.21:

$$\mathcal{D} f = \mu\left(\sum_x c_x \operatorname{Var}_x f\right) \le \mu\left(\sum_x c_x q \mathbb{1}_{G_x}\right)$$
$$\le q 16L \binom{L^2}{L/2} q^{L/2} = C_L q^{L/2+1}.$$

The expected value is bounded by the probability that all sites are occupied:

$$\mu f \ge (1-q)^{(2L+1)^2}.$$

Now consider for some $\lambda \in \mathbb{R}$ the rescaled function $\overline{f} = \lambda f$.

$$\mathcal{T}\overline{f} = 2\mu\overline{f} - \mathcal{D}\overline{f}$$
$$\ge 2\lambda(1-q)^{(2L+1)^2} - \lambda^2 C_L q^{L/2+1}.$$

The optimal choice of λ is $\frac{(1-q)^{(2L+1)^2}}{C_L} q^{-L/2-1}$, which yields

$$\mathcal{T}\overline{f} \ge \frac{(1-q)^{(2L+1)^2}}{C_L} q^{-L/2-1}.$$

Proposition 4.7 and the fact that $\overline{f}$ vanishes on $\{\eta_0 = 0\}$ imply that $\mu(\tau_0) \geq C'_L q^{-L/2-1}$, and therefore $\mathbb{E}_\mu(\tau_0) \geq q^{-\underline{\alpha}}$ for q small enough.

Concerning Remark 2.3, note that for every α

$$\nu(\underline{\alpha} \geq \alpha) \geq \nu(L \geq 2\alpha + 4) = (1-\pi)^{(4\alpha+9)^2}.$$

In particular, for α_0 defined in the proof of the upper bound $\nu(\underline{\alpha} \geq \alpha_0) > 0$.

5.3. *Mixed threshold models on* $\mathbb{Z}^d$. The argument above, for the case of $\mathbb{Z}^2$, works also in more general settings, as long as the probability to be easy (i.e., threshold 1) is strictly positive.

The lower bound of the bootstrap percolation is immediate, since only at scale $q^{-1/d}$ it is possible to find an easy site. The lower bound for the kinetically constrained model could be analyzed similarly to the two dimensional case using the methods of [6], but in order to keep things simple we could take $\overline{\alpha} < 1$, which suffices since the rate at which the origin becomes empty is always at most q.

For the upper bound of both the bootstrap percolation and the kinetically constrained model we need to construct a path that will empty the origin. First, note that we may assume that sites have either threshold 1 (easy) or threshold d (difficult) with probabilities π and $1-\pi$, for some $\pi > 0$. In this case the path is described explicitly in [26]. It is constructed for the FAdf model, but we will only need to adapt the definitions there in order to take into account the easy sites.

Fix L that will be equal to n defined in the beginning of Section 5.1 of [26], replacing q by π and taking ϵ such that good boxes (see Definition 5.23 that follows) percolate, and the origin belongs to the infinite cluster. We then consider, just as before, an infinite path of good boxes starting at the origin.

DEFINITION 5.22. The easy bootstrap percolation will be the threshold d bootstrap percolation defined on $\mathbb{Z}^d$, with initial conditions in which easy sites are empty and difficult sites are filled. A set $V \in \mathbb{Z}^d$ is *easy internally spanned* if it is internally spanned for this process.

DEFINITION 5.23. A *good* box $V = x + [L]^d$ is a box for which the event G_1 in Definition 5.4 of [26] occurs, replacing “internally spanned” by “easy internally spanned.”

DEFINITION 5.24. As *excellent* box is a box that, by adding a single easy site at its corner, will be easy internally spanned. p_L will be the probability that the box $[L]^d$ is excellent. Note that (as for the two-dimensional case) p_L is nonzero, and that it does not depend on q.

DEFINITION 5.25. An *essentially empty* box $V = x + [L]^d$ will correspond to the event G_2 in Definition 5.4 of [26]—it is a good box in which the first slice in any direction is empty.

Being good and being excellent are events measurable with respect to the disorder ω, whereas being essentially empty depends also on the configuration of the empty and filled sites. The definition of the bad event B remains the same as in the previous section, and taking $l = q^{-dL^{d-1}-1}$ its probability has the same decay.

With these definitions, replacing “supergood” by “essentially empty,” the proof in Section 5 of [26] shows how to propagate an essentially empty box along the path of good boxes, corresponding to Figures 2 and 3 in the two-dimensional case. We may then consider a path as the one of Lemma 5.15. That is, for any configuration η there is a path $\eta_0, \dots, \eta_N$ with flips $x_0, \dots, x_{N-1}$ such that:

1. $\eta_0 = \eta$,

2. $\eta_N \in A$,
3. $\eta_{i+1} \in \eta_i^{x_i}$,
4. $c_{x_i}(\eta_i) = 1$,
5. $N \leq cL^d l$ for some $c > 0$,
6. For all $i \leq N$, η_i differs from η at at most cL^{d-1} points, contained in at most two neighboring boxes.

Applying the exact same argument as for the two dimensional case yields the upper bounds.

5.4. *Mixed North-East and FA1f.*

5.4.1. *Spectral gap.* This is the same argument as for the previous model—one can always find arbitrarily large regions of difficult sites, so the gap is bounded by that of the North-East model. Since for the parameters that we have chosen the North-East model is not ergodic, it has 0 gap [12].

5.4.2. *Hitting time.* Let A be the event $\{\eta_0 = 0\}$. Recall Definition 4.3 and let

$$\tau = \overline{\tau}_A.$$

The exponential tail of τ_0 is a consequence of Proposition 4.4, so we are left with proving that $\nu(\tau \geq t) \leq t^{\frac{c}{\log q}}$ for some constant c. We will do that by choosing a subgraph on which we can estimate the gap, and then apply Claim 4.11.

Since π is greater than the critical probability for the Bernoulli site percolation, there will be an infinite cluster of easy sites $\mathcal{C}$. We denote by $\mathcal{C}_0$ the cluster of the origin surrounded by a path in $\mathcal{C}$. $\partial\mathcal{C}_0$ will be the outer boundary of $\mathcal{C}_0$, that is, the sites in $\mathcal{C}$ that have a neighbor in $\mathcal{C}_0$. Then, we fix a self avoiding infinite path of easy sites $v_0, v_1, \ldots$ starting with the sites of $\partial\mathcal{C}_0$. That is, $v_0, \ldots, v_{|\partial\mathcal{C}_0|}$ is a path that encircles $\mathcal{C}_0$, and then $v_{|\partial\mathcal{C}_0|+1}, \ldots$ continues to infinity. We will denote $\mathcal{V} = \{v_i\}_{i\in\mathbb{N}}$. Let $H = \mathcal{V} \cup \mathcal{C}_0$, and consider the restricted dynamics $\mathcal{L}_H$ introduced in Definition 4.9. We split the dynamics in two—for some local function f on H

$$\mathcal{L}_H f = \mathcal{L}^{\mathcal{C}_0} f + \mathcal{L}^{\mathcal{V}} f,$$

$$\mathcal{L}^{\mathcal{V}} f = \sum_{i\in\mathbb{N}} c_{v_i}^H (\mu_{v_i} f - f),$$

$$\mathcal{L}^{\mathcal{C}_0} f = \sum_{x\in\mathcal{C}_0} c_x^H (\mu_x f - f).$$

Note that the boundary conditions of the $\mathcal{C}_0$ dynamics depend on the state of the vertices in $\mathcal{V}$ and vice versa. We will denote by $\mathcal{L}_0^{\mathcal{C}_0}$ the $\mathcal{C}_0$ dynamics with empty boundary conditions and by $\mathcal{L}_1^{\mathcal{V}}$ the $\mathcal{V}$ dynamics with occupied boundary conditions. All generators come with their Dirichlet forms carrying the same superscript and subscript.

We will bound the gap of $\mathcal{L}_H$ using the gaps of $\mathcal{L}_1^{\mathcal{V}}$, $\mathcal{L}_0^{\mathcal{C}_0}$ and the following block dynamics:

$$\mathcal{L}^b f = \big(\mu_{\mathcal{V}}(f) - f\big) + \mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}}(\mu_{\mathcal{C}} f - f).$$

Denote the spectral gaps of $\mathcal{L}_1^{\mathcal{V}}$, $\mathcal{L}_0^{\mathcal{C}_0}$, $\mathcal{L}^b$, $\mathcal{L}_H$ by $\gamma_1^{\mathcal{V}}$, $\gamma_0^{\mathcal{C}_0}$, γ^b, γ_H.

By Proposition 4.4 of [12]:

CLAIM 5.26.

$$\gamma^b = 1 - \sqrt{1 - q^{|\partial\mathcal{C}_0|}},$$

that is, $\operatorname{Var} f \leq \frac{1}{1-\sqrt{1-q^{|\partial\mathcal{C}_0|}}} \mathcal{D}^b f$ *for any local function* f.

Let us now use this gap in order to relate γ_H to $\gamma^{\mathcal{V}}$ and $\gamma^{\mathcal{C}_0}$:

CLAIM 5.27.

$$\gamma_H \geq \gamma^b \min\{\gamma_1^{\mathcal{V}}, \gamma_0^{\mathcal{C}_0}\}.$$

PROOF. Fix a nonconstant local function f,

$$\begin{aligned}\operatorname{Var} f &\leq \frac{1}{\gamma^b}\mathcal{D}^b f = \frac{1}{\gamma^b}\big[\mu(\operatorname{Var}_{\mathcal{V}} f) + \mu(\mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}} \operatorname{Var}_{\mathcal{C}} f)\big] \\ &\leq \frac{1}{\gamma^b}\left[\frac{1}{\gamma_1^{\mathcal{V}}}\mu(\mathcal{D}_1^{\mathcal{V}} f) + \frac{1}{\gamma_0^{\mathcal{C}_0}}\mu(\mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}}\mathcal{D}^{\mathcal{C}_0} f)\right] \\ &\leq \frac{1}{\gamma^b}\max\left\{\frac{1}{\gamma_1^{\mathcal{V}}}, \frac{1}{\gamma_0^{\mathcal{C}_0}}\right\}\mathcal{D}_H f.\end{aligned}$$

□

We are left with estimating $\gamma_1^{\mathcal{V}}$ and $\gamma_0^{\mathcal{C}_0}$.

CLAIM 5.28. *There exists $C > 0$ such that $\gamma_1^{\mathcal{V}} \geq Cq^3$.*

PROOF. The Dirichlet form $\mathcal{D}_1^{\mathcal{V}}$ is dominated by the Dirichlet form of FA1f on $\mathbb{Z}_+$, and that dynamics has a spectral gap which is proportional to q^3 (see [12]). □

For $\gamma_0^{\mathcal{C}_0}$ we will use the bisection method, comparing the gap on a box to that of a smaller box. For $L \in \mathbb{N}$, let $\mathcal{L}_L^{\mathrm{NE}}$ be the generator of the North-East dynamics in the box $[L]^2$ with empty boundary (for the North-East model this is equivalent to putting empty boundary only above and to the right). Denote its gap by $\gamma_{[L]^2}^{\mathrm{NE}}$. By monotonicity we can restrict the discussion to this dynamics, that is,

$$\gamma_0^{\mathcal{C}_0} \geq \gamma_{\operatorname{diam}\mathcal{C}_0}^{\mathrm{NE}}. \tag{5.2}$$

We will now bound γ^{NE} (see also Theorem 6.16 of [12]).

CLAIM 5.29. $\gamma_{[L]^2}^{\mathrm{NE}} \geq e^{3\log qL}$.

PROOF. We will prove the result for $L_k = 2^k$ by induction on k. Then monotonicity will complete the argument for all L. Consider the box $[L_k]^2$, and divide it in two rectangles—$R_- = [L_{k-1}] \times [L_k]$ and $R_+ = [L_{k-1} + 1, L_k] \times [L_k]$. We will run the following block dynamics:

$$\mathcal{L}^{b\mathrm{NE}} f = (\mu_{R_+} f - f) + \mathbb{1}_{\partial_- R_+ \text{ is empty}}(\mu_{R_-} f - f),$$

where $\partial_- R_+$ is the inner left boundary of R_+. Again, by Proposition 4.4 of [12],

$$\begin{aligned}\operatorname{gap}(\mathcal{L}^{b\mathrm{NE}}) &= 1 - \sqrt{1 - \mu(\mathbb{1}_{\partial_- R_+ \text{ is empty}})} \\ &= 1 - \sqrt{1 - q^{L_k}}.\end{aligned}$$

Therefore, for every local function f,

$$\operatorname{Var} f \leq \frac{1}{1 - \sqrt{1 - q^{L_k}}}\mathcal{D}^{b\mathrm{NE}} f$$

$$= \frac{1}{1-\sqrt{1-q^{L_k}}}\mu(\mathrm{Var}_{R_+} f + \mathbb{1}_{\partial_- R_+ \text{ is empty}} \mathrm{Var}_{R_-} f)$$

$$\leq \frac{1}{1-\sqrt{1-q^{L_k}}}\mu\left(\frac{1}{\gamma_{R_+}^{\mathrm{NE}}}\mathcal{D}_{R_+}^{\mathrm{NE}} f + \frac{1}{\gamma_{R_-}^{\mathrm{NE}}}\mathcal{D}_{R_-}^{\mathrm{NE}} f\right),$$

where γ_R^{NE}, $\mathcal{D}_R^{\mathrm{NE}}$ are the spectral gap and Dirichlet form of the North-East dynamics in R with empty boundary conditions for any fixed rectangle R. We see that

$$\gamma_{[L_k]^2}^{\mathrm{NE}} \geq (1-\sqrt{1-q^{L_k}})\gamma_{[L_{k-1}]\times[L_k]}^{\mathrm{NE}}.$$

If we repeat the same argument dividing $[L_{k-1}]\times[L_k]$ into the rectangles $[L_{k-1}]\times[L_{k-1}]$ and $[L_{k-1}]\times[L_{k-1}+1, L_k]$, we obtain

$$\gamma_{L_{k-1}\times L_k}^{\mathrm{NE}} \geq (1-\sqrt{1-q^{L_{k-1}}})\gamma_{[L_{k-1}]^2}^{\mathrm{NE}}.$$

Hence,

$$\log\gamma_{[L_k]^2}^{\mathrm{NE}} \geq \log\gamma_{[L_{k-1}]^2}^{\mathrm{NE}} + 2^k \log q - \log 4,$$

yielding

$$\log\gamma_{[L_k]^2}^{\mathrm{NE}} \geq \log q \sum_{n=1}^{k} 2^n - k\log 4$$

which finishes the proof. □

We can now put everything together. Let L be the diameter of $\mathcal{C}_0$. By the second part of Claim 4.11,

$$\tau \leq \frac{1+q}{q}\frac{1}{\gamma^r} \leq \frac{1+q}{q}\frac{1}{(1-\sqrt{1-q^{|\mathcal{C}|}})\min\{\gamma_1^{\mathcal{V}}, \gamma_0^{\mathcal{C}_0}\}} \tag{5.3}$$
$$\leq q^{-4L-1}.$$

Finally, we will use the sharpness of the phase transition for the site percolation on the dual graph (see [1, 13]):

CLAIM 5.30. *There exists a positive constant c_2 that depends on π such that $\nu(L \geq D) \leq e^{-c_2 D}$ for any $D \in \mathbb{N}$.*

Using this claim and equation (5.3),

$$\nu(\tau \geq t) \leq \nu(q^{-4L-1} \geq t) = \nu\left(L \geq \frac{\log t}{4\log\frac{1}{q}} - \frac{1}{4}\right)$$
$$\leq Ct^{c/\log q}.$$

6. Conclusions and further questions. We have seen here two simple examples of KCMs in random environments. These examples show that when the environment has some rare remote “bad” regions the relaxation time fails to describe the true observed time scales of the system. Since the dynamics are not attractive, techniques such as monotone coupling and censoring cannot be applied to these models. In order to overcome this difficulty we considered the hitting time τ_0, which on one hand describes a physically measurable observable, and on the other hand could be studied using variational principles. We formulated some tools based on these variational principles and used them in order to understand the behavior of τ_0 in both models.

For future research, one may try to apply these tools on kinetically constrained models in more types of random environment, such as the polluted lattice, more general mixture of constraints, and models on random graphs.

There are also some questions left open considering the models studied here. For the first model, it is natural to conjecture that τ_0 scales as $q^{-\alpha}$ for some random α. We can also look at the π dependence of α—we know that when there are not many easy sites this time should become larger, until it reaches the FA2f time when $\pi = 0$. Looking at the proof of Theorem 2.2, we can see that $\overline{\alpha}$ scales like $\frac{\log 1/\pi}{\pi}$, and $\underline{\alpha}$ scales like $\pi^{-1/2}$. It seems more likely, however, that the actual exponent α behaves like $\frac{1}{\pi}$—the $\log \frac{1}{\pi}$ of the lower bound comes up also in the proof bounding the gap of the FA2f model [26], and also there it is conjectured that it does not appear in the true scaling. In fact, if we use the path that we have chosen in order to bound τ_0 also for the bootstrap percolation, we will have the same $\log \frac{1}{q}$ factor, and in this case it is known that it does not appear in the correct scaling. The lower bound of $\pi^{-1/2}$ could be improved, with the price of complicating the proof. The refined argument appears in [29], obtaining a bound that scales as $\frac{1}{\pi}$. It is worth noting here that for the bootstrap percolation, by repeating the arguments of [2, 20] with some minor adaptations, we can show that the scaling of the prefactor a with π is between $e^{c/\pi}$ and $e^{C/\pi}$ for $c, C > 0$ (see [29]).

CONJECTURE 6.1. *For the mixed threshold FA model*, ν-*almost surely the limit* $\lim_{q\to 0} \frac{\log \tau_0}{\log 1/q}$ *exists. Its value* α *is a random variable whose law depends on* π. *Moreover, the law of* $\pi\alpha$ *converges* (*in some sense*) *to a nontrivial law as* π *tends to* 0.

The mixed North-East and FA1f model also raises many questions, among them finding the critical probability q_c, and characterizing the behavior of both the bootstrap percolation and the KCM in the different parameter regimes.

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