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Intensity approximation for pairwise interaction Gibbs point processes using determinantal point processes

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Abstract: The intensity of a Gibbs point process is usually an intractable function of the model parameters. For repulsive pairwise interaction point processes, this intensity can be expressed as the Laplace transform of some particular function. Baddeley and Nair (2012) developped the Poisson-saddlepoint approximation which consists, for basic models, in calculating this Laplace transform with respect to a homogeneous Poisson point process. In this paper, we develop an approximation which consists in calculating the same Laplace transform with respect to a specific determinantal point process. This new approximation is efficiently implemented and turns out to be more accurate than the Poisson-saddlepoint approximation, as demonstrated by some numerical examples.

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Contents

1. Introduction

Due to their simple interpretation, Gibbs point processes and in particular pairwise interaction point processes play a central role in the analysis of spatial point patterns (see van Lieshout (2000); Møller and Waagepetersen (2004); Baddeley et al. (2015)). In a nutshell, such models (in the homogeneous case) are defined in a bounded domain by a density with respect to the unit rate Poisson point process which takes the form

$$f(\mathbf{x}) \propto \beta^{|\mathbf{x}|} \prod_{u \in \mathbf{x}} g(v-u),$$

where $\mathbf{x}$ is a finite configuration of points, $\beta > 0$ represents the activity parameter, $|\mathbf{x}|$ is the number of elements of $\mathbf{x}$ and where $g : \mathbb{R}^d \to \mathbb{R}^+$ is the pairwise interaction function. Here $\propto$ means "proportional to".

However, many important theoretical properties of these models are in general intractable, like for instance the simplest one, the intensity $\lambda \in \mathbb{R}^+$, representing the mean number of points per unit volume. It is known (see e.g. Section 2.2) that

$$\lambda = \beta \, \mathrm{E}\left(\prod_{u \in \mathbf{x}} g(u) \right).$$

Such an expectation is in general intractable. As clearly outlined by Baddeley and Nair (2012), this intractability constitutes a severe drawback. For example, simulating a Gibbs point process with a prescribed value of λ cannot be done beforehand even for simple models such as Strauss models. Baddeley and Nair (2012) suggest to evaluate the expectation with respect to a homogeneous Poisson point process with intensity λ. This results in the Poisson-saddlepoint approximation, denoted by λ_{PS}, obtained as the solution of

$$\log \lambda_{\mathrm{PS}} = \log \beta - \lambda_{\mathrm{PS}} \, G$$

where $G = \int_{\mathbb{R}^d} (1 - g(u)) \, \mathrm{d}u$ (provided this integral is finite).

The general idea of the present paper is to evaluate the same expectation with respect to a determinantal point process with intensity λ. Determinantal point processes (DPP), see e.g. Lavancier et al. (2015), are a class of repulsive models which is more tractable than Gibbs models. For example all moments are explicit. If g has a finite range $R > 0$ and a possible hard-core distance $0 \le \delta \le R$, in the sense that $g(u) = 0$ for $\|u\| \le \delta$, our approximation denoted by λ_{DPP} is the solution of

$$\log \lambda_{\mathrm{DPP}} = \log \beta + (1 + \lambda_{\mathrm{DPP}} |B_\delta|) \log \left(1 - \frac{\lambda_{\mathrm{DPP}} |B_\delta|}{1 + \lambda_{\mathrm{DPP}} |B_\delta|} \right)$$

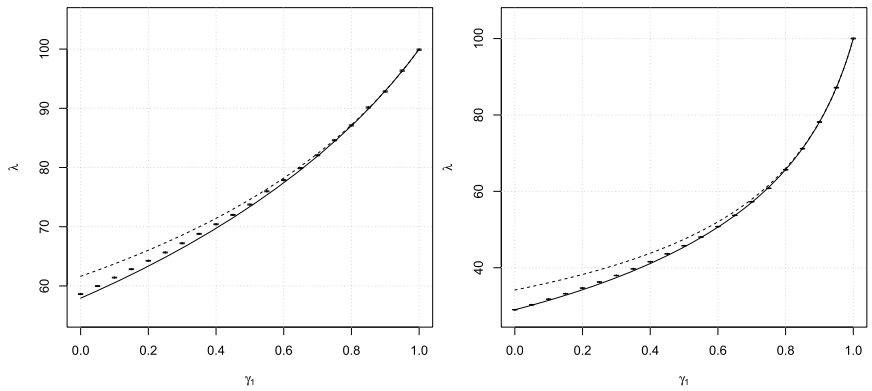
$$+ (1 + \lambda_{\mathrm{DPP}}(G - |B_\delta|)/\kappa) \log \left(1 - \frac{\lambda_{\mathrm{DPP}}(G - |B_\delta|)}{1 + \lambda_{\mathrm{DPP}}(G - |B_\delta|)/\kappa}\right), \quad (1.1)$$

where

$$\kappa = \frac{\int (1-g)^2 - |B_\delta|}{|B_R| - |B_\delta|}, \quad (1.2)$$

B_ρ denotes the Euclidean ball centered at 0 with radius ρ and $|B_\rho|$ is its volume. Here we agree that if $\delta = R$ then $\kappa = 1$, which implies $\kappa > 0$.

Both approximations λ_{DPP} and λ_{PS} can be obtained very quickly with a unit-root search algorithm. Figure 1 reports λ_{DPP} and λ_{PS} as well as the true intensity λ (obtained by Monte-Carlo simulations) for Strauss models in terms of the interaction parameter $\gamma_1 \in [0, 1]$. This setting is considered by Baddeley and Nair (2012). The DPP approximation outperforms the Poisson-saddlepoint approximation especially when γ_1 is close to zero, i.e. for very repulsive point processes. More numerical illustrations are displayed in Section 4.



(a) Strauss: $R = 0.05$, $\beta = 100$ (b) Strauss: $R = 0.1$, $\beta = 100$

FIG 1. *Comparison of the exact intensity (small boxplots), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for homogeneous Strauss models with activity parameter β and range of interaction R. Curves and boxplots are reported in terms of the interaction parameter $\gamma_1 \in [0, 1]$.*

The rest of the paper is organized as follows. Section 2 provides necessary notation and background material on point processes, Gibbs point processes and determinantal point processes. Intensity approximations are discussed in detail in Section 3. Finally, Section 4 presents numerical experiments for several classes of pairwise interaction point processes.

2. Gibbs point processes and determinantal point processes

2.1. Background and Poisson point processes

For $d \geq 1$, let $\mathbf{X}$ be a spatial point process defined on $\mathbb{R}^d$, which we see as a random locally finite subset of $\mathbb{R}^d$. Local finiteness of $\mathbf{X}$ means that $\mathbf{X}_B = \mathbf{X} \cap B$

is finite almost surely (a.s.), that is the number of points $N(B)$ of $\mathbf{X}_B$ is finite a.s., whenever $B \subset \mathbb{R}^d$ is bounded. We let $\mathcal{N}$ stand for the state space consisting of the locally finite subsets (or point configurations) of $\mathbb{R}^d$. Let $\mathcal{B}(\mathbb{R}^d)$ denote the class of bounded Borel sets in $\mathbb{R}^d$. For any $B \in \mathcal{B}(\mathbb{R}^d)$, we denote by $|B|$ its Lebesgue measure. A realization of $\mathbf{X}_B$ is of the form $\mathbf{x} = \{x_1, \ldots, x_m\} \subset B$ for some nonnegative finite integer m and we sometimes denote its cardinal by $|\mathbf{x}|$. For further details about point processes, we refer to Daley and Vere-Jones (2003) and Møller and Waagepetersen (2004).

A spatial point process is said to have an nth order intensity function $\rho^{(n)}$ if for any nonnegative measurable function $h : (\mathbb{R}^d)^n \to \mathbb{R}^+$, the following formula referred to as Campbell-Mecke formula holds

$$\mathrm{E} \sum_{u_1,\ldots,u_n \in \mathbf{X}}^{\neq} h(u_1, \ldots, u_n) = \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} h(u_1, \ldots, u_n) \rho^{(n)}(u_1, \ldots, u_n) \, \mathrm{d}u_1 \ldots \mathrm{d}u_n, \tag{2.1}$$

where the sign $\neq$ over the sum means that $u_1, \ldots, u_n$ are pairwise distinct. Then, $\rho^{(n)}(u_1, \ldots, u_n) \, \mathrm{d}u_1 \cdots \mathrm{d}u_n$ can be interpreted as the approximate probability for $\mathbf{X}$ having a point in each of infinitesimally small regions around $u_1, \ldots, u_n$ of volumes $\mathrm{d}u_1, \ldots, \mathrm{d}u_n$, respectively. We also write $\rho(u)$ for the intensity function $\rho^{(1)}(u)$. A spatial point process $\mathbf{X}$ in $\mathbb{R}^d$ is said to be stationary (respectively isotropic) if its distribution is invariant under translations (respectively under rotations). When $\mathbf{X}$ is stationary, the intensity function reduces to a constant denoted by λ in the rest of this paper. As a matter of fact, λ measures the mean number of points per unit volume.

The Poisson point process, often defined as follows (see e.g. Møller and Waagepetersen (2004)), serves as the reference model.

Definition 2.1. *Let ρ be a locally integrable function on S, for $S \subseteq \mathbb{R}^d$. A point process $\mathbf{X}$ satisfying the following statements is called the Poisson point process on S with intensity function ρ:*

- *for any $m \geq 1$, and for any disjoint and bounded $A_1, \ldots, A_m \subset S$, the random variables $\mathbf{X}_{A_1}, \ldots, \mathbf{X}_{A_m}$ are independent;*
- *$N(A)$ follows a Poisson distribution with parameter $\int_A \rho(u) \, \mathrm{d}u$ for any bounded $A \subset S$.*

Among the many properties of Poisson point processes, it is to be noticed that the nth order intensity function writes $\rho^{(n)}(u_1, \ldots, u_n) = \prod_{i=1}^n \rho(u_i)$, for any pairwise distinct $u_1, \ldots, u_n \in S$.

Let $\mathbf{Z}$ be a unit rate Poisson point process on S, which means that its intensity is constant and equal to one. Assume, first, that S is bounded ($|S| < \infty$). We say that a spatial point process $\mathbf{X}$ has a density f if the distribution of $\mathbf{X}$ is absolutely continuous with respect to the one of $\mathbf{Z}$ and with density f. Thus, for any nonnegative measurable function h defined on $\mathcal{N}$, $\mathrm{E}\, h(\mathbf{X}) = \mathrm{E}(f(\mathbf{Z})h(\mathbf{Z}))$. Now, suppose that f is *hereditary*, i.e., for any pairwise distinct $u_0, u_1, \ldots, u_n \in S$, $f(\{u_1, \ldots, u_n\}) > 0$ whenever $f(\{u_0, u_1, \ldots, u_n\}) > 0$. We can then define

the so-called *Papangelou conditional intensity* by

$$\lambda(u, \mathbf{x}) = f(\mathbf{x} \cup u)/f(\mathbf{x}) \tag{2.2}$$

for any $u \in S$ and $\mathbf{x} \in \mathcal{N}$, setting $0/0 = 0$. By the interpretation of f, $\lambda(u, \mathbf{x})\,\mathrm{d}u$ can be considered as the conditional probability of observing one event in a small ball, say B, centered at u with volume $\mathrm{d}u$, given that $\mathbf{X}$ outside B agrees with $\mathbf{x}$. When f is hereditary, there is a one-to-one correspondence between f and λ.

Because the notion of density for $\mathbf{Z}$ when $S = \mathbb{R}^d$ makes no sense, the Papangelou conditional intensity cannot be defined through a ratio of densities in $\mathbb{R}^d$. But it still makes sense as the Papangelou conditional intensity can actually be defined at the Radon-Nykodym derivative of $\mathrm{P}^!_u$ the reduced Palm distribution of $\mathbf{X}$ with respect to P, the distribution of $\mathbf{X}$ (see Daley and Vere-Jones (2003)). We do not want to enter in too much detail here and prefer to refer the interested reader to Coeurjolly et al. (2017).

Finally, we mention the celebrated Georgii-Nguyen-Zessin formula (see Georgii, 1976; Nguyen and Zessin, 1979), which states that for any $h : \mathbb{R}^d \times \mathcal{N} \to \mathbb{R}$ (such that the following expectations are finite)

$$\mathrm{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \int_{\mathbb{R}^d} \mathrm{E}\left(h(u, \mathbf{X})\lambda(u, \mathbf{X})\right) \mathrm{d}u. \tag{2.3}$$

By identification of (2.1) and (2.3), we see a link between the intensity function of a point process and the Papangelou conditional intensity: for any $u \in \mathbb{R}^d$

$$\rho(u) = \mathrm{E}\left(\lambda(u, \mathbf{X})\right),$$

which in the stationary case reduces to

$$\lambda = \mathrm{E}\left(\lambda(0, \mathbf{X})\right). \tag{2.4}$$

2.2. Gibbs point processes

For a recent and detailed presentation, we refer to Dereudre (2017). Gibbs processes are characterized by an energy function H (or Hamiltonian) that maps any finite point configuration to $\mathbb{R} \cup \{\infty\}$. Specifically, if $|S| < \infty$, a Gibbs point process on S associated to H and with activity $\beta > 0$ admits the following density with respect to the unit rate Poisson process:

$$f(\mathbf{x}) \propto \beta^{|\mathbf{x}|} e^{-H(\mathbf{x})}. \tag{2.5}$$

This definition makes sense under some regularity conditions on H, typically non degeneracy ($H(\emptyset) < \infty$) and stability (there exists $a \in \mathbb{R}$ such that $H(\mathbf{x}) \geq a|\mathbf{x}|$ for any $\mathbf{x} \in \mathcal{N}$). Consequently, configurations $\mathbf{x}$ having a small energy $H(\mathbf{x})$ are more likely to be generated by a Gibbs point process than by a Poisson point process, and conversely for configurations having a high energy. In the extreme case where $H(\mathbf{x}) = \infty$, then $\mathbf{x}$ cannot, almost surely, be the realization of a Gibbs point process associated to H.

In this paper, we focus on pairwise interaction point processes. To be close to the original paper by Baddeley and Nair (2012) the present contribution is based on, we use their notation: a Gibbs point process in S is said to be a pairwise interaction point process with pairwise interaction function $g : \mathbb{R}^d \to \mathbb{R}^+$, if its density writes

$$f(\mathbf{x}) \propto \beta^{|\mathbf{x}|} \prod_{u,v \in \mathbf{x}} g(u-v).$$

If $|S| = \infty$, this definition and more generally Definition (2.5) do not make sense since $H(\mathbf{x})$ can be infinite or even undefined if $|\mathbf{x}| = \infty$. In this case, Gibbs point processes have to be defined via their conditional specifications and for pairwise interactions Gibbs point processes, restrictions on g have to be imposed for existence (see again Dereudre (2017) and the references therein for details). Nonetheless, as mentioned in the previous section, the concept of Papangelou conditional intensity applies whenever $|S| < \infty$ or $|S| = \infty$, and in either case it has the explicit form

$$\lambda(u, \mathbf{x}) = \beta \prod_{v \in \mathbf{x}} g(u-v), \tag{2.6}$$

for any $u \in S$. Note that when $S = \mathbb{R}^d$, a pairwise interaction Gibbs point process is stationary if g is symmetric (i.e. $g(w) = g(-w)$ for any $w \in \mathbb{R}^d$) and it is further isotropic if $g(u-v)$ depends simply on $\|u-v\|$. Moreover, we say that the interaction involves a hard-core radius $\delta \geq 0$ if $g(u) = 0$ for $\|u\| \leq \delta$.

From (2.4), we deduce that the intensity parameter of a stationary pairwise interaction process writes

$$\lambda = \mathrm{E}\left(\lambda(0, \mathbf{X})\right) = \beta\, \mathrm{E}\left(\prod_{v \in \mathbf{X}} g(v)\right). \tag{2.7}$$

Let us give a few examples (which are in particular well-defined in $\mathbb{R}^d$). Many other examples can be found e.g. in the recent monograph by Baddeley et al. (2015).

- *Strauss model*: let $\gamma \in [0,1]$ and $0 < R < \infty$

$$g(u) = \begin{cases} \gamma & \text{if } \|u\| \leq R \\ 1 & \text{otherwise.} \end{cases} \tag{2.8}$$

- *Strauss Hard-core model*: let $\gamma \in \mathbb{R}^+$ and $0 < \delta < R < \infty$

$$g(u) = \begin{cases} 0 & \text{if } \|u\| < \delta \\ \gamma & \text{if } \delta \leq \|u\| \leq R \\ 1 & \text{otherwise.} \end{cases}$$

- *Piecewise Strauss Hard-core model*:

$$g(u) = \begin{cases} 0 & \text{if } \|u\| < \delta \\ \gamma_i & \text{if } R_i \leq \|u\| \leq R_{i+1},\ i = 1, \ldots, I \\ 1 & \text{otherwise,} \end{cases}$$

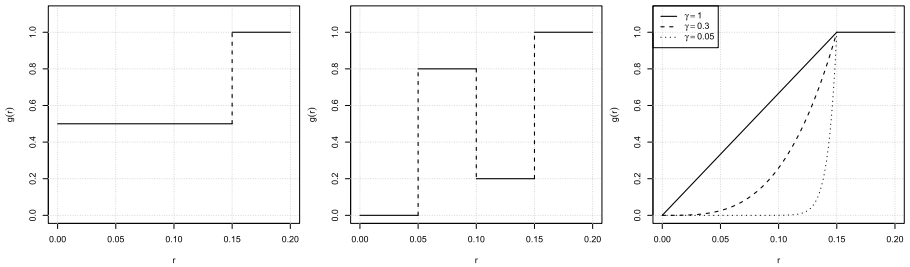
where $I \geq 1$, $0 \leq R_1 = \delta < R_2 < \cdots < R_{I+1} = R < \infty$ and $\gamma_1, \ldots, \gamma_I \in \mathbb{R}^+$ if $\delta > 0$, otherwise $\gamma_1, \ldots, \gamma_I \in [0, 1]$.

- *Diggle-Graton model*: let $\gamma \in [0, 1]$

$$g(u) = \begin{cases} \left(\frac{\|u\|}{R}\right)^{1/\gamma} & \text{if } \|u\| \leq R \\ 1 & \text{otherwise,} \end{cases}$$

where for $t \in (0, 1)$, $t^\infty = 0$ and $1^\infty = 1$ by convention.

Let us note that a Strauss model with $\gamma = 0$ and radius R is actually a hard-core model with radius R. The Diggle-Graton potential can be found in Baddeley et al. (2015) in a slightly different parameterization. The one chosen here makes comparisons with the Strauss model easier. For instance, when $\gamma = 0$ the model reduces to a Strauss model with $\gamma = 0$ and radius R. When $\gamma = 1$, the function g grows linearly from 0 to 1. Figure 2 depicts the form of some of the pairwise interaction functions presented above.



(a) Strauss model (b) Piecewise Strauss hard-core model (c) Diggle-Graton model

FIG 2. *Examples of pairwise interaction functions for the Strauss model ($\gamma = 0.5$, $R = 0.15$), the piecewise Strauss hard-core model ($\delta = R_1 = 0.05$, $R_2 = 0.1$, $R_3 = R = 0.15$, $\gamma_1 = 0.8$, $\gamma_2 = 0.2$), and the Diggle-Graton model ($\gamma = 0.05$, 0.3 and 1, $R = 0.15$).*

A Gibbs point process has a finite range R if for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in \mathcal{N}$, $\lambda(u, \mathbf{x}) = \lambda(u, \mathbf{x} \cap B(u, R))$. For pairwise interaction point processes, this property translates to $g(u) = 1$ for any $u \in \mathbb{R}^d$ such that $\|u\| > R$. All previous models have a finite range $R < \infty$. An example of infinite range pairwise interaction point process which will not be considered in this paper is the Lennard-Jones model (see e.g. Ruelle (1969); Baddeley et al. (2015)).

2.3. Determinantal point processes

Determinantal point processes (DPPs) are models for inhibitive point patterns. We refer to Lavancier et al. (2015) for their main statistical properties. The interested reader is also referred to Macchi (1975); Shirai and Takahashi (2003); Hough et al. (2009) for historical and theoretical aspects.

DPPs are defined through a kernel function K which is a function from $S \times S$ to $\mathbb{C}$. A point process is a DPP on S with kernel K, denoted by DPP(K), if for any n, its nth order intensity function takes the form

$$\rho^{(n)}(u_1, \ldots, u_n) = \det[K](u_1, \ldots, u_n), \tag{2.9}$$

for every $(u_1, \ldots, u_n) \in S^n$, where $[K](u_1, \ldots, u_n)$ denotes the matrix with entries $K(u_i, u_j)$, $1 \leq i, j \leq n$. In particular, the intensity function of DPP(K) is $K(u, u)$.

Conditions on the kernel K are required to ensure the existence of DPP(K). For our purpose, we will only consider DPPs on a compact set. So let us assume that S is compact and suppose that K is a continuous real-valued covariance function on $S \times S$. In this setting, by the Mercer theorem (see Riesz and Nagy (1990)), K admits the spectral expansion

$$K(u, v) = \sum_{i=1}^{\infty} \lambda_i \phi_i(u) \phi_i(v), \quad \forall u, v \in S, \tag{2.10}$$

where $\{\phi_i\}_{i \geq 1}$ is an orthonormal basis of $L^2(S)$ and where λ_i, $i \geq 1$, are referred to as the eigenvalues of K. Under the above assumptions, DPP(K) exists if and only if $\lambda_i \leq 1$ for all i.

Due to their tractability, DPPs have many interesting properties. Many of them have been obtained by Shirai and Takahashi (2003), from which we derive the following key-equation used by our intensity approximation. Its proof is postponed to Appendix A.

Proposition 2.2. *Let* $\mathbf{X}$ *be a* DPP *on a compact set* S *with kernel* K. *Assume that* K *is a continuous real-valued covariance function on* $S \times S$ *whose all eigenvalues are not greater than 1. Then, for any bounded function* $g : S \to \mathbb{R}^+$,

$$\mathrm{E}\left(\prod_{v \in \mathbf{X}} g(v) \right) = \prod_{i \geq 1} (1 - \tilde{\lambda}_i) \tag{2.11}$$

where $\tilde{\lambda}_i$, *for* $i \geq 1$, *are the eigenvalues of the integral operator with kernel* $\tilde{K} : S \times S \to \mathbb{R}$ *given by*

$$\tilde{K}(u, v) = (1 - g(u)) K(u, v). \tag{2.12}$$

If $g \leq 1$ *we can equivalently choose for* $\tilde{K}$ *the symmetric kernel*

$$\sqrt{1 - g(u)} K(u, v) \sqrt{1 - g(v)},$$

in which case the $\tilde{\lambda}_i$*'s are simply the eigenvalues of* $\tilde{K}$ *in its spectral representation given by the Mercer theorem.*

3. Intensity approximation

3.1. Poisson-saddlepoint approximation

We remind that the intensity parameter of a Gibbs point process, and in particular a pairwise interaction point process satisfies (2.7). The expectaction in (2.7) is to be regarded with respect to P the distribution of the Gibbs point process $\mathbf{X}$. Baddeley and Nair (2012) suggest to replace P by a simpler distribution, say Q, for which the right-hand-side of (2.7) becomes tractable. The Poisson-saddlepoint approximation consists in choosing $\Pi(\lambda)$, the Poisson distribution with parameter λ, as distribution Q. As a result, the Poisson-saddlepoint approximation consists in resolving the equation

$$\lambda = \beta \, \mathrm{E}_{\Pi(\lambda)} \left(\prod_{v \in \mathbf{Y}} g(v) \right) = \beta \, \mathrm{E}_{\Pi(\lambda)} \left(\exp \left(\sum_{v \in \mathbf{Y}} \log g(v) \right) \right), \tag{3.1}$$

with the convention that $\log 0 = -\infty$ and where, to avoid any ambiguity, we denote by $\mathbf{Y}$ a Poisson point process with intensity λ defined on $\mathbb{R}^d$ and stress also this by indexing the E with the distribution $\Pi(\lambda)$. It turns out that if $g(u) \in [0,1]$ for any $u \in \mathbb{R}^d$, the right-hand side of (3.1) is the Laplace transform of some Poisson functional and equals $\beta \exp(-\lambda G)$ where $G = \int_{\mathbb{R}^d} (1 - g(u)) \, \mathrm{d}u$, see e.g. Møller and Waagepetersen (2004, Proposition 3.3). As noticed in Baddeley and Nair (2012), this formula extends to more general functions g, provided $G > -\infty$. Hence, the Poisson-saddlepoint approximation, denoted by λ_{PS} in this paper, is defined as the solution, when it exists, of

$$\lambda_{\mathrm{PS}} = \beta \exp(-\lambda_{\mathrm{PS}} \, G). \tag{3.2}$$

For stationary pairwise Gibbs models with finite range R, and such that $\lambda(u, \mathbf{x}) \leq \beta$ (or equivalently such that $g \leq 1$), then $0 \leq G \leq |B_R|$. In this case, Baddeley and Nair (2012, Theorem 2) states, among other properties, that λ_{PS} exists uniquely and is an increasing function of β. Actually, when $G < 0$, which corresponds to an attractive pattern, we can say more: if $-e^{-1}/\beta < G < 0$, λ_{PS} exists but is not unique (there are two solutions) and if $G < -e^{-1}/\beta$, λ_{PS} does not exist (if $G = -e^{-1}/\beta$, a case which is not really interesting, there is a unique solution), see Section 4 for an illustration of this result in the case of a Strauss hard-core interaction.

From a numerical point of view, λ_{PS} can be very efficiently and quickly estimated using root-finding algorithms.

3.2. DPP approximation

Following the same idea as the Poisson-saddlepoint approximation, for a stationary pairwise interaction point process with finite range R, we suggest to substitute the measure P involved in the expectation (2.7) by the measure Q corresponding to a DPP defined on B_R with some kernel K (to be chosen)

and intensity λ, i.e. $K(u,u) = \lambda$. Similarly to the previous section, by letting $\mathrm{DPP}(K;\lambda)$ denote the distribution of such a DPP and $\mathbf{Y} \sim \mathrm{DPP}(K;\lambda)$, the DPP approximation of the intensity λ is the solution of

$$\lambda = \beta \, \mathrm{E}_{\mathrm{DPP}(K;\lambda)} \left(\prod_{v \in \mathbf{Y}} g(v) \right). \tag{3.3}$$

From Proposition 2.2 and in particular from (2.11), this yields the estimating equation

$$\log \lambda = \log \beta + \sum_{i \geq 1} \log(1 - \tilde{\lambda}_i), \tag{3.4}$$

where the eigenvalues $\tilde{\lambda}_i$ of the integral operator with kernel $\tilde{K}$ are related to λ by the relation

$$\tilde{K}(u,v) = (1 - g(u))K(u,v) \quad \text{with} \quad K(u,u) = \lambda.$$

To complete this approximation, the eigenvalues $\tilde{\lambda}_i$ need to be specified. We first consider in the next section the situation where there is no hard-core, i.e. $\delta = 0$, before turning to the general case in Section 3.2.2. Some theoretical properties of our DPP approximation are presented in Section 3.2.4.

3.2.1. DPP approximation without hard-core

We assume in this section that the pairwise interaction function g does not involve a hard-core part, i.e. $\delta = 0$. We also assume that $G = \int (1 - g) > 0$, which is the typical situation of a repulsive interaction. In this case we choose the eigenvalues $\tilde{\lambda}_i$ in (3.4) to be zero except a finite number N of them that are all equal. Given that

$$\sum_{i \geq 1} \tilde{\lambda}_i = \int_{\mathbb{R}^d} \tilde{K}(u,u) \mathrm{d}u = \int_{\mathbb{R}^d} (1 - g(u))K(u,u) \mathrm{d}u = \lambda G, \tag{3.5}$$

this means that for some $N \geq \lambda G$,

$$\tilde{\lambda}_i = \frac{\lambda G}{N}, \quad \text{for } i = 1, \ldots, N \tag{3.6}$$

and $\tilde{\lambda}_i = 0$ for $i \geq N + 1$. With this choice, the integer N remains the single parameter to choose in our approximation. Note that $N \geq \lambda G$ is a necessary condition to ensure $\tilde{\lambda}_i \leq 1$ as imposed by relation (2.11), but it is in general not sufficient to ensure the existence of $\mathrm{DPP}(K)$ where K is related to $\tilde{K}$ through (2.12). This will be clearly illustrated below when g is the Strauss interaction function. For the choice (3.6), the DPP approximation of the intensity, denoted by λ_{DPP}, becomes the solution of

$$\log \lambda_{\mathrm{DPP}} = \log \beta + N \log \left(1 - \frac{\lambda_{\mathrm{DPP}} G}{N} \right). \tag{3.7}$$

To motivate (3.6) and how we should set N, assume for a moment that g is the interaction function of a Strauss model with range R and interaction parameter $\gamma \in [0,1]$, see (2.8). In this case $\tilde{K}(u,v) = (1-\gamma)K(u,v)$ for any $u, v \in B_R$ and the eigenvalues of K and $\tilde{K}$ satisfy $\tilde{\lambda}_i = (1-\gamma)\lambda_i$. In the approximation (3.3), we start by choosing a kernel K with a finite number of non-vanishing eigenvalues λ_i that are all equal. In view of $\sum \lambda_i = \int K(u,u)\mathrm{d}u = \lambda|B_R|$, this leads to $\lambda_i = \lambda|B_R|/N$ for $i = 1, \dots, N$ and $N \geq \lambda|B_R|$. Note that the latter inequality is necessary to ensure the existence of DPP(K). Going back to $\tilde{\lambda}_i$, this means that (3.6) follows with the necessary and sufficient condition $N \geq \lambda|B_R| = \lambda G/(1-\gamma)$ which is greater than λG.

In order to set N precisely for the Strauss model, remember that a homogeneous DPP is more repulsive when its eigenvalues are close to 1, see Lavancier et al. (2015); Biscio and Lavancier (2016), and at the opposite a DPP is close to a Poisson point process when its eigenvalues are all close to 0. This suggests that in order to make the approximation (3.3) efficient, we should choose λ_i close to 1 when the Gibbs process we want to approximate is very repulsive, that is when γ is close to 0. Moreover the eigenvalues should decrease to 0 when γ increases to 1. If $\lambda_i = \lambda|B_R|/N$, this is equivalent to choosing N an integer that increases from $\lambda|B_R|$ to infinity when γ increases from 0 to 1. A natural option is thus to choose N as the smallest integer larger than $\lambda|B_R|/(1-\gamma)$. Our final choice for the Strauss model is therefore $N = \lceil \lambda|B_R|/(1-\gamma) \rceil$, where $\lceil . \rceil$ denotes the ceiling function, which we may write, for later purposes, $N = \lceil \lambda G/(1-\gamma)^2 \rceil$.

However, with the latter choice, the function in the right-hand side of equation (3.7) is not continuous in λ, which may lead to none or several solutions to this equation. As a last step in our approximation, we therefore consider the upper convex envelope of this function, ensuring a unique solution to (3.7). This finally leads for the Strauss interaction process to the approximation λ_{DPP} defined as the solution, when it exists, of

$$\log \lambda_{\text{DPP}} = \log \beta + (1 + \lambda_{\text{DPP}} G/(1-\gamma)^2) \log \left(1 - \frac{\lambda_{\text{DPP}} G}{1 + \lambda_{\text{DPP}} G/(1-\gamma)^2} \right).$$

Let us now discuss the case of a general pairwise interaction function g without hard-core. In this setting, it is generally not possible to relate the eigenvalues λ_i with the eigenvalues $\tilde{\lambda}_i$. Motivated by the Strauss case, we choose $\tilde{\lambda}_i$ as in (3.6) where $N = \lceil \lambda G/\kappa \rceil$ and $\kappa \in [0,1]$ is a parameter that takes into account the repulsiveness encoded in g. In general κ must be close to 0 when g is close to 1 (the Poisson case), and close to 1 when g is close to a pure hard-core interaction. We decide to quantify the repulsiveness of the model by

$$\kappa = \frac{\int (1-g)^2}{|B_R|},$$

in agreement with our choice for the Strauss model for which $\kappa = (1-\gamma)^2$.

Plugging $N = \lceil \lambda G/\kappa \rceil$ into (3.7) and considering the upper convex envelope to ensure the existence of a unique solution, we finally end up with our DPP

approximation, when there is no hard-core, being the solution, when it exists, of

$$\log \lambda_{\text{DPP}} = \log \beta + (1 + \lambda_{\text{DPP}} G/\kappa) \log \left(1 - \frac{\lambda_{\text{DPP}} G}{1 + \lambda G/\kappa}\right). \tag{3.8}$$

3.2.2. DPP approximation in presence of a hard-core part

A pure hard-core interaction with radius $\delta > 0$ is a particular instance of a Strauss interaction where $\gamma = 0$ and $R = \delta$. Following the approximation of the intensity of a Strauss point process discussed in the previous section, we obtain in this case the DPP approximation:

$$\log \lambda_{\text{DPP}} = \log \beta + (1 + \lambda_{\text{DPP}} |B_\delta|) \log \left(1 - \frac{\lambda_{\text{DPP}} |B_\delta|}{1 + \lambda_{\text{DPP}} |B_\delta|}\right), \tag{3.9}$$

which is just (3.8) with $G = |B_\delta|$ and $\kappa = 1$. This approximation is associated with the initial choice of $N = \lceil \lambda |B_\delta| \rceil$ non vanishing eigenvalues $\tilde{\lambda}_i$ in (3.6), taking the common value $\lambda |B_\delta|/N$.

Assume now that the finite range pairwise interaction with range R is not a pure hard-core interaction, but still involves some hard-core part. This means that there exists $\delta > 0$ with $\delta < R$ such that $g(u) = 0$ for $\|u\| \leq \delta$ and $g(u) > 0$ for $\|u\| > \delta$. In this situation, we do not choose the same common value for all eigenvalues $\tilde{\lambda}_i$, but we consider two groups of eigenvalues. The first group accounts for the hard-core part. It contains $\lceil \lambda |B_\delta| \rceil$ eigenvalues that take the common value $\lambda |B_\delta| / \lceil \lambda |B_\delta| \rceil$, just as for a pure hard-core interaction. The second group accounts for the rest of interaction, quantified by $g_\delta(u) = g(u)$ for $\|u\| \geq \delta$ while $g_\delta(u) = 0$ if $\|u\| < \delta$. Following the previous section, we fix their common value to $\lambda G_\delta / N_\delta$ where $G_\delta = \int (1 - g_\delta)$ and $N_\delta = \lceil \lambda G_\delta / \kappa_\delta \rceil$ with $\kappa_\delta = \int (1 - g_\delta)^2 / (|B_R| - |B_\delta|)$. The choice of κ_δ follows the same motivation as before: It is 1 when $g_\delta(u) = 1$ for $\|u\| > \delta$ (the case of no interaction beyond δ) and 0 when $g_\delta(u) = 0$ for $\delta < \|u\| < R$ (the case of a pure hard-core interaction). Note that this choice guarantees that (3.5) is satisfied.

Plugging the above choice into (3.4) and considering the upper convex enveloppe as in the previous section, we obtain our general approximation (1.1), which can be viewed as a compromise between (3.8) and (3.9).

3.2.3. A remark in the case $G < 0$

In the case $G < 0$, corresponding to an attractive pairwise interaction, it might seem unnatural to use a DPP approximation for the intensity, since DPPs are models for inhibitive point patterns. Nonetheless, DPPs generalise to α-DPPs, see Shirai and Takahashi (2003), that induce repulsiveness when $\alpha < 0$ and attraction when $\alpha > 0$. The particular case $\alpha = -1$ corresponds to standard DPPs. For an α-DPP $\mathbf{Y}$ with kernel K and under some assumptions ensuring

existence, (2.11) becomes

$$\mathrm{E}\left(\prod_{v\in\mathbf{Y}} g(v)\right) = \prod_{i\geq 1}(1+\alpha\tilde{\lambda}_i)^{-1/\alpha} \tag{3.10}$$

where the $\tilde{\lambda}_i$'s are the same as in Proposition 2.2. In this case (3.5) remains valid and if, just as in Section 3.2.1, we choose N non-vanishing eigenvalues $\tilde{\lambda}_i$ that take the same value, we get the approximation

$$\log\lambda_{\mathrm{DPP}} = \log\beta - \frac{N}{\alpha}\log\left(1+\alpha\frac{\lambda_{\mathrm{DPP}}G}{N}\right). \tag{3.11}$$

We then obtain the approximation (3.8) for the choice $N/\alpha = -(1+\lambda_{\mathrm{DPP}}G/\kappa)$. This remark shows that our approximation can be viewed as an approximation by an α-DPP, for a specific value of α, which is adapted to both inhibitive and attractive interactions. Since DPPs are more familiar models than α-DPPs and the theory is more comprehensive for DPPs than for α-DPPs, we still prefer to take a DPP point of view.

3.2.4. Theoretical properties

The existence and uniqueness of a solution λ_{DPP} to the DPP approximation (3.4) are established in Theorem 3.1. This is to be compared with Baddeley and Nair (2012, Theorem 2), see also the end of Section 3.1.

Theorem 3.1. *Consider a stationary pairwise interaction process in $\mathbb{R}^d$ with Papangelou conditional intensity given by (2.6), with finite range R and a possible hard-core distance $0 \leq \delta \leq R$.*

- *If $\int(1-g) \geq |B_\delta|$ then λ_{DPP} exists uniquely, is an increasing function of β and is such that $\lambda_{\mathrm{DPP}} \leq \lambda_{\mathrm{PS}}$.*
- *If $\int(1-g) < |B_\delta|$ and $\int(1-g)^2 < |B_R|$, λ_{DPP} exists but is not necessarily unique.*
- *If $\int(1-g) < |B_\delta|$ and $\int(1-g)^2 \geq |B_R|$, λ_{DPP} does not necessarily exist.*

Proof. Let $f_{\mathrm{PS}}(\lambda) = \log\beta - \lambda G$ and

$$f_{\mathrm{DPP}}(\lambda) = \log\beta + (1+\lambda_{\mathrm{DPP}}|B_\delta|)\log\left(1-\frac{\lambda_{\mathrm{DPP}}|B_\delta|}{1+\lambda_{\mathrm{DPP}}|B_\delta|}\right) + (1+\lambda_{\mathrm{DPP}}(G-|B_\delta|)/\kappa)\log\left(1-\frac{\lambda_{\mathrm{DPP}}(G-|B_\delta|)}{1+\lambda_{\mathrm{DPP}}(G-|B_\delta|)/\kappa}\right),$$

where κ is given by (1.2). The approximations λ_{PS} and λ_{DPP} are defined by the fixed point equations $\log\lambda_{\mathrm{PS}} = f_{\mathrm{PS}}(\lambda_{\mathrm{PS}})$ and $\log\lambda_{\mathrm{DPP}} = f_{\mathrm{DPP}}(\lambda_{\mathrm{DPP}})$.

If $G \geq |B_\delta|$, it can be verified that f_{DPP} is a concave decreasing function from $[0,\infty[$ into $[\log(\beta),-\infty[$. So there exists a unique solution to $\log\lambda_{\mathrm{DPP}} = f_{\mathrm{DPP}}(\lambda_{\mathrm{DPP}})$. Moreover, concavity of f_{DPP} implies that f'_{DPP} is decreasing and we have $f'_{\mathrm{DPP}}(0) = -G$. This shows that $f_{\mathrm{DPP}} - f_{\mathrm{PS}}$ is also decreasing on $[0,\infty[$,

whereby $f_{\mathrm{DPP}} - f_{\mathrm{PS}} \leq f_{\mathrm{DPP}}(0) - f_{\mathrm{PS}}(0) = 0$ on $[0, \infty[$. This implies that $\lambda_{\mathrm{DPP}} \leq \lambda_{\mathrm{PS}}$ when $G \geq |B_\delta|$. The fact that f_{DPP} is increasing in β shows moreover that λ_{DPP} is an increasing function of β.

If $G < |B_\delta|$ and $\int (1-g)^2 < |B_R|$, implying $\kappa < 1$, f_{DPP} is only defined on $[0, c] \cup]c/(1-\kappa), \infty[$ where $c = \kappa/(|B_\delta| - G)$. On the interval $]c/(1-\kappa), \infty[$, f_{DPP} is continuous with $\lim_{\lambda \to c/(1-\kappa)} f_{\mathrm{DPP}}(\lambda) = \infty$ and $\lim_{\lambda \to \infty} f_{\mathrm{DPP}}(\lambda) = -\infty$. This shows that there is at least one solution to $\log \lambda_{\mathrm{DPP}} = f_{\mathrm{DPP}}(\lambda_{\mathrm{DPP}})$ on $]c/(1-\kappa), \infty[$. However, this solution is not unique in general: If $c > 1$ and $0 < \log(\beta) < (1 + |B_\delta| c) \log(1 + |B_\delta| c)$, then f_{DPP} is continuous on $[0, c]$ with $f_{\mathrm{DPP}}(0) > 0$ and $f_{\mathrm{DPP}}(c) < 0$, so there is another solution to $\log \lambda_{\mathrm{DPP}} = f_{\mathrm{DPP}}(\lambda_{\mathrm{DPP}})$ on $[0, c]$ in this case.

If $G < |B_\delta|$ and $\int (1-g)^2 \geq |B_R|$, implying $\kappa \geq 1$, f_{DPP} is only defined on $[0, c]$. Since f_{DPP} is bounded on this interval and the bounds are increasing functions of β, no solution to $\log \lambda_{\mathrm{DPP}} = f_{\mathrm{DPP}}(\lambda_{\mathrm{DPP}})$ can be guaranteed in general. □

4. Numerical study

In this section, we focus on the planar case to investigate the performances of the DPP approximation and compare it with the initial one proposed by Baddeley and Nair (2012). All computations were performed in the `R` language (R development core team, 2011). The Poisson-saddlepoint approximation as well as the DPP approximation are implemented using root-finding algorithms and in particular we use the `R` function `uniroot` for this task. From a computational point of view, both approximations are quickly and easily obtained.

We have considered 14 different numerical experiments involving Strauss models (S), Strauss hard-core models (SHC), Diggle-Graton models (DG), piecewise Strauss models (PS) and piecewise Strauss hard-core (PSHC) models. The pairwise interaction functions of these models are detailed in Section 2.2. To sum up here are the parameters, that include a continuously varying parameter $\gamma_1 \in [0, 1]$:

- Strauss (S): $\beta = 100$ with $R = 0.05$ or 0.1; $\beta = 50$ with $R = 0.1$ or 0.15; $\beta = 200$ with $R = 0.05, 0.1$. For all these models $\gamma = \gamma_1$.
- Strauss hard-core (SHC): $\beta = 200$, $(\delta, R) \in \Delta$ where Δ is the set $\{(.025, .05), (.025, .1), (.05, .1), (.05, .15)\}$. For this model $\gamma = \gamma_1$.
- Diggle-Graton (DG): $\beta = 200$, $R = 0.025, 0.05$ or 0.075 and $\beta = 50$ and $R = 0.15$. For all these models $\gamma = \gamma_1$.
- Piecewise Strauss and Strauss hard-core (PS and PSHC): $\beta = 200$, $\delta = 0$ or 0.025, $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_2 = 0$ or 0.5. The vector of breaks is $R = (0.05, 0.1)$.

For all these models, $G > |B_\delta|$ for any $\gamma_1 \in [0, 1]$. Hence according to Baddeley and Nair (2012, Theorem 2) and Theorem 3.1, λ_{PS} and λ_{DPP} exist and are unique.

We also investigate the case $G < 0$ with a Strauss hard-core process with $\gamma_1 \in [1, 1.3]$. In particular, we consider the parameters:

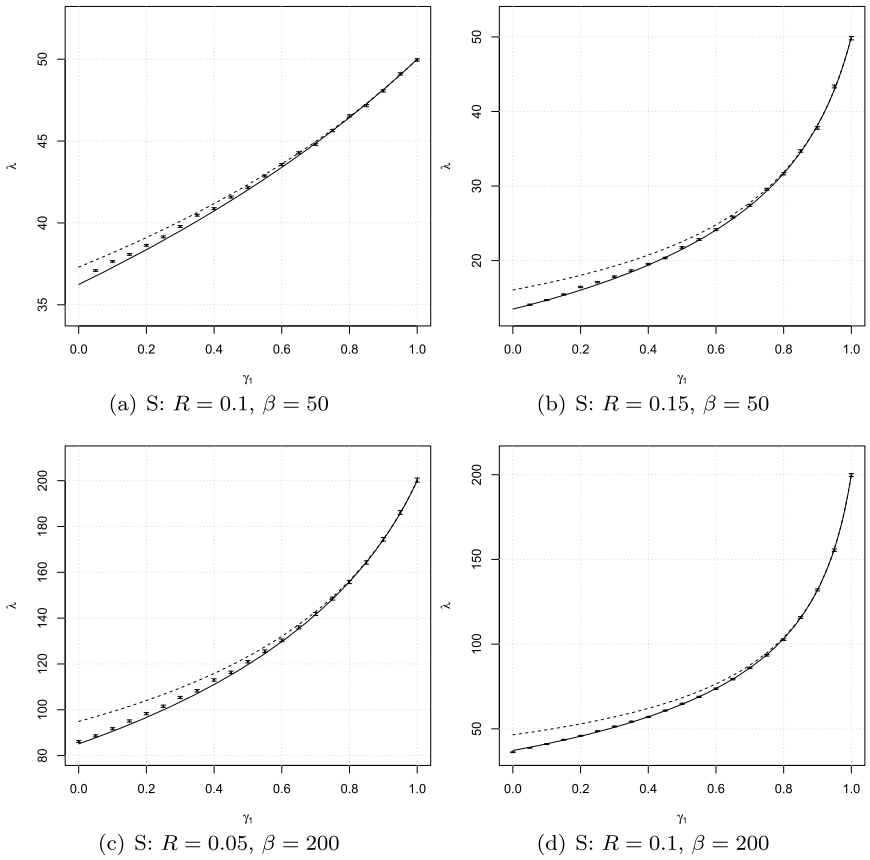
- Attractive Strauss hard-core (ASHC): $\beta = 200$, $(\delta, R) \in \Delta$.

In this setting, we look for a solution for the intensity λ in the interval [80, 100000], 80 being approximately the solution found by both approximations when $\gamma_1 = 1$. According to our remark in Section 3.1, we observe that λ_{PS} exists and is unique for any $\gamma_1 \in [1, 1.3]$ when $(\delta, R) = (.025, .05)$ or $(.05, .1)$. When $(\delta, R) = (.025, .1)$ (resp. when $(\delta, R) = (.05, .15)$), the solution exists and is unique only when $\gamma_1 \leq 1.067$ approximatively (resp. $\gamma_1 \leq 1.124$). Moreover, there is no solution when $\gamma_1 > 1 + (|B_\delta| + e^{-1}/\beta)/(|B_R| - |B_\delta|)$, i.e. 1.129 (resp. 1.154) and when $1.067 < \gamma_1 < 1.129$ (resp. $1.124 < \gamma_1 < 1.154$), two solutions occur. On the other hand, for this model, the DPP approximation λ_{DPP} exists and is unique for any $\gamma_1 \in [1, 1.3]$ and any $(\delta, R) \in \Delta$.

For each numerical experiment, we therefore obtain curves of intensity approximation in terms of γ_1. For γ_1 varying from 0 to 1 (or from 1 to 1.3 for the ASHC model) by step of 0.05 (the value 0 is exluded for DG models to save time), the true intensity λ is estimated by Monte-Carlo methods. For each set of parameters m realizations of the model are generated on the square $[-2R, 1 + 2R]^2$ and then clipped to the unit square. That strategy is detailed and justified by Baddeley and Nair (2012). Specifically, the number of points in each realization is averaged to obtain the estimated intensity and its standard error. The simulation results for the Strauss models with $\beta = 50$ or 100 were obtained by Baddeley and Nair (2012), where $m = 10000$ realizations were generated and the exact simulation algorithm was used, implemented in the `R` function `rStrauss` of the `spatstat` package (see Baddeley et al. (2015)). For the Strauss models with $\beta = 200$, SHC models, PS and PSHC models and ASHC models, we generate $m = 1000$ replications and use the `rmh` function in the `spatstat` package which implements a Metropolis-Hastings algorithm. Even if we use 10^6 iterations of the algorithm, the results may be slightly biased. For the DG models, the `R` package `spatstat` provides an exact simulation algorithm (function `rDiggleGraton`) and for such models we generate 10000 replications when $\beta = 200$ and $R = 0.025, 0.05$ and when $\beta = 50$ and $R = 0.15$. We used 1000 replications when $\beta = 200$ and $R = 0.075$ to save time.

All results can be found in Figures 3–7. Plots provide the same information: we depict intensity approximation λ (when they exist) based on different methods in terms of γ_1. The dashed curve represents the Poisson-saddlepoint approximation proposed by Baddeley and Nair (2012) and detailed in Section 3.1. The solid curve is the DPP approximation we propose in this paper and is given by (1.1).

Let us first comment Figures 3-4 dealing with Strauss models. As expected the Poisson-saddlepoint approximation is not efficient when γ_1 is small, i.e. for very repulsive models. This is very significant in particular for the Strauss hard-core model, see Figure 4. The DPP aproximation we propose is more likely able to capture the repulsiveness of the Strauss models. Figure 5 also clearly shows that our approximation is particulalry efficient and outperforms unambiguously the Poisson-saddlepoint approximation. Note that replications for the Diggle-Graton models are generated using an exact algorithm; so the numerical results seem to be exact, except the slight bias induced by clipping the pattern from $[-2R, 1 + 2R]^2$ to the unit square.



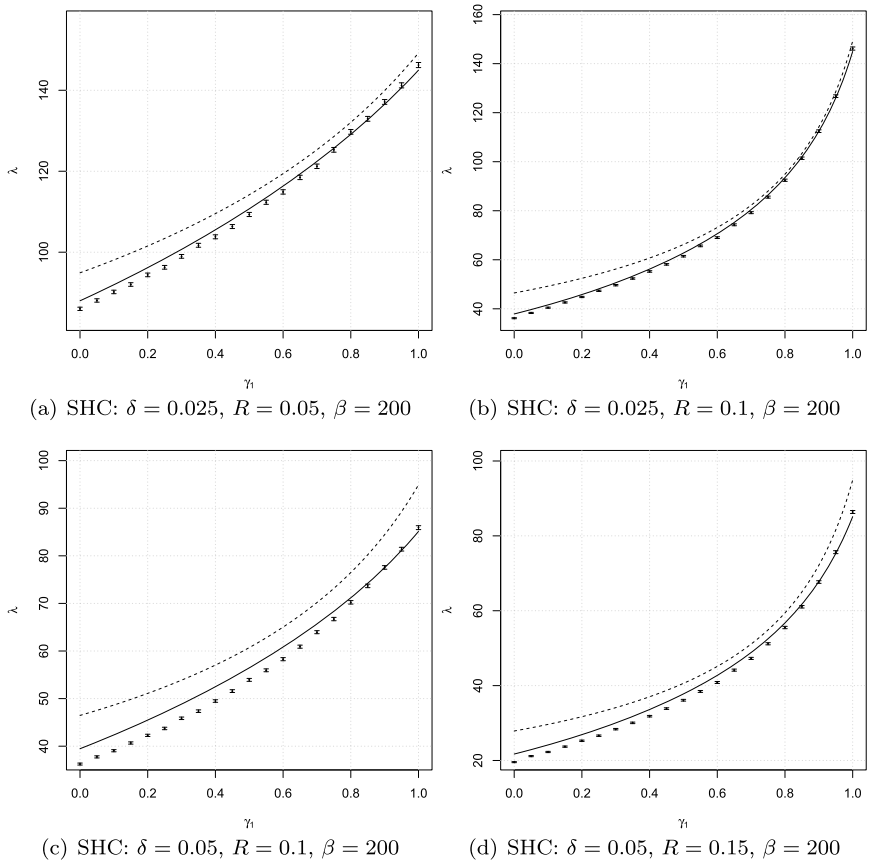
(a) S: $R = 0.1$, $\beta = 50$ (b) S: $R = 0.15$, $\beta = 50$

(c) S: $R = 0.05$, $\beta = 200$ (d) S: $R = 0.1$, $\beta = 200$

FIG 3. *Comparison of the exact intensity (small boxplots obtained by Monte-Carlo method), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for homogeneous Strauss models with activity parameter β, range of interaction R. Curves and boxplots are reported in terms of the interaction parameter $\gamma_1 \in [0, 1]$.*

Let us now comment Figure 6. When $\gamma_2 = 0.5$, i.e. Figures 6 (a)–(b), the results are very satisfactory. Our approximation is able to approximate λ very efficiently for any value of γ_1. For Figures 6 (c)–(d), $\gamma_2 = 0$ which means that points within a distance comprised between 0.05 and 0.1 are forbidden. Such a parameterization tends to create repulsive clusters. When $\gamma_1 = 1$ and $\delta = 0$, such a piecewise Strauss model was called annulus model by Stucki and Schuhmacher (2014). This model demonstrates the limitations of our approximation even if when γ_1 is close to zero which means that the model is close to a hard-core process with radius 0.1 our approximation remains satisfactory.

Figure 7 investigates the situation $G < 0$. Figures 7 (a) and (c) are related to a relative attractive situation. In these situations, both approximations exist; clearly λ_{DPP} outperforms the Poisson-saddlepoint approximation and is quite efficient. In the two other situations (Figures 7 (b) and (d)) for which



(a) SHC: $\delta = 0.025$, $R = 0.05$, $\beta = 200$ (b) SHC: $\delta = 0.025$, $R = 0.1$, $\beta = 200$

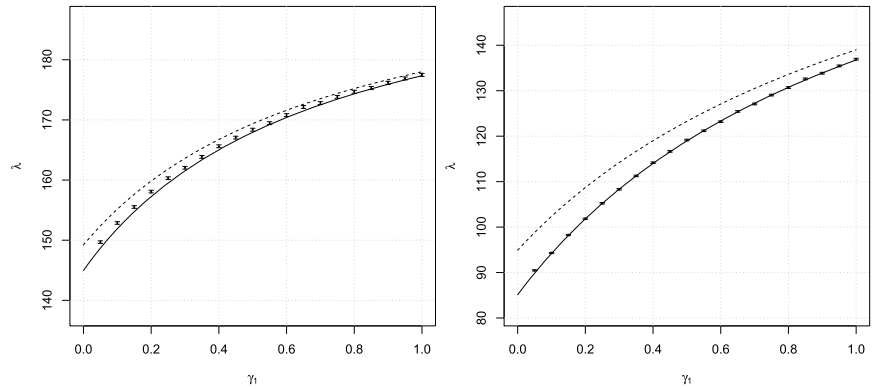
(c) SHC: $\delta = 0.05$, $R = 0.1$, $\beta = 200$ (d) SHC: $\delta = 0.05$, $R = 0.15$, $\beta = 200$

FIG 4. *Comparison of the exact intensity (small boxplots obtained by Monte-Carlo method), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for homogeneous Strauss hard-core models with activity parameter β, range of interaction R and hard-core distance δ. Curves and boxplots are reported in terms of the interaction parameter $\gamma_1 \in [0, 1]$.*

the model is very attractive, the approximations show some limitations. The Poisson-saddlepoint approximation does not always exist whereas λ_{DPP} always exists but reaches very high values when γ_1 is close to 1.3. Nevertheless when $\gamma_1 \leq 1.15$, the DPP approximation remains efficient.

5. Conclusion

Gibbs point processes constitute a very flexible class of spatial point processes models for applications. However, one of their main drawback is that moments are not expressible in a closed form, even the simplest one which is the intensity parameter. So for instance, when generating a Strauss model with parameters



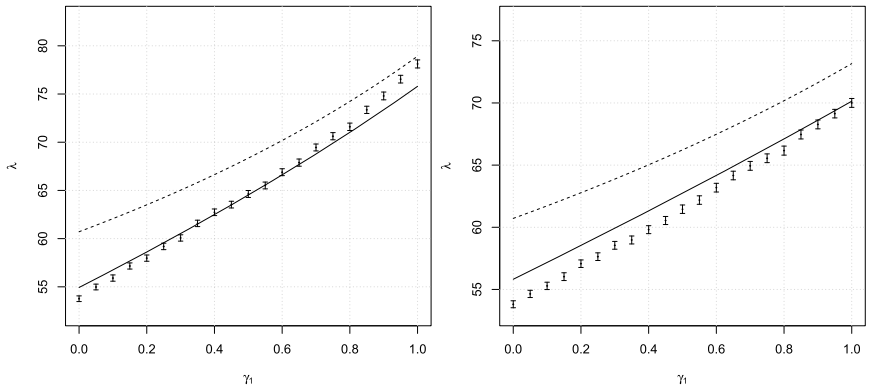
(a) DG: $R = 0.025$, $\beta = 200$ (b) DG: $R = 0.05$, $\beta = 200$

(c) DG: $R = 0.075$, $\beta = 200$ (d) DG: $R = 0.15$, $\beta = 50$

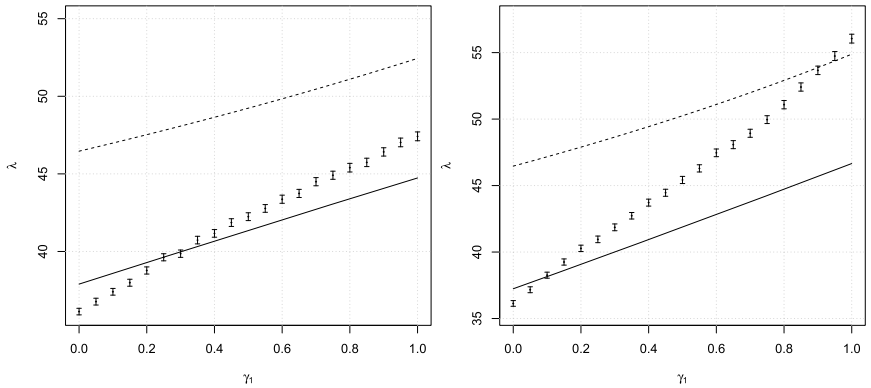
FIG 5. *Comparison of the exact intensity (small boxplots obtained by Monte-Carlo method), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for Diggle-Graton models. Curves and boxplots are reported in terms of the interaction parameter* $\gamma_1 \in [0, 1]$.

β, γ and R, it is impossible to guess how many points the realization will have in average before running it. In particular, for the generation of a Strauss model with parameters γ and R fixed, how should we adjust the parameter β to get in average n number of points, without running many simulations? This paper offers an accurate solution. We have followed the initial idea by Baddeley and Nair (2012) who proposed a Poisson-saddlepoint approximation and suggest to substitute the Poisson point process with a well-chosen determinantal point process. The approximation we propose is simple, fast and more accurate than the Poisson-saddlepoint approximation, especially when the underlying pairwise interaction exhibits a strong repulsion.

It would be tempting to extend the methodology proposed in this paper to approximate higher order moments, in order to get approximations for the pair correlation function for instance, or to approximate the intensity function of an



(a) PS: $R = (0.05, 0.1)$, $\beta = 200$, $\gamma_2 = 0.5$ (b) PSHC: $\delta = 0.025$, $R = (0.05, 0.1)$, $\beta = 200$, $\gamma_2 = 0.5$

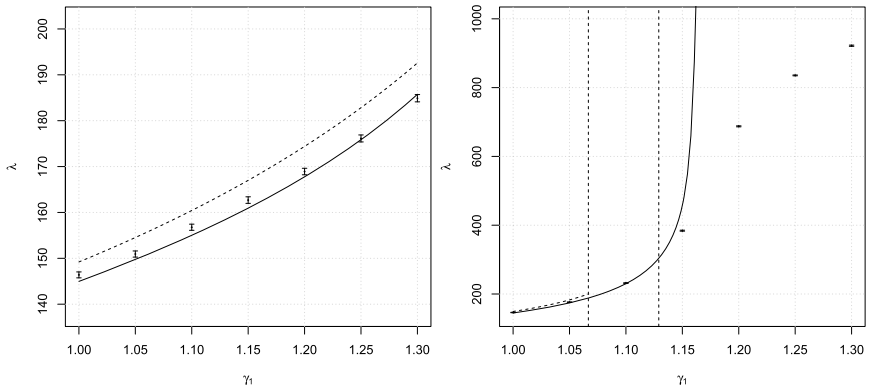


(c) PS: $R = (0.05, 0.1)$, $\beta = 200$, $\gamma_2 = 0$ (d) PSHC: $\delta = 0.025$, $R = (0.05, 0.1)$, $\beta = 200$, $\gamma_2 = 0$

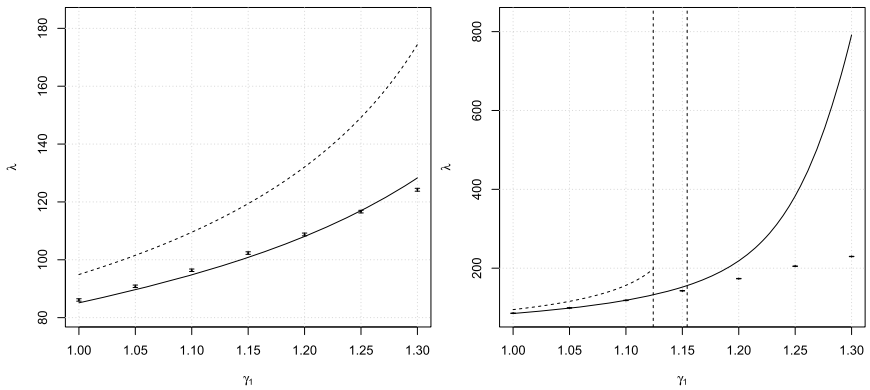
FIG 6. *Comparison of the exact intensity (small boxplots obtained by Monte-Carlo method), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for piecewise Strauss and piecewise Strauss hard-core models. Curves and boxplots are reported in terms of the (remaining) interaction parameter* $\gamma_1 \in [0, 1]$.

inhomogeneous pairwise interaction point process. Our first attempts convinced us that both from a theoretical and practical points of view, the extension is not straightforward.

This paper reveals also an open question. Is it possible to give some theoretical guarantees for the approximation λ_{DPP} we propose, like bounds for $|\lambda_{\text{DPP}} - \lambda|$? The paper by Stucki and Schuhmacher (2014) is an excellent starting point. The authors develop a Stein-type equation and propose bounds for the generating functional for Gibbs point processes. These bounds are applied to derive bounds for the Poisson-saddlepoint approximation λ_{PS}. However, their work cannot be directly applied to the DPP approximation. This definitely constitutes an interesting research perspective.



(a) ASHC: $\delta = 0.025$, $R = 0.05$, $\beta = 200$ (b) ASHC: $\delta = 0.025$, $R = 0.1$, $\beta = 200$



(c) ASHC: $\delta = 0.05$, $R = 0.1$, $\beta = 200$ (d) ASHC: $\delta = 0.05$, $R = 0.15$, $\beta = 200$

FIG 7. *Comparison of the exact intensity (small boxplots obtained by Monte-Carlo method), the Poisson-saddlepoint approximation (dashed line) and the DPP approximation (solid line) for homogeneous Strauss hard-core models with activity parameter β, range of interaction R and hard-core distance δ. Curves and boxplots are reported in terms of the interaction parameter $\gamma_1 \in [1, 1.3]$. For Figures (b) and (d), the Poisson-saddlepoint exists and is unique when $\gamma_1 \leq 1.067$ and $\gamma_1 \leq 1.124$ respectively (corresponding to the most left vertical dashed lines). It does not exist when $\gamma_1 > 1.129$ and 1.154 respectively (corresponding to the most right vertical dashed lines) and in between two solutions occur which are not reported. In all these cases, the DPP approximation exists and is unique. However, for Figure (b), the approximation reaches a value close to 50000 when $\gamma_1 = 1.3$ and the graph is therefore restricted to values $\lambda \leq 1000$.*

Acknowledgements

The authors are sincerely grateful to Adrian Baddeley and Gopalan Nair for sharing the Monte-carlo replications produced in Baddeley and Nair (2012) we used to compare the Poisson-saddlepoint approximation and the DPP approximation (Figure 3 (a), (b) and (d)). The authors thank also the reviewers and associate editor for their suggestions and comments which invited us to propose

a more accurate approximation than the one proposed in a previous version of this mauscript. The research of J-F. Coeurjolly is supported by the Natural Sciences and Engineering Research Council of Canada.

Appendix A: Proof of Proposition 2.2

Proof. Assume first that $g \leq 1$ and note that

$$\mathrm{E}\left(\prod_{v\in\mathbf{X}} g(v)\right) = L_{\mathbf{X}}(-\log g)$$

where $L_{\mathbf{X}}$ denotes the Laplace transform of $\mathbf{X}$. From Theorem 1.2 in Shirai and Takahashi (2003), for any nonnegative measurable function f on S

$$L_{\mathbf{X}}(f) = \mathrm{Det}(I - \tilde{\mathcal{K}})$$

where Det denotes the Fredholm determinant of an operator and $\tilde{\mathcal{K}}$ is the integral operator associated to the kernel

$$\tilde{K}(u,v) = \sqrt{1-\exp(-f(u))}K(u,v)\sqrt{1-\exp(-f(v))}. \tag{A.1}$$

From (a variant of) Lidskii's theorem, see Theorem 3.12.2 in Simon (2015), we know that

$$\mathrm{Det}(I - \tilde{\mathcal{K}}) = \prod_{i\geq 1}(1-\tilde{\lambda}_i) \tag{A.2}$$

where $\tilde{\lambda}_i$ are the eigenvalues of $\tilde{\mathcal{K}}$, that correspond to the eigenvalues of $\tilde{K}$ given by (A.1), since it is a semi-positive definite continuous kernel and Mercer theorem applies. This completes the proof in the case $g \leq 1$.

The extension to any bounded function g is in essence the statement of Theorem 1.5 in Shirai and Takahashi (2003), except that in our setting we do need that $\|g\|_\infty$ is sufficiently small. The main point is to verify that $\mathrm{E}\left(\prod_{v\in\mathbf{X}} g(v)\right)$ is well-defined in this case. For this, we start from the identity

$$\prod_{v\in\mathbf{X}} g(v) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{u_1,\ldots,u_n\in\mathbf{X}}^{\neq} \prod_{i=1}^{n}(1-g(u_i)), \tag{A.3}$$

where the second sum is understood as 0 when $n = 0$, see (4.2) in Shirai and Takahashi (2003). We observe that

$$\mathrm{E}\left|\sum_{u_1,\ldots,u_n\in\mathbf{X}}^{\neq}\prod_{i=1}^{n}(1-g(u_i))\right| \leq \|1-g\|_\infty^n \,\mathrm{E}(N(N-1)\ldots(N-n+1)\mathbf{1}(N\geq n)),$$

where $N = N(S)$ denotes the number of points of $\mathbf{X}$. By Lemma 4.2 in Shirai and Takahashi (2003), there exists $\beta > 0$ such that $P(N = k) < \beta^k/k!$, whence $\mathrm{E}(N(N-1)\ldots(N-n+1)\mathbf{1}(N\geq n)) < \beta^n e^\beta$ and

$$\frac{1}{n!}\,\mathrm{E}\left|\sum_{u_1,\ldots,u_n\in\mathbf{X}}^{\neq}\prod_{i=1}^{n}(1-g(u_i))\right| \leq \frac{1}{n!}\|1-g\|_\infty^n \beta^n e^\beta.$$

This proves that the series whose generic term is the expression on the left hand side above is absolutely convergent and from (A.3), this proves that $\mathrm{E}\left(\prod_{v\in\mathbf{X}} g(v)\right)$ is well-defined. We can now compute this expectation and by definition of $\rho^{(n)}$,

$$\begin{aligned}\mathrm{E}\prod_{v\in\mathbf{X}} g(v) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\mathrm{E}\left(\sum_{u_1,\ldots,u_n\in\mathbf{X}}^{\neq}\prod_{i=1}^{n}(1-g(u_i))\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int\prod_{i=1}^{n}(1-g(u_i))\rho^{(n)}(u_1,\ldots,u_n)\mathrm{d}u_1\ldots\mathrm{d}u_n.\end{aligned}$$

Given (2.9), this last expression is exactly the Fredholm determinant of $(I-\tilde{\mathcal{K}})$, where $\tilde{\mathcal{K}}$ is the integral operator with kernel (2.12), see (2.12) in Shirai and Takahashi (2003). The result follows by Lidskii's theorem, see (A.2). □

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