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The convex hull of a planar random walk: perimeter, diameter, and shape

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Abstract

We study the convex hull of the first n steps of a planar random walk, and present large-n asymptotic results on its perimeter length L_n, diameter D_n, and shape. In the case where the walk has a non-zero mean drift, we show that $L_n/D_n \to 2$ a.s., and give distributional limit theorems and variance asymptotics for D_n, and in the zero-drift case we show that the convex hull is infinitely often arbitrarily well-approximated in shape by any unit-diameter compact convex set containing the origin, and then $\liminf_{n\to\infty} L_n/D_n = 2$ and $\limsup_{n\to\infty} L_n/D_n = \pi$, a.s. Among the tools that we use is a zero-one law for convex hulls of random walks.

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1 Model and main results

1.1 Introduction and notation

The geometry of random walks in Euclidean space is a topic of persistent interest (see e.g. [17]). The *convex hull* of a random walk is a classical geometrical characteristic of the walk [19, 18] that has received renewed attention recently [5, 6, 10, 11, 20, 22, 23, 21, 24]; see [14] for a recent survey, including motivation in terms of modelling the home range of roaming animals. The present paper studies the asymptotic behaviour of the convex hull of a random walk in $\mathbb{R}^2$, focusing on its *shape*, its *perimeter length*, and its *diameter*.

Let $Z, Z_1, Z_2, \ldots$ be i.i.d. random variables in $\mathbb{R}^2$, and let $S_0, S_1, S_2, \ldots$ be the associated random walk, defined by $S_0 := \mathbf{0}$ (the origin in $\mathbb{R}^2$) and $S_n := \sum_{k=1}^n Z_k$ for $n \in \mathbb{N}$. Let $\mathcal{H}_n := \operatorname{hull}\{S_0, S_1, \ldots, S_n\}$, where $\operatorname{hull} A$ denotes the convex hull of $A \subseteq \mathbb{R}^2$. Write L_n for the perimeter length of $\mathcal{H}_n$, and let

$$D_n := \operatorname{diam}\{S_0, S_1, \ldots, S_n\} = \operatorname{diam} \mathcal{H}_n.$$

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(Note that $\mathcal{H}_n$, L_n, D_n are all random variables on the appropriate spaces: see the comments at the start of Section 3.)

A striking early result on L_n is the formula of Spitzer and Widom [19]:

$$\mathbb{E}\, L_n = 2 \sum_{k=1}^{n} \frac{1}{k}\, \mathbb{E}\, \|S_k\|, \tag{1.1}$$

where, and subsequently, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. For $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we set $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$. If $\mathbb{E}\,\|Z\| < \infty$, we write $\mu := \mathbb{E}\, Z$. If $\mathbb{E}(\|Z\|^2) < \infty$, we write $\sigma^2 := \mathbb{E}(\|Z - \mu\|^2)$, and if $\mu \neq \mathbf{0}$ we write

$$\sigma_\mu^2 := \mathbb{E}[((Z-\mu)\cdot\hat{\mu})^2], \text{ and } \sigma_{\mu_\perp}^2 := \sigma^2 - \sigma_\mu^2.$$

The results in this paper concern the asymptotics of L_n, D_n, and the shape of $\mathcal{H}_n$. The cases where $\mu = \mathbf{0}$ and $\mu \neq \mathbf{0}$ are, as is no surprise, quite different. When $\mu \neq \mathbf{0}$, law of large numbers behaviour dominates, and the n-step walk scales linearly in n in the direction of the mean. Of interest are the fluctuations around this behaviour. When $\mu = \mathbf{0}$ and $\|Z\|$ has two moments, the walk scales as $n^{1/2}$ in all directions.

Rough information about the shape of $\mathcal{H}_n$ is given by the ratio L_n/D_n; provided that $\mathbb{P}(Z = \mathbf{0}) < 1$, convexity implies that a.s., for all but finitely many n,

$$2 \leq L_n/D_n \leq \pi, \tag{1.2}$$

the extrema being the line segment and shapes of constant width (such as the disc).

1.2 Laws of large numbers

We present first laws of large numbers for L_n (extending a result of [18]) and D_n. The intuition behind these results is that, on the law of large numbers scale, the convex hull of the walk exhibits the properties of the line segment from the origin to μ. Our first result is the following law of large numbers for L_n.

Theorem 1.1. *Suppose that $\mathbb{E}\,\|Z\| < \infty$. Then*

$$\lim_{n\to\infty} n^{-1} L_n = 2\|\mu\|, \text{ a.s. and in } L^1.$$

On the other hand, if $\mathbb{E}\,\|Z\| = \infty$ then $\limsup_{n\to\infty} n^{-1} L_n = \infty$, a.s.

Remark 1.2. Snyder and Steele [18] obtained the almost-sure convergence result under the stronger condition $\mathbb{E}(\|Z\|^2) < \infty$ as a consequence of an upper bound on $\mathbb{V}\mathrm{ar}\, L_n$ deduced from Steele's version of the Efron–Stein inequality. (Snyder and Steele state their result for the case $\mu \neq \mathbf{0}$, but their proof works equally well when $\mu = \mathbf{0}$.)

Similarly, we have a law of large numbers for D_n.

Theorem 1.3. *Suppose that $\mathbb{E}\,\|Z\| < \infty$. Then*

$$\lim_{n\to\infty} n^{-1} D_n = \|\mu\|, \text{ a.s. and in } L^1.$$

On the other hand, if $\mathbb{E}\,\|Z\| = \infty$ then $\limsup_{n\to\infty} n^{-1} D_n = \infty$, a.s.

In the case $\mu \neq \mathbf{0}$, Theorems 1.1 and 1.3 have the following immediate consequence, which is also consistent with the intuition that the behaviour in the case with $\mu \neq \mathbf{0}$ is governed by that of the line segment from $\mathbf{0}$ to μ, whose perimeter length is twice its diameter.

Corollary 1.4. *Suppose that $\mathbb{E}\,\|Z\| < \infty$ and that $\mu \neq \mathbf{0}$. Then*

$$\lim_{n\to\infty} L_n/D_n = 2, \text{ a.s.}$$

1.3 Zero-drift case

In the zero-drift case, we need the extra condition $\mathbb{E}(\|Z\|^2) < \infty$. Let $\Sigma := \mathbb{E}(ZZ^\top)$, viewing Z as a column vector; note that if $\mu = \mathbf{0}$ then $\operatorname{tr} \Sigma = \sigma^2$. Let ρ_H denote the Hausdorff distance between non-empty compact sets; see (3.1) below for a definition. Our first result concerns the sequence of the convex hulls and their shape, indicating that, if rescaled by diameter, the convex hulls will infinitely often be arbitrarily close to any unit-diameter compact convex set in the sense of Hausdorff distance. In other words, the set of limit points of the sequence of (normalized) convex hulls contains all (normalized) compact convex sets. A trivial consequence of this result is that, in contrast to the case with drift, there is no almost-sure limit for the shape of the convex hull.

Theorem 1.5. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$, Σ is positive definite, and $\mu = \mathbf{0}$. Then, for any compact convex set $K \subset \mathbb{R}^2$ with $\operatorname{diam} K = 1$,*

$$\liminf_{n\to\infty} \rho_H(D_n^{-1}\mathcal{H}_n, K) = 0, \text{ a.s.}$$

Note that under the hypotheses of Theorem 1.5, $\mathbb{P}(Z = \mathbf{0}) < 1$, so that $D_n > 0$ for all but finitely many n, a.s. A consequence of Theorem 1.5 is that the rescaled convex hull will both be close to a unit-length line segment infinitely often and be close to a unit-diameter disc infinitely often; this is the intuition behind the following result, which should be contrasted with Corollary 1.4.

Corollary 1.6. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$, Σ is positive definite, and $\mu = \mathbf{0}$. Then,*

$$2 = \liminf_{n\to\infty} \frac{L_n}{D_n} < \limsup_{n\to\infty} \frac{L_n}{D_n} = \pi, \text{ a.s.}$$

1.4 Case with drift

Now we turn to second-order asymptotics for L_n and D_n in the case with non-zero drift. The behaviour of L_n was studied in [22], where it was shown (Theorem 1.3 of [22]) that if $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$, then, as $n \to \infty$,

$$n^{-1/2}|L_n - \mathbb{E}\, L_n - 2(S_n - \mathbb{E}\, S_n) \cdot \hat{\mu}| \to 0, \text{ in } L^2. \tag{1.3}$$

This result shows that $L_n - \mathbb{E}\, L_n$ is well-approximated by a sum of i.i.d. random variables, which, as shown in [22], implies variance asymptotics for L_n as well as a central limit theorem when $\sigma_\mu^2 > 0$. We show that (1.3) may be recast in the following stronger form, which reinforces the intuition that the perimeter length is well approximated by the perimeter length of the line segment connecting the origin to $S_n \cdot \hat{\mu}$, the end point of the walk projected onto the unit vector in the direction of the mean.

Theorem 1.7. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. Then, as $n \to \infty$,*

$$n^{-1/2}|L_n - 2S_n \cdot \hat{\mu}| \to 0, \text{ in } L^2.$$

It follows from the Spitzer–Widom formula (1.1) (or Theorem 1.1) that $\mathbb{E}\, L_n \sim 2\|\mu\| n$. The following asymptotic expansion of $\mathbb{E}\, L_n$ tells us that in the case with two moments and a non-zero drift, the second term in the expansion is of order $\log n$. Theorem 1.8 is the key additional component in the proof of Theorem 1.7; its proof uses (1.1).

Theorem 1.8. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. Then, as $n \to \infty$,*

$$\mathbb{E}\, L_n = 2\|\mu\| n + \left(\frac{\sigma_{\mu_\perp}^2}{\|\mu\|} + o(1) \right) \log n.$$

Such asymptotic expansions are of interest in their own right; analogous results for $\mathbb{E} L_n$ in the case $\mu = \mathbf{0}$ have been presented recently in [7]. For the diameter D_n, we have the following analogue of Theorem 1.7 which likewise reinforces the notion that the diameter is well approximated by the length of the line segment from the origin to $S_n \cdot \hat{\mu}$.

Theorem 1.9. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. Then, as $n \to \infty$,*

$$n^{-1/2} |D_n - S_n \cdot \hat{\mu}| \to 0, \text{ in } L^2.$$

Denote by $\mathcal{N}(0,1)$ the standard normal distribution, and by $\xrightarrow{\mathrm{d}}$ convergence in distribution. Theorem 1.9 yields variance asymptotics and a central limit theorem when $\sigma_\mu^2 > 0$, as follows.

Corollary 1.10. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. Then $\lim_{n\to\infty} n^{-1} \mathbb{V}\mathrm{ar}\, D_n = \sigma_\mu^2$. Moreover, if $\sigma_\mu^2 > 0$, for $\zeta \sim \mathcal{N}(0,1)$, as $n \to \infty$,*

$$\frac{D_n - \mathbb{E} D_n}{\sqrt{\mathbb{V}\mathrm{ar}\, D_n}} \xrightarrow{\mathrm{d}} \zeta, \text{ and } \frac{D_n - n\|\mu\|}{\sqrt{n\sigma_\mu^2}} \xrightarrow{\mathrm{d}} \zeta.$$

The degenerate case $\sigma_\mu^2 = 0$ corresponds to the case where $Z \cdot \hat{\mu} = \|\mu\|$ a.s., and is of its own interest. It includes, for example, the case where $Z = (1,1)$ or $(1,-1)$, each with probability $1/2$, in which the two-dimensional walk S_n corresponds to the space-time diagram of a one-dimensional simple symmetric random walk. In the case $\sigma_\mu^2 = 0$, Corollary 1.10 says only that $\mathbb{V}\mathrm{ar}\, D_n = o(n)$. We prove the following sharper result which shows that the variance in fact converges to a constant, only depending on the distribution of Z, and the errors $D_n - \|\mu\| n$ converge to a rescaled square of a normal distribution. Note that the restriction $\sigma_\mu^2 = 0$ implies $D_n \geq \|\mu\| n$, which is why the limit distribution is supported on $[0, \infty)$. The result requires some additional conditions.

Theorem 1.11. *Suppose that $\mathbb{E}(\|Z\|^p) < \infty$ for some $p > 2$, $\mu \neq \mathbf{0}$, and $\sigma_\mu^2 = 0$. Then,*

$$D_n - \|\mu\| n \xrightarrow{\mathrm{d}} \frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|}, \tag{1.4}$$

where $\zeta \sim \mathcal{N}(0,1)$. Further, if, in addition, $\mathbb{E}(\|Z\|^p) < \infty$ for some $p > 4$, then

$$\lim_{n\to\infty} \mathbb{V}\mathrm{ar}\, D_n = \frac{\sigma_{\mu_\perp}^4}{2\|\mu\|^2}. \tag{1.5}$$

Remark 1.12. (i) The higher moments conditions required in Theorem 1.11 are necessary for the proofs that we employ; see also Remark A.3 below.

(ii) The statement (1.4) may be written as

$$D_n - S_n \cdot \hat{\mu} \xrightarrow{\mathrm{d}} \frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|}. \tag{1.6}$$

It is natural to ask whether (1.6) also holds in the case where $\sigma_\mu^2 > 0$; if it did, then it would provide an alternative proof of the central limit theorem in Corollary 1.10. Simulations suggest that when $\sigma_\mu^2 > 0$, equation (1.6) holds in some, but not all cases.

1.5 Open problems, extensions and paper outline

When $\mathbb{E}(\|Z\|^2) < \infty$, $\mu \neq \mathbf{0}$, and $\sigma_\mu^2 = 0$, Theorem 1.7 (see also Theorem 1 in [22]) shows that $\mathbb{V}\mathrm{ar}\, L_n = o(n)$. It was conjectured in [22] that $\mathbb{V}\mathrm{ar}\, L_n = O(\log n)$ in this case, which is the subject of ongoing work. We make the following stronger conjecture.

Conjecture 1.13. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$, $\mu \neq \mathbf{0}$, $\sigma_\mu^2 = 0$, *and* $\sigma_{\mu_\perp}^2 > 0$. *Then*

$$\lim_{n\to\infty} \frac{\mathbb{V}\mathrm{ar}\, L_n}{\log n} \text{ exists in } (0, \infty).$$

Another direction in which this work may be extended is to higher dimensions. An alternative approach to the almost-sure laws of large numbers in Theorems 1.1 and 1.3 is via the functional strong law and the result, valid in any dimension d, that $n^{-1}\mathcal{H}_n \to s_\mu$ a.s. as convex compact sets containing the origin, where s_μ is the line segment from $\mathbf{0}$ to μ: see [13] for details. Thus Theorem 1.3 extends to any dimension [13]. For $d \geq 3$ however, s_μ has no non-trivial intrinsic volumes, so there is no non-trivial analogue of Theorem 1.1.

It seems likely that, in the case with drift, Theorem 1.9, Corollary 1.10 and Theorem 1.11 could be extended to higher dimensions using the methods of the present paper. In the zero-drift case, it seems likely that the shape theorem, Theorem 1.5, could be extended to higher dimensions. In $d \geq 3$, the zero-drift case is the most interesting for other intrinsic volumes, such as the $(d-1)$-dimensional surface area; this is addressed in part in [13].

The outline of the remainder of the paper is as follows. In Section 2 we give the proofs of the laws of large numbers Theorems 1.1 and 1.3. In Section 3 we present a zero-one law for the convex hull of random walk (Theorem 3.1), which we then use to prove Theorem 1.5 and Corollary 1.6. Section 4 presents the proofs of Theorems 1.7 and 1.8. Sections 5 and 6 give the proofs of Theorems 1.9 and 1.11 respectively. Finally, rather than interrupting the flow of the main arguments, we present in Appendix A a couple of auxiliary technical results.

2 Laws of large numbers

Throughout we use the notation $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$ for the unit vector in direction θ. We recall (see e.g. equation (2.1) of [18]) that *Cauchy's formula* states that for a finite point set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbb{R}^2$, the perimeter length of $\mathrm{hull}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is given by

$$\int_0^{2\pi} \max_{0\leq k\leq n} (\mathbf{x}_k \cdot \mathbf{e}_\theta) \mathrm{d}\theta.$$

Proof of Theorem 1.1. Cauchy's formula applied to our random walk implies that

$$L_n = \int_0^{2\pi} \max_{0\leq k\leq n} (S_k \cdot \mathbf{e}_\theta) \mathrm{d}\theta. \tag{2.1}$$

First suppose that $\mathbb{E}\,\|Z\| < \infty$. Then the strong law of large numbers says that for any $\varepsilon > 0$ there exists a random variable N_ε, measurable with respect to $Z_1, Z_2, \ldots$, with $\mathbb{P}(N_\varepsilon < \infty) = 1$, such that

$$\|S_n - n\mu\| < n\varepsilon, \text{ for all } n \geq N_\varepsilon. \tag{2.2}$$

Since $S_0 = \mathbf{0}$, taking $k = 0$ and $k = n$ in (2.1) and writing $x^+ := x\mathbb{1}\{x > 0\}$, we have

$$L_n \geq \int_0^{2\pi} (S_n \cdot \mathbf{e}_\theta)^+ \mathrm{d}\theta = 2\|S_n\|, \tag{2.3}$$

by Cauchy's formula for $\mathrm{hull}\{\mathbf{0}, S_n\}$. For $n \geq N_\varepsilon$ we have from (2.2) that

$$\|S_n\| \geq \|n\mu\| - \|S_n - n\mu\| \geq n\|\mu\| - n\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\liminf_{n\to\infty} n^{-1} L_n \geq 2\|\mu\|$, a.s.

On the other hand, for any $\varepsilon > 0$, we have from (2.2) that

$$\begin{aligned}\max_{0\le k\le n}(S_k\cdot\mathbf{e}_\theta) &\le \max_{0\le k\le N_\varepsilon}(S_k\cdot\mathbf{e}_\theta)+\max_{N_\varepsilon\le k\le n}(S_k\cdot\mathbf{e}_\theta)\\ &\le \max_{0\le k\le N_\varepsilon}\|S_k\|+\max_{0\le k\le n}(k(\mu\cdot\mathbf{e}_\theta+\varepsilon))\\ &= \max_{0\le k\le N_\varepsilon}\|S_k\|+n(\mu\cdot\mathbf{e}_\theta+\varepsilon)^+.\end{aligned}$$

Let $A_\varepsilon := \{\theta\in[0,2\pi] : \mu\cdot\mathbf{e}_\theta > -\varepsilon\}$. Then

$$\int_0^{2\pi}(\mu\cdot\mathbf{e}_\theta+\varepsilon)^+\mathrm{d}\theta=\int_{A_\varepsilon}(\mu\cdot\mathbf{e}_\theta+\varepsilon)\mathrm{d}\theta\le\int_{A_\varepsilon}\mu\cdot\mathbf{e}_\theta\mathrm{d}\theta+2\pi\varepsilon.$$

But

$$\begin{aligned}\int_{A_\varepsilon}\mu\cdot\mathbf{e}_\theta\mathrm{d}\theta &= \int_{A_0}\mu\cdot\mathbf{e}_\theta\mathrm{d}\theta+\int_{A_\varepsilon\setminus A_0}\mu\cdot\mathbf{e}_\theta\mathrm{d}\theta\\ &\le \int_0^{2\pi}(\mu\cdot\mathbf{e}_\theta)^+\mathrm{d}\theta+\|\mu\||A_\varepsilon\setminus A_0|.\end{aligned}$$

Hence, from (2.1) we obtain

$$L_n\le 2\pi\max_{0\le k\le N_\varepsilon}\|S_k\|+n\int_0^{2\pi}(\mu\cdot\mathbf{e}_\theta)^+\mathrm{d}\theta+2\pi n\varepsilon+n\|\mu\||A_\varepsilon\setminus A_0|.$$

Since $\mathbb{P}(N_\varepsilon<\infty)=1$, it follows from Cauchy's formula for $\mathrm{hull}\{\mathbf{0},\mu\}$ that, a.s.,

$$\limsup_{n\to\infty}n^{-1}L_n\le 2\|\mu\|+2\pi\varepsilon+\|\mu\||A_\varepsilon\setminus A_0|.$$

Since $\varepsilon>0$ was arbitrary, and $|A_\varepsilon\setminus A_0|\to 0$ as $\varepsilon\to 0$, we get $\limsup_{n\to\infty}n^{-1}L_n\le 2\|\mu\|$, a.s. Thus the almost sure convergence statement is established.

Moreover, from (2.1),

$$\begin{aligned}L_n &\le \int_0^{2\pi}\max_{0\le k\le n}\|S_k\|\mathrm{d}\theta\\ &\le 2\pi\max_{0\le k\le n}\sum_{j=1}^k\|Z_j\|\\ &\le 2\pi\sum_{j=1}^n\|Z_j\|.\end{aligned}$$

The strong law shows that, a.s., $n^{-1}\sum_{j=1}^n\|Z_j\|\to\mathbb{E}\|Z\|<\infty$, while $\mathbb{E}(n^{-1}\sum_{j=1}^n\|Z_j\|)=\mathbb{E}\|Z\|$; hence Pratt's lemma [9, p. 221] implies that $n^{-1}L_n\to 2\|\mu\|$ in L^1.

Finally, suppose that $\mathbb{E}\|Z\|=\infty$. From (2.3), it suffices to show that

$$\limsup_{n\to\infty}n^{-1}\|S_n\|=\infty,\ \text{a.s.}$$

To this end we follow [9, p. 297]. First (see e.g. [9, p. 75]) $\mathbb{E}\|Z\|=\infty$ implies that for any $c>0$, we have $\sum_{n=1}^\infty\mathbb{P}(\|Z_n\|\ge cn)=\infty$, which, by the Borel–Cantelli lemma, implies that $\mathbb{P}(\|Z_n\|\ge cn\text{ i.o.})=1$. But $\|Z_n\|\le\|S_n\|+\|S_{n-1}\|$, so it follows that $\mathbb{P}(\|S_n\|\ge cn/2\text{ i.o.})=1$. In other words, $\limsup_{n\to\infty}n^{-1}\|S_n\|\ge c/2$, a.s., and, since $c>0$ was arbitrary, we get the result. □

Proof of Theorem 1.3. Since $\|S_n\|\le D_n\le L_n/2$ we can apply the strong law for S_n, which implies that $n^{-1}\|S_n\|\to\|\mu\|$, and Theorem 1.1, to deduce that $n^{-1}D_n\to\|\mu\|$, a.s. Since $n^{-1}D_n\le n^{-1}L_n/2$ we may again apply Pratt's lemma [9, p. 221] to deduce the L^1 convergence. Finally, if $\mathbb{E}\|Z\|=\infty$ we use the bound $D_n\ge L_n/\pi$ and the final statement in Theorem 1.1 to deduce that $\limsup_{n\to\infty}n^{-1}D_n=\infty$, a.s. □

3 A zero-one law for convex hulls

A key ingredient in the proof of Theorem 1.5 is a *zero-one law* (Theorem 3.1 below). The statement says that events in the tail σ-algebra corresponding to the sequence of convex hulls are trivial as long as the hull grows in all directions. While reminiscent of the Kolmogorov zero-one law, despite the sequence of convex hulls being neither independent nor identically distributed, we will see below that the result is more closely related to the Hewitt–Savage zero-one law.

Before we state the result, we need some notation. Define σ-algebras $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$ for $n \geq 1$; also set $\mathcal{F}_\infty := \sigma(\cup_{n\geq 0} \mathcal{F}_n)$. Let ρ_d denote the Euclidean metric on $\mathbb{R}^d$, and for $A \subseteq \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$, let $\rho_d(\mathbf{x}, A) := \inf_{\mathbf{y}\in A} \rho_d(\mathbf{x}, \mathbf{y})$.

Let $\mathcal{K}$ denote the set of compact convex subsets of $\mathbb{R}^2$ containing the origin, endowed with the Hausdorff metric ρ_H defined for $K_1, K_2 \in \mathcal{K}$ by

$$\rho_H(K_1, K_2) = \inf\{\varepsilon \geq 0 : K_1 \subseteq K_2^\varepsilon \text{ and } K_2 \subseteq K_1^\varepsilon\}, \tag{3.1}$$

where $K^\varepsilon := \{\mathbf{x} \in \mathbb{R}^2 : \rho_2(\mathbf{x}, K) \leq \varepsilon\}$. The metric ρ_H generates the associated Borel σ-algebra $\mathcal{B}(\mathcal{K})$. Since the function $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \mapsto \operatorname{hull}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ (with $\mathbf{x}_0 := \mathbf{0}$) is continuous from $(\mathbb{R}^{2(n+1)}, \rho_{2(n+1)})$ to $(\mathcal{K}, \rho_H)$, it is measurable from $(\mathbb{R}^{2(n+1)}, \mathcal{B}(\mathbb{R}^{2(n+1)}))$ to $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$; thus $\mathcal{H}_n$ is a $\mathcal{K}$-valued random variable, and $\mathcal{H}_n$ is $\mathcal{F}_n$-measurable.

For $n \geq 0$, set $\mathcal{T}_n := \sigma(\mathcal{H}_n, \mathcal{H}_{n+1}, \ldots)$ and define $\mathcal{T} := \cap_{n\geq 0} \mathcal{T}_n$. Also, for $n \geq 0$ define

$$r_n := \inf\{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^2 \setminus \mathcal{H}_n\}.$$

Note that r_n is non-decreasing. Here is the zero-one law.

Theorem 3.1. *Suppose that $r_n \to \infty$ a.s. Then if $A \in \mathcal{T}$, $\mathbb{P}(A) \in \{0, 1\}$.*

The intuition behind this result is as follows. Since $r_n \to \infty$, eventually any initial segment $S_0, S_1, \ldots, S_k$ of the trajectory ends up in the interior of the convex hull. A stronger version of this fact (Lemma 3.4) shows that for any k there is a random time after which the convex hull is invariant under permutations of $Z_1, \ldots, Z_k$. The proof of Theorem 3.1 is then a variation on the Hewitt–Savage zero-one law.

Next we give a sufficient condition for $r_n \to \infty$. Recall [4, p. 190] that S_n is *recurrent* if there is a non-empty set $\mathcal{R}$ of points $\mathbf{x} \in \mathbb{R}^2$ (the recurrent values) such that, for any $\varepsilon > 0$, $\|S_n - \mathbf{x}\| < \varepsilon$ i.o., a.s.

Proposition 3.2. *If S_n is genuinely 2-dimensional and recurrent, then $r_n \to \infty$ a.s.*

Remark 3.3. One may also have $r_n \to \infty$ a.s. in the case of a transient walk, provided it visits all angles. However, $\lim_{n\to\infty} r_n < \infty$ a.s. may occur if the walk has a limiting direction, such as if there is a finite non-zero drift.

Let $B(\mathbf{x}; r)$ denote the closed Euclidean ball centred at $\mathbf{x} \in \mathbb{R}^2$ with radius r.

Proof of Proposition 3.2. Since S_n is recurrent, the set $\mathcal{R}$ of recurrent values is a closed subgroup of $\mathbb{R}^2$ and coincides with the set of *possible values* for the walk: see [4, p. 190]. Since S_n is genuinely 2-dimensional, it follows from e.g. Theorem 21.2 of [1, p. 225] that $\mathcal{R}$ contains a further closed subgroup $\mathcal{R}'$ of the form $H\mathbb{Z}^2$ where H is a non-singular 2 by 2 matrix. Hence there exists $h > 0$ such that for every $\mathbf{x} \in \mathbb{R}^2$ there exists $\mathbf{y} \in \mathcal{R}'$ with $\|\mathbf{x} - \mathbf{y}\| < h/2$. In particular, for any $\mathbf{x} \in \mathbb{R}^2$, $\mathbb{P}(S_n \in B(\mathbf{x}; h) \text{ i.o.}) = 1$.

Fix $r > h$, and consider 4 discs, D_1, D_2, D_3, D_4, each of radius h, centred at $(\pm 2r, \pm 2r)$. Define T_r to be the first time at which the walk has visited all 4 discs, i.e.,

$$T_r := \min\{n \geq 0 : \exists\, i_1, i_2, i_3, i_4 \in [0, n] \text{ with } S_{i_j} \in D_j \text{ for } j = 1, 2, 3, 4\}.$$

The first paragraph of this proof shows that $T_r < \infty$ a.s. By construction, for $n \geq T_r$ we have that $\mathcal{H}_n$ contains the square $[-r, r]^2$, and so $n \geq T_r$ implies $r_n \geq r$. Hence,

$$\mathbb{P}\left(\liminf_{m\to\infty} r_m \geq r\right) \geq \mathbb{P}(T_r \leq n) \to 1,$$

as $n \to \infty$, and so $\liminf_{n\to\infty} r_n \geq r$, a.s. Since $r > h$ was arbitrary, the result follows. □

The first step in the proof of Theorem 3.1 is the following result, which uses the fact that $r_n \to \infty$ to show that any initial segment of the trajectory is eventually contained in the interior of the convex hull, uniformly over permutations of the initial increments.

Lemma 3.4. *Suppose that $r_n \to \infty$ a.s. Let $k \in \mathbb{N}$. Then there exists a random variable N_k with $\mathbb{P}(k < N_k < \infty) = 1$ such that (i) N_k is invariant under permutations of $Z_1, \ldots, Z_k$, and (ii) $\mathcal{H}_n = \text{hull}\{S_{k+1}, \ldots, S_n\}$ for all $n \geq N_k$ and all permutations of $Z_1, \ldots, Z_k$.*

Proof. Fix $k \in \mathbb{N}$. Let $R_k := \sum_{i=1}^{k} \|Z_i\|$ and define $N_k := \min\{n > k : r_n > R_k\}$. Note that since r_n is non-decreasing, $n \geq N_k$ implies $r_n > R_k$. Since $R_k < \infty$ a.s. and $r_n \to \infty$ a.s., we have $N_k < \infty$ a.s. Observe that if $r_n > R_k$ for $n > k$, then $S_0, S_1, \ldots, S_k$ are all contained in the interior of $\mathcal{H}_n$, for all permutations of $Z_1, \ldots, Z_k$, so that $\mathcal{H}_n = \mathcal{H}_{n,k} := \text{hull}\{S_{k+1}, \ldots, S_n\}$. So statement (ii) holds. Moreover, if $r_{n,k} := \inf\{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^2 \setminus \mathcal{H}_{n,k}\}$ we have that $\{r_n > R_k\} = \{r_{n,k} > R_k\}$. But the events $\{r_{n,k} > R_k\}$, $n > k$, which determine N_k, depend only on R_k and $S_{k+1}, S_{k+2}, \ldots$, and so statement (i) holds. □

Now we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We adapt one of the standard proofs of the Hewitt–Savage zero-one law; see e.g. [4, pp. 180–181]. Let $A \in \mathcal{T}$ and fix $\varepsilon > 0$. Recall a fact from measure theory: if $\mathcal{A}$ is an algebra and $A \in \sigma(\mathcal{A})$, then we can find $A' \in \mathcal{A}$ such that $\mathbb{P}(A \triangle A') < \varepsilon$ (see e.g. [3, p. 179]). Applied to the algebra $\cup_{n\geq 0}\mathcal{F}_n$ which generates $\mathcal{F}_\infty \supseteq \mathcal{T}$, this result implies that we can find $k \geq 0$ and $A_k \in \mathcal{F}_k$ such that $\mathbb{P}(A \triangle A_k) < \varepsilon$. Fix this k, and fix n such that $\mathbb{P}(N_{2k} > n) < \varepsilon$, where N_{2k} is as given in Lemma 3.4. Applied to the algebra $\mathcal{A}_n := \cup_{m\geq 0}\sigma(\mathcal{H}_n, \mathcal{H}_{n+1}, \ldots, \mathcal{H}_{n+m})$, which has $\sigma(\mathcal{A}_n) \supseteq \mathcal{T}_n \supseteq \mathcal{T}$, the same measure-theoretic result shows that we can find $E_n \in \mathcal{A}_n$ such that $\mathbb{P}(A \triangle E_n) < \varepsilon$.

Now $A_k \in \mathcal{F}_k$ can be expressed as $A_k = \{Z_1 \in C_{k,1}, \ldots, Z_k \in C_{k,k}\}$ for Borel sets $C_{k,1}, \ldots, C_{k,k}$. Set $A'_k := \{Z_{k+1} \in C_{k,1}, \ldots, Z_{2k} \in C_{k,k}\}$; since the Z_i are i.i.d., $\mathbb{P}(A'_k) = \mathbb{P}(A_k)$, and A_k and A'_k are independent. We claim that

$$\mathbb{P}((A'_k \triangle E_n) \cap \{N_{2k} \leq n\}) = \mathbb{P}((A_k \triangle E_n) \cap \{N_{2k} \leq n\}) \leq 2\varepsilon. \tag{3.2}$$

To see the equality in (3.2), observe that Lemma 3.4 shows that $E_n \cap \{N_{2k} \leq n\}$ is invariant under permutations of $Z_1, \ldots, Z_{2k}$, and the Z_i are i.i.d. For the inequality in (3.2), we use the fact that $\mathbb{P}(A \triangle B) \leq \mathbb{P}(A \triangle C) + \mathbb{P}(B \triangle C)$ to get

$$\begin{aligned}\mathbb{P}((A_k \triangle E_n) \cap \{N_{2k} \leq n\}) &\leq \mathbb{P}(A_k \triangle E_n) \\ &\leq \mathbb{P}(A_k \triangle A) + \mathbb{P}(E_n \triangle A) \leq 2\varepsilon.\end{aligned}$$

Hence the claim (3.2) is verified. Since $\mathbb{P}((A \triangle B) \cap D) \leq \mathbb{P}((A \triangle C) \cap D) + \mathbb{P}(B \triangle C)$, we also get that

$$\mathbb{P}((A \triangle A'_k) \cap \{N_{2k} \leq n\}) \leq \mathbb{P}((A'_k \triangle E_n) \cap \{N_{2k} \leq n\}) + \mathbb{P}(A \triangle E_n) \leq 3\varepsilon,$$

by (3.2). Hence

$$\mathbb{P}(A \triangle A'_k) \leq \mathbb{P}(N_{2k} > n) + \mathbb{P}((A \triangle A'_k) \cap \{N_{2k} \leq n\}) \leq 4\varepsilon.$$

The final sequence of the proof is a variation on the standard argument. First note that

$$|\mathbb{P}(A)^2 - \mathbb{P}(A)| \le |\mathbb{P}(A)^2 - \mathbb{P}(A_k \cap A'_k)| + |\mathbb{P}(A_k \cap A'_k) - \mathbb{P}(A)|. \tag{3.3}$$

For the first term on the right-hand side of (3.3), we use the fact that A_k and A'_k are independent with $\mathbb{P}(A_k) = \mathbb{P}(A'_k)$, along with the property of the symmetric difference operator that $|\mathbb{P}(A) - \mathbb{P}(B)| \le \mathbb{P}(A \triangle B)$, to get

$$\begin{aligned} |\mathbb{P}(A)^2 - \mathbb{P}(A_k \cap A'_k)| &= |\mathbb{P}(A)^2 - \mathbb{P}(A_k)^2| \\ &\le |\mathbb{P}(A) + \mathbb{P}(A_k)||\mathbb{P}(A) - \mathbb{P}(A_k)| \\ &\le 2\mathbb{P}(A \triangle A_k) \le 2\varepsilon. \end{aligned}$$

Now considering the second term on the right-hand side of (3.3) and using the fact that $\mathbb{P}(A \triangle (B \cap C)) \le \mathbb{P}(A \triangle B) + \mathbb{P}(A \triangle C)$, we have

$$\begin{aligned} |\mathbb{P}(A_k \cap A'_k) - \mathbb{P}(A)| &\le \mathbb{P}(A \triangle (A_k \cap A'_k)) \\ &\le \mathbb{P}(A \triangle A_k) + \mathbb{P}(A \triangle A'_k) \le 5\varepsilon. \end{aligned}$$

Combining these two bounds, we obtain from (3.3) that $|\mathbb{P}(A)^2 - \mathbb{P}(A)| \le 7\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get the result. □

The strategy of the proof of Theorem 1.5, carried out in the remainder of this section, is as follows. We use Donsker's theorem and the mapping theorem to show that $D_n^{-1}\mathcal{H}_n$ converges weakly to the convex hull of an appropriate Brownian motion, scaled to have unit diameter (Lemma 3.7). This limiting set has positive probability of being an arbitrarily good approximation to any given unit-diameter convex compact set K. An application of the zero-one law (Theorem 3.1) then completes the proof.

For $K \in \mathcal{K}$ let $\mathcal{D}(K) := \operatorname{diam} K$. The next result shows that the map $K \mapsto \mathcal{D}(K)$ is continuous from $(\mathcal{K}, \rho_H)$ to $(\mathbb{R}_+, \rho_1)$.

Lemma 3.5. *For $K_1, K_2 \in \mathcal{K}$, $|\mathcal{D}(K_1) - \mathcal{D}(K_2)| \le 2\rho_H(K_1, K_2)$.*

Proof. Let $\rho_H(K_1, K_2) = r$. From (3.1) we have that for any $\mathbf{x}_1, \mathbf{x}_2 \in K_1$ and any $s > r$, there exist $\mathbf{y}_1, \mathbf{y}_2 \in K_2$ such that $\|\mathbf{x}_i - \mathbf{y}_i\| \le s$. Then,

$$\|\mathbf{x}_1 - \mathbf{x}_2\| \le \|\mathbf{x}_1 - \mathbf{y}_1\| + \|\mathbf{y}_1 - \mathbf{y}_2\| + \|\mathbf{y}_2 - \mathbf{x}_2\| \le 2s + \mathcal{D}(K_2).$$

Hence $\mathcal{D}(K_1) \le 2s + \mathcal{D}(K_2)$, and since $s > r$ was arbitrary we get $\mathcal{D}(K_1) - \mathcal{D}(K_2) \le 2r$. A symmetric argument gives $\mathcal{D}(K_2) - \mathcal{D}(K_1) \le 2r$. □

For $K \in \mathcal{K}$ and $\mathbf{x} \in \mathbb{S} := \{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y}\| = 1\}$, define $h_K(\mathbf{x}) := \sup_{\mathbf{y} \in K}(\mathbf{y} \cdot \mathbf{x})$. Equivalent to (3.1) for $K_1, K_2 \in \mathcal{K}$ is the formula [8, p. 84]

$$\rho_H(K_1, K_2) = \sup_{\mathbf{x} \in \mathbb{S}} |h_{K_1}(\mathbf{x}) - h_{K_2}(\mathbf{x})|. \tag{3.4}$$

Let $\mathcal{K}^\star := \{K \in \mathcal{K} : \mathcal{D}(K) > 0\} = \mathcal{K} \setminus \{\{\mathbf{0}\}\}$.

Lemma 3.6. *Suppose that $K_1, K_2 \in \mathcal{K}^\star$. Then*

$$\rho_H(K_1/\mathcal{D}(K_1), K_2/\mathcal{D}(K_2)) \le \frac{3\rho_H(K_1, K_2)}{\mathcal{D}(K_1)}. \tag{3.5}$$

In particular, the map $K \mapsto K/\mathcal{D}(K)$ is continuous from $(\mathcal{K}^\star, \rho_H)$ to $(\mathcal{K}^\star, \rho_H)$.

Proof. We first claim that for $K_1, K_2 \in \mathcal{K}$ and $\alpha_1, \alpha_2 > 0$,

$$\rho_H(\alpha_1 K_1, \alpha_2 K_2) \leq \alpha_1 \rho_H(K_1, K_2) + |\alpha_1 - \alpha_2| \mathcal{D}(K_2). \tag{3.6}$$

Suppose that $K_1, K_2 \in \mathcal{K}^\star$. Applying (3.6) with $\alpha_i = 1/\mathcal{D}(K_i)$, we get

$$\rho_H(K_1/\mathcal{D}(K_1), K_2/\mathcal{D}(K_2)) \leq \frac{\rho_H(K_1, K_2)}{\mathcal{D}(K_1)} + \frac{|\mathcal{D}(K_1) - \mathcal{D}(K_2)|}{\mathcal{D}(K_1)},$$

from which (3.5) follows by Lemma 3.5. This gives the desired continuity.

It remains to verify the claim (3.6). From (3.4), with the observation that, for $\alpha > 0$, $h_{\alpha K}(\mathbf{x}) = \alpha h_K(\mathbf{x})$, it follows that

$$\begin{aligned} \rho_H(\alpha_1 K_1, \alpha_2 K_2) &= \sup_{\mathbf{x} \in \mathbb{S}} |\alpha_1 h_{K_1}(\mathbf{x}) - \alpha_1 h_{K_2}(\mathbf{x}) + (\alpha_1 - \alpha_2) h_{K_2}(\mathbf{x})| \\ &\leq \alpha_1 \sup_{\mathbf{x} \in \mathbb{S}} |h_{K_1}(\mathbf{x}) - h_{K_2}(\mathbf{x})| + |\alpha_1 - \alpha_2| \sup_{\mathbf{x} \in \mathbb{S}} h_{K_2}(\mathbf{x}), \end{aligned}$$

from which the claim (3.6) follows. □

Suppose that $\Sigma := \mathbb{E}(ZZ^\top)$ is positive definite. Let $(b(t), t \geq 0)$ be standard Brownian motion in $\mathbb{R}^2$. Let $h_1 := \operatorname{hull} b[0,1]$, the convex hull of Brownian motion run for unit time. Let $\Sigma^{1/2}$ denote the (unique) positive-definite symmetric matrix such that $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. The map $\mathbf{x} \mapsto \Sigma^{1/2}\mathbf{x}$ is an affine transformation of $\mathbb{R}^2$, such that $\Sigma^{1/2} b$ is Brownian motion with covariance matrix Σ, and $\Sigma^{1/2} h_1 = \operatorname{hull} \Sigma^{1/2} b[0,1]$ is the corresponding convex hull.

Lemma 3.7. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$, $\mu = \mathbf{0}$, and Σ is positive definite. Then*

$$D_n^{-1} \mathcal{H}_n \Rightarrow \frac{\Sigma^{1/2} h_1}{\mathcal{D}(\Sigma^{1/2} h_1)},$$

in the sense of weak convergence on $(\mathcal{K}, \rho_H)$.

Proof. The convergence $n^{-1/2}\mathcal{H}_n \Rightarrow \Sigma^{1/2} h_1$ is given in Theorem 2.5 of [23]. Since (by Lemma 3.6) $K \mapsto K/\mathcal{D}(K)$ is continuous on $\mathcal{K}^\star$, and $\mathbb{P}(\Sigma^{1/2} h_1 \in \mathcal{K}^\star) = 1$, we may apply the mapping theorem [2, p. 21] to deduce the result. □

Proof of Theorem 1.5. Fix $K \in \mathcal{K}$ with $\mathcal{D}(K) = 1$. We claim that, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\liminf_{n \to \infty} \rho_H\left(D_n^{-1}\mathcal{H}_n, K\right) \leq \varepsilon \right) > 0. \tag{3.7}$$

Under the conditions of the theorem, S_n is genuinely 2-dimensional and recurrent [4, p. 195], and so, by Proposition 3.2, $r_n \to \infty$ a.s. Since the event in (3.7) is in $\mathcal{T}$, the zero-one law (Theorem 3.1) shows that the probability in (3.7) must be equal to 1. Since $\varepsilon > 0$ was arbitrary, the statement of the theorem follows.

Thus it remains to prove the claim (3.7). To this end, observe that, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\liminf_{n \to \infty} \rho_H\left(D_n^{-1}\mathcal{H}_n, K\right) \leq \varepsilon \right) &\geq \mathbb{P}\left(\rho_H\left(D_n^{-1}\mathcal{H}_n, K\right) < \varepsilon \text{ i.o.} \right) \\ &= \mathbb{P}\left(\limsup_{n \to \infty} \{ \rho_H\left(D_n^{-1}\mathcal{H}_n, K\right) < \varepsilon \} \right) \\ &\geq \lim_{n \to \infty} \mathbb{P}\left(\rho_H(D_n^{-1}\mathcal{H}_n, K) < \varepsilon \right), \end{aligned}$$

by the reverse Fatou lemma. By the triangle inequality, $|\rho_H(K, K_1) - \rho_H(K, K_2)| \leq \rho_H(K_1, K_2)$, i.e., for fixed K, the function $K_1 \mapsto \rho_H(K, K_1)$ is continuous. Thus by Lemma 3.7 and the mapping theorem

$$\lim_{n \to \infty} \mathbb{P}\left(\rho_H(D_n^{-1}\mathcal{H}_n, K) < \varepsilon \right) = \mathbb{P}\left(\rho_H\left(\frac{\Sigma^{1/2} h_1}{\mathcal{D}(\Sigma^{1/2} h_1)}, K \right) < \varepsilon \right). \tag{3.8}$$

Let $\delta \in (0, \varepsilon/6)$. For convenience, set $A = \Sigma^{1/2} h_1$. First suppose that $\mathbf{0}$ is in the interior of K. Then, it is not hard to see that $K \subseteq A \subseteq (1+\delta)K$ occurs with positive probability (one can force the Brownian motion to make a loop in $((1+\delta)K) \setminus K$). On this event, we have $h_K(\mathbf{x}) \leq h_A(\mathbf{x}) \leq (1+\delta) h_K(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}$, so that, by (3.4), recalling $\mathcal{D}(K) = 1$,

$$\rho_H(A, K) = \sup_{\mathbf{x} \in \mathbb{S}} |h_A(\mathbf{x}) - h_K(\mathbf{x})| \leq \delta \sup_{\mathbf{x} \in \mathbb{S}} h_K(\mathbf{x}) \leq \delta \mathcal{D}(K) = \delta.$$

It follows from taking $K_1 = K$ and $K_2 = A$ in (3.5) that

$$\rho_H(A/\mathcal{D}(A), K) \leq 3\rho_H(A, K) \leq 3\delta < \varepsilon/2.$$

If $\mathbf{0}$ is not in the interior of K, then we can find $K' \in \mathcal{K}$ with $K \subset K'$ such that $\mathbf{0}$ is in the interior of K' and $\rho_H(K, K') < \varepsilon/2$. Then

$$\rho_H(A/\mathcal{D}(A), K) \leq \rho_H(A/\mathcal{D}(A), K') + \rho_H(K, K') < \varepsilon,$$

on the event $K' \subseteq A \subseteq (1+\delta)K'$, which has positive probability. Hence, in either case, the probability on the right-hand side of (3.8) is strictly positive, establishing (3.7). □

Proof of Corollary 1.6. For $K \in \mathcal{K}$, let $\mathcal{L}(K)$ denote the perimeter length of K; then, Lemma 2.4 of [23] shows that

$$|\mathcal{L}(K_1) - \mathcal{L}(K_2)| \leq 2\pi \rho_H(K_1, K_2), \text{ for any } K_1, K_2 \in \mathcal{K}. \tag{3.9}$$

First, take K to be a unit-length line segment in $\mathbb{R}^2$ containing $\mathbf{0}$ and note that $\mathcal{L}(K) = 2$. Theorem 1.5 shows that, for any $\varepsilon > 0$, $\rho_H(D_n^{-1}\mathcal{H}_n, K) < \varepsilon$ i.o., a.s. Hence, by (3.9),

$$L_n/D_n = \mathcal{L}(D_n^{-1}\mathcal{H}_n) \leq \mathcal{L}(K) + 2\pi\varepsilon, \text{ i.o.}$$

Since $\varepsilon > 0$ was arbitrary, we get $\liminf_{n\to\infty} L_n/D_n \leq 2$, and the first inequality in (1.2) shows that this latter inequality is in fact an equality.

Now take K to be a unit-diameter disc in $\mathbb{R}^2$ containing $\mathbf{0}$ and note that $\mathcal{L}(K) = \pi$. Again, Theorem 1.5 shows that, for any $\varepsilon > 0$, $\rho_H(D_n^{-1}\mathcal{H}_n, K) < \varepsilon$ i.o., a.s. Hence, by (3.9),

$$L_n/D_n = \mathcal{L}(D_n^{-1}\mathcal{H}_n) \geq \mathcal{L}(K) - 2\pi\varepsilon, \text{ i.o.},$$

and now we get $\limsup_{n\to\infty} L_n/D_n \geq \pi$, which combined with the second inequality in (1.2) completes the proof. □

4 Perimeter in the case with drift

Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. We work towards the proof of Theorem 1.8. Write $X_n := S_n \cdot \hat{\mu}$ and $Y_n := S_n \cdot \hat{\mu}_\perp$, where $\hat{\mu}_\perp$ is any fixed unit vector orthogonal to μ. Then X_n and Y_n are one-dimensional random walks with increment distributions $Z \cdot \hat{\mu}$ and $Z \cdot \hat{\mu}_\perp$ respectively; note that $\mathbb{E}(Z \cdot \hat{\mu}) = \|\mu\|$, $\mathbb{E}(Z \cdot \hat{\mu}_\perp) = 0$, $\mathbb{V}\mathrm{ar}(Z \cdot \hat{\mu}) = \sigma_\mu^2$, and

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z \cdot \hat{\mu}_\perp) = \mathbb{E}[((Z-\mu) \cdot \hat{\mu}_\perp)^2] &= \mathbb{E}[\|Z-\mu\|^2] - \mathbb{E}[((Z-\mu) \cdot \hat{\mu})^2] \\ &= \sigma^2 - \sigma_\mu^2 = \sigma_{\mu_\perp}^2.\end{aligned}$$

The first step towards the proof of Theorem 1.8 is the following result.

Lemma 4.1. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$ *and* $\mu \neq \mathbf{0}$*. Then* $\|S_n\| - |S_n \cdot \hat{\mu}|$ *is uniformly integrable.*

Proof. The central limit theorem shows that $n^{-1} Y_n^2 \xrightarrow{\mathrm{d}} \sigma_{\mu_\perp}^2 \zeta^2$ where $\zeta \sim \mathcal{N}(0,1)$. Also, since $\mathbb{E}[Y_n^2] = n\sigma_{\mu_\perp}^2$, $n^{-1}\,\mathbb{E}(Y_n^2) \to \sigma_{\mu_\perp}^2 = \mathbb{E}(\sigma_{\mu_\perp}^2 \zeta^2)$. It is known that if $\theta, \theta_1, \theta_2, \ldots$ are $\mathbb{R}_+$-valued random variables with $\theta_n \xrightarrow{\mathrm{d}} \theta$, then $\mathbb{E}\,\theta_n \to \mathbb{E}\,\theta < \infty$ if and only if θ_n is uniformly integrable: see [12, Lemma 4.11]. Hence we conclude that

$$n^{-1} Y_n^2 \text{ is uniformly integrable.} \tag{4.1}$$

Fix $\varepsilon > 0$. Let $\delta \in (0, \|\mu\|)$ to be chosen later. For ease of notation, write $T_n = \|S_n\| - |X_n|$. Then since $T_n \le \|S_n\|$ and $|X_n| \le \|S_n\|$, we have that for any $M > 0$,

$$\begin{aligned} \mathbb{E}\left[T_n \mathbb{1}\{T_n > M\} \mathbb{1}\{\|S_n\| \le \delta n\}\right] &\le \delta n \mathbb{P}(\|S_n\| \le \delta n) \\ &\le \delta n \mathbb{P}(|X_n| \le \delta n) \\ &\le \delta n \mathbb{P}(|X_n - \|\mu\| n| > (\|\mu\| - \delta) n). \end{aligned}$$

Since $\mathbb{E}\,X_n = n\|\mu\|$ and $\mathbb{V}\mathrm{ar}\,X_n = n\sigma_\mu^2$, Chebyshev's inequality then yields

$$\mathbb{E}\left[T_n \mathbb{1}\{T_n > M\} \mathbb{1}\{\|S_n\| \le \delta n\}\right] \le \delta n \frac{n\sigma_\mu^2}{(\|\mu\| - \delta)^2 n^2}.$$

It follows that, for suitable choice of δ (depending only on σ_μ^2, $\|\mu\|$ and ε) and any $M \in (0, \infty)$,

$$\sup_n \mathbb{E}\left[T_n \mathbb{1}\{T_n > M\} \mathbb{1}\{\|S_n\| \le \delta n\}\right] \le \varepsilon.$$

On the other hand, we use the fact that

$$0 \le \|S_n\| - |X_n| = T_n = \frac{\|S_n\|^2 - X_n^2}{\|S_n\| + |X_n|} = \frac{Y_n^2}{\|S_n\| + |X_n|}. \tag{4.2}$$

Hence

$$\begin{aligned} \mathbb{E}\left[T_n \mathbb{1}\{T_n > M\} \mathbb{1}\{\|S_n\| > \delta n\}\right] &= \mathbb{E}\left[\tfrac{Y_n^2}{\|S_n\| + |X_n|} \mathbb{1}\left\{\tfrac{Y_n^2}{\|S_n\| + |X_n|} > M\right\} \mathbb{1}\{\|S_n\| > \delta n\}\right] \\ &\le \frac{1}{\delta n}\,\mathbb{E}\left[Y_n^2 \mathbb{1}\{Y_n^2 > M\delta n\}\right]. \end{aligned}$$

It follows that

$$\sup_n \mathbb{E}\left[T_n \mathbb{1}\{T_n > M\} \mathbb{1}\{\|S_n\| > \delta n\}\right] \le \frac{1}{\delta} \sup_n \mathbb{E}\left[n^{-1} Y_n^2 \mathbb{1}\{n^{-1} Y_n^2 > M\delta\}\right],$$

which, for fixed δ, tends to 0 as $M \to \infty$ by (4.1).

Thus for any $\varepsilon > 0$ we have that $\sup_n \mathbb{E}\left[T_n \mathbb{1}\{T_n > M\}\right] \le \varepsilon$, for all M sufficiently large, which completes the proof. □

The next result is of some independent interest, and may be known, although we could find no reference.

Lemma 4.2. *Suppose that $\mathbb{E}(\|Z\|^2) < \infty$ and $\mu \neq \mathbf{0}$. Then*

$$0 \le \|S_n\| - S_n \cdot \hat{\mu} \to \frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|}, \text{ in } L^1, \text{ as } n \to \infty,$$

for $\zeta \sim \mathcal{N}(0,1)$. In particular,

$$0 \le \mathbb{E}\,\|S_n\| - \|\mu\| n = \frac{\sigma_{\mu_\perp}^2}{2\|\mu\|} + o(1), \text{ as } n \to \infty.$$

Proof. We have from (4.2) that

$$\|S_n\| - |X_n| = \frac{Y_n^2}{\|S_n\| + |X_n|} = \frac{n^{-1}Y_n^2}{n^{-1}\|S_n\| + n^{-1}|X_n|},$$

where $n^{-1}Y_n^2 \xrightarrow{\mathrm{d}} \sigma_{\mu_\perp}^2 \zeta^2$ for $\zeta \sim \mathcal{N}(0,1)$, and, by the strong law of large numbers, both $n^{-1}\|S_n\|$ and $n^{-1}|X_n|$ tend to $\|\mu\|$ a.s. Hence $0 \le \|S_n\| - |X_n| \xrightarrow{\mathrm{d}} \frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|}$, and by Lemma 4.1 we can conclude that

$$\|S_n\| - |X_n| \to \frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|} \text{ in } L^1 \text{ as } n \to \infty. \tag{4.3}$$

Now, as above, for $x \in \mathbb{R}$ set $x^+ := x\mathbb{1}\{x > 0\}$, and also set $x^- = -x\mathbb{1}\{x < 0\}$. Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$, so $x = |x| - 2x^-$; thus $|X_n| - 2X_n^- = X_n \le |X_n|$, and

$$0 \le \|S_n\| - |X_n| \le \|S_n\| - X_n = \|S_n\| - |X_n| + 2X_n^-;$$

in particular $\mathbb{E}\,\|S_n\| \ge \mathbb{E}\,X_n = \|\mu\|n$. Moreover, Lemma A.1 shows that $X_n^- \to 0$ in L^1, thus the result follows from (4.3). □

We can now complete the proof of Theorem 1.8 and then the proof of Theorem 1.7.

Proof of Theorem 1.8. From the Spitzer–Widom formula (1.1) and Lemma 4.2, we have

$$\mathbb{E}\,L_n = 2\sum_{k=1}^{n} \frac{1}{k}\left(\|\mu\|k + \frac{\sigma_{\mu_\perp}^2}{2\|\mu\|} + o(1)\right) = 2\|\mu\|n + \frac{\sigma_{\mu_\perp}^2}{\|\mu\|}\log n + o(\log n),$$

as claimed. □

Proof of Theorem 1.7. Theorem 1.8 shows that

$$n^{-1/2}|\,\mathbb{E}\,L_n - 2\,\mathbb{E}\,S_n \cdot \hat{\mu}| \to 0. \tag{4.4}$$

Then by the triangle inequality

$$n^{-1/2}|L_n - 2S_n \cdot \hat{\mu}| \le n^{-1/2}|L_n - \mathbb{E}\,L_n - 2(S_n - \mathbb{E}\,S_n)\cdot\hat{\mu}| + n^{-1/2}|\,\mathbb{E}\,L_n - 2\,\mathbb{E}\,S_n \cdot \hat{\mu}|,$$

which tends to 0 in L^2 by (1.3) and (4.4). □

5 Diameter in the case with drift

Now we turn to the diameter D_n. The main aim of this section is to establish the following result, from which we will deduce Theorem 1.9.

Theorem 5.1. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$ *and* $\mu \neq \mathbf{0}$. *Then, as* $n \to \infty$,

$$n^{-1/2}\,|D_n - \mathbb{E}\,D_n - (S_n - \mathbb{E}\,S_n)\cdot\hat{\mu}| \to 0, \text{ in } L^2. \tag{5.1}$$

Theorem 5.1 is the analogue for D_n of the result (1.3) for L_n, established in Theorem 1.3 of [22]. Our approach to proving Theorem 5.1 is similar in outline to that in [22], where a martingale difference idea (which we explain below in the present context) was combined with Cauchy's formula for the perimeter length. Here, the use of Cauchy's formula is replaced by the formula

$$\operatorname{diam} A = \sup_{0 \le \theta \le \pi} \rho_A(\theta), \tag{5.2}$$

where $A \subset \mathbb{R}^d$ is a non-empty compact set, and $\rho_A(\theta) := \sup_{\mathbf{x}\in A}(\mathbf{x}\cdot\mathbf{e}_\theta) - \inf_{\mathbf{x}\in A}(\mathbf{x}\cdot\mathbf{e}_\theta)$; see Lemma 6 of [15] for a derivation of (5.2).

Before embarking on the proof of Theorem 5.1, we observe the following result.

Lemma 5.2. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$ *and* $\mu \neq \mathbf{0}$*. There exists* $C < \infty$ *such that*

$$0 \leq \mathbb{E}\, D_n - \|\mu\| n \leq C(1 + \log n), \text{ for all } n \geq 1.$$

Proof. The lower bound follows from Lemma 4.2 and the fact that $D_n \geq \|S_n\|$. The upper bound follows from the fact that $D_n \leq L_n/2$ and the fact that, by Theorem 1.8, $\mathbb{E}\, L_n \leq 2\|\mu\| n + C(1 + \log n)$. □

Now we describe the martingale difference construction, which is standard. Recall that $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$ for $n \geq 1$. Let $Z_1', Z_2', \ldots$ be an independent copy of the sequence $Z_1, Z_2, \ldots$. Fix $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$, define

$$S_j^{(i)} := \begin{cases} S_j & \text{if } j < i, \\ S_j - Z_i + Z_i' & \text{if } j \geq i; \end{cases}$$

then $(S_j^{(i)}; 0 \leq j \leq n)$ is the random walk $(S_j; 0 \leq j \leq n)$ but with Z_i resampled and replaced by Z_i'. For $i \in \{1, \ldots, n\}$, define

$$D_n^{(i)} := \operatorname{diam}\{S_0^{(i)}, \ldots, S_n^{(i)}\}, \text{ and } \Delta_{n,i} := \mathbb{E}(D_n - D_n^{(i)} \mid \mathcal{F}_i). \tag{5.3}$$

Observe that we also have the representation $\Delta_{n,i} = \mathbb{E}(D_n \mid \mathcal{F}_i) - \mathbb{E}(D_n \mid \mathcal{F}_{i-1})$ and hence $\Delta_{n,i}$ is a martingale difference sequence, i.e., $\Delta_{n,i}$ is $\mathcal{F}_i$-measurable with $\mathbb{E}(\Delta_{n,i} \mid \mathcal{F}_{i-1}) = 0$. The utility of this construction is the following result (see e.g. Lemma 2.1 of [22]).

Lemma 5.3. *Let* $n \in \mathbb{N}$*. Then* $D_n - \mathbb{E}\, D_n = \sum_{i=1}^n \Delta_{n,i}$*, and* $\mathbb{V}\mathrm{ar}\, D_n = \sum_{i=1}^n \mathbb{E}(\Delta_{n,i}^2)$.

Recall that $\mathbf{e}_\theta$ denotes the unit vector in direction θ. For $\theta \in [0, \pi]$, define

$$M_n(\theta) := \max_{0 \leq j \leq n} (S_j \cdot \mathbf{e}_\theta), \text{ and } m_n(\theta) := \min_{0 \leq j \leq n} (S_j \cdot \mathbf{e}_\theta),$$

and define $R_n(\theta) := M_n(\theta) - m_n(\theta)$. Note that since $S_0 = \mathbf{0}$, we have $M_n(\theta) \geq 0$ and $m_n(\theta) \leq 0$, a.s. It follows from (5.2) that $D_n = \sup_{0 \leq \theta \leq \pi} R_n(\theta)$.

Similarly, when the ith increment is resampled, $D_n^{(i)} = \sup_{0 \leq \theta \leq \pi} R_n^{(i)}(\theta)$, where $R_n^{(i)}(\theta) := M_n^{(i)}(\theta) - m_n^{(i)}(\theta)$, with

$$M_n^{(i)}(\theta) := \max_{0 \leq j \leq n} (S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } m_n^{(i)}(\theta) := \min_{0 \leq j \leq n} (S_j^{(i)} \cdot \mathbf{e}_\theta).$$

Thus to study $\Delta_{n,i}$ as defined at (5.3), we are interested in

$$D_n - D_n^{(i)} = \sup_{0 \leq \theta \leq \pi} R_n(\theta) - \sup_{0 \leq \theta \leq \pi} R_n^{(i)}(\theta). \tag{5.4}$$

For the remainder of this section we suppose, without loss of generality, that $\mu = \|\mu\| \mathbf{e}_{\pi/2}$ with $\|\mu\| \in (0, \infty)$. An important observation is that the diameter does not deviate far from the direction of the drift. For $\delta \in (0, \pi/2)$ and $i \in \{1, \ldots, n\}$, define the event

$$A_{n,i}(\delta) := \left\{ \left| \frac{\pi}{2} - \arg\max_{0 \leq \theta \leq \pi} R_n(\theta) \right| < \delta \right\} \cap \left\{ \left| \frac{\pi}{2} - \arg\max_{0 \leq \theta \leq \pi} R_n^{(i)}(\theta) \right| < \delta \right\}.$$

Lemma 5.4. *Suppose that* $\mathbb{E}\,\|Z\| < \infty$ *and* $\mu = \|\mu\| \mathbf{e}_{\pi/2} \neq \mathbf{0}$*. Then for any* $\delta \in (0, \pi/2)$*,* $\lim_{n \to \infty} \min_{1 \leq i \leq n} \mathbb{P}(A_{n,i}(\delta)) = 1$.

Proof. Fix $\delta \in (0, \pi/2)$. Note that $S_j \cdot \mathbf{e}_0$ is a random walk on $\mathbb{R}$ with mean increment $\mathbb{E}(Z \cdot \mathbf{e}_0) = \mu \cdot \mathbf{e}_0 = 0$. Hence the strong law of large numbers implies that for any $\varepsilon > 0$, there exists N_ε with $\mathbb{P}(N_\varepsilon < \infty) = 1$ such that, for all $n \geq N_\varepsilon$,

$$\max_{0 \leq j \leq n} |S_j \cdot \mathbf{e}_0| \leq \varepsilon n.$$

Similarly, since $S_j \cdot \mathbf{e}_{\pi/2}$ is a random walk on $\mathbb{R}$ with mean increment $\|\mu\| > 0$, there exists N' with $\mathbb{P}(N' < \infty) = 1$ such that

$$S_j \cdot \mathbf{e}_{\pi/2} \geq \tfrac{1}{2}\|\mu\| j, \text{ for all } j \geq N'.$$

Let $A'_n(\varepsilon)$ denote the event

$$\left\{ \max_{0 \leq j \leq n} |S_j \cdot \mathbf{e}_0| \leq \varepsilon n \right\} \cap \left\{ S_n \cdot \mathbf{e}_{\pi/2} \geq \tfrac{1}{2}\|\mu\| n \right\}.$$

Then if $A'_n(\varepsilon)$ occurs, any line segment that achieves the diameter has length at least $\frac{1}{2}\|\mu\| n$ and horizontal component at most $2\varepsilon n$. Thus if $\theta_n = \arg\max_{0 \leq \theta \leq \pi} R_n(\theta)$ we have

$$|\cos \theta_n| \leq \frac{4\varepsilon}{\|\mu\|}, \text{ on } A'_n(\varepsilon).$$

Thus for ε sufficiently small, depending only on δ and $\|\mu\|$, we have that $A'_n(\varepsilon)$ implies $|\theta_n - \pi/2| < \delta$. Hence

$$\mathbb{P}(|\theta_n - \pi/2| < \delta) \geq \mathbb{P}(A'_n(\varepsilon)) \geq \mathbb{P}\left(n \geq \max\{N_\varepsilon, N'\}\right) \to \mathbb{P}\left(\max\{N_\varepsilon, N'\} < \infty\right) = 1.$$

But $\theta_n^{(i)} = \arg\max_{0 \leq \theta \leq \pi} R_n^{(i)}(\theta)$ has the same distribution as θ_n, so

$$\min_{1 \leq i \leq n} \mathbb{P}(\{|\theta_n - \pi/2| < \delta\} \cap \{|\theta_n^{(i)} - \pi/2| < \delta\}) \geq 1 - 2\mathbb{P}(|\theta_n - \pi/2| \geq \delta),$$

and the result follows. □

Lemma 5.4 tells us that the key to understanding (5.4) is to understand what is happening with $R_n(\theta)$ and $R_n^{(i)}(\theta)$ for $\theta \approx \pi/2$. The next important observation is that for $\theta \in (0, \pi)$, the one-dimensional random walk $S_j \cdot \mathbf{e}_\theta$ has drift $\mu \cdot \mathbf{e}_\theta = \mu \sin\theta > 0$, so, with very high probability, $M_n(\theta)$ is attained somewhere near the end of the walk, and $m_n(\theta)$ somewhere near the start.

To formalize this statement, and its consequence for $R_n(\theta) - R_n^{(i)}(\theta)$, define

$$\bar{J}_n(\theta) := \arg\max_{0 \leq j \leq n}(S_j \cdot \mathbf{e}_\theta), \text{ and } \underline{J}_n(\theta) := \arg\min_{0 \leq j \leq n}(S_j \cdot \mathbf{e}_\theta);$$
$$\bar{J}_n^{(i)}(\theta) := \arg\max_{0 \leq j \leq n}(S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } \underline{J}_n^{(i)}(\theta) := \arg\min_{0 \leq j \leq n}(S_j^{(i)} \cdot \mathbf{e}_\theta).$$

For $\gamma \in (0, 1/2)$ (a constant that will be chosen to be suitably small later in our argument), we denote by $E_{n,i}(\gamma)$ the event that the following occur:

- for all $\theta \in [\pi/4, 3\pi/4]$, $\underline{J}_n(\theta) < \gamma n$ and $\bar{J}_n(\theta) > (1-\gamma)n$;
- for all $\theta \in [\pi/4, 3\pi/4]$, $\underline{J}_n^{(i)}(\theta) < \gamma n$ and $\bar{J}_n^{(i)}(\theta) > (1-\gamma)n$;

note that the choice of the interval $[\pi/4, 3\pi/4]$ could be replaced by any other interval containing $\pi/2$ and bounded away from 0 and π. Define $I_{n,\gamma} := \{1, \ldots, n\} \cap [\gamma n, (1-\gamma)n]$. The next result is contained in Lemma 4.1 of [22].

Lemma 5.5. *For any $\gamma \in (0, 1/2)$ the following hold.*

(i) If $i \in I_{n,\gamma}$, then, on the event $E_{n,i}(\gamma)$,

$$R_n(\theta) - R_n^{(i)}(\theta) = (Z_i - Z'_i) \cdot \mathbf{e}_\theta, \text{ for any } \theta \in [\pi/4, 3\pi/4]. \quad (5.5)$$

(ii) If $\mathbb{E}\|Z\| < \infty$ and $\mu \neq \mathbf{0}$ then $\lim_{n\to\infty} \min_{1 \leq i \leq n} \mathbb{P}(E_{n,i}(\gamma)) = 1$.

In light of Lemma 5.4, the key to estimating (5.4) is provided by the following.

Lemma 5.6. *Let $\gamma \in (0, 1/2)$. Then for any $\delta \in (0, \pi/4)$ and any $i \in I_{n,\gamma}$, on $E_{n,i}(\gamma)$,*

$$\left| \sup_{|\theta - \pi/2| \leq \delta} R_n(\theta) - \sup_{|\theta - \pi/2| \leq \delta} R_n^{(i)}(\theta) - (Z_i - Z_i') \cdot \mathbf{e}_{\pi/2} \right| \leq \delta \|Z_i - Z_i'\|.$$

Proof. We claim that with $i \in I_{n,\gamma}$, for any $\theta_1, \theta_2 \in [\pi/4, 3\pi/4]$ with $\theta_1 < \theta_2$, on the event $E_{n,i}(\gamma)$, it holds that

$$\inf_{\theta_1 \leq \theta \leq \theta_2} (Z_i - Z_i') \cdot \mathbf{e}_\theta \leq \sup_{\theta_1 \leq \theta \leq \theta_2} R_n(\theta) - \sup_{\theta_1 \leq \theta \leq \theta_2} R_n^{(i)}(\theta) \leq \sup_{\theta_1 \leq \theta \leq \theta_2} (Z_i - Z_i') \cdot \mathbf{e}_\theta. \tag{5.6}$$

Also, it is easy to check that, for any $\mathbf{x} \in \mathbb{R}^2$ and $\theta_1, \theta_2 \in \mathbb{R}$,

$$|\mathbf{x} \cdot \mathbf{e}_{\theta_1} - \mathbf{x} \cdot \mathbf{e}_{\theta_2}| \leq \|\mathbf{x}\| |\theta_1 - \theta_2|. \tag{5.7}$$

Given the claim (5.6), and that, as follows from (5.7),

$$\sup_{|\theta - \pi/2| \leq \delta} (Z_i - Z_i') \cdot \mathbf{e}_\theta \leq (Z_i - Z_i') \cdot \mathbf{e}_{\pi/2} + \delta \|Z_i - Z_i'\|, \text{ and}$$
$$\inf_{|\theta - \pi/2| \leq \delta} (Z_i - Z_i') \cdot \mathbf{e}_\theta \geq (Z_i - Z_i') \cdot \mathbf{e}_{\pi/2} - \delta \|Z_i - Z_i'\|,$$

the statement of the lemma follows by taking $\theta_1 = \pi/2 - \delta$ and $\theta_2 = \pi/2 + \delta$.

It remains to establish the claim (5.6). First we note that for $f, g : \mathbb{R} \to \mathbb{R}$ with $\sup_{\theta \in I} |f(\theta)| < \infty$ and $\sup_{\theta \in I} |g(\theta)| < \infty$,

$$\inf_{\theta \in I} (f(\theta) - g(\theta)) \leq \sup_{\theta \in I} f(\theta) - \sup_{\theta \in I} g(\theta) \leq \sup_{\theta \in I} (f(\theta) - g(\theta)). \tag{5.8}$$

In particular, taking $I = [\theta_1, \theta_2]$, with $\theta_1, \theta_2 \in [\pi/3, 3\pi/4]$, we have

$$\inf_{\theta_1 \leq \theta \leq \theta_2} \left(R_n(\theta) - R_n^{(i)}(\theta) \right) \leq \sup_{\theta_1 \leq \theta \leq \theta_2} R_n(\theta) - \sup_{\theta_1 \leq \theta \leq \theta_2} R_n^{(i)}(\theta) \leq \sup_{\theta_1 \leq \theta \leq \theta_2} \left(R_n(\theta) - R_n^{(i)}(\theta) \right),$$

and, on the event $E_{n,i}(\gamma)$, we have from (5.5) that

$$R_n(\theta) - R_n^{(i)}(\theta) = (Z_i - Z_i') \cdot \mathbf{e}_\theta, \text{ for all } \theta \in [\theta_1, \theta_2],$$

which establishes the claim (5.6). □

To obtain rough estimates when the events $A_{n,i}(\delta)$ and $E_{n,i}(\gamma)$ do not occur, we need the following bound.

Lemma 5.7. *For any $i \in \{1, 2, \dots, n\}$, a.s.,*

$$|D_n^{(i)} - D_n| \leq 2\|Z_i\| + 2\|Z_i'\|.$$

Proof. Lemma 3.1 from [22] states that, for any $i \in \{1, 2, \dots, n\}$, a.s.,

$$\sup_{0 \leq \theta \leq \pi} \left| R_n(\theta) - R_n^{(i)}(\theta) \right| \leq 2\|Z_i\| + 2\|Z_i'\|.$$

Now from (5.4) and (5.8) we obtain the result. □

Now define the event $B_{n,i}(\gamma, \delta) := E_{n,i}(\gamma) \cap A_{n,i}(\delta)$. Let $B_{n,i}^{\mathrm{c}}(\gamma, \delta)$ denote the complementary event. The preceding results in this section can now be combined to obtain the following approximation lemma for $\Delta_{n,i}$ as given by (5.3).

Lemma 5.8. *Suppose that* $\mathbb{E}\,\|Z\| < \infty$ *and* $\mu \neq \mathbf{0}$*. For any* $\gamma \in (0, 1/2)$*,* $\delta \in (0, \pi/4)$*, and* $i \in I_{n,\gamma}$*, we have, a.s.,*

$$|\Delta_{n,i} - (Z_i - \mu) \cdot \hat{\mu}| \leq 3\|Z_i\|\mathbb{P}(B^{\mathrm{c}}_{n,i}(\gamma, \delta) \mid \mathcal{F}_i) + 3\,\mathbb{E}[\|Z'_i\|\mathbf{1}(B^{\mathrm{c}}_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i] + \delta\,(\|Z_i\| + \mathbb{E}\,\|Z\|)\,.$$

Proof. First observe that, since Z_i is $\mathcal{F}_i$-measurable and Z'_i is independent of $\mathcal{F}_i$,

$$\Delta_{n,i} - (Z_i - \mu) \cdot \hat{\mu} = \mathbb{E}[D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu} \mid \mathcal{F}_i].$$

Hence, by the triangle inequality,

$$|\Delta_{n,i} - (Z_i - \mu) \cdot \hat{\mu}| \leq \mathbb{E}\left[\left|D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu}\right| \mathbf{1}(B_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i\right] + \mathbb{E}\left[\left|D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu}\right| \mathbf{1}(B^{\mathrm{c}}_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i\right].$$

Here, by Lemma 5.7, we have that

$$\mathbb{E}\left[\left|D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu}\right| \mathbf{1}(B^{\mathrm{c}}_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i\right] \leq 3\,\mathbb{E}\left[(\|Z_i\| + \|Z'_i\|)\mathbf{1}(B^{\mathrm{c}}_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i\right].$$

Now, on $A_{n,i}(\delta)$ we have that

$$D_n = \sup_{|\theta - \pi/2| \leq \delta} R_n(\theta), \text{ and } D_n^{(i)} = \sup_{|\theta - \pi/2| \leq \delta} R_n^{(i)}(\theta),$$

and hence, by Lemma 5.6, on $A_{n,i}(\delta) \cap E_{n,i}(\gamma)$,

$$|D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu}| \leq \delta\|Z_i - Z'_i\|.$$

Hence

$$\mathbb{E}\left[\left|D_n - D_n^{(i)} - (Z_i - Z'_i) \cdot \hat{\mu}\right| \mathbf{1}(B_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i\right] \leq \delta\,\mathbb{E}[\|Z_i\| + \|Z'_i\| \mid \mathcal{F}_i].$$

Combining these bounds, and using the fact that Z_i is $\mathcal{F}_i$-measurable and Z'_i is independent of $\mathcal{F}_i$, we obtain the result. □

We are now almost ready to complete the proof of Theorem 5.1. To do so, we present an analogue of Lemma 6.1 from [22]; we set $V_i := (Z_i - \mu) \cdot \hat{\mu}$, and $W_{n,i} := \Delta_{n,i} - V_i$.

Lemma 5.9. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$ *and* $\mu \neq \mathbf{0}$*. Then*

$$\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} \mathbb{E}(W_{n,i}^2) = 0.$$

Proof. The proof is similar to that of Lemma 6.1 of [22]. Fix $\varepsilon \in (0, 1)$. Take $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/4)$, to be specified later. Note that from Lemma 5.7 we have $|W_{n,i}| \leq 3(\|Z_i\| + \mathbb{E}\,\|Z\|)$, so that, provided $\mathbb{E}(\|Z\|^2) < \infty$, we have $\mathbb{E}(W_{n,i}^2) \leq C_0$ for all n and all i, for some constant $C_0 < \infty$, depending only on the distribution of Z. Hence

$$\frac{1}{n} \sum_{i \notin I_{n,\gamma}} \mathbb{E}(W_{n,i}^2) \leq 2\gamma C_0.$$

From now on choose and fix $\gamma > 0$ small enough so that $2\gamma C_0 < \varepsilon$.

Now consider $i \in I_{n,\gamma}$. For such i, Lemma 5.8 yields an upper bound for $|W_{n,i}|$. Note that, for any $C_1 < \infty$, since Z'_i is independent of $\mathcal{F}_i$,

$$\mathbb{E}[\|Z'_i\|\mathbf{1}(B^{\mathrm{c}}_{n,i}(\gamma, \delta)) \mid \mathcal{F}_i] \leq \mathbb{E}[\|Z\|\mathbb{1}\{\|Z\| \geq C_1\}] + C_1\mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma, \delta) \mid \mathcal{F}_i].$$

Given $\varepsilon \in (0,1)$ we can take $C_1 = C_1(\varepsilon)$ large enough such that $\mathbb{E}[\|Z\| \mathbb{1}\{\|Z\| \geq C_1\}] \leq \varepsilon$, by dominated convergence; for convenience we take $C_1 > 1$ and $C_1 > \mathbb{E}\,\|Z\|$. Hence from Lemma 5.8 we obtain

$$|W_{n,i}| \leq 3(\|Z_i\| + C_1)\mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma,\delta) \mid \mathcal{F}_i] + 3\varepsilon + \delta\left(\|Z_i\| + \mathbb{E}\,\|Z\|\right).$$

Using the fact that $\mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma,\delta) \mid \mathcal{F}_i] \leq 1$, $\varepsilon \leq 1$, $\delta \leq 1$, and $C_1 > 1$, $C_1 > \mathbb{E}\,\|Z\|$, we can square both sides of the last display and collect terms to obtain

$$W_{n,i}^2 \leq 27 C_1^2 (1 + \|Z_i\|)^2 \mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma,\delta) \mid \mathcal{F}_i] + 9\varepsilon + 13 C_1^2 \delta\left(1 + \|Z_i\|\right)^2.$$

Since $\mathbb{E}(\|Z\|^2) < \infty$, it follows that, given ε and hence C_1, we can choose $\delta \in (0,\pi/4)$ sufficiently small so that $13 C_1^2 \delta\,\mathbb{E}[(1+\|Z_i\|)^2] < \varepsilon$; fix such a δ from now on. Then

$$\mathbb{E}(W_{n,i}^2) \leq 27 C_1^2\,\mathbb{E}[(1+\|Z_i\|)^2 \mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma,\delta) \mid \mathcal{F}_i]] + 10\varepsilon.$$

Here we have that, for any $C_2 > 0$,

$$\mathbb{E}[(1+\|Z_i\|)^2 \mathbb{P}[B^{\mathrm{c}}_{n,i}(\gamma,\delta) \mid \mathcal{F}_i]] \leq (1 + C_2)^2 \mathbb{P}(B^{\mathrm{c}}_{n,i}(\gamma,\delta)) + \mathbb{E}[(1+\|Z\|)^2 \mathbb{1}\{\|Z\| \geq C_2\}],$$

where dominated convergence shows that we may choose C_2 large enough so that the last term is less than ε/C_1^2, say. Then,

$$\mathbb{E}(W_{n,i}^2) \leq 37\varepsilon + 27 C_1^2 (1 + C_2)^2 \mathbb{P}(B^{\mathrm{c}}_{n,i}(\gamma,\delta)).$$

Finally, we see from Lemmas 5.4 and 5.5 that $\max_{1 \leq i \leq n} \mathbb{P}(B^{\mathrm{c}}_{n,i}(\gamma,\delta)) \to 0$, so that, for given $\varepsilon > 0$ (and hence C_1 and C_2) we may choose $n \geq n_0$ sufficiently large so that $\max_{i \in I_{n,\gamma}} \mathbb{E}(W_{n,i}^2) \leq 38\varepsilon$. Hence

$$\frac{1}{n} \sum_{i \in I_{n,\gamma}} \mathbb{E}(W_{n,i}^2) \leq 38\varepsilon,$$

for all $n \geq n_0$. Combining this result with the estimate for $i \notin I_{n,\gamma}$, we see that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(W_{n,i}^2) \leq 39\varepsilon,$$

for all $n \geq n_0$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

Proof of Theorem 5.1. First note that $W_{n,i}$ is $\mathcal{F}_i$-measurable with $\mathbb{E}(W_{n,i} \mid \mathcal{F}_{i-1}) = \mathbb{E}(\Delta_{n,i} \mid \mathcal{F}_{i-1}) - \mathbb{E}\,V_i = 0$, so that $W_{n,i}$ is a martingale difference sequence. Therefore by orthogonality, $n^{-1}\,\mathbb{E}[(\sum_{i=1}^n W_{n,i})^2] = n^{-1} \sum_{i=1}^n \mathbb{E}(W_{n,i}^2) \to 0$ as $n \to \infty$, by Lemma 5.9. In other words, $n^{-1/2} \sum_{i=1}^n W_{n,i} \to 0$ in L^2. But, by Lemma 5.3,

$$\sum_{i=1}^{n} W_{n,i} = \sum_{i=1}^{n} \Delta_{n,i} - \sum_{i=1}^{n} (Z_i - \mu) \cdot \hat{\mu} = D_n - \mathbb{E}\,D_n - (S_n - \mathbb{E}\,S_n) \cdot \hat{\mu}.$$

This yields the statement in the theorem. □

Finally we can give the proof of Theorem 1.9.

Proof of Theorem 1.9. Lemma 5.2 shows that

$$n^{-1/2} |\,\mathbb{E}\,D_n - \mathbb{E}\,S_n \cdot \hat{\mu}| \to 0. \tag{5.9}$$

Then by the triangle inequality

$$n^{-1/2} |D_n - S_n \cdot \hat{\mu}| \leq n^{-1/2} |D_n - \mathbb{E}\,D_n - (S_n - \mathbb{E}\,S_n) \cdot \hat{\mu}| + n^{-1/2} |\,\mathbb{E}\,D_n - \mathbb{E}\,S_n \cdot \hat{\mu}|,$$

which tends to 0 in L^2 by (5.1) and (5.9). □

Proof of Corollary 1.10. Corollary 1.10 is deduced from Theorem 1.9 in a very similar manner to how Theorems 1.1 and 1.2 in [22] were deduced from Theorem 1.3 there, so we omit the details. □

6 Diameter in the degenerate case

The aim of this section is to prove Theorem 1.11; thus we assume $\mu \neq \mathbf{0}$. First we state a result that will enable us to obtain the second statement in Theorem 1.11 from the first.

Lemma 6.1. *Suppose that* $\mathbb{E}(\|Z\|^p) < \infty$ *for some* $p > 4$, $\mu \neq \mathbf{0}$, *and* $\sigma_\mu^2 = 0$. *Then* $(D_n - \|\mu\|n)^2$ *is uniformly integrable.*

As in Section 4, we write $X_n := S_n \cdot \hat{\mu}$ and $Y_n := S_n \cdot \hat{\mu}_\perp$, where $\hat{\mu}_\perp$ is any fixed unit vector orthogonal to μ. Note that if $\sigma_\mu^2 = 0$, then $X_n = n\|\mu\|$ is deterministic.

Proof of Lemma 6.1. For $i \leq j$, we have $\|S_j - S_i\|^2 = (Y_j - Y_i)^2 + (X_j - X_i)^2$, so that

$$\begin{aligned}(D_n - \|\mu\|n)^2 &= \left(\max_{0 \leq i \leq j \leq n} \left((Y_j - Y_i)^2 + \|\mu\|^2 (j-i)^2 \right)^{1/2} - \|\mu\|n \right)^2 \\ &\leq \left(\|\mu\|n \max_{0 \leq i \leq j \leq n} \left(1 + \frac{(Y_j - Y_i)^2}{\|\mu\|^2 n^2} \right)^{1/2} - \|\mu\|n \right)^2 .\end{aligned}$$

Since $(1+y)^{1/2} \leq 1 + (y/2)$ for $y \geq 0$, and $(a-b)^2 \leq 2(a^2+b^2)$ for $a, b \in \mathbb{R}$, we obtain

$$(D_n - \|\mu\|n)^2 \leq \left(\|\mu\|n \max_{0 \leq i \leq j \leq n} \frac{(Y_j - Y_i)^2}{2\|\mu\|^2 n^2} \right)^2 \leq \frac{4}{\|\mu\|^2} \max_{1 \leq i \leq n} \frac{Y_i^4}{n^2}.$$

Now, $|Y_n|$ is a non-negative submartingale, so Doob's L^p inequality [9, p. 505] yields

$$\mathbb{E}\left[\left(\max_{1 \leq i \leq n} \frac{Y_i^4}{n^2} \right)^{p/4} \right] = n^{-p/2}\, \mathbb{E}\left(\max_{1 \leq i \leq n} |Y_i|^p \right) \leq C_p n^{-p/2}\, \mathbb{E}(|Y_n|^p),$$

for any $p > 1$ and some constant $C_p < \infty$. Assuming that $\mathbb{E}(\|Z\|^p) < \infty$ for $p > 4$, Y_n is a random walk on $\mathbb{R}$ whose increments have zero mean and finite pth moments, so, by the Marcinkiewicz–Zygmund inequality [9, p. 151], $\mathbb{E}(|Y_n|^p) \leq Cn^{p/2}$. Hence

$$\sup_{n \geq 0} \mathbb{E}\left[\left((D_n - \|\mu\|n)^2 \right)^{p/4} \right] < \infty,$$

which, since $p/4 > 1$, establishes uniform integrability. □

Next we show that, under the conditions of Theorem 1.11, the diameter must be attained by a point close to the start and one close to the end of the walk.

Lemma 6.2. *Suppose that* $\mathbb{E}(\|Z\|^2) < \infty$, $\mu \neq \mathbf{0}$, *and* $\sigma_\mu^2 = 0$. *Let* $\beta \in (0,1)$. *Then, a.s., for all but finitely many* n,

$$D_n = \max_{\substack{0 \leq i \leq n^\beta \\ n - n^\beta \leq j \leq n}} \|S_j - S_i\|.$$

Proof. Fix $\beta \in (0,1)$. Since $D_n = \max_{0 \leq i,j \leq n} \|S_j - S_i\|$, we have

$$D_n = \max \left\{ \max_{\substack{0 \leq i \leq n^\beta \\ n - n^\beta \leq j \leq n}} \|S_j - S_i\|, \max_{\substack{0 \leq i \leq n^\beta \\ 0 \leq j \leq n - n^\beta}} \|S_j - S_i\|, \max_{n^\beta \leq i,j \leq n} \|S_j - S_i\| \right\}. \tag{6.1}$$

It is clear that

$$\max_{\substack{0 \leq i \leq n^\beta \\ n - n^\beta \leq j \leq n}} \|S_j - S_i\| \geq \|S_n\| \geq |X_n| = \|\mu\|n.$$

We aim to show that the other two terms on the right-hand side of (6.1) are strictly less than $\|\mu\|n$ for all but finitely many n.

A consequence of the law of the iterated logarithm is that, for any $\varepsilon > 0$, a.s., for all but finitely many n, $\max_{0\le i\le n} Y_i^2 \le n^{1+\varepsilon}$; see e.g. [9, p. 384]. Take $\varepsilon \in (0,\beta)$. Then,

$$\begin{aligned}\max_{\substack{0\le i\le n^\beta\\ 0\le j\le n-n^\beta}} \|S_j - S_i\|^2 &\le \max_{\substack{0\le i\le n^\beta\\ 0\le j\le n-n^\beta}} |X_j - X_i|^2 + \max_{\substack{0\le i\le n^\beta\\ 0\le j\le n-n^\beta}} |Y_j - Y_i|^2 \\ &\le \|\mu\|^2(n-n^\beta)^2 + \max_{0\le j\le n-n^\beta} Y_j^2 + \max_{0\le i\le n^\beta} Y_i^2 + 2 \max_{\substack{0\le i\le n^\beta\\ 0\le j\le n-n^\beta}} |Y_j||Y_i| \\ &\le \|\mu\|^2 n^2 - 2\|\mu\|^2 n^{1+\beta} + \|\mu\|^2 n^{2\beta} + n^{1+\varepsilon},\end{aligned}$$

for all but finitely many n. Since $\varepsilon < \beta < 1$, this last expression is strictly less than $\|\mu\|^2 n^2$ for all n sufficiently large. Similarly,

$$\begin{aligned}\max_{n^\beta\le i,j\le n} \|S_j - S_i\|^2 &\le \|\mu\|^2(n-n^\beta)^2 + \max_{n^\beta\le j\le n} Y_j^2 + \max_{n^\beta\le i\le n} Y_i^2 + 2\max_{n^\beta\le i,j\le n} |Y_j||Y_i| \\ &\le \|\mu\|^2 n^2 - 2\|\mu\|^2 n^{1+\beta} + \|\mu\|^2 n^{2\beta} + n^{1+\varepsilon},\end{aligned}$$

for all but finitely many n, and, as before, this is strictly less than $\|\mu\|^2 n^2$ for all n sufficiently large. Then (6.1) yields the result. □

The main remaining step in the proof of Theorem 1.11 is the following result.

Lemma 6.3. *Suppose that $\mathbb{E}(\|Z\|^p) < \infty$ for some $p > 2$, $\mu \ne \mathbf{0}$, and $\sigma_\mu^2 = 0$. Then, as $n \to \infty$, $D_n - \|S_n\| \to 0$, a.s.*

Proof. Using the fact that $\|S_n\|^2 = \|\mu\|^2 n^2 + Y_n^2$, we have that, for $j \le n$,

$$\begin{aligned}\|S_j - S_i\|^2 &= \|\mu\|^2 (j-i)^2 + (Y_j - Y_i)^2 \\ &= \|S_n\|^2 + \|\mu\|^2 i^2 + \|\mu\|^2 j^2 - 2\|\mu\|^2 ij - \|\mu\|^2 n^2 + Y_i^2 + Y_j^2 - 2Y_iY_j - Y_n^2 \\ &\le \|S_n\|^2 + \|\mu\|^2 i^2 - (Y_n - Y_j)(Y_n + Y_j) + 2Y_i(Y_n - Y_j) - 2Y_iY_n + Y_i^2.\end{aligned}$$

Here, as before, we have that as a consequence of the law of the iterated logarithm, for any $\varepsilon > 0$, $\max_{0\le i\le n^\beta} |Y_iY_n| \le n^{\frac{1+\beta}{2}+\varepsilon}$ and $\max_{0\le i\le n^\beta} Y_i^2 \le n^{\beta+\varepsilon}$ for all but finitely many n. For the terms involving Y_j, Lemma A.2 shows that we may choose $\beta \in (0, 1/2)$ such that, for any sufficiently small $\varepsilon > 0$,

$$\max_{n-n^\beta\le j\le n} |Y_n - Y_j| \le n^{\frac12-\varepsilon}, \text{ and } \max_{n-n^\beta\le j\le n} |Y_n - Y_j||Y_n + Y_j| \le n^{1-\varepsilon},$$

for all but finitely many n. With this choice of β and sufficiently small ε, we combine these bounds to obtain

$$\max_{\substack{0\le i\le n^\beta\\ n-n^\beta\le j\le n}} \|S_j - S_i\|^2 \le \|S_n\|^2 + \|\mu\|^2 n^{2\beta} + n^{1-\varepsilon} + n^{\frac{1+\beta}{2}+\varepsilon} + n^{\beta+\varepsilon},$$

for all but finitely many n. Since $\beta \in (0, 1/2)$, we may apply Lemma 6.2 and choose $\varepsilon > 0$ sufficiently small to see that $D_n^2 \le \|S_n\|^2 + n^{1-\varepsilon}$, for all but finitely many n. Hence

$$D_n \le \|S_n\| \left(1 + \|S_n\|^{-2} n^{1-\varepsilon}\right)^{1/2} \le \|S_n\| \left(1 + \|\mu\|^{-2} n^{-1-\varepsilon}\right)^{1/2},$$

since $\|S_n\| \ge n\|\mu\|$. Using the fact that $(1+x)^{1/2} \le 1 + (x/2)$ for $x \ge 0$, we get

$$D_n \le \|S_n\| \left(1 + \tfrac12 \|\mu\|^{-2} n^{-1-\varepsilon}\right) \le \|S_n\| + \|\mu\|^{-1} n^{-\varepsilon},$$

for all but finitely many n, since, by the strong law of large numbers, $\|S_n\| \le 2\|\mu\|n$ for all but finitely many n. Combined with the bound $D_n \ge \|S_n\|$, this completes the proof. □

Proof of Theorem 1.11. Combining Lemmas 6.3 and 4.2 with Slutsky's theorem [9, p. 249] and the fact that, in this case, $X_n = \|\mu\| n$, we obtain (1.4).

From Lemma 6.1 we have that, if $\mathbb{E}(\|Z\|^p) < \infty$ for $p > 4$, both $D_n - \|\mu\| n$ and $(D_n - \|\mu\| n)^2$ are uniformly integrable. Thus from (1.4) we obtain

$$\lim_{n\to\infty} \mathbb{E}(D_n - \|\mu\| n) = \mathbb{E}\left[\frac{\sigma_{\mu_\perp}^2 \zeta^2}{2\|\mu\|}\right] = \frac{\sigma_{\mu_\perp}^2}{2\|\mu\|}, \text{ and}$$
$$\lim_{n\to\infty} \mathbb{E}[(D_n - \|\mu\| n)^2] = \mathbb{E}\left[\frac{\sigma_{\mu_\perp}^4 \zeta^4}{4\|\mu\|^2}\right] = \frac{3\sigma_{\mu_\perp}^4}{4\|\mu\|^2}.$$

Using the fact that

$$\mathbb{V}\mathrm{ar}\, D_n = \mathbb{V}\mathrm{ar}(D_n - \|\mu\| n) = \mathbb{E}[(D_n - \|\mu\| n)^2] - \mathbb{E}[D_n - \|\mu\| n]^2,$$

we obtain (1.5) by letting $n \to \infty$. □

A Auxiliary results

In this appendix we present two technical results on sums of i.i.d. random variables that are needed in the body of the paper. The first is used in the proof of Lemma 4.2.

Lemma A.1. *Let $\xi, \xi_1, \xi_2, \dots$ be i.i.d. random variables with $\mathbb{E}(\xi^2) < \infty$ and $\mathbb{E}\,\xi > 0$. Let $X_n = \sum_{k=1}^n \xi_k$. Then $\lim_{n\to\infty} \mathbb{E}\, X_n^- = 0$.*

Proof. Let $\mathbb{E}\,\xi = m > 0$ and $\mathbb{V}\mathrm{ar}\,\xi = s^2 < \infty$. Fix $\varepsilon > 0$. Note that

$$\mathbb{E}\, X_n^- = \int_0^\infty \mathbb{P}(X_n^- > r)\mathrm{d}r = \int_0^{\varepsilon n} \mathbb{P}(X_n^- > r)\mathrm{d}r + \int_{\varepsilon n}^\infty \mathbb{P}(X_n^- > r)\mathrm{d}r.$$

Here we have that, by Chebyshev's inequality,

$$\mathbb{P}(X_n^- > r) \le \mathbb{P}(|X_n - mn| > mn + r) \le \frac{\mathbb{V}\mathrm{ar}\, X_n}{(mn+r)^2} = \frac{s^2 n}{(mn+r)^2}.$$

It follows that

$$\int_0^{\varepsilon n} \mathbb{P}(X_n^- > r)\mathrm{d}r \le s^2 n \int_0^{\varepsilon n} \frac{\mathrm{d}r}{(mn+r)^2} \le \frac{s^2 \varepsilon}{m^2}. \tag{A.1}$$

For $B \in (0,\infty)$ let $\xi_k' := \xi_k \mathbb{1}\{|\xi_k| \le B\}$ and $\xi_k'' := \xi_k \mathbb{1}\{|\xi_k| > B\}$. Set $X_n' := \sum_{k=1}^n \xi_k'$ and $X_n'' := \sum_{k=1}^n \xi_k''$. By dominated convergence, we have that as $B \to \infty$, $\mathbb{E}\,\xi_1' \to m$, $\mathbb{V}\mathrm{ar}\,\xi_1' \to s^2$, $\mathbb{E}\,|\xi_1''| \to 0$, and $\mathbb{V}\mathrm{ar}\,\xi_1'' \to 0$, so in particular we may (and do) choose B large enough so that $\mathbb{E}\,\xi_1' > m/2$, $\mathbb{E}\,|\xi_1''| < \varepsilon/4$, and $\mathbb{V}\mathrm{ar}\,\xi_1'' < \varepsilon^2$.

Since $X_n = X_n' + X_n''$, for any $r > 0$ we have

$$\mathbb{P}(X_n < -r) \le \mathbb{P}(X_n' < -r/2) + \mathbb{P}(X_n'' < -r/2). \tag{A.2}$$

Here since $\mathbb{E}((\xi_k')^4) \le B^4 < \infty$ it follows from Markov's inequality and the Marcinkiewicz–Zygmund inequality [9, p. 151] that for some constant $C < \infty$ (depending on B),

$$\mathbb{P}(X_n' < -r) \le \mathbb{P}(|X_n' - \mathbb{E}\, X_n'|^4 > (\mathbb{E}\, X_n' + r)^4) \le \frac{Cn^2}{((m/2)n + r)^4}.$$

So

$$\int_{\varepsilon n}^\infty \mathbb{P}(X_n' < -r/2)\mathrm{d}r \le 16Cn^2 \int_0^\infty \frac{\mathrm{d}r}{(mn+r)^4} = O(1/n). \tag{A.3}$$

On the other hand, by Chebyshev's inequality, for $r > (\varepsilon/4)n$,

$$\mathbb{P}(X_n'' < -r) \le \mathbb{P}(|X_n'' - \mathbb{E}\, X_n''| > \mathbb{E}\, X_n'' + r) \le \frac{\mathbb{V}\mathrm{ar}\, X_n''}{(r-(\varepsilon/4)n)^2} \le \frac{\varepsilon^2 n}{(r-(\varepsilon/4)n)^2}.$$

Hence

$$\int_{\varepsilon n}^{\infty} \mathbb{P}(X_n'' < -r/2) \le 4\varepsilon^2 n \int_{\varepsilon n}^{\infty} \frac{\mathrm{d}r}{(r-(\varepsilon/2)n)^2} = 8\varepsilon. \tag{A.4}$$

So from (A.2) with (A.3) and (A.4), we have

$$\limsup_{n\to\infty} \int_{\varepsilon n}^{\infty} \mathbb{P}(X_n < -r)\mathrm{d}r \le 8\varepsilon,$$

which combined with (A.1) implies that

$$\limsup_{n\to\infty} \mathbb{E}\, X_n^- \le \frac{s^2\varepsilon}{m^2} + 8\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

The next result is used in the proof of Lemma 6.3.

Lemma A.2. *Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(|\xi|^p) < \infty$ for some $p > 2$, and $\mathbb{E}\,\xi = 0$. For $0 \le j \le n$, let $T_{n,j} := \sum_{i=n-j}^{n} \xi_i$. Then there exist $\beta_0 \in (0, 1/2)$ and $\varepsilon_0 \in (0, 1/2)$ such that for any $\beta \in (0, \beta_0)$ and any $\varepsilon \in (0, \varepsilon_0)$,*

$$\lim_{n\to\infty} \max_{0\le j\le n^\beta} \frac{|T_{n,j}|}{n^{(1/2)-\varepsilon}} = 0, \text{ a.s.}$$

Remark A.3. On first sight, by the fact that there are $O(n^\beta)$ terms in the sum $T_{n,j}$, one's intuition may be misled to conclude that $T_{n,j}$ should be only of size about $n^{\beta/2}$. However, note that assuming only $\mathbb{E}(\xi^2) < \infty$, $\max_{0\le i\le n} \xi_i$ can be essentially as big as $n^{1/2}$, and with probability at least $1/n$ this maximal value is a member of $T_{n,j}$, and so it seems reasonable to expect that $T_{n,j}$ should be as big as $n^{1/2}$ infinitely often. Thus our $p > 2$ moments condition seems to be necessary.

Proof. Let $\xi_i' = \xi_i \mathbb{1}\{|\xi_i| \le i^{1/2-\delta}\}$ and $\xi_i'' = \xi_i \mathbb{1}\{|\xi_i| > i^{1/2-\delta}\}$ for some $\delta \in (0, 1/2)$ to be chosen later. Then we use the subadditivity of the supremum, the triangle inequality, and the condition $\varepsilon \in (0, \varepsilon_0)$ to get

$$\max_{0\le j\le n^\beta} \frac{|T_{n,j}|}{n^{1/2-\varepsilon}} \le \max_{0\le j\le n^\beta} \frac{|\sum_{i=n-j}^{n}(\xi_i' - \mathbb{E}\,\xi_i')|}{n^{1/2-\varepsilon}} + \frac{\sum_{i=n-n^\beta}^{n} |\mathbb{E}\,\xi_i'|}{n^{1/2-\varepsilon_0}} + \frac{\sum_{i=n-n^\beta}^{n} |\xi_i''|}{n^{1/2-\varepsilon_0}}, \tag{A.5}$$

where, and for the rest of this proof, if n^β appears in the index of a sum, we understand it to be shorthand for $\lfloor n^\beta \rfloor$. By Markov's inequality, since $\mathbb{E}(|\xi|^p) < \infty$ for $p > 2$ we have

$$\mathbb{P}\left(|\xi_i| > i^{1/2-\delta}\right) \le \frac{\mathbb{E}(|\xi|^p)}{i^{(1/2-\delta)p}} = O(i^{\delta p - p/2}).$$

Suppose that $\delta \in (0, (p-2)/2p)$, so that $\delta p - p/2 < -1$, and thus the Borel–Cantelli lemma implies that $\xi_i'' = 0$ for all but finitely many i. Thus, for any $\beta, \varepsilon_0 \in (0, 1/2)$,

$$\lim_{n\to\infty} \frac{\sum_{i=n-n^\beta}^{n} |\xi_i''|}{n^{1/2-\varepsilon_0}} = 0, \text{ a.s.}$$

For the second term on the right-hand side of (A.5), $\mathbb{E}\,\xi = 0$ implies $|\mathbb{E}\,\xi_i'| = |\mathbb{E}\,\xi_i''|$, so

$$\sum_{i=n-n^\beta}^{n} |\mathbb{E}\,\xi_i'| = \sum_{i=n-n^\beta}^{n} |\mathbb{E}\,\xi_i''| \le (n^\beta + 1)\,\mathbb{E}\left(|\xi| \mathbb{1}\{|\xi| > (n/2)^{1/2-\delta}\}\right),$$

for all n large enough so that $n - n^\beta > n/2$. Here

$$\mathbb{E}\left(|\xi|\mathbb{1}\{|\xi| > (n/2)^{1/2-\delta}\}\right) = \mathbb{E}\left(|\xi|^2|\xi|^{-1}\mathbb{1}\{|\xi| > (n/2)^{1/2-\delta}\}\right) \leq Cn^{\delta-1/2},$$

for some constant C depending only on $\mathbb{E}(\xi^2)$. Suppose that $\delta \leq 1/4$. Then we get $\sum_{i=n-n^\beta}^{n} |\mathbb{E}\,\xi_i'| = O(n^{\beta-1/4})$, so that, for any $\beta \in (0, 1/2)$ and $\varepsilon_0 \in (0, 1/4)$,

$$\lim_{n\to\infty} \frac{\sum_{i=n-n^\beta}^{n} |\mathbb{E}\,\xi_i'|}{n^{1/2-\varepsilon_0}} = 0, \text{ a.s.}$$

Finally, we consider the first term on the right-hand side of (A.5), with the truncated, centralised sum, which we denote as $T'_{n,j} := \sum_{i=n-j}^{n}(\xi_i' - \mathbb{E}\,\xi_i')$. The $\xi_i' - \mathbb{E}\,\xi_i'$ are independent, zero-mean random variables with $|\xi_i' - \mathbb{E}\,\xi_i'| \leq 2n^{1/2-\delta}$ for $i \leq n$, so we may apply the Azuma–Hoeffding inequality [16, p. 33] to obtain, for any $t \geq 0$,

$$\mathbb{P}\left(|T'_{n,j}| \geq t\right) \leq 2\exp\left(-\frac{t^2}{8(j+1)n^{1-2\delta}}\right).$$

In particular, taking $t = n^{1/2-\varepsilon_0}$ we obtain

$$\begin{aligned}\mathbb{P}\left(\max_{0\leq j\leq n^\beta} |T'_{n,j}| \geq n^{1/2-\varepsilon_0}\right) &\leq (n^\beta+1)\max_{0\leq j\leq n^\beta}\mathbb{P}\left(|T'_{n,j}| \geq n^{1/2-\varepsilon_0}\right)\\ &\leq 2(n^\beta+1)\exp\left(-\frac{n^{1-2\varepsilon_0}}{16n^{1+\beta-2\delta}}\right), \qquad \text{(A.6)}\end{aligned}$$

for all n sufficiently large. Now choose and fix $\delta = \delta(p) := \min\{1/4, (p-2)/4p\}$, so $\delta > 0$ satisfies the bounds earlier in this proof, and then choose $\beta < \beta_0 := \delta$ such that

$$\frac{n^{1-2\varepsilon_0}}{n^{1+\beta-2\delta}} = n^{2\delta-2\varepsilon_0-\beta} \geq n^{\delta-2\varepsilon_0}.$$

So choosing $\varepsilon_0 = \delta/4$ we have that the probability bound in (A.6) is summable. Thus by the Borel–Cantelli lemma, we have that $\max_{0\leq j\leq n^\beta}|T'_{n,j}| \leq n^{1/2-\varepsilon_0}$ for all but finitely many n, a.s. It follows that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{n\to\infty}\frac{\max_{0\leq j\leq n^\beta}|\sum_{i=n-j}^{n}(\xi_i' - \mathbb{E}\,\xi_i')|}{n^{1/2-\varepsilon}} = 0, \text{ a.s.},$$

which completes the proof. □

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