BOOTSTRAP PERCOLATION ON THE PRODUCT OF THE TWO-DIMENSIONAL LATTICE WITH A HAMMING SQUARE

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Bootstrap percolation on a graph is a deterministic process that iteratively enlarges a set of occupied sites by adjoining points with at least θ occupied neighbors. The initially occupied set is random, given by a uniform product measure with a low density p. Our main focus is on this process on the product graph $\mathbb{Z}^2 \times K_n^2$, where K_n is a complete graph. We investigate how p scales with n so that a typical site is eventually occupied. Under critical scaling, the dynamics with even θ exhibits a sharp phase transition, while odd θ yields a gradual percolation transition. We also establish a gradual transition for bootstrap percolation on $\mathbb{Z}^2 \times K_n$. The community structure of the product graphs connects our process to a heterogeneous bootstrap percolation on $\mathbb{Z}^2$. This natural relation with a generalization of polluted bootstrap percolation is the leading theme in our analysis.

1. Introduction.

1.1. *Background.* Spread of signals—information, say, or infection—on graphs with community structure has attracted interest in the mathematical literature recently [7, 27, 28, 31, 34]. The idea is that any single community is densely connected, while the connections between communities are much more sparse. This naturally leads to multiscale phenomena, as the spread of the signal within a community is much faster then between different communities. Often, communities are modeled as cliques, that is, the intracommunity graph is complete, but in other cases some close-knit structure is assumed. By contrast, the intercommunity graph may, for example, impose spatial proximity as a precondition for connectivity. See [33] for an applications-oriented recent survey.

The principal graph under study in this paper is $G = \mathbb{Z}^2 \times K_n^2$, the Cartesian product between the lattice $\mathbb{Z}^2$ and two copies of the complete graph K_n on n points. Thus "community" consists of "individuals" determined by two characteristics, and two individuals within the community only communicate if they have one of the characteristics in common. Between the communities, communication is between like individuals that are also neighbors in the lattice. For comparison, we also address the case where each community is a clique, that is, the graph $\mathbb{Z}^2 \times K_n$.

The particular dynamics we use for spread of signals is *bootstrap percolation* with integer *threshold* parameter $\theta \geq 1$. In this very simple deterministic process, one starts with an initial configuration ω_0 of 0s (or *empty* sites) and 1s (or *occupied* sites) on vertices of G, and iteratively enlarges the set of occupied sites in discrete time as follows. Assume ω_t is given for some $t \geq 0$, and fix a vertex v of G. If $\omega_t(v) = 1$, then $\omega_{t+1}(v) = 1$. If $\omega_t(v) = 0$, and v has θ or more neighboring vertices v' with $\omega_t(v') = 1$, then $\omega_{t+1}(v) = 1$; otherwise $\omega_{t+1}(v) = 0$. We will typically identify the configuration ω_t with the set of its occupied sites $\{v : \omega_t(v) = 1\}$. Thus ω_t increases to the set $\omega_\infty = \bigcup_{t \geq 0} \omega_t$ of eventually occupied vertices.

As is typical, we assume that the initial state ω_0 is a uniform product measure with some small density $p \in (0, 1)$. This makes the set ω_∞ random as well, and it is natural to ask how to choose p to make ω_∞ large, that is, to make the initially sparse signal widespread. Observe that, if $\theta \geq 3$, ω_∞ cannot comprise *all* vertices of G with nonzero probability for any $p < 1$, as a block of neighboring empty copies of K_n^2 (e.g., $\{(0, 0), (0, 1), (1, 0), (1, 1)\} \times K_n^2$) cannot be invaded by occupied sites, and the infinite lattice will contain such a block with probability 1. We therefore ask a weaker question: how large should p be, in terms of n, so that ω_∞ comprises a substantial proportion of points? That is, we are interested in the size of the *final density*, $\mathbb{P}_p(v_0 \in \omega_\infty)$, which is independent of $v_0 \in G$ by vertex-transitivity of G.

Bootstrap percolation was introduced on trees in [10], but it has received by far the most attention on lattices $\mathbb{Z}^d$. In this case, $\mathbb{P}_p(\omega_\infty = \mathbb{Z}^d) = 1$, as proved in [36] for $d = 2$ and in [32] for $d \geq 3$. Many deep and surprising results originated from the study of metastability properties of the model on finite regions (see, e.g., [2, 4, 9, 15, 24]). We refer to the recent survey [30] for a comprehensive review.

Study of bootstrap percolation and related dynamics on graphs with long-range connectivity is a more recent undertaking [3, 13, 18, 20, 35] and has a fundamentally different flavor: while on sparse graphs, the dominant mechanism is formation of small nuclei that are likely to grow indefinitely, the relevant events in densely connected graphs tend to depend on the configuration on the whole space. It is therefore tempting to consider graphs that combine aspects of both, and we continue here our work started in [19].

As already remarked, ω_∞ cannot cover all vertices of our graph G due to the presence of local configurations of sparsely occupied copies of Hamming squares, K_n^2. Other copies, of course, have higher initial occupation, get fully occupied and spread their occupation to the neighboring squares. Thus we have a competition between densely occupied copies of K_n^2 that act as nuclei, and sparsely occupied ones that function as obstacles to growth. This invites comparison with polluted bootstrap percolation [14, 16, 17] on $\mathbb{Z}^2$, which is indeed the main source of our tools. However, by contrast with the model in the cited papers, which has only three states (empty and occupied sites, and permanent obstacles), the dynamics that arise from our process has more types corresponding to all possible thresholds (0, 1, 2, 3, 4, 5) that different sites in $\mathbb{Z}^2$ require to become occupied. Moreover, we need different variants for the case $\theta = 3$ and the graph $\mathbb{Z}^2 \times K_n$. We call these comparison dynamics *heterogeneous bootstrap percolation*. We also encounter a technical difficulty in the form of correlations in the initial state, which are handled by coupling and other related perturbation methods.

After its introduction in [17], the basic polluted version of heterogeneous bootstrap percolation was further analyzed in [14, 16]; it is the recent techniques developed in these two papers that will be useful to us. Related models include processes on a complete graph with excluded edges [25], Glauber dynamics with "frozen" vertices [11], dynamics on complex networks with "damaged" vertices [5, 6] and on inhomogeneous geometric random graphs [26].

1.2. *Statements of main theorems.* Our main results determine a critical scaling for prevalent occupation on $\mathbb{Z}^2 \times K_n^2$: we exhibit functions $f_\theta(n)$ so that, when $p = af_\theta(n)$, the limit as $n \to \infty$ of the final density $\mathbb{P}_p(v_0 \in \omega_\infty)$ is low for small a and high for large a. In fact, for all θ, this limit vanishes for $a < a_c$, where $a_c = a_c(\theta)$ is a critical value that we are able to identify (and in fact compute explicitly for even θ). The behavior for $a > a_c$ is however not the same for all θ: if θ is even, the limit is 1, while if θ is odd the final density is bounded away from 1 for any finite a and only approaches 1 as $a \to \infty$. We already encountered the nonintuitive qualitative difference between odd and even θ in our earlier work [19], in which the lattice factor was one-dimensional. This, and the connection with heterogeneous bootstrap percolation, are the most inviting features of our present model.

We now proceed to formal statements of our results. We first remark that for $\theta \leq 2$ we have no obstacles and $\mathbb{P}_p(\omega_\infty \equiv 1) = 1$ for any $p > 0$ by standard bootstrap percolation arguments [32, 36]; therefore, we assume that $\theta \geq 3$ throughout the paper. As we have so far, we denote by v_0 an arbitrary fixed vertex of the graph in question, and we use the notation $\mathbf{0} = (0, 0)$ for the origin in $\mathbb{Z}^2$. We begin with our main result for even thresholds.

THEOREM 1.1. *Consider bootstrap percolation on $\mathbb{Z}^2 \times K_n^2$ with threshold $\theta = 2\ell + 2$, for some $\ell \geq 1$. Assume that*

$$(1.1) \qquad p = a \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}},$$

for some $a > 0$.
    If $a^\ell < 2(\ell - 1)!$, then

$$(1.2) \qquad \mathbb{P}_p(v_0 \in \omega_\infty) = n^{-2/\ell + o(1)} \quad \text{as } n \to \infty.$$

Conversely, if $a^\ell \geq 2(\ell - 1)!$, then

$$(1.3) \qquad \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \subset \omega_\infty) \to 1 \quad \text{as } n \to \infty.$$

Moreover, if $a^\ell > 2(\ell - 1)!$, then

$$(1.4) \qquad \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \not\subset \omega_\infty) = \begin{cases} n^{4/\ell - 4a^\ell/\ell! + o(1)}, & \ell \geq 2, \\ n^{-2a + o(1)}, & \ell = 1 \end{cases} \quad \text{as } n \to \infty,$$

and $\mathbb{P}_p(\omega_0 = \omega_\infty \text{ on } \{\mathbf{0}\} \times K_n^2)$ satisfies the same asymptotics.

Our results for odd thresholds are somewhat less precise, but suffice to provide the announced distinction from even θ.

THEOREM 1.2. *Consider bootstrap percolation on $\mathbb{Z}^2 \times K_n^2$ with threshold $\theta = 2\ell + 1$, for some $\ell \geq 1$. Assume that*

$$(1.5) \qquad p = \frac{a}{n^{1+1/\ell}},$$

for some $a > 0$.
    There exists a critical value $a_c = a_c(\ell) \in (0, \infty)$ so that the following holds. If $a < a_c$, then

$$(1.6) \qquad \mathbb{P}_p(v_0 \in \omega_\infty) \to 0 \quad \text{as } n \to \infty.$$

Conversely, if $a > a_c$, then

$$(1.7) \qquad 0 < \liminf_{n \to \infty} \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \subset \omega_\infty) \leq \limsup_{n \to \infty} \mathbb{P}_p(v_0 \in \omega_\infty) < 1.$$

Furthermore,

$$(1.8) \qquad \liminf_{n \to \infty} \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \subset \omega_\infty) \to 1 \quad \text{as } a \to \infty.$$

Finally, we state our result for the case of clique community, in which there is no difference between odd and even θ and no phase transition as in Theorems 1.1 and 1.2.

THEOREM 1.3. *Consider bootstrap percolation on $\mathbb{Z}^2 \times K_n$ with threshold $\theta \geq 3$. Assume that $p = a/n$ for some $a \in (0, \infty)$. Then both*

$$\liminf_n \mathbb{P}_p(\omega_\infty(v_0) = 1) \quad \text{and} \quad \limsup_n \mathbb{P}_p(\omega_\infty(v_0) = 1)$$

are in $(0, 1)$ and converge to 0 (resp., 1) as $a \to 0$ (resp., $a \to \infty$). If $\theta \geq 14$, then $\lim_n \mathbb{P}_p(\omega_\infty(v_0) = 1)$ exists and is continuous in a.

A similar result to Theorem 1.3 holds for $\mathbb{Z}^d \times K_n$ for all $d \geq 3$, but extension of our results to $\mathbb{Z}^d \times K_n^2$ is much more challenging (see Section 7 on open problems).

1.3. *Sketch of the main ideas and organization.* The main purpose of this subsection is to outline our strategy for proving Theorems 1.1 and 1.2. For any $x \in \mathbb{Z}^2$, find the smallest integer $k \in [0, \theta]$, so that the bootstrap dynamics restricted to $\{x\} \times K_n^2$, and using threshold $\theta - k$, eventually fully occupies this copy of the Hamming square. (Below, we introduce a technical term, internal spanning, for the ability to fill a part of the space without outside help.) Then let $\xi_0(x) = k$. Thus, $\xi_0(x) = 0$ means that $\{x\} \times K_n^2$ will get occupied regardless of the configuration on the surrounding copies of K_n^2, $\xi_0(x) = 1$ implies that $\{x\} \times K_n^2$ will get occupied provided at least one neighboring copy of K_n^2 gets fully occupied, and so on.

By definition, $\xi_0(x) = 0$ implies that $\{x\} \times K_n^2 \subset \omega_\infty$. Iteratively, for $t = 1, 2, \ldots$, we define ξ_t as follows: if $\xi_0(x) = k$ and x has at least k lattice neighbors y with $\xi_{t-1}(y) = 0$, then let $\xi_t(x) = 0$. By induction, if $\xi_t(x) = 0$ for some t, then $\{x\} \times K_n^2 \subset \omega_\infty$. This heterogeneous bootstrap percolation process is discussed in Section 2.2, while two other variants are used in Section 5.4 and Section 6. The decreasing sequence ξ_t of configurations provides a lower bound on ω_∞ (see Lemma 2.6 for a formal statement), which also turns out to be sufficiently close to an upper bound (provided by Lemma 2.7).

The dynamics ξ_t is rather similar to the polluted bootstrap percolation [17]. To explain the connection, let us call *active* sites those x with $\xi_0(x) \in \{0, 1\}$. Although 1s must be activated by neighboring 0s, it turns out that we can treat the two states as equivalent, provided 0s are not too rare. The active sites spread, using the bootstrap percolation rule with threshold 2, over the *background* sites, the 2s in ξ_0. The remaining sites, those x with $\xi_0(x) \geq 3$, are *obstacles* that are able to stop the growth of active sites.

To estimate the densities of active sites and obstacles, we need a fairly detailed understanding of the bootstrap dynamics on a single copy of a Hamming square, provided in Section 2.1, which is mostly a review the results from [13, 19]. In this outline, we will use informal versions of these results.

For even $\theta = 2\ell + 2$, $\ell \geq 2$, a necessary, and asymptotically sufficient, condition for x to be active is that $\{x\} \times K_n^2$ contains either a horizontal or a vertical line with at least $\ell + 1$ occupied points. This happens with probability about $n(np)^{\ell+1}$. (Multiplicative constants are not important in this case.) On the other hand, the asymptotically necessary and sufficient condition for x *not* to be an obstacle is that $\{x\} \times K_n^2$ contains both a horizontal and a vertical line with at least ℓ occupied points, which results in the density of obstacles about $(1 - n^\ell p^\ell / \ell!)^n \approx \exp(-n^{\ell+1} p^\ell / \ell!)$. According to [17], the critical transition is when

$$\text{density of obstacles} \approx (\text{density of active sites})^2,$$

which forces the choice of (1.1) as the critical scaling, the critical a to satisfy $a^\ell / \ell! = 2/\ell$, and the sharp transition in Theorem 1.1.

In Section 3, we prove the subcritical rate (1.2). Our argument closely follows that of [16], but we give a substantial amount of details due to the differences in the assumptions and conclusions. In Section 4, we focus on the supercritical part of Theorem 1.1, which is handled by the method from [17], and then involves finding the most likely configuration that prevents occupation from spreading inwards from a circuit of fully occupied copies of the Hamming square.

When $\theta = 2\ell + 1$, $\ell \geq 2$, is odd, the active density is approximated by the probability that both a horizontal and a vertical line with at least ℓ occupied points exist in $\{x\} \times K_n^2$, which is about $(1 - (1 - n^\ell p^\ell / \ell!)^n)^2 \approx (1 - \exp(-n^{\ell+1} p^\ell / \ell!))^2$. Moreover, now the density of obstacles, approximated by a probability that there is no line in $\{x\} \times K_n^2$ with ℓ occupied points, is about $(1 - n^\ell p^\ell / \ell!)^{2n} \approx \exp(-2n^{\ell+1} p^\ell / \ell!)$. Observe the crucial difference from the case of even θ: the number of required sites on a line in $\{x\} \times K_n^2$ is the same, namely ℓ, for both active sites and nonobstacles. The two probability estimates now force the critical scaling (1.5), under which in this case both probabilities converge to a constant depending on a. As a changes, the dynamics experiences a percolation transition at some critical value a_c.

Section 5 contains the proof Theorem 1.2, in which we characterize a_c through the limiting dynamics (as $n \to \infty$), which can be appropriately coupled to the dynamics for finite n. A different (but related) limiting dynamics is similarly used in Section 6, which is devoted to the proof of Theorem 1.3. We conclude with a list of open problems in Section 7.

2. Preliminaries.

2.1. *Copies of Hamming squares.* Fix an initial state ω_0 for our bootstrap dynamics on $\mathbb{Z}^2 \times K_n^2$. For a set $A \subset \mathbb{Z}^2 \times K_n^2$, the dynamics *restricted to A* uses the bootstrap rule on the subgraph induced by A, with the initial state ω_0 on A. As in [19], we call a copy $\{x\} \times K_n^2$, $x \in \mathbb{Z}^2$:

- *internally spanned at threshold r* (*r-IS*) if the bootstrap dynamics with threshold r, restricted to $\{x\} \times K_n^2$, eventually results in full occupation of $\{x\} \times K_n^2$;
- *internally inert at threshold r* (*r-II*) if the bootstrap dynamics with threshold r, restricted to $\{x\} \times K_n^2$, never changes the state of any vertex in $\{x\} \times K_n^2$; and
- *inert* at threshold r (*r-inert*) if the (unrestricted) bootstrap dynamics with threshold r does not occupy any point in $\{x\} \times K_n^2$ in the first time step.

In the rest of this subsection, we mostly summarize the results from [19] and [13]. We begin with the results for even θ, which were essentially proved in [19].

LEMMA 2.1. *Assume that p is given by* (1.1).

1. *If $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 2)\text{-IS}) = \mathcal{O}(n^{-L}),$$

for any constant $L > 0$.

2. *If $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 1)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } (2\ell - 1)\text{-II}) \sim n^{-2a^\ell/\ell!},$$

and for $\ell = 1$ we have

$$\mathbb{P}_p(K_n^2 \text{ is not } 1\text{-IS}) = \mathbb{P}_p(K_n^2 \text{ is } 1\text{-II}) \sim \frac{1}{n^a}.$$

3. *If $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } (2\ell)\text{-II}) \sim 2n^{-a^\ell/\ell!},$$

and for $\ell = 1$ we have

$$\mathbb{P}_p(K_n^2 \text{ is not } 2\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } 2\text{-II}) \sim \frac{a \log n}{n^a}.$$

4. *If $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell + 1)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is not } (2\ell + 1)\text{-II})$$

$$\sim \frac{2a^{\ell+1}}{(\ell + 1)!} \cdot \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}.$$

5. *If $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell + 2)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is not } (2\ell + 2)\text{-II})$$

$$\sim \left(\frac{a^{\ell+1}}{(\ell + 1)!}\right)^2 \cdot \frac{(\log n)^{2+2/\ell}}{n^{2/\ell}}.$$

PROOF. Statements 1 through 4 are Lemmas 3.6, 3.3, 3.4 and 3.5 in [19], and the proof of the last statement is similar to the proof of the 4th, so we omit it. □

The next lemma compares probabilities for inertness and internal inertness for $\ell \geq 2$.

LEMMA 2.2. *Assume $\theta = 2\ell + 2$, $\ell \geq 2$, and p is given by* (1.1). *If $\frac{a^\ell}{\ell!} < 1$, then for any $x \in \mathbb{Z}^2$*

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - 2)\text{-inert}) \sim \mathbb{P}_p(K_n^2 \text{ is } (\theta - 2)\text{-II}) \sim 2n^{-a^\ell/\ell!},$$

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is not } (\theta - 1)\text{-inert}) \sim \mathbb{P}_p(K_n^2 \text{ is not } (\theta - 1)\text{-II})$$

$$\sim \frac{2a^{\ell+1}}{(\ell + 1)!} \cdot \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}} \quad \text{and}$$

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is not } \theta\text{-inert}) \sim \mathbb{P}_p(K_n^2 \text{ is not } \theta\text{-II})$$

$$\sim \left(\frac{a^{\ell+1}}{(\ell + 1)!}\right)^2 \cdot \frac{(\log n)^{2+2/\ell}}{n^{2/\ell}}.$$

PROOF. Fix an $r = 0, 1, 2$. Then the probability that any fixed copy of K_n^2 has a site with exactly $k \geq 1$ occupied $\mathbb{Z}^2$-neighbors and at least $\theta - r - k$ occupied K_n^2-neighbors is

$$\mathcal{O}(n^2 p^k (np)^{\theta - r - k}) = \mathcal{O}(n^{-k - (2-r)/\ell} (\log n)^{(2\ell + 2 - r)/\ell}).$$

Therefore,

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - r)\text{-II but not } (\theta - r)\text{-inert}) = n^{-1 - (2-r)/\ell + o(1)}.$$

The rest follows from Lemma 2.1 parts 3, 4 and 5 and the assumptions put on a and ℓ. □

We need a slightly more involved argument for $\ell = 1$.

LEMMA 2.3. *Assume $\theta = 4$ and $p = a\frac{\log n}{n^2}$. We have*

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } 2\text{-inert}) \geq a\frac{\log n}{n^a}(1 - o(1))$$

PROOF. Let G_1 be the event that $\{x\} \times K_n^2$ contains at least two occupied vertices, and G_2 the event that a point in $\{x\} \times K_n^2$ has both an occupied $\mathbb{Z}^2$-neighbor and an occupied K_n-neighbor. Note that these are increasing events and that

$$\{\{x\} \times K_n^2 \text{ is not } 2\text{-inert}\} \subset G_1 \cup G_2.$$

Therefore, by FKG inequality,

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is not } 2\text{-inert}) \leq \mathbb{P}_p(G_1) + \mathbb{P}_p(G_2) - \mathbb{P}_p(G_1)\mathbb{P}_p(G_2),$$

and so

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } 2\text{-inert}) \geq \mathbb{P}_p(G_1^c) - \mathbb{P}_p(G_1^c)\mathbb{P}_p(G_2).$$

Finally, we use that $\mathbb{P}_p(G_1^c) \sim a\frac{\log n}{n^a}$ and $\mathbb{P}_p(G_2) \leq 8n^3 p^2 = \mathcal{O}(\log n/n)$. □

We proceed with the analogous results for odd θ, which mostly follow from [13], and we again omit the detailed proofs.

LEMMA 2.4. *Assume that p is given by* (1.5).

1. *If $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 2)\text{-IS}) = \mathcal{O}(n^{-L}),$$

for any constant $L > 0$.

2. *If $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 1)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } (2\ell - 1)\text{-II}) \sim \exp\left[-\frac{2a^\ell}{\ell!}\right].$$

3. *If $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is not } (2\ell)\text{-II}) \sim (1 - e^{-a^\ell/\ell!})^2.$$

4. *If $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell + 1)\text{-II}) \sim 2 \cdot \frac{a^{\ell+1}}{(\ell + 1)!} \cdot (1 - e^{-a^\ell/\ell!}) \cdot \frac{1}{n^{1/\ell}},$$

and

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell + 1)\text{-IS}) \sim 2 \cdot \frac{a^{\ell+1}}{(\ell + 1)!} \cdot (1 - e^{-a^\ell/\ell!})^2 \cdot \frac{1}{n^{1/\ell}}.$$

PROOF. Parts 2 and 3 follow from Theorem 2.1 in [13]. Part 1 is proved in the same fashion as Lemma 3.6 in [19]. The proof of part 4 is similar to the proof of parts 2 and 3 and is omitted; in fact, we only need in our arguments in Section 5 that the two probabilities are positive for all n and go to 0 as $n \to \infty$, which is very easy to show. $\square$

We conclude with an analogue of Lemma 2.2.

LEMMA 2.5. *Assume that $\theta = 2\ell + 1$, $\ell \geq 1$, and that p is given by* (1.5). *Fix an $x \in \mathbb{Z}^2$. Then, for $\ell \geq 2$,*

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - 2)\text{-II but not } (\theta - 2)\text{-inert}) = \mathcal{O}(n^{-1})$$

and, for $\ell \geq 1$,

$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - 1)\text{-II but not } (\theta - 1)\text{-inert}) = \mathcal{O}(n^{-1}),$$
$$\mathbb{P}_p(\{x\} \times K_n^2 \text{ is } \theta\text{-II but not } \theta\text{-inert}) = \mathcal{O}(n^{-1-1/\ell}).$$

PROOF. Observe that, for $r \in \{0, 1, 2\}$, the probability that any fixed copy of K_n^2 has a site with exactly $k \geq 1$ occupied $\mathbb{Z}^2$-neighbors and at least $\theta - r - k$ occupied K_n^2-neighbors is

$$\mathcal{O}(n^2 p^k (np)^{\theta-r-k}) = \mathcal{O}(n^{-k+(r-1)/\ell}),$$

and the desired estimates follow. $\square$

2.2. *Heterogeneous bootstrap percolation.* We now introduce a comparison bootstrap dynamics ξ_t on $\mathbb{Z}^2$, which is a generalization of polluted bootstrap percolation introduced in [17]. We assume that $\xi_t \in \{0, 1, 2, 3, 4, 5\}^{\mathbb{Z}^2}$, $t \in \mathbb{Z}_+$, and that ξ_0 is given. The rules mandate that a state can only change to 0 by contact with sufficient number of 0s. More precisely, if $Z_t(x)$ is the cardinality of $\{y : y \sim x \text{ and } \xi_t(y) = 0\}$, where $x \sim y$ means that x and y are nearest neighbors in $\mathbb{Z}^2$, then

$$\xi_{t+1}(x) = \begin{cases} 0, & Z_t(x) \geq \xi_t(x), \\ \xi_t(x), & \text{otherwise.} \end{cases}$$

If $\xi_0 \in \{0, 2\}^{\mathbb{Z}^2}$, this is the usual threshold-2 bootstrap percolation. Adding 1s adds sites which need to be "switched on" by neighboring 0s. Finally, 3s, 4s and 5s act like "obstacles," which prevent the spread of 0s at sufficient density.

The next two lemmas establish upper and lower-bounding couplings between ξ_t and ω_t. Their proofs are similar, so we only provide details for the second one.

LEMMA 2.6. *Assume $\xi_0(x) = 0$ whenever the Hamming plane $\{x\} \times K_n^2$ is θ-IS; $\xi_0(x) = k \in \{1, 2, 3, 4\}$ whenever $\{x\} \times K_n^2$ is $(\theta - k)$-IS, but is not $(\theta - k + 1)$-IS; and that $\xi_0(x) = 5$ if $\{x\} \times K_n^2$ is not $(\theta - 4)$-IS. Then*

$$\bigcup \{\{x\} \times K_n^2 : \xi_\infty(x) = 0\} \subset \omega_\infty.$$

LEMMA 2.7. *Assume $\xi_0(x) = 0$ whenever the Hamming plane $\{x\} \times K_n^2$ is not θ-inert; that $\xi_0(x) = k \in \{1, 2, 3, 4\}$ whenever $\{x\} \times K_n^2$ is not $(\theta - k)$-inert, but is $(\theta - k + 1)$-inert; and that $\xi_0(x) = 5$ if $\{x\} \times K_n^2$ is $(\theta - 4)$-inert. Then*

$$\omega_\infty \subset \bigcup \{\{x\} \times K_n^2 : \xi_\infty(x) = 0\} \cup \omega_0.$$

PROOF. We will prove the following stronger statement by induction. We claim that for every $t \geq 0$,

$$(2.1) \qquad\qquad \omega_t \subset \bigcup \{\{x\} \times K_n^2 : \xi_t(x) = 0\} \cup \omega_0.$$

Suppose that (2.1) holds through time $t - 1 \geq 0$, and let $x \in \mathbb{Z}^2$ be a point such that $\xi_t(x) \neq 0$. Suppose x has exactly k neighbors $y \in \mathbb{Z}^2$ with $\xi_{t-1}(y) = 0$. Therefore, $\xi_0(x) \geq k + 1$, so $\{x\} \times K_n^2$ is $(\theta - k)$-inert. Every vertex in $(\{x\} \times K_n^2) \setminus \omega_0$ has at most $\theta - k - 1$ neighbors in ω_0, so every vertex in $(\{x\} \times K_n^2) \setminus \omega_0$ has at most $\theta - 1$ neighbors in

$$\bigcup \{\{x\} \times K_n^2 : \xi_{t-1}(x) = 0\} \cup \omega_0.$$

Therefore, by the induction hypothesis, every vertex in $(\{x\} \times K_n^2) \setminus \omega_0$ has at most $\theta - 1$ neighbors in ω_{t-1}, so no vertex in $\{x\} \times K_n^2$ becomes occupied at time t. $\quad\square$

3. The subcritical regime for even threshold. This section contains the proof of (1.2). Our argument is a suitable modification of the methods from [16], which are in turn based on duality-based construction of random surfaces [12, 22, 23]. We cannot immediately apply the result from [17], as we need to handle short-range dependence in the initial state.

3.1. *Bootstrap percolation with obstacles.* Our focus will be the heterogeneous bootstrap percolation ξ_t, with a random initial set ξ_0. We will call such initial set a *positively correlated random field* if increasing events are positively correlated (i.e., the FKG inequality holds), and *1-dependent* if $\xi_0(x)$ and $\xi_0(y)$ are independent for $\|x - y\|_1 \geq 2$.

THEOREM 3.1. *Let* $\mathsf{p}, \mathsf{q} > 0$ *be such that* $\mathsf{p} + \mathsf{q} < 1$. *Suppose* ξ_0 *has the following properties*: *for every* $x \in \mathbb{Z}^2$,

(3.1)
$$\mathbb{P}(\xi_0(x) = 0) = \mathsf{p},$$
$$\mathbb{P}(\xi_0(x) = 2) = 1 - \mathsf{p} - \mathsf{q},$$
$$\mathbb{P}(\xi_0(x) = 3) = \mathsf{q},$$

and ξ_0 *is a 1-dependent, positively correlated random field. Let* $C > 0$, *and suppose that* $\mathsf{q} > C\mathsf{p}^2$. *Then for C sufficiently large, we have that with probability at least* $1 - C\mathsf{p}^3$ *either* $\xi_\infty(\mathbf{0}) \geq 2$, *or else* $\mathbf{0}$ *is contained in a cluster* (*maximal connected set*) *of sites* $x \in \mathbb{Z}^2$ *with* $\xi_\infty(x) = 0$ *that has* ℓ^∞-*diameter at most* 1000.

We first explain how Theorem 3.1 accomplishes the goal of this section.

PROOF OF THEOREM 1.1 EQUATION (1.2). Initialize ξ_0 using inertness as in Lemma 2.7, then convert all 1s to 0s, and all 4s and 5s to 3s. Suppose $v_0 \in \mathbf{0} \times K_n^2$. If $v_0 \in \omega_\infty$, then either $v_0 \in \omega_0$, or some Hamming square in $\{\{x\} \times K_n^2 : x \in [-1000, 1000]^2\}$ is not θ-inert, or else $\mathbf{0}$ is in a cluster of state-0 sites in ξ_∞ that has diameter larger than 1000. Therefore, by Theorem 3.1 and Lemma 2.2

$$\mathbb{P}_p(v_0 \in \omega_\infty)$$
$$\leq \mathbb{P}_p(v_0 \in \omega_0) + 10^7 \mathbb{P}_p(\mathbf{0} \times K_n^2 \text{ is not } \theta\text{-inert})$$
$$\quad + C\mathbb{P}_p(\mathbf{0} \times K_n^2 \text{ is not } (\theta - 1)\text{-inert})^3$$
$$= n^{-2/\ell + o(1)}.$$

The lower bound is easy: by Lemma 2.1 part 5,

$$\mathbb{P}_p(v_0 \in \omega_\infty) \geq \mathbb{P}_p(\mathbf{0} \times K_n^2 \text{ is } \theta\text{-IS}) = n^{-2/\ell + o(1)},$$

and (1.2) is thus proved. $\square$

We will complete the proof of Theorem 3.1 in Section 3.4. Throughout this section, we will assume that p is sufficiently small to make certain estimates work.

For a set $A \subset \mathbb{Z}^2$, a configuration $\xi_0 \in \{0, \ldots, 5\}^{\mathbb{Z}^2}$, and $k \in \{0, \ldots, 5\}$, define $\xi_0^{(A,k)}$ by

$$\xi_0^{(A,k)}(x) = \begin{cases} \xi_0(x) & \text{for } x \in A, \\ k & \text{for } x \in A^c. \end{cases}$$

The resulting bootstrap dynamics, with initial configuration $\xi_0^{(A,k)}$, is denoted by $(\xi_t^{(A,k)})_{t \geq 0}$. Observe that $(\xi_t^{(A,5)})_{t \geq 0}$ is the heterogeneous bootstrap dynamics restricted to A, that is, run on the subgraph of $\mathbb{Z}^2$ induced by A. Also, for an $x \in \mathbb{Z}^2$, let $\mathtt{Nbrs}(x, A)$ denote the number of neighbors of x that lie in A. The next proposition gives a sufficient condition under which the configuration outside a set Z does not to influence the final set of 0s inside Z.

PROPOSITION 3.2.   *Assume that $\xi_0 \in \{0, 2, 3\}^{\mathbb{Z}^2}$. Fix an integer $m \geq 1$. Fix a finite set $Z \subset \mathbb{Z}^2$ with $\mathtt{Nbrs}(x, Z^c) \leq 2$ for every $x \in Z$, and run two heterogeneous bootstrap perco-lation dynamics: the first with initial configuration $\xi_0^{(Z,0)}$; the second with initial configura-tion $\xi_0^{(Z,5)}$. Assume that the configuration ξ_0 on Z satisfies the following conditions:*

(i) *Any $x \in Z$ with $\mathtt{Nbrs}(x, Z^c) = 2$ has $\xi_0(x) = 3$.*

(ii) *For any $x \in Z$ with $\mathtt{Nbrs}(x, Z^c) \geq 1$, there is no vertex y with $\xi_0(y) = 0$ within ℓ^∞-distance m of x.*

(iii) *The final configuration in the dynamics started from the initial configuration $\xi_0^{(Z,5)}$ has no connected set of vertices in state 0 with ℓ^∞-diameter larger than $m/2$.*

Then, for all $t \geq 0$, we have

$$\{x \in Z : \xi_t^{(Z,0)}(x) = 0\} = \{x \in Z : \xi_t^{(Z,5)}(x) = 0\}.$$

PROOF.   Assume the conclusion does not hold, and consider the first time t at which there exists a vertex $x \in Z$ such that $\xi_t^{(Z,0)}(x) = 0$ but $\xi_t^{(Z,5)}(x) > 0$. As the two dynamics have the same initial configuration on Z, we have $t > 0$. By minimality of t, and properties (ii) and (iii), at time $t - 1$ every $y \in Z$ such that $\mathtt{Nbrs}(y, Z^c) \geq 1$ has no neighbors in Z with state 0 in either dynamics. So, we cannot have $\mathtt{Nbrs}(x, Z^c) = 2$, since by (i), $\xi_0^{(Z,0)}(x) = 3$, and x has at most two neighbors in state 0 through time $t - 1$, so the state of x could not change at time t. We cannot have $\mathtt{Nbrs}(x, Z^c) = 1$ either, since $\xi_{t-1}^{(Z,0)}(x) \geq 2$. Thus $\mathtt{Nbrs}(x, Z^c) = 0$, but then x sees the same states among its neighbors in both dynamics at time $t - 1$ and, therefore, x has the same state in both dynamics at time t, a contradiction. $\square$

We now present a number of lemmas that all assume the conditions in the statement of Theorem 3.1, and are followed by the proof of this theorem in Section 3.4.

We will search for a set Z satisfying Proposition 3.2 within a square of size polynomial in p^{-1}. The following lemma will guarantee that Z satisfies condition (iii) of Proposition 3.2 with high probability.

LEMMA 3.3.   *Fix an integer $s > 0$, and let $N = \lfloor \mathsf{p}^{-s} \rfloor$. Let $A = [-N, N]^2$. With prob-ability at least $1 - C\mathsf{p}^s$, where $C = C(s)$ is a constant, all connected clusters (maximal connected sets) of state 0 vertices in $\xi_\infty^{(A,5)}$ have ℓ^∞-diameter at most $24s$.*

PROOF.   First, replace all 3s by 2s in the initial configuration $\xi_0^{(A,5)}$; then all connected clusters of 0s in $\xi_\infty^{(A,5)}$ are rectangles. Fix an integer $k > 0$, and let E_k be the event that the final configuration contains a rectangle of 0s whose longest side has length at least k. If E_k occurs, A contains an internally spanned rectangle R whose longest side length is in the interval $[k/2, k]$ [2]. Then any pair of neighboring lines, each perpendicular to the longest side of R, and such that both intersect R, must contain a state 0 vertex within R initially. Moreover, two pairs of neighboring lines that are at distance at least 2 from one another satisfy this requirement independently (since ξ_0 is 1-dependent). There are at most $(2N + 1)^2 k^2$ possible selections of the rectangle R. Therefore,

(3.2) $$\mathbb{P}(E_k) \leq 5N^2 k^2 (2k\mathsf{p})^{k/6-1} \leq \mathsf{p}^{(k-12s)/6-1}(2k)^{k/6+2},$$

and the claim follows by choosing $k = 24s$. $\square$

Let

$$L = \lfloor \delta/(m\mathsf{p}) \rfloor,$$

where $\delta > 0$ is a small constant to be fixed later. Also let $M = 12L$. Define the set

$$(3.3) \qquad J = \big([-m, m] \times [-M, M]\big) \cup \big([-M, M] \times [-m, m]\big).$$

Call a vertex $x \in \mathbb{Z}^2$ *nice* if $\xi_0(x) = 3$ and every vertex $y \in x + J$ has $\xi_0(y) \geq 2$. For each $u \in \mathbb{Z}^2$, define the rescaled box at u to be

$$Q_u := (2L + 1)u + [-L, L]^2.$$

We call a box Q_u *good* if it contains a nice vertex. We will give a lower bound on the probability that a box is good. Call a vertex $x \in \mathbb{Z}^2$ *viable* if every vertex $y \in x + J$ has $\xi_0(y) \geq 2$, and note that a viable vertex x with $\xi_0(x) = 3$ is nice.

LEMMA 3.4. *Fix a vertex $x \in \mathbb{Z}^2$ and an $\epsilon > 0$. Assume $\delta \leq \epsilon/10^3$. Then*

$$(3.4) \qquad \mathbb{P}(x \text{ is viable}) \geq 1 - \epsilon.$$

PROOF. The argument is a simple estimate, where the first inequality below follows from the positive correlation assumption on ξ_0,

$$
(3.5) \qquad
\begin{aligned}
\mathbb{P}\big(\xi_0(y) &\geq 2 \text{ for all } y \in x + J\big) \\
&\geq [1 - \mathsf{p}]^{2(2M+1)(2m+1)} \\
&\geq \exp[-36 m M \mathsf{p}] \\
&\geq \exp(-500\delta),
\end{aligned}
$$

provided p is small enough. Thus we can choose any $\delta < \epsilon/500$ to make the probability in (3.5) larger than $1 - \epsilon$. $\square$

LEMMA 3.5. *Fix any $\epsilon > 0$, and assume $\delta \leq 1/(4 \cdot 10^3)$. Then there exists a constant $C = C(m, \epsilon, \delta)$, such that $\mathsf{q} \geq C\mathsf{p}^2$ implies that the probability that the box Q_0 is good is at least $1 - \epsilon$.*

PROOF. For $k = 1, \dots, \lfloor \frac{2L+1}{3m} \rfloor - 1$, let

$$\text{Row}_k = \big((-M + 3km) + [-m, m]\big) \times [-M, M]$$

and

$$\text{Col}_k = [-M, M] \times \big((-M + 3km) + [-m, m]\big).$$

Define events

$$G_r = \big\{ \text{For at least } L/2m \text{ values of } k, \text{ every } y \in \text{Row}_k \text{ has } \xi_0(y) \geq 2 \big\} \quad \text{and}$$

$$G_c = \big\{ \text{For at least } L/2m \text{ values of } k, \text{ every } y \in \text{Col}_k \text{ has } \xi_0(y) \geq 2 \big\}$$

The probability that Row_k has no 0s is at least $3/4$, which can be proved by applying Lemma 3.4 with $\epsilon \leq 1/4$. By large deviations for binomial random variables (noting that Row_k and Row_{k+1} are at least distance 2 apart), we have

$$\mathbb{P}(G_r) = \mathbb{P}(G_c) \geq 1 - \epsilon/4,$$

for small enough p. By the assumed positive correlations in ξ_0, we have

$$\mathbb{P}(G_r \cap G_c) \geq \mathbb{P}(G_r)\mathbb{P}(G_c) \geq 1 - \epsilon/2,$$

and

$$\mathbb{P}(Q_0 \text{ is good} \mid G_r \cap G_c) \geq \mathbb{P}\left(\text{Binomial}\left[\left(\frac{L}{2m}\right)^2, \mathsf{q}\right] \geq 1\right)$$

$$\geq 1 - \exp(-\mathsf{q}(L/2m)^2)$$

$$\geq 1 - \exp(-C\mathsf{p}^2(\delta/(4m^2\mathsf{p}))^2)$$

$$\geq 1 - \epsilon/2$$

provided C is large enough. The claim follows from the last two estimates. $\square$

3.2. *Construction of a shell of good boxes.* Let $B \subset \mathbb{Z}^2$. A site $u \in \mathbb{Z}^2$ off the coordinate axes is called *protected by B* provided that:

- if $u \in [1, \infty)^2 \cup (-\infty, -1]^2$ then both $u + [-2, -1] \times [1, 2]$ and $u + [1, 2] \times [-2, -1]$ intersect B; and
- if $u \in (-\infty, -1] \times [1, \infty) \cup [1, \infty) \times (-\infty, -1]$, then both $u + [-2, -1] \times [-2, -1]$ and $u + [1, 2] \times [1, 2]$ intersect B.

If u lies on one of the coordinate axes, we will not need to refer to u as being protected.

A *shell S of radius $r \in \mathbb{N}$* is defined to be a subset of $\mathbb{Z}^2$ that satisfies the following properties:

(S1) The shell S contains all sites u such that $\|u\|_1 = r$ and $\|u\|_\infty \geq r - 3$. (This implies that S contains portions of the $\|\cdot\|_1$-sphere of radius r in neighborhoods of each of the four sites $(\pm r, 0)$ and $(0, \pm r)$.)

(S2) For each $u \in S$, we have $r \leq \|u\|_1 \leq r + \sqrt{r}$ and $\|u\|_\infty \leq r$.

(S3) For each of the four directions $\varphi \in \{(\pm 1, \pm 1)\}$, there exists an integer $k = k(\varphi) \geq r/2$ such that $k\varphi \in S$.

(S4) If $u = (u_1, u_2) \in S$, and $|u_1| \geq 3$ and $|u_2| \geq 3$, then u is protected by S.

Let sites in the lattice $\mathbb{Z}^2$ be independently marked black with probability b and white otherwise. We wish to consider paths of a certain type, and we start by defining two types of steps. An ordered pair $u \rightarrowtail v$ of distinct sites $u, v \in \mathbb{Z}^2$ is called:

1. a *taxed step* if each nonzero coordinate of u increases in absolute value by 1 to obtain the corresponding coordinate of v, while each zero coordinate of u changes to -1, 0 or 1 to obtain the corresponding coordinate of v;

2. a *free step* if $\|v\|_1 < \|u\|_1$ and $v - u \in F$, where F is the set of all vectors obtained by permuting coordinates and flipping signs from any of

$$(1, 0) \quad \text{and} \quad (2, 1).$$

(For example, $(-1, 2) \in F$.)

Observe that, in a taxed step $u \rightarrowtail v$, we have $\|v\|_1 > \|u\|_1$. We call $v - u$ the *direction* of either type of step.

A *permissible path from u_0 to u_k* is a finite sequence of distinct sites $u_0, u_1, \ldots, u_k$ such that for every $i = 1, \ldots, k$, $u_{i-1} \rightarrowtail u_i$ is either a free step or a taxed step, and in the latter case, u_i is white.

To obtain a (random) shell S of radius r, we let

(3.6) $\qquad A = \{v \in \mathbb{Z}^2 : \exists u \in \mathbb{Z}^2 \text{ with } \|u\|_1 < r \text{ and a permissible path from } u \text{ to } v\}$,

and we define

(3.7) $\qquad S = \{v \in \mathbb{Z}^2 \setminus A : \exists u \in A \text{ such that } u \rightarrowtail v \text{ is a taxed step}\}$.
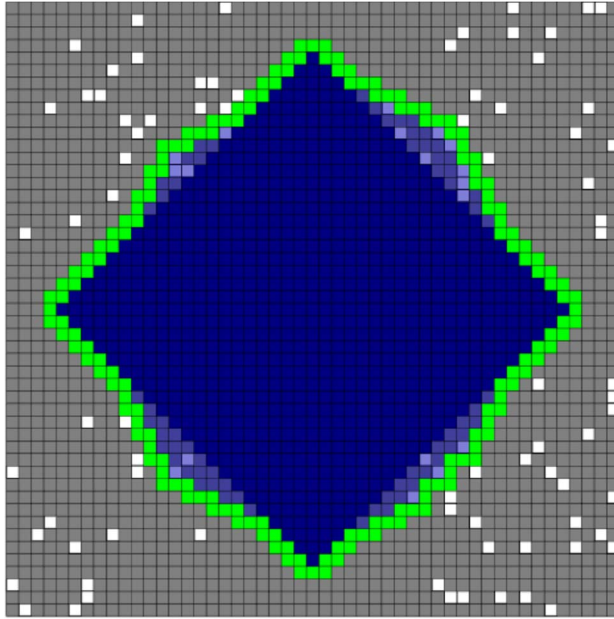
FIG. 1. *A shell of radius* 21. *Sites in the shell S are highlighted in green, while sites in A are shades of blue. The random field of black and white sites are shown in grey and white for those sites outside of $A \cup S$. The darkest blue sites are in the $\| \cdot \|_1$-ball of radius* 20; *the lightest blue sites are the initially white sites outside of this ball, to which there exist permissible paths originating from dark blue sites; the remaining blue sites are initially black sites outside of this ball, to which there exist permissible paths originating from dark blue sites. Note that the sites highlighted in green are black in the random field, and they form a circuit that takes at most two consecutive steps in the same direction.*

Note that if S is nonempty, then all sites in S must be black, since there are no permissible paths from A to A^c. For a picture of a realization of A and S, see Figure 1. This oriented surface construction, which was originally devised in [22], is the key to proving the next result.

PROPOSITION 3.6. *Let E_r be the event that there exists a shell of radius r consisting of black sites. There exists $b_1 \in (0, 1)$ such that for any $b > b_1$ and $r \geq 1$, we have $\mathbb{P}(E_r) \geq 1/2$.*

Note that the event E_r depends only on the colors of sites in $\{u \in \mathbb{Z}^2 : r \leq \|u\|_1 \leq r + \sqrt{r}\}$. However, in proving Proposition 3.6, we show that the set S defined in (3.7) is, in fact, the desired shell with large probability. The proof of the first lemma below, based on path counting, is nearly identical to the proofs of Lemmas 8, 9 and 10 in [16], so we omit the details.

LEMMA 3.7. *There exists $b_2 < 1$ such that if $b > b_2$, then for each $r \geq 1$, the set S defined by (3.6) and (3.7) satisfies properties (S1), (S2) and (S3) with probability at least 1/2.*

LEMMA 3.8. *The set S defined by (3.6) and (3.7) satisfies property (S4).*

PROOF. Without loss of generality, suppose $u = (u_1, u_2) \in S$ is such that $u_i \geq 3$ for $i = 1, 2$, and by symmetry it suffices to show that $u + [1, 2] \times [-2, -1]$ intersects S. By the definition of S in (3.7), u must be reachable from A by a taxed step. Since u is not on a coordinate axis, the only site from which we can reach u via a taxed step is $u + (-1, -1)$, so

$u + (-1, -1) \in A$. Taking a free step in the direction $(1, -2)$ implies $u + (0, -3) \in A$ (this is where we require $|u_1| \geq 3$ and $|u_2| \geq 3$, to guarantee that direction $(1, -2)$ is, in fact, a free step). Observe that $u + (2, -1) \in A^c$, otherwise we would have $u \in A$, since it is reachable from this point by the free step in the direction $(-2, 1)$.

Now there are two cases. If $u + (1, -2) \in A$, then $u + (2, -1) \in S$, since it is reachable from $u + (1, -2)$ along the taxed step in the direction $(1, 1)$. Otherwise, if $u + (1, -2) \in A^c$, then $u + (1, -2) \in S$, since it is reachable from $u + (0, -3) \in A$ along the taxed step in the direction $(1, 1)$. In either case, we have found a site in $(u + [1, 2] \times [-2, -1]) \cap S$. $\quad\square$

PROOF OF PROPOSITION 3.6.   The claim follows from Lemmas 3.7 and 3.8. $\quad\square$

3.3. *Construction of a protected set Z.*   In this section, we construct a set $Z \subset \mathbb{Z}^2$, which is our candidate for the set satisfying the assumptions of Proposition 3.2.

Suppose that there exists a shell S of radius r so that Q_u is a good box for every $u \in S$. For every $u \in S$ with both coordinates at least 3 in absolute value, select a nice vertex from Q_u and gather the selected vertices into the set U. (No nice vertices are chosen from Q_u if at least one coordinate of $u \in S$ is less than 3 in absolute value.)

A *fortress* is a square of side length $12L + 1$ (this is the reason for our choice of $M = 12L$ in the definition of J at (3.3)), all four of whose corners are nice. Suppose that there is a fortress centered at each of the four vertices $(\pm r(2L + 1), 0), (0, \pm r(2L + 1))$. Let K be the set of all corner vertices of all fortresses (16 in all). For $x \in \mathbb{Z}^2$, define $\mathrm{Rect}(x)$ to be the rectangle with opposite corners at x and $\mathbf{0}$ (e.g., if $x = (x_1, x_2)$ with $x_1 \geq 0$ and $x_2 \leq 0$, then $\mathrm{Rect}(x) = [0, x_1] \times [x_2, 0]$). Now define Z by

$$(3.8) \qquad Z = \bigcup_{x \in U \cup K} \mathrm{Rect}(x).$$

Note that by construction, all convex corners of Z are nice vertices, and near each of the coordinate axes, there are two nice vertices on the line orthogonal to the nearby axis that are at distance $12L + 1$. In addition, the fact that the slope of S is locally bounded above and below (by property (S4)) makes the following proposition geometrically transparent. The formal proof is very similar to the proofs of Lemmas 20 through 26 in [16], though it is much simpler, and is omitted. See Figure 2 for a realization of Z.

LEMMA 3.9.   *Suppose Z is defined as in* (3.8). *If p is sufficiently small (depending on δ and m) to make L sufficiently large, then Z satisfies assumptions* (i) *and* (ii) *of Proposition* 3.2.

3.4. *Existence of a protected set Z.*   Assume $N_0 = 3\lfloor \mathsf{p}^{-36} \rfloor$, $n_0 = \lfloor \mathsf{p}^{-19} \rfloor$, $T = \lfloor \mathsf{p}^{-17} \rfloor$, and $\Delta = \lfloor \mathsf{p}^{-19} \rfloor$. Define the sequence of separated annuli

$$A_i = \{x \in \mathbb{Z}^2 : n_0 + (2i - 1)\Delta \leq \|x\|_1 \leq n_0 + 2i\Delta\},$$

for $i = 1, \ldots, T$.

LEMMA 3.10.   *Fix an m. For a small enough $\epsilon > 0$ and $\delta > 0$, and $\mathsf{q} \geq C\mathsf{p}^2$, where C is given in Lemma* 3.5, *the following holds. With probability at least $1 - \exp(-1/(4\mathsf{p}))$, there exists a protected set Z satisfying assumptions* (i) *and* (ii) *of Proposition* 3.2, *and such that Z contains the origin and is contained in $\{x \in \mathbb{Z}^2 : \|x\|_1 \leq N_0\}$.*
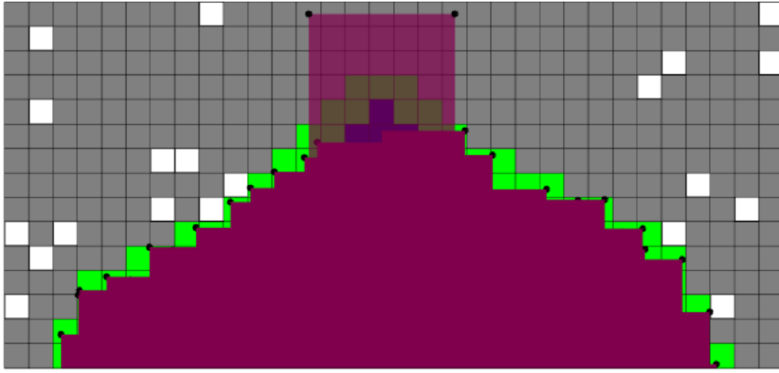
PROOF.   Note that $n_0 + 2T\Delta \leq N_0$.

FIG. 2. *The top portion of the protected set Z consists of the dark magenta region, with the part belonging only to the fortress made transparent. Green boxes are good boxes, which correspond to sites in the shell S from Figure 1. Black dots are the nice vertices, which are selected from each good box corresponding to a site in the shell S with both coordinates at least 3 in absolute value. The top two black dots correspond to the nice vertices in the fortress. Some of the selected nice vertices are hidden within the magenta region, as they are not extremal. Note that due to the slope condition on the shell S (essentially, no two consecutive steps are in the same direction), a similar slope condition holds for the nice vertices, which easily implies the set Z satisfies assumptions* (i) *and* (ii) *in Proposition* 3.2.

Paint each site $x \in \mathbb{Z}^2$ *black* if the box Q_x is good. Let
$$r_i = \lfloor (n_0 + (2i - 1)\Delta)/(2L + 1) \rfloor + 11,$$
so $(2L + 1)r_i - 20L \geq n_0 + (2i - 1)\Delta$, and observe that $\sqrt{r_i} \leq \sqrt{N_0/L} \ll \Delta/(2L + 1)$ for p small. Therefore, existence of a shell of good boxes of radius r_i depends only on the states of vertices within the annulus A_i. Moreover, we have that sites x_1 and x_2 with $\|x_1 - x_2\|_\infty \geq 30$ are painted independently, and so by [29] the configuration of black sites dominates a product measure of density b_1 (chosen from Proposition 3.6) provided $\epsilon > 0$ in Lemma 3.5 is small enough, and δ is chosen appropriately. It follows that, when p is small enough, by Proposition 3.6, a shell of good boxes of radius r_i exists with probability at least $1/2$. The existence of a shell of good boxes of radius r_i is an increasing event (in ξ_0), and so it is positively correlated with existence of nice vertices at the 16 locations comprising the set K ($\subset A_i$) in (3.8). Therefore, the set Z given by (3.8) exists with convex corners $U \cup K \subset A_i$ with probability at least $\mathsf{p}^{16}/2$. Due to the separation of shells, the probability that such a Z does not exist in A_i for all $i = 1, \ldots, T$ is then at most $(1 - \mathsf{p}^{16}/2)^{\mathsf{p}^{-17}/2} \leq \exp(-1/(4\mathsf{p}))$. By Lemma 3.9, if Z constructed in this manner exists, then it satisfies assumptions (i) and (ii) of Proposition 3.2. $\square$

PROOF OF THEOREM 3.1. Choose $s = 37$ in Lemma 3.3. That determines $m = 48s < 2000$. The proof is concluded by Lemma 3.10, Lemma 3.3 and Proposition 3.2. $\square$

4. The supercritical regime for even threshold. In this section, we prove the claims of Theorem 1.1 when $a^\ell \geq 2(\ell - 1)!$. In the following subsections, we prove, in order: (1.3), upper bound on the rate (1.4) for $\ell \geq 2$, lower bound on the same rate for $\ell \geq 2$, and the asymptotics for the exceptional case $\ell = 1$.

4.1. *Comparison process and rescaling.* Initialize the comparison process, ξ_t, as follows. For $x \in \mathbb{Z}^2$, let

(4.1) $\quad \xi_0(x) = \begin{cases} 0 & \text{if } \{x\} \times K_n^2 \text{ is } \theta\text{-IS}, \\ k & \text{if } k \in \{1, 2\} \text{ and } \{x\} \times K_n^2 \text{ is } (\theta - k)\text{-IS but not } (\theta - k + 1)\text{-IS}, \\ 5 & \text{if } \{x\} \times K_n^2 \text{ is not } (\theta - 2)\text{-IS}. \end{cases}$

In other words, initialize ξ_t as in Lemma 2.6, but replace all 3s and 4s with 5s.

To apply Lemma 2.6, we need a method to show that $\mathbb{P}_p(\xi_\infty(\mathbf{0}) = 0)$ is close to 1, and for that, we adapt the rescaling from [17] to our purposes; in particular, we need to account for the existence of 1s, which require activation from 0s, and to prove high final density at the critical value (when $a^\ell = 2(\ell - 1)!$). We let

$$(4.2) \qquad N = \begin{cases} \lfloor n^{1/\ell}(\log n)^{-1/2\ell} \rfloor, & \ell \geq 2, \\ \lfloor n(\log n)^{-3/4} \rfloor, & \ell = 1 \end{cases}$$

and, for $x \in \mathbb{Z}^2$, let $\Lambda_x = N \cdot x + [0, N-1]^2$ be the $N \times N$ box in $\mathbb{Z}^2$ with lower-left corner at Nx. Call the box Λ_x *good* if $\xi_0(y) \leq 2$ for every $y \in \Lambda_x$ and, in addition, every row and column of Λ_x contains at least one y such that $\xi_0(y) \leq 1$. Call a box *very good* if $\xi_0(y) \leq 1$ for every $y \in \Lambda_x$ and $\xi_0(y) = 0$ for some $y \in \Lambda_x$.

LEMMA 4.1.   *For $\ell \geq 1$ and large enough n,*

$$\mathbb{P}_p(\Lambda_x \text{ is not good}) \leq 6n^{(2/\ell)-(a^\ell/\ell!)} \cdot (\log n)^{-(1/\ell \wedge 1/2)}.$$

PROOF.   By Lemma 2.1, for $\ell \geq 2$,

$$\mathbb{P}_p(\Lambda_x \text{ is not good}) \leq N^2 \mathbb{P}_p(\xi_0(\mathbf{0}) = 5) + 2N \cdot (1 - \mathbb{P}_p(\xi_0(\mathbf{0}) \leq 1))^N$$

$$\leq 3N^2 n^{-a^\ell/\ell!}$$

$$+ 2N \exp\left[-N \cdot \frac{2a^{\ell+1}}{(\ell+1)!} \cdot \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}(1 + o(1))\right]$$

$$\leq 3n^{(2/\ell)-(a^\ell/\ell!)} \cdot (\log n)^{-1/\ell}$$

$$+ n^{1/\ell} \exp[-C(\log n)^{1+1/2\ell}].$$

When $\ell = 1$, repeat the above computation using the bound $\mathbb{P}_p(\xi_0(\mathbf{0}) = 5) \leq 3an^{-a}\log n$. □

PROOF OF (1.3).   It follows from Lemma 2.6 that

$$\bigcup\{\{x\} \times K_n^2 : \xi_\infty(x) = 0\} \subset \omega_\infty,$$

so we need only to show that $\mathbb{P}_p(\xi_\infty(\mathbf{0}) = 0) \to 1$ when $a^\ell \geq 2(\ell-1)!$. Let $\mathcal{C}_0$ denote the cluster of good boxes containing the box Λ_0. Observe that

$$\mathbb{P}_p(|\mathcal{C}_0| = \infty) = \mathbb{P}_p(\{|\mathcal{C}_0| = \infty\} \cap \{\mathcal{C}_0 \text{ contains a very good box}\})$$

$$\leq \mathbb{P}_p(\xi_\infty(\mathbf{0}) = 0).$$

The last inequality follows from the fact that a very good box in $\mathcal{C}_0$ sets off a cascade resulting in all vertices in $\mathcal{C}_0$ eventually flipping to 0. Now, Lemma 4.1 implies $\mathbb{P}_p(|\mathcal{C}_0| = \infty) \to 1$. □

4.2. *Upper bound in* (1.4) *for $\ell \geq 2$.*   Throughout this subsection, assume that $\ell \geq 2$, $a^\ell/\ell! > 2/\ell$ and that ξ_0 is built by internal spanning properties, as in Lemma 2.6.

We will prove first the upper bound on the rate.

LEMMA 4.2.   *The probability that the Hamming square based at the origin is not completely filled satisfies the following bound*:

$$(4.3) \qquad \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \not\subset \omega_\infty) \leq n^{4/\ell - 4a^\ell/\ell! + o(1)}.$$

For a deterministic or random set $A \subset \mathbb{Z}^2$, we say that the event `Blocking_In` A occurs if there exists a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ so that: $\mathbf{0} \in R$; R is nondegenerate, that is, $a_1 < a_2$ and $b_1 < b_2$; and each of the four sides of R, $\{a_1\} \times [b_1, b_2]$, $\{a_2\} \times [b_1, b_2]$, $[a_1, a_2] \times \{b_1\}$ and $[a_1, a_2] \times \{b_2\}$, either contains two distinct sites in A with ξ_0-state 3 or a site in A with ξ_0-state 4.

LEMMA 4.3. *Suppose that $\xi_\infty(\mathbf{0}) \neq 0$. Assume that there is circuit of 0s around $\mathbf{0}$ in ξ_t, for some t. Denote by A the set of sites in the strict interior of this circuit. Assume that there are no sites in A with ξ_0-state 5, and there is at most one site in A with ξ_0-state 4. Then the event* `Blocking_In` A *happens.*

PROOF. We may assume that all sites in A^c are 0s in ξ_0. Let A' be the set of sites which are nonzero in ξ_∞. Then the leftmost and the rightmost site on the top line of A' must either be the same site with ξ_0-state 4, or be two distinct sites which both have ξ_0-state at least 3. To check nondegeneracy, assume that, say, $b_1 = b_2$. As there are no sites in ξ_0-state 5 in A, there then must be two sites at ξ_0-state 4 on either side of $\mathbf{0}$ on the x-axis, but by the assumption there can be at most one such site. $\square$

Now we pick N as in (4.2) and also keep the definition of good boxes from the previous subsection. For a constant D, let $G_1(D)$ be the event that there is a circuit of good boxes that encircles $\mathbf{0}$, is contained in $[-DN, DN]^2$, and is connected to the infinite cluster of good boxes.

LEMMA 4.4. *For any L there is a constant $D = D(a, L)$ so that*

(4.4) $$\mathbb{P}_p\big(G_1(D)^c\big) \leq n^{-L}.$$

PROOF. This follows from Lemma 4.1, together with a standard percolation argument (see, e.g., Chapter 11 of [21]). $\square$

LEMMA 4.5. *The probability that $[-DN, DN]^2$ contains at least one site with ξ_0-state 5 or at least two sites in A with ξ_0-state 4 is $n^{4/\ell - 4a^\ell/\ell! + o(1)}$.*

PROOF. This follows from Lemma 2.1. $\square$

LEMMA 4.6. *Assume D is a fixed constant. Then*

$$\mathbb{P}_p\big(\texttt{Blocking\_In}\,[-DN, DN]^2\big) \leq n^{4/\ell - 4a^\ell/\ell! + o(1)}.$$

PROOF. Define λ so that $DN = n^\lambda$, so that $\lambda = 1/\ell + o(1)$, and let $\alpha = a^\ell/\ell!$. Note that $2\lambda < \alpha$. We will restrict all our sites to the region $[-DN, DN]^2$.

A *frame* is a nondegenerate rectangle whose four corners are all in ξ_0-state 3. Let `Frame` be the event that a frame exists (which thus by definition means existence in $[-DN, DN]^2$). Then $\mathbb{P}_p(\texttt{Frame}) = \Theta(n^{4\lambda - 4\alpha})$. We will show below that all other possibilities for the event `Blocking_In` $[-DN, DN]^2$ to happen have much smaller probabilities. We group these possibilities according to whether the rectangle required by this event does not have, or does have, a boundary site with ξ_0-state 4.

The event that there exists a nondegenerate rectangle R that has at least two sites with ξ_0-state 3 on all sides can be split into the following events, according to additional properties of the configuration on R:

- R is a frame;

- R has no 3s at the corners (i.e., there is no sharing), which happens with probability at most a constant times

$$n^{4\lambda}(n^{2\lambda}n^{-2\alpha})^4 = n^{12\lambda-8\alpha} = o(\mathbb{P}_p(\text{Frame}))$$

(we give these probabilities as products, reflecting successive choices: four lines determining R, pairs of points on the same line away from corners; single points on lines away from corners, states at corners);

- R has exactly one 3 at a corner, with probability at most a constant times

$$n^{4\lambda}(n^{2\lambda}n^{-2\alpha})^2(n^\lambda n^{-\alpha})^2 n^{-\alpha} = n^{10\lambda}n^{-7\alpha} = o(\mathbb{P}_p(\text{Frame}));$$

- R has exactly two corner 3s on the same line, with probability at most a constant times

$$n^{4\lambda}(n^{2\lambda}n^{-2\alpha})(n^\lambda n^{-\alpha})^2 n^{-2\alpha} = n^{8\lambda}n^{-6\alpha} = o(\mathbb{P}_p(\text{Frame}));$$

- R has exactly two corner 3s not on the same line, with probability at most a constant times

$$n^{4\lambda}(n^\lambda n^{-\alpha})^4 n^{-2\alpha} = n^{8\lambda}n^{-6\alpha} = o(\mathbb{P}_p(\text{Frame}));$$

- R has exactly three corner 3s, with probability at most a constant times

$$n^{4\lambda}(n^\lambda n^{-\alpha})^2 n^{-3\alpha} = n^{6\lambda}n^{-5\alpha} = o(\mathbb{P}_p(\text{Frame})).$$

Next, we consider the event that a rectangle R has exactly one 4 on its boundary, and either two 3s or a 4 on each of its sides. Again, we split this event according to additional properties:

- 4 is not at a corner of R and neither are 3s, with probability at most a constant times

$$n^{4\lambda}(n^{2\lambda}n^{-2\alpha})^3 n^\lambda n^{-2\alpha} = n^{11\lambda}n^{-8\alpha} = o(\mathbb{P}_p(\text{Frame}));$$

- the 4 is at a corner of R, but no 3s are at corners, with probability at most a constant times

$$n^{4\lambda}(n^{2\lambda}n^{-2\alpha})^2 n^{-2\alpha} = n^{8\lambda}n^{-6\alpha} = o(\mathbb{P}(\text{Frame}));$$

- the 4 is at a corner of R, and a 3 is at the opposite corner, with probability at most a constant times

$$n^{4\lambda}(n^\lambda n^{-\alpha})^2 n^{-2\alpha}n^{-\alpha} = n^{6\lambda}n^{-5\alpha} = o(\mathbb{P}_p(\text{Frame})).$$

Together with Lemma 4.5, these calculations complete the proof. □

PROOF OF LEMMA 4.2. Choose the constant D in Lemma 4.4 so that L in (4.4) satisfies $L > 4a^\ell/\ell! - 4/\ell$. Then (4.3) follows from Lemmas 4.3–4.6. □

4.3. *Lower bound in* (1.4) *for* $\ell \geq 2$. In this subsection also, we assume that $a^\ell/\ell! > 2/\ell$ but now ξ_0 is built by inertness properties, as in Lemma 2.7. In this section, we prove the lower bound on the rate.

LEMMA 4.7. *The probability that the configuration on the Hamming square based at the origin never changes satisfies the following bound*:

$$(4.5) \qquad \mathbb{P}_p(\omega_\infty = \omega_0 \text{ on } \{0\} \times K_n^2) \geq n^{4/\ell-4a^\ell/\ell!+o(1)}.$$

Fix a nondegenerate rectangle R. Let ξ_0^0 be obtained from ξ_0 by converting all 4s and 5s to 3s on R, and changing all sites to 0 off R. Let ξ_t^0 be the bootstrap dynamics started from this initial state. We say that R is *protected* if R has its four corners in ξ_0^0-state 3, no site in R has ξ_0^0-state 0 and no site on the boundary of R has ξ_0^0-state 1.

LEMMA 4.8. *Assume a nondegenerate rectangle R is protected. Then no site ever changes state in ξ_t^0, and therefore ξ_t never changes any state in R.*

PROOF. The first site to change state would have to be on the boundary of R, which is clearly impossible. $\square$

Assume now $N = \lfloor n^{1/\ell}/\log^5 n \rfloor$. Define the following two events:

$G_1 = \{$there exists a rectangle R with $\mathbf{0} \in R \subset [-N, N]^2$, four corners in

$\quad \xi_0^0$-state 3, and no site on the boundary of R is in ξ_0^0-state 0 or 1$\}$,

$G_2 = \{$there is no $x \in [-N, N]^2$ with $\xi_0(x) = 0\}$.

LEMMA 4.9. *With our choice of N,*

$$\mathbb{P}_p(G_1) \geq n^{4/\ell - 4a^\ell/\ell! + o(1)}.$$

PROOF. This follows from an argument that is very similar to the one for Lemma 3.5. $\square$

LEMMA 4.10. *With our choice of N,*

$$\mathbb{P}_p(G_2^c) \to 0,$$

as $n \to \infty$.

PROOF. This follows from Lemma 2.1 and Lemma 2.2. $\square$

PROOF OF LEMMA 4.7. Observe that G_1 and G_2 are increasing events, therefore by FKG and Lemmas 4.9 and 4.10,

$$\mathbb{P}_p(G_1 \cap G_2) \geq n^{4/\ell - 4a^\ell/\ell! + o(1)},$$

and the result follows from Lemma 4.8. $\square$

4.4. *The exceptional case*: $\theta = 4$. We assume that $\theta = 4$ throughout this section, and that, in accordance with (1.1),

$$p = a\frac{\log n}{n^2},$$

with $a > 2$. We first prove an analogue of Lemma 4.2. We will again assume that ξ_0 is built by internal spanning properties, as in Lemma 2.6, and observe that the sites with ξ_0-state 4 and ξ_0-state 3, both of which we call *4-obstacles*, are comparably improbable at our precision level. (Also note that there are no sites with ξ_0-state 5.) As a result, the convergence rate changes.

LEMMA 4.11. *The probability that the Hamming square based at the origin is not completely filled satisfies the following bound*:

(4.6)
$$\mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \not\subset \omega_\infty) \leq n^{-2a + o(1)}.$$

If $R = [a_1, a_2] \times [b_1, b_2]$ is a nondegenerate rectangle (i.e., $a_1 < a_2$ and $b_1 < b_2$), then its two-layer boundary rectangles are denoted by $R^\ell = [a_1, a_1 + 1] \times [b_1, b_2]$, $R^r = [a_2 - 1, a_2] \times [b_1, b_2]$, $R^b = [a_1, a_2] \times [b_1, b_1 + 1]$ and $R^t = [a_1, a_2] \times [b_2 - 1, b_2]$.

For a set $A \subset \mathbb{Z}^2$, we say that the event 4_Blocking_In A happens if there exists a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ so that $\mathbf{0} \in R$ and either:

- $a_2 - a_1 \geq 3$ and $b_2 - b_1 \geq 3$, and each of the four rectangles R^ℓ, R^r, R^b, R^t contains at least two 4-obstacles in A;
- $0 \leq a_2 - a_1 \leq 2$, $b_2 - b_1 \geq 3$, and R contains 4 or more 4-obstacles in A;
- $a_2 - a_1 \geq 3$, $0 \leq b_2 - b_1 \leq 2$, and R contains 4 or more 4-obstacles in A; or
- $0 \leq a_2 - a_1 \leq 2$, $0 \leq b_2 - b_1 \leq 2$, and R contains 2 or more 4-obstacles in A.

LEMMA 4.12. *Suppose that $\xi_\infty(\mathbf{0}) \neq 0$. Assume that there is circuit of 0s around $\mathbf{0}$ in ξ_t, for some t. Denote by A the set of sites in the strict interior of this circuit. Then the event* `4_Blocking_In` *A happens.*

PROOF. As before, we may assume that all sites in A^c are 0s in ξ_0 and let A' be the set of sites which are nonzero in ξ_∞. If the top line of A' consists of a single 4-obstacle, then the next line from the top must also contain a 4-obstacle. (Otherwise, the next line from the top would eventually turn into all 0s, causing the solitary 4-obstacle on the top line to be surrounded by 0s.) Finally, if there is a single 4-obstacle within R, then all sites in R eventually turn into 0s. □

We next note that Lemma 4.4 still holds, with N given by (4.2) with $\ell = 1$, and proceed with our final lemma.

LEMMA 4.13. *Assume D is a fixed constant. Then*

$$\mathbb{P}_p\big(\texttt{4\_Blocking\_In}\,[-DN, DN]^2\big) \leq n^{-2a+o(1)}.$$

PROOF. For the event $\{\texttt{4\_Blocking\_In}\,[-DN, DN]^2\}$ to happen, one of the four events, corresponding to the four items in its definition, must happen. The event in the first item happens with probability at most $n^{4-4a+o(1)}$, as in the proof of Lemma 4.6. The events in the second and third item also happen with probability at most $n^{4-4a+o(1)}$. The event in the last item happens with probability $n^{-2a+o(1)}$, and this last probability is the largest, as $a > 2$. □

PROOF OF LEMMA 4.11. Analogously to the case of even $\theta \geq 6$, choose the constant D in Lemma 4.4 so that L in (4.4) satisfies $L > 2a$, and use Lemmas 4.12 and 4.13 to conclude (4.6). □

We conclude this section by the simple observation that gives the matching lower bound.

LEMMA 4.14. *The Hamming square based at the origin remains unoccupied forever with probability bounded below as follows*:

(4.7) $$\mathbb{P}_p(\omega_\infty \equiv 0 \ on \ \{\mathbf{0}\} \times K_n^2) \geq n^{-2a}(1 + o(1)).$$

PROOF. The inclusion

$$\{\omega_0 \equiv 0 \ \text{on} \ \{\mathbf{0}, (0, 1)\} \times K_n^2\} \subset \{\omega_\infty \equiv 0 \ \text{on} \ \{\mathbf{0}\} \times K_n^2\},$$

gives the desired bound. □

5. The odd threshold. In this section, we prove Theorem 1.2. In the first three subsections, we handle the case $\ell \geq 2$: first we define, and give bounds for, the critical value a_c, then we prove (1.7), and then (1.6). In the last, fourth subsection, we sketch the argument for the case $\ell = 1$ in lesser detail.

5.1. *The critical value of a for $\ell \geq 2$.* Pick an $a > 0$ and an $\epsilon \in (0, 2\exp[-\frac{a^\ell}{\ell!}] - 2\exp[-\frac{2a^\ell}{\ell!}])$. Consider the initial state $\xi_0^{(a,\epsilon)}$ given by the product measure with

$$\mathbb{P}(\xi_0^{(a,\epsilon)}(x) = 0) = \epsilon,$$

$$\mathbb{P}(\xi_0^{(a,\epsilon)}(x) = 1) = (1 - e^{-a^\ell/\ell!})^2,$$

$$\mathbb{P}(\xi_0^{(a,\epsilon)}(x) = 3) = \exp\left[-\frac{2a^\ell}{\ell!}\right],$$

$$\mathbb{P}(\xi_0^{(a,\epsilon)}(x) = 2) = 1 - P(\xi_0^{(a,\epsilon)}(x) = 0) - P(\xi_0^{(a,\epsilon)}(x) = 1) - P(\xi_0^{(a,\epsilon)}(x) = 3)$$

for every $x \in \mathbb{Z}^2$. We will call this an (a, ϵ)-*initialization* and denote the resulting bootstrap dynamics by $\xi_t^{(a,\epsilon)}$.

Define $a_c \in [0, \infty]$ as follows:

$$a_c = \inf\left\{a > 0 : \lim_{\epsilon \to 0} \mathbb{P}(\xi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0) > 0\right\}.$$

Observe that $\mathbb{P}(\xi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0)$ is a nonincreasing function of ϵ and, therefore, its limit as $\epsilon \to 0$ exists. Furthermore, this limit is a nondecreasing function of a, and therefore it vanishes on $[0, a_c)$ and is strictly positive on (a_c, ∞).

The next two lemmas establish that a_c is nontrivial, that is, $a_c \in (0, \infty)$, by comparison to the critical value p_c^{site} of site percolation on $\mathbb{Z}^2$, and to the critical value of the site percolation on the triangular lattice. Nonstrict inequalities in both lemmas have much simpler proofs, but we prefer the strict versions as they indicate that this percolation problem is not a standard one.

LEMMA 5.1. *The following strict inequality holds*:

$$(5.1) \qquad (1 - e^{-a_c^\ell/\ell!})^2 < p_c^{\text{site}}.$$

In particular, $a_c < \infty$. Furthermore, $\lim_{\epsilon \to 0} \mathbb{P}(\xi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0) \to 1$ as $a \to \infty$.

PROOF. Given a configuration $\xi_0 = \xi_0^{(a,\epsilon)}$, form the following set of *green* sites. Any site x with $\xi_0(x) \leq 1$ is green. Also make green any site x such that $\xi_0(x) = 2$ and $\xi_0(y) \leq 1$ for all sites y among the 8 nearest neighbors of x, except possibly for two diagonally opposite neighbors. That is, if the local configuration in ξ_0 around a site x is

$$(5.2) \qquad \begin{matrix} 1\,1\,* & & *\,1\,1 \\ 1\,2\,1 & \text{or} & 1\,2\,1, \\ *\,1\,1 & & 1\,1\,* \end{matrix}$$

where $*$ denotes an arbitrary state, then x is green, and it is also green if its local configuration has 0s in place of any of the 1s in (5.2). Let `Green_Percolation` be the event that $\mathbf{0}$ is in an infinite connected set of green sites, and `Green_Connection` the event that $\mathbf{0}$ is green and connected to a vertex with state 0 in ξ_0 through green sites. Then

$$(5.3) \qquad \mathbb{P}(\texttt{Green\_Percolation} \setminus \texttt{Green\_Connection}) = 0.$$

Moreover, we claim that

$$(5.4) \qquad \texttt{Green\_Connection} \subset \{\xi_\infty(\mathbf{0}) = 0\}.$$

To see this, consider the set of all sites in a connected cluster $\mathcal{C}$ of $\mathbf{0}$ of green sites that includes a 0 in ξ_0. Let $\mathcal{C}_0$ be the set of all sites in $\mathcal{C}$ that eventually assume state 0. If $\mathcal{C}_0 \subsetneq \mathcal{C}$, then there

exist neighbors x and y with $x \in \mathcal{C}_0$ and $y \in \mathcal{C} \setminus \mathcal{C}_0$. But then $\xi_0(y) = 2$, and by inspection of the configurations in (5.2), we see that y must have at least 2 neighbors in $\mathcal{C}_0$, a contradiction. Therefore, $\mathcal{C}_0 = \mathcal{C}$ and (5.4) holds.

Finally, it follows from [1] (see also [8]) that there exists an a with $(1 - e^{-a^\ell/\ell!})^2 < p_c^{\text{site}}$, so that $\mathbb{P}(\texttt{Green\_Percolation}) > 0$. This, together with (5.2)–(5.4), establishes (5.1). Moreover, it follows from standard percolation arguments that $\mathbb{P}(\texttt{Green\_Percolation}) \to 1$ as $a \to \infty$, and then (5.3) implies the last claim. □

LEMMA 5.2. *The critical value a_c satisfies the following strict inequality*:

$$\exp[-2a_c^\ell/\ell!] < 1/2.$$

In particular, $a_c > 0$.

PROOF.   Pick an $\alpha > 0$. Given a configuration $\xi_0 = \xi_0^{(a,\epsilon)}$, declare a site x *red* if $\xi_0(x) = 3$, or $\xi_0(x) = 2$ and the local configuration in ξ_0 around x is

(5.5)
$$
\begin{array}{ccc}
3 & 3 & * \\
3 & 2 & 3, \\
* & 3 & 3
\end{array}
$$

where $*$ denotes an arbitrary state.

The triangular lattice $\mathbb{T}$ is obtained by adding SW-NE edges to the nearest neighbor edges in $\mathbb{Z}^2$. (When we say that $x, y \in \mathbb{Z}^2$ are neighbors without specifying the lattice, we still mean nearest neighbors.) Recall that $\mathbb{T}$ is (site-)self-dual and so the site percolation on $\mathbb{T}$ has critical density $1/2$. We call a $\mathbb{T}$-circuit ζ a sequence of distinct points $y_0, y_1, \ldots, y_n = y_0$ such that y_i and y_{i-1} are $\mathbb{T}$-neighbors for $i = 1, \ldots, n$. We will also assume that ζ is a boundary of its connected interior, that is, its sites are all points, which are outside some nonempty $\mathbb{T}$-connected set S, but have a $\mathbb{T}$-neighbor in S (this is possible, again, because $\mathbb{T}$ is site-self-dual); we call S the *interior* of ζ. Observe that every site on ζ has at least two neighbors in the set obtained as the union of sites on ζ and its interior.

Let $\texttt{Red\_Circuit}_N$ be the event that there exists a $\mathbb{T}$-circuit of red sites, with the origin in its interior, and inside $[-N, N]^2$. Moreover, let $\texttt{No\_Zero}_N$ be the event that no site $x \in [-N, N]^2$ has $\xi_0(x) = 0$. It follows from [1, 8], and standard arguments from percolation theory (see Chapter 11 of [21]), that there exists an a with $\exp[-2a^\ell/\ell!] < 1/2$, with the following property. For every $\alpha > 0$, there exists an $N = N(\alpha)$ so that

(5.6)
$$\mathbb{P}(\texttt{Red\_Circuit}_N) > 1 - \alpha.$$

Pick any $\mathbb{T}$-circuit ζ of red states. Form the set of sites R that consists of: all sites of ζ; all sites in the interior of ζ; and all sites required to be in ξ_0-state 3 in (5.5) around any site with state 2 on ζ. Assume that there is no site in ξ_0-state 0 in R. Then we claim that no site in R ever changes its state to 0. Indeed, to get a contradiction, let $x \in R$ be the first such site to change its state to 0 (chosen arbitrarily in case of a tie). Clearly, x cannot be in the interior of ζ, as then x has no neighbor outside R. The site x cannot have ξ_0-state 3 and be on ζ, as x then has at least two neighbors in R, and hence at most two outside R. Furthermore, x cannot be a site with ξ_0-state 2 on ζ, as x must then have all neighbors in R in accordance with (5.5). The final possibility is that x is one of the sites with ξ_0-state 3 in (5.5). But each of those sites clearly also has two neighbors in R.

So we have, for every N,

(5.7)
$$\texttt{Red\_Circuit}_N \cap \texttt{No\_Zero}_N \subset \{\xi_\infty(\mathbf{0}) = 0\}^c.$$

It follows from (5.6) and (5.7) that there exists an $N = N(\alpha)$ so that

$$(5.8) \qquad \mathbb{P}(\xi_\infty(\mathbf{0}) = 0) \leq \alpha + (2N + 1)^2 \epsilon.$$

Now in (5.8), we send $\epsilon \to 0$ first, and then send $\alpha \to 0$ to conclude that $\mathbb{P}(\xi_\infty(\mathbf{0}) = 0) \to 0$ as $\epsilon \to 0$ and, therefore, $a \leq a_c$. $\quad \square$

5.2. *The supercritical regime for $\ell \geq 2$.*

LEMMA 5.3. *Assume $\vec{X} = (X_1, X_2, X_3, X_4)$ and $\vec{Y} = (Y_1, Y_2, Y_3, Y_4)$ are 4-tuples of i.i.d. Bernoulli random variables with $\mathbb{P}(X_i = 1) = \alpha_1$ and $P(Y_i = 1) = \alpha_2$ for all i. If $1 - (1 - \alpha_1)^4 \leq \alpha_2^4$, then $\vec{X}$ and $\vec{Y}$ can be coupled so that $\{\exists i : X_i = 1\} \subset \{\forall i : Y_i = 1\}$.*

PROOF. This follows from an elementary argument and we omit the details. $\quad \square$

LEMMA 5.4. *If $a > a_c$, then (1.7) holds. Moreover, (1.8) holds.*

PROOF. Fix an $a' \in (a_c, a)$. Fix also a small $\delta > 0$, to be chosen later dependent on a'. For $i = 0, \ldots, 5$, we define probabilities $p_i^{(n)}$ as follows. For $i = 1, 2, 3, 4$, let

$$p_i^{(n)} = \mathbb{P}_p(K_n^2 \text{ is } (\theta - i)\text{-IS but not } (\theta - i + 1)\text{-IS}),$$

and

$$p_0^{(n)} = \mathbb{P}_p(K_n^2 \text{ is } \theta\text{-IS}), \qquad p_5^{(n)} = \mathbb{P}_p(K_n^2 \text{ is not } (\theta - 4)\text{-IS}).$$

Denote by $\pi(\alpha)$ the Bernoulli product measure of *active* and *inactive* sites with density α of active sites. Build the initial state $\overline{\xi}_0$ in four steps as follows. In the first step, choose active sites according to $\pi(p_4^{(n)} + p_5^{(n)})$ and fill them with 5s. In the second step, choose active sites according to $\pi(p_0^{(n)}/(1 - p_4^{(n)} - p_5^{(n)}))$ and fill them with 0s, provided they are not already filled. Continue in the third step with $\pi(p_3^{(n)}/(1 - p_0^{(n)} - p_4^{(n)} - p_5^{(n)}))$ to fill some unfilled sites with 3s, and then in the fourth step analogously with 2s, and then finally 1s fill all the remaining unfilled sites.

Divide $\mathbb{Z}^2$ into 2×2 boxes and couple product measures $\pi(p_4^{(n)} + p_5^{(n)})$ and $\pi(\delta)$ on the space of pairs $(\eta_1, \eta_2) \in 2^{\mathbb{Z}^2} \times 2^{\mathbb{Z}^2}$ so that any box with at least one active site in η_1 is fully activated in η_2. This coupling is possible, for large enough n, by Lemmas 2.4 and 5.3.

Use this to couple $\overline{\xi}_0$ with another initial state $\widehat{\xi}_0$. To build this configuration, keep all selected product measures used to define $\overline{\xi}_0$, but change the first step above as follows: replace $\pi(p_4^{(n)} + p_5^{(n)})$ by $\pi(\delta)$ (coupled as above), and fill the active sites by 3s (instead of 5s). Note that we now fill by 3s twice, and that some 0s, 1s and 2s in $\overline{\xi}_0$ are converted to 3s in $\widehat{\xi}_0$.

Denote the resulting bootstrap dynamics by $\overline{\xi}_t$ and $\widehat{\xi}_t$. The important observation is that no site that is 5 in $\overline{\xi}_0$ can ever turn to 0 in $\widehat{\xi}_t$, as it is covered by a 2×2 block of 3s that cannot change. Therefore, by Lemma 2.6 and the coupling between $\overline{\xi}_t$ and $\widehat{\xi}_t$,

$$(5.9) \qquad \mathbb{P}_p(\{\mathbf{0}\} \times K_n^2 \subset \omega_\infty) \geq \mathbb{P}(\overline{\xi}_\infty(\mathbf{0}) = 0) \geq \mathbb{P}(\widehat{\xi}_\infty(\mathbf{0}) = 0).$$

Now if $\delta = \delta(a')$ is small enough, then for large enough n,

$$\epsilon_n = \mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 0) > 0,$$

$$(5.10) \qquad \mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 1) \geq \mathbb{P}(\xi^{(a', \epsilon_n)}(\mathbf{0}) = 1),$$

$$\mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 3) \leq \mathbb{P}(\xi^{(a', \epsilon_n)}(\mathbf{0}) = 3).$$

As $a' > a_c$, the inequalities (5.10) guarantee that $\liminf_n \mathbb{P}(\widehat{\xi}_\infty(\mathbf{0}) = 0) > 0$. Therefore, by (5.9), the leftmost inequality in (1.7) holds. When $a \to \infty$, we can send $a' \to \infty$ as well, and then Lemma 5.1 gives (1.8).

Finally, we prove the rightmost inequality in (1.7), which states that $\mathbb{P}_p(v_0 \in \omega_\infty)$ is bounded away from 1 for any finite a. Let $\texttt{Obstacle\_Box}$ be the event that $\{x\} \times K_n^2$ is $(\theta - 2)$-inert for all $x \in \{\mathbf{0}, (0, 1), (1, 0), (1, 1)\}$. Then

$$\texttt{Obstacle\_Box} \subset \{\omega_\infty = \omega_0 \text{ on } \{\mathbf{0}\} \times K_n^2\}$$

and, therefore, for any $a > 0$, by Lemmas 2.4 and 2.5,

$$\limsup_{n \to \infty} \mathbb{P}_p(v_0 \in \omega_\infty) \leq \lim_{n \to \infty} \mathbb{P}_p(\texttt{Obstacle\_Box}^c) = 1 - \exp(-8a^\ell/\ell!) < 1,$$

which completes the proof of (1.7). $\quad\square$

5.3. The subcritical regime for $\ell \geq 2$.

LEMMA 5.5. *Assume that $a < a_c$ and $\ell \geq 2$. Then (1.6) holds.*

PROOF. Pick now an $a' \in (a, a_c)$ and $\alpha > 0$, and again also fix $\delta > 0$, to be chosen later to be appropriately dependent on a' and α. We will redefine $p_i^{(n)}, \overline{\xi}_0$ and $\widehat{\xi}_0$ from the previous proof. Let

$$p_0^{(n)} = \mathbb{P}_p(K_n^2 \text{ is not } \theta\text{-II}),$$

$$p_1^{(n)} = \mathbb{P}_p(K_n^2 \text{ is not } (\theta - 1)\text{-II but is } \theta\text{-II}),$$

$$p_2^{(n)} = \mathbb{P}_p(K_n^2 \text{ is not } (\theta - 2)\text{-II but is } (\theta - 1)\text{-II}),$$

$$p_3^{(n)} = \mathbb{P}_p(K_n^2 \text{ is } (\theta - 2)\text{-II}).$$

Next, we will build the initial state $\overline{\xi}_0$. We emphasize that $\overline{\xi}_0$ is *not* a product measure, as we need to take account of the possibility that some copies of the Hamming plane are internally inert but not inert. However, such copies are rare, and the bounded range of dependence allows for the coupling with a low-density product measure.

The construction of $\overline{\xi}_0$ proceeds in three steps. In the first step, choose active sites according to $\pi(p_3^{(n)})$ and fill them by 3s. In the second step, choose active sites according to $\pi(p_2^{(n)}/(1 - p_3^{(n)}))$ and fill them by 2s, provided they are not already filled. In the third step, choose the configuration of *bad* sites: those are sites that:

- are not θ-II; or
- are internally inert but not inert for some threshold in $[\theta - 2, \theta]$.

Observe that the conditional distribution of bad sites given the configuration of 3s and 2s has finite range of dependence: if $\|x - y\|_1 \geq 3$, then x and y are bad independently. Furthermore, by Lemma 2.5, the conditional probability that any site is bad is, uniformly over the configurations of 2s and 3s, $n^{-1+1/\ell+o(1)}$, and thus goes to 0 if $\ell \geq 2$. Finally, finish the construction of $\overline{\xi}_0$ by filling all bad sites with 0's and the remaining unfilled sites with 1s.

By [29], the configuration of bad sites can be coupled with a product measure $\pi(\delta)$ that dominates it, and is independent of the configuration of 2s and 3s. As in the previous proof, we now couple $\overline{\xi}_0$ with another initial state $\widehat{\xi}_0$. To build $\widehat{\xi}_0$, keep the selected product measures used in the first two steps. The third step is changed by using the $\pi(\delta)$, obtained from the domination coupling, as active sites, all of which are filled by 0s, possibly replacing some 2s and 3s. This way, some of the 1s, 2s and 3s in $\overline{\xi}_0$ are changed to 0s in $\widehat{\xi}_0$.

Denote again the resulting bootstrap dynamics by $\overline{\xi}_t$ and $\widehat{\xi}_t$. The construction of $\overline{\xi}_0$ results in a 0 at the location of every noninert internally inert copy of the Hamming plane, for all relevant thresholds. Therefore, $\overline{\xi}_\infty$ provides an upper bound for the comparison configuration ξ_∞ in Lemma 2.7, and this lemma then implies that

$$(5.11) \qquad \mathbb{P}_p(\omega_\infty \neq \omega_0 \text{ on } \{\mathbf{0}\} \times K_n^2) \leq \mathbb{P}(\overline{\xi}_\infty(\mathbf{0}) = 0).$$

Next, by the properties of the coupling we constructed,

$$(5.12) \qquad \mathbb{P}(\overline{\xi}_\infty(\mathbf{0}) = 0) \leq \mathbb{P}(\widehat{\xi}_\infty(\mathbf{0}) = 0).$$

Now if $\delta = \delta(a')$ is small enough, then for large enough n,

$$\mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 0) \leq \delta,$$

$$(5.13) \qquad \mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 1) \leq \mathbb{P}(\xi^{(a',\epsilon)}(\mathbf{0}) = 1),$$

$$\mathbb{P}(\widehat{\xi}_0(\mathbf{0}) = 3) \geq \mathbb{P}(\xi^{(a',\epsilon)}(\mathbf{0}) = 3).$$

As $a' < a_c$, the inequalities (5.13) guarantee that $\mathbb{P}(\widehat{\xi}_\infty(\mathbf{0}) = 0) < \alpha$ if $\delta = \delta(a', \alpha)$ is small enough. Therefore, by (5.11) and (5.12), (1.6) holds. $\square$

5.4. *The exceptional case*: $\theta = 3$.

We assume here that $p = a/n^2$, in accordance with (1.5). In this case, we need another version of the heterogeneous bootstrap dynamics, somewhere between ξ_t used when $\ell \geq 2$ and ζ_t used later for the graph $\mathbb{Z}^2 \times K_n$. Indeed, observe that the obstacles are now empty Hamming planes, but they become completely occupied by contact with two fully occupied neighboring planes and another neighboring plane that is merely nonempty. Clearly, the probability of having a nonempty neighboring plane does not go to 0, and so this possibility now cannot be handled by a coupling with a low-density measure.

We denote the new rule by $\chi_t \in \{0, 1, 2, 3\}^{\mathbb{Z}^2}$, $t \in \mathbb{Z}_+$. Assume that χ_0 is given. For a given $t \geq 0$, let as before $Z_t(x)$ be the cardinality of $\{y : y \sim x \text{ and } \chi_t(y) = 0\}$ and let $W_t(x) = \mathbb{1}(\{y : y \sim x \text{ and } 0 < \chi_t(y) < 3\} \neq \varnothing)$ then

$$\chi_{t+1}(x) = \begin{cases} 0, & Z_t(x) \geq \chi_t(x) \text{ or } (\chi_t(x) = 3, Z_t(x) = 2, \text{ and } W_t(x) = 1), \\ \chi_t(x), & \text{otherwise.} \end{cases}$$

For a small $\epsilon > 0$, we consider the initial state $\chi_0^{(a,\epsilon)}$ given by the product measure with

$$\mathbb{P}(\chi_0^{(a,\epsilon)}(x) = 0) = \epsilon,$$

$$\mathbb{P}(\chi_0^{(a,\epsilon)}(x) = 1) = 1 - (a+1)e^{-a},$$

$$\mathbb{P}(\chi_0^{(a,\epsilon)}(x) = 3) = e^{-a},$$

$$\mathbb{P}(\chi_0^{(a,\epsilon)}(x) = 2) = 1 - P(\chi_0^{(a,\epsilon)}(x) = 0) - P(\chi_0^{(a,\epsilon)}(x) = 1) - P(\chi_0^{(a,\epsilon)}(x) = 3)$$

for every $x \in \mathbb{Z}^2$, denote the resulting bootstrap dynamics by $\chi_t^{(a,\epsilon)}$, and for $\theta = 3$ define $a_c \in [0, \infty]$ by

$$a_c = \inf\left\{ a > 0 : \lim_{\epsilon \to 0} \mathbb{P}(\chi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0) > 0 \right\}.$$

We will not provide complete proofs of the next three lemmas, but only point to previous arguments that apply with simplifications and minor modifications.

LEMMA 5.6. *The following strict inequalities hold*:

$$1 - (a_c + 1)e^{-a_c} < p_c^{\text{site}}, \quad e^{-a_c} < p_c^{\text{site}}.$$

In particular, $a_c \in (0, \infty)$. Also, $\lim_{\epsilon \to 0} \mathbb{P}(\chi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0) \to 1$ as $a \to \infty$.

PROOF. The argument is very similar to that for Lemmas 5.1 and 5.2. □

LEMMA 5.7. *If $a > a_c$, then* (1.7) *holds. Also,* (1.8) *holds.*

PROOF. This follows from the proof of Lemma 5.4, simplified by the absence of states 4 and 5, which eliminates the need for a coupling domination. □

LEMMA 5.8. *Assume that $a < a_c$. Then* (1.6) *holds.*

PROOF. The difference from the proof of Lemma 5.5 is the definition of *bad* sites, which in this case are those that are not 3-inert, and those that are 2-II but not 2-inert. As the density of bad sites goes to 0 by Lemma 2.5, the proof of Lemma 5.5 can be easily adapted. □

6. Bootstrap percolation on $\mathbb{Z}^2 \times K_n$. In this section, we prove Theorem 1.3, which follows from Lemmas 6.3 and 6.4 below.

As already announced, we need yet another heterogeneous bootstrap rule in which sites in $\mathbb{Z}^2$ receive more help from their neighbors than in ξ_t. In this case, we have a new state, labeled by θ and representing an empty site that has no contribution to make. We denote this rule by $\zeta_t \in \{0, 1, 2, 3, 4, 5, \theta\}^{\mathbb{Z}^2}$, $t \in \mathbb{Z}_+$. Assume that ζ_0 is given. For a given $t \geq 0$, let as before $Z_t(x)$ be the cardinality of $\{y : y \sim x$ and $\zeta_t(y) = 0\}$ and $W_t(x) = \mathbb{1}(\{y : y \sim x$ and $0 < \zeta_t(y) < \theta\} \neq \varnothing)$ then

$$\zeta_{t+1}(x) = \begin{cases} 0, & Z_t(x) + W_t(x) \geq \zeta_t(x), \\ \zeta_t(x), & \text{otherwise.} \end{cases}$$

For an initially occupied set ω_0, we create two initial states ζ_0 as follows. For $x \in \mathbb{Z}^2$, let

$$N_x = \left|\{y \in \{x\} \times K_n : \omega_0(y) = 1\}\right|.$$

Call x a *clash site* if $N_x < \theta$ and $\omega_0(y_1, u) = \omega_0(y_2, u) = 1$ for some $y_1 \neq y_2$ in $\{x\} \cup \{y : y \sim x\}$ and some $u \in K_n$, such that $N_{y_1} < \theta$ and $N_{y_2} < \theta$. We define the *favoring initialization* $\zeta_0^{\text{fv}}(x)$ and the *restricting initialization* $\zeta_0^{\text{rs}}(x)$ as follows. If x is a clash site, then $\zeta_0^{\text{fv}}(x) = 0$, while $\zeta_0^{\text{rs}}(x) = \theta$. If x is not a clash site, the two initializations are equal: $\zeta_0^{\text{fv}}(x) = \zeta_0^{\text{rs}}(x) = \text{nz}(N_x)$, where $\text{nz} : \mathbb{Z}_+ \to \{0, \ldots, 5, \theta\}$ is given by

(6.1)
$$\text{nz}(m) = \begin{cases} 0, & m \geq \theta, \\ k, & m = \theta - k \text{ for some } k \in \{1, 2, 3, 4\}, \\ 5, & 0 < m < \theta - 4, \\ \theta, & m = 0. \end{cases}$$

These initializations determine their respective dynamics ζ_t^{rs} and ζ_t^{fv}, $0 \leq t \leq \infty$. We next state the comparison lemma whose simple proof is omitted.

LEMMA 6.1. *We have*

$$\bigcup \{\{x\} \times K_n : \zeta_\infty^{\text{rs}}(x) = 0\} \subset \omega_\infty \subset \bigcup \{\{x\} \times K_n : \zeta_\infty^{\text{fv}}(x) = 0\} \cup \omega_0.$$

Consider $\mathbb{Z}^2 \times [0, \infty)$ and equip each $\{x\} \times [0, \infty)$, $x \in \mathbb{Z}^2$ with an independent Poisson point location of unit intensity. Then we define the *a-initialization* $\zeta_0^{(a)}$ obtained by $\zeta_0^{(a)}(x) = \mathrm{nz}(N_x^a)$, where now N_x^a is the number of location points in $\{x\} \times [0, a]$ and the function nz is defined in (6.1).

For the rest of this section, we assume that ω_0 is a product measure with density $p = a/n$.

LEMMA 6.2. *Assume $a' > a$. Then, for large enough n, ω_0 and the a'-initialization $\zeta_0^{(a')}$ can be coupled so that $\zeta_0^{\mathrm{fv}} \geq \zeta_0^{(a')}$*

Conversely, assume $a' < a$. Then, for large enough n, ω_0 and $\zeta_0^{(a')}$ can be coupled so that $\zeta_0^{\mathrm{rs}} \leq \zeta_0^{(a')}$.

PROOF. We will prove only the first statement; the second is proved similarly. Observe that the random variables N_x are i.i.d. Binomial(n,p). Fix an $\epsilon > 0$ such that $a + \epsilon < a'$.

Assume that first the i.i.d. random field of truncated random variables $N_x \wedge \theta$, $x \in \mathbb{Z}^2$, is selected. Conditional on this selection, any site $x \in \mathbb{Z}^2$ is a clash site with probability at most C/n, where $C = C(\theta)$ is a constant. Furthermore, if $\|x - x'\|_1 \geq 3$, then x and x' are clash sites independently. Therefore, by [29], there exists an i.i.d. random field η_x, $x \in \mathbb{Z}^2$ of Bernoulli random variables, independent also of the field $N_x \wedge \theta$, $x \in \mathbb{Z}^2$, so that $\eta_x = 1$ whenever x is a clash site and $\mathbb{P}(\eta_x = 1) = \epsilon$.

If n is large enough, we can, for a fixed x, find a coupling between (N_x, η_x) and a Poisson(a) random variable M_x so that $(N_x \wedge \theta)\mathbb{1}(\eta_x = 0) \geq (M_x \wedge \theta)$. Thus we can construct an independent field M_x, $x \in \mathbb{Z}^2$ with this property, which concludes the proof. □

Define now

(6.2) $$\phi(a) = \mathbb{P}(\zeta_\infty^{(a)}(\mathbf{0}) = 0).$$

Observe that $\phi : (0, \infty) \to [0, 1]$ is a nondecreasing limit of nondecreasing continuous functions ϕ_t given by $\phi_t(a) = \mathbb{P}(\zeta_t^{(a)}(\mathbf{0}) = 0)$. Therefore, ϕ is left continuous and nondecreasing.

LEMMA 6.3. *Assume $\theta \geq 3$. Fix any $a \in (0, \infty)$ and $v \in \mathbb{Z}^2 \times K_n$. As $n \to \infty$,*

$$\mathbb{P}(\mathrm{Poisson}(a) \geq \theta) \leq \phi(a) \leq \liminf_n \mathbb{P}_p(\omega_\infty(v) = 1)$$

$$\leq \limsup_n \mathbb{P}_p(\omega_\infty(v) = 1) \leq \phi(a+) \leq 1 - e^{-4a}.$$

PROOF. We have

$$\{N_{\mathbf{0}}^a \geq \theta\} = \{\zeta_0^{(a)}(\mathbf{0}) = 0\} \subset \{\zeta_\infty^{(a)}(\mathbf{0}) = 0\},$$

and, for any 2×2 block $B \subset \mathbb{Z}^2$ including $\mathbf{0}$,

$$\bigcap_{x \in B}\{N_x^a = 0\} = \bigcap_{x \in B}\{\zeta_0^{(a)}(x) = \theta\} \subset \{\zeta_\infty^{(a)}(\mathbf{0}) = \theta\},$$

which gives the two extreme bounds. The remainder follows from Lemmas 6.1 and 6.2. □

LEMMA 6.4. *For $\theta \geq 14$, ϕ is continuous on $(0, \infty)$.*

PROOF. Recall that by the construction, $\zeta_t^{(a)}$ are coupled for all a. Let

$$E_a = \bigcap_{a' > a}\{\zeta_\infty^{(a')}(\mathbf{0}) = 0\},$$

so that $\phi(a+) = \mathbb{P}(E_a)$. Let also F_a be the event that there is an ℓ^∞-circuit C around the origin, consisting of sites x with $N_x^a \notin [\theta - 5, \theta - 1]$. As no site in C ever changes its state in the $\zeta_t^{(a)}$ dynamics,

$$E_a \cap F_a \subset \{\zeta_\infty^{(a)}(\mathbf{0}) = 0\}.$$

It remains to show that, for $\theta \geq 14$, $\mathbb{P}(F_a) = 1$ for all $a \in (0, \infty)$, that is,

$$\mathbb{P}(\text{Poisson}(a) \in [\theta - 5, \theta - 1]) \leq p_c^{\text{site}}.$$

Using the rigorous lower bound $p_c^{\text{site}} > 0.556$ [37], a numerical computation shows that the above bound indeed holds for $\theta \geq 14$. □

7. Open problems.   We conclude with a selection of a few natural questions.

QUESTION 7.1.   Is the function ϕ defined in (6.2) continuous on $(0, \infty)$ for all θ? Is it analytic for all, or at least large enough, θ?

QUESTION 7.2.   Is the function $a \mapsto \lim_{\epsilon \to 0} \mathbb{P}(\xi_\infty^{(a,\epsilon)}(\mathbf{0}) = 0)$, where $\xi_\infty^{(a,\epsilon)}$ is defined in Section 5.1, continuous for all a? A related question is whether $\lim_{n \to \infty} \mathbb{P}_p(v_0 \in \omega_\infty)$ exists for odd θ and all a when p is given by (1.5)?

In both question above, arguments similar to that for Lemma 6.4 imply continuity for large enough a and for small enough a.

QUESTION 7.3.   When $a < a_c$ in Theorem 1.2, what is the rate of convergence in (1.6)?

Our last three questions are more open-ended, and their answers likely require development of new techniques. We first propose a closer look into the critical scaling in Theorem 1.1.

QUESTION 7.4.   Assume θ is even, as in Theorem 1.1. Assume that

$$p = (2(\ell - 1)!)^{1/\ell} \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}} + bf(n).$$

Can the function $f(n)$ be chosen so that the limit of the final density as $n \to \infty$ exists and is neither a constant nor a step function of $b \in \mathbb{R}$?

We conclude with two questions on larger dimensions of the lattice factor or the Hamming torus factor (see also [13, 19]).

QUESTION 7.5.   What are the analogues of our main theorems for bootstrap percolation on $\mathbb{Z}^d \times K_n^2$, for $d \geq 3$?

To approach this question using the methods of our present paper would require a much deeper understanding of heterogeneous bootstrap percolation on $\mathbb{Z}^d$ (see [16]).

QUESTION 7.6.   What are the analogues of our main theorems for bootstrap percolation on $\mathbb{Z}^2 \times K_n^d$, $d \geq 3$?

This question poses a significant challenge at present, as the bootstrap percolation on K_n^d, $d \geq 3$, alone is poorly understood [13], except for $\theta = 2$ [35].

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