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The incipient infinite cluster of the uniform infinite half-planar triangulation

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Abstract

We introduce the Incipient Infinite Cluster (IIC) in the critical Bernoulli site percolation model on the Uniform Infinite Half-Planar Triangulation (UIHPT), which is the local limit of large random triangulations with a boundary. The IIC is defined from the UIHPT by conditioning the open percolation cluster of the origin to be infinite. We prove that the IIC can be obtained by adding within the UIHPT an infinite triangulation with a boundary whose distribution is explicit.

Keywords: random planar maps; local limits; percolation; incipient infinite cluster.
AMS MSC 2010: 05C80; 60K35; 60K37.
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1 Introduction

The purpose of this work is to describe the geometry of a large critical percolation cluster in the (type 2) Uniform Infinite Half-Planar Triangulation (UIHPT for short), which is the *local limit* of random triangulations with a boundary, upon letting first the volume and then the perimeter tend to infinity. Roughly speaking, rooted planar maps are close in the local sense if they have the same graph distance ball of a large radius around the root. The study of local limits of large planar maps goes back to Angel & Schramm, who introduced in [5] the Uniform Infinite Planar Triangulation (UIPT), while the half-plane model was defined later on by Angel in [3]. Given a planar map, the Bernoulli site percolation model consists in declaring independently every site open with probability p and closed otherwise.

Local limits of large planar maps equipped with a percolation model have been studied extensively. Critical thresholds and critical exponents were provided for the UIPT [2, 20] and the UIHPT [3, 4] as well as for their quadrangular equivalents [27, 4, 29]. The central idea of these papers is a Markovian exploration of the maps introduced by Angel and called the *peeling process*, which turns out to be much simpler in half-plane

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models. In this setting, the scaling limits of (half-plane) crossing probabilities [3, 29] can also be derived.

A natural goal in percolation theory is the description of the geometry of percolation clusters at criticality. In the UIPT, such a description has been achieved by Curien & Kortchemski in [14]. They identified the *scaling limit* of the boundary of a critical percolation cluster conditioned to be large as a random *stable looptree* with parameter $3/2$, previously introduced in [15]. Here, our aim is to understand not only the local limit of a percolation cluster conditioned to be large, but also the local limit of the whole UIHPT under this conditioning. This is inspired by the work of Kesten [23] in the two-dimensional square lattice.

Precisely, we consider a random map distributed as the UIHPT, equipped with a site percolation model with parameter $p \in [0,1]$, and denote the resulting probability measure by $\mathbf{P}_p$ (details are postponed to Section 2). Angel proved in [3] that the critical threshold p_c equals $1/2$, and that there is no infinite connected component at the critical point almost surely. We also work conditionally on a "White-Black-White" boundary condition, meaning that all the vertices on the infinite simple boundary of the map are closed, except the origin which is open. We denote by $\mathcal{C}$ the open cluster of the origin, and by $|\mathcal{C}|$ its number of vertices or *volume*. We will also define a natural notion of *height* $h(\mathcal{C})$ of the cluster $\mathcal{C}$ by using the exploration of the percolation interface in the UIHPT (see Section 5.1 for details). Theorem 2.3 states the existence of a probability measure $\mathbf{P}_{\mathsf{IIC}}$ such that

$$\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty) \underset{p \downarrow p_c}{\Longrightarrow} \mathbf{P}_{\mathsf{IIC}} \quad \text{and} \quad \mathbf{P}_{p_c}(\cdot \mid h(\mathcal{C}) \geq n) \underset{n\to\infty}{\Longrightarrow} \mathbf{P}_{\mathsf{IIC}}$$

in the sense of weak convergence, for the local topology. The probability measure $\mathbf{P}_{\mathsf{IIC}}$ is called (the law of) the Incipient Infinite Cluster of the UIHPT (IIC for short) and is supported on triangulations of the half-plane. As in [23], the limit is universal in the sense that it arises under at least two distinct and natural ways of conditioning $\mathcal{C}$ to be large. We emphasize that $\mathbf{P}_{\mathsf{IIC}}$ is a construction of the Incipient Infinite Cluster that is *annealed* over the map.

The proof of Theorem 2.3 unveils a decomposition of the IIC into independent sub-maps with an explicit distribution. We first consider the percolation clusters of the origin (open) and its neighbours on the boundary (closed). By filling in their finite holes, we obtain the associated percolation *hulls*. The boundaries of the percolation hulls are random infinite *looptrees*, that is, a collection of cycles glued along a tree structure introduced in [15]. The percolation hulls are rebuilt from their boundaries by filling in the cycles with independent *Boltzmann triangulations* with a simple boundary. Finally, the IIC is recovered by gluing the percolation hulls along *uniform infinite necklaces*, which are random triangulations of a semi-infinite strip first introduced in [11].

In Theorem 2.1, we decompose the UIHPT into two infinite sub-maps distributed as the closed percolation hulls of the IIC, and glued along a uniform necklace. The idea of such a decomposition goes back to [18]. Together with Theorem 2.3, this describes how the geometry of the UIHPT is altered by the conditioning to have an infinite open percolation cluster. The IIC is obtained by cutting the UIHPT along the uniform necklace, and gluing inside, *ex-nihilo*, the infinite open percolation hull.

2 Definitions and results

Notation. We will use the notation $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_- := \{0, -1, \ldots\}$.

2.1 Random planar maps and percolation

Maps. A planar map is the proper embedding of a finite connected graph in the two-dimensional sphere, up to orientation-preserving homeomorphisms. For technical reasons, the planar maps we consider are always *rooted*, meaning that an oriented edge called the *root* is distinguished. The *origin* is the tail vertex of the root. The faces of a planar map are the connected components of the complement of the embedding of the edges. The degree of a face is the number of its incident oriented edges (with the convention that the face incident to an oriented edge lies on its left). The face incident to the right of the root edge is called the *root face*, and the other faces are called *internal*. The set of all planar maps is denoted by $\mathcal{M}_f$, and a generic element of $\mathcal{M}_f$ is usually denoted by $\mathbf{m}$.

In the next part, we will consider planar maps *with a boundary*. This means that the embedding of the edges and vertices of the root face is interpreted as the boundary $\partial\mathbf{m}$ of $\mathbf{m}$. When the edges of the root face form a cycle without self-intersection, the planar map is said to have a *simple boundary*. The degree of the root face is then the *perimeter* of the map. Any vertex that does not belong to the root face is an *inner* vertex. The set of planar maps with a boundary of perimeter k is denoted by $\mathcal{M}_k$ (resp. $\widehat{\mathcal{M}}_k$ if the boundary is simple).

In this paper, we deal with *triangulations*, which are planar maps whose faces all have degree three. We will also consider *triangulations with a boundary*, in which all the faces are triangles except possibly the root face. We make the technical assumption that triangulations are 2-connected (or type 2), meaning that multiple edges are allowed but self-loops are not.

Local topology. The *local topology* on $\mathcal{M}_f$ is induced by the distance d_{loc} defined by

$$d_{\mathrm{loc}}(\mathbf{m},\mathbf{m}') := \left(1+\sup\left\{R\geq 0 : \mathbf{B}_R(\mathbf{m})=\mathbf{B}_R(\mathbf{m}')\right\}\right)^{-1}, \quad \mathbf{m},\mathbf{m}'\in\mathcal{M}_f.$$

Here, $\mathbf{B}_R(\mathbf{m})$ is the ball of radius R in $\mathbf{m}$ for the graph distance, centered at the origin vertex. Precisely, $\mathbf{B}_0(\mathbf{m})$ is the origin of the map, and for every $R>0$, $\mathbf{B}_R(\mathbf{m})$ contains vertices at graph distance less than R from the origin, and all the edges whose endpoints are in this set.

Equipped with the distance d_{loc}, $\mathcal{M}_f$ is a metric space whose completion is denoted by $\mathcal{M}$. The elements of $\mathcal{M}_\infty := \mathcal{M}\backslash\mathcal{M}_f$ are called infinite planar maps. The boundary of an infinite planar map is the embedding of edges and vertices of its root face. When the root face is infinite, its vertices and edges on the left (resp. right) of the origin form the left (resp. right) boundary of the map, and the boundary is called simple if it is isomorphic to $\mathbb{Z}$. An infinite planar map whose faces are all triangles (except the root face) is called an infinite triangulation (with a boundary).

The uniform infinite half-planar triangulation. The study of the convergence of random planar triangulations in the local topology goes back to Angel and Schramm [5, Theorem 1.8], who introduced the Uniform Infinite Planar Triangulation (UIPT). Later on, Angel defined in [3] a model of infinite triangulation with an infinite boundary that has nicer properties. For $n\geq 0$ and $k\geq 2$, let $\widehat{\mathcal{M}}_{n,k}^{\triangle}$ be the set of rooted triangulations of the k-gon (i.e., with a simple boundary of perimeter k) having n inner vertices. Let $\mathbb{P}_{n,k}$ be the uniform probability measure on $\widehat{\mathcal{M}}_{n,k}^{\triangle}$. Then, first by [5, Theorem 5.1] and then by [3, Theorem 2.1], in the sense of weak convergence for the local topology,

$$\mathbb{P}_{n,k} \underset{n\to\infty}{\Longrightarrow} \mathbb{P}_{\infty,k} \qquad \text{and} \qquad \mathbb{P}_{\infty,k} \underset{k\to\infty}{\Longrightarrow} \mathbb{P}_{\infty,\infty}.$$

The probability measure $\mathbb{P}_{\infty,k}$ is called (the law of) the UIPT of the k-gon, while $\mathbb{P}_{\infty,\infty}$ is (the law of) the Uniform Infinite Half-Planar Triangulation (UIHPT). The measure $\mathbb{P}_{\infty,\infty}$ is supported on infinite triangulations of the upper half-plane (that is, infinite one-ended triangulations with an infinite simple boundary, see Figure 2). It also enjoys a re-rooting invariance property, in the sense that it is preserved under the natural shift operation for the root edge along the boundary.

The properties of the UIHPT are best understood using a probability measure supported on triangulations with fixed perimeter called the *Boltzmann measure*. Let $k \geq 2$ and introduce the *partition function*

$$Z_k := \sum_{n \in \mathbb{Z}_+} \#\widehat{\mathcal{M}}^{\triangle}_{n,k} \left(\frac{2}{27}\right)^n. \tag{2.1}$$

The Boltzmann measure on the set $\widehat{\mathcal{M}}^{\triangle}_k$ of triangulations with a simple boundary of perimeter k is defined as follows. For every $n \in \mathbb{Z}_+$ and $\mathbf{m} \in \widehat{\mathcal{M}}^{\triangle}_{n,k}$,

$$\mathbb{W}_k(\mathbf{m}) := \frac{1}{Z_k} \left(\frac{2}{27}\right)^n. \tag{2.2}$$

This object is of particular importance because it satisfies a branching property.

The spatial Markov property. In this paragraph, we detail the so-called *peeling* technique introduced by Angel. The general idea is to suppose the whole map unknown and to reveal its faces one after another. To do so, consider a map M with law $\mathbb{P}_{\infty,\infty}$ and the face A of M incident to the root. To *reveal* or *peel* the face A means that we suppose the whole map unknown and work conditionally on the *configuration* of this face (see the definition below). We now consider the map $M \backslash$A, obtained by removing (or "peeling") the root edge of M. This map has at most one cut-vertex on the boundary, which defines sub-maps that we call the (connected) components of $M \backslash$A.

The spatial Markov property [3, Theorem 2.2] shows that $M \backslash$A has a unique infinite component M' with law $\mathbb{P}_{\infty,\infty}$, and at most one finite component $\tilde{M}$ with law $\mathbb{W}_l$ (and perimeter $l \geq 2$ given by the configuration of the face A). Moreover, $\tilde{M}$ is independent of M'.

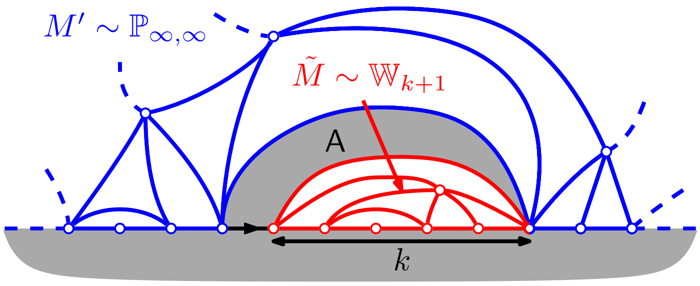

Figure 1: The spatial Markov property.

The peeling technique is extended to a *peeling process* by successively revealing a new face in the unique infinite component. This is the key idea to study percolation, see [3, 4, 29].

We now describe the possible configurations of the face A incident to the root in the UIHPT. On the one hand, some edges of A, called *exposed*, belong to the boundary of the infinite component of $M \backslash$A. On the other hand, some edges of the boundary, called *swallowed*, may be enclosed in a finite component of $M \backslash$A. The number of exposed and swallowed edges are denoted by $\mathcal{E}$ and $\mathcal{R}$. We may use the notations $\mathcal{R}_l$ and $\mathcal{R}_r$ for the number of swallowed edges on the left and right of the root edge. The probabilities of the two possible configurations are provided in [4, Section 2.3.1]:

1. The third vertex of A is an inner vertex $(\mathcal{E} = 2, \mathcal{R} = 0)$ with probability $q_{-1} = 2/3$.
2. The third vertex of A is on the boundary of the map, $k \in \mathbb{N}$ edges on the left (or right) of the root $(\mathcal{E} = 1, \mathcal{R} = k)$ with probability $q_k = Z_{k+1} 9^{-k}$.

Percolation. We now equip the UIHPT with a Bernoulli site percolation model, meaning that every site is open (coloured black, taking value 1) with probability p and closed (coloured white, taking value 0) otherwise, independently of every other site. The probability measure induced by this model is denoted by $\mathbf{P}_p$. We emphasize that this probability measure is annealed, so that conditioning on events depending only on the colouring may still affect the geometry of the underlying random lattice. We will often work conditionally on the colouring of the boundary of the map, which we call the *boundary condition*.

The open *percolation cluster* of a vertex v is the set of open vertices connected to v by an open path, together with the edges connecting them (and similarly for closed clusters). If $\mathcal{C}$ is the open percolation cluster of the origin and $|\mathcal{C}|$ its number of vertices, the percolation probability is $\Theta(p) := \mathbf{P}_p(|\mathcal{C}| = \infty)$, for $p \in [0,1]$. The percolation threshold p_c is such that $\Theta(p) > 0$ if $p > p_c$ and $\Theta(p) = 0$ if $p < p_c$. Angel proved in [3] (see also [4, Theorem 5]) that $p_c = 1/2$. We will regularly work at criticality and use the notation $\mathbf{P}$ instead of $\mathbf{P}_{p_c}$. We slightly abuse notation here and use $\mathbf{P}$ for several boundary conditions. For every $k \geq 2$, we also denote by $\mathbf{W}_k$ the measure induced by the Bernoulli site percolation model with parameter $1/2$ on a map with law $\mathbb{W}_k$ (and a boundary condition to be defined).

We end with some definition. Let $\mathcal{C}$ be a percolation cluster of a vertex v of the boundary in the UIHPT. The *hull* $\mathcal{H}$ of $\mathcal{C}$ is the coloured map obtained by filling in the finite holes of $\mathcal{C}$.

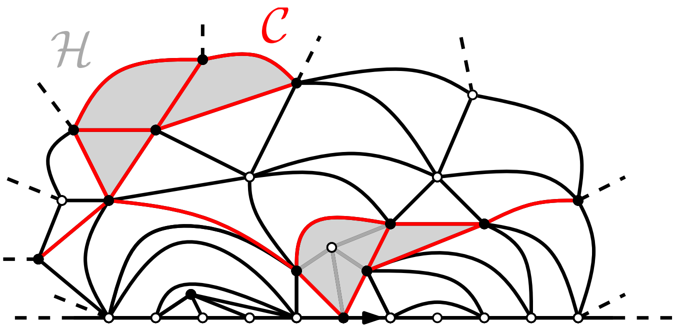

Figure 2: The percolation cluster of the origin and its hull.

2.2 Random trees and looptrees

Plane trees. We use the formalism of [28]. A finite plane tree $\mathbf{t}$ is a finite subset of $\bigcup_{n=0}^{\infty} \mathbb{N}^n$ satisfying the following properties. First, the empty word $\emptyset$ is an element of $\mathbf{t}$ (the root). Next, for every $n \in \mathbb{N}$, if $v = (v_1, \ldots, v_n) \in \mathbf{t}$, then $(v_1, \ldots v_{n-1}) \in \mathbf{t}$ (the parent of v). Finally, for every $v = (v_1, \ldots, v_n) \in \mathbf{t}$, there exists $k_v(\mathbf{t}) \in \mathbb{Z}_+$ such that $(v_1, \ldots, v_n, j) \in \mathbf{t}$ iff $1 \leq j \leq k_v(\mathbf{t})$ (the number of children of v).

For every $v = (v_1, \ldots, v_n) \in \mathbf{t}$, $|v| = n$ is the height of v in $\mathbf{t}$. Vertices of $\mathbf{t}$ at even height are called white, and those at odd height are called black. We let $\mathbf{t}_\circ$ and $\mathbf{t}_\bullet$ denote the corresponding sets of vertices. The set of finite plane trees is denoted by $\mathcal{T}_f$.

We will also deal with the set $\mathcal{T}_{\mathrm{loc}}$ of *locally finite* plane trees which is the completion of $\mathcal{T}_f$ with respect to the local topology. A *spine* in a tree $\mathbf{t}$ is an infinite sequence $\{s_k : k \in \mathbb{Z}_+\}$ of vertices of $\mathbf{t}$ such that $s_0 = \emptyset$ and for every $k \in \mathbb{Z}_+$, s_k is the parent of s_{k+1}.

Looptrees. Looptrees have been introduced in [15, 14], of which we closely follow the presentation. A (finite) *looptree* is a (finite) planar map whose edges are incident to two distinct faces, one of them being the root face. The set of finite looptrees is denoted by $\mathcal{L}_f$. Informally, a looptree is a collection of simple cycles glued along a tree structure.

To every plane tree $\mathbf{t} \in \mathcal{T}_f$ we associate a looptree $\mathbf{l} := \mathrm{Loop}(\mathbf{t})$ as follows. Vertices of $\mathbf{l}$ are those of $\mathbf{t}_\circ$, and around each vertex $u \in \mathbf{t}_\bullet$, we connect the incident (white) vertices with edges in cyclic order. The looptree $\mathbf{l}$ is the planar map obtained by discarding the edges of $\mathbf{t}$ and its black vertices. The inverse mapping associates to a looptree $\mathbf{l} \in \mathcal{L}_f$ the plane tree $\mathbf{t} := \mathrm{Tree}(\mathbf{l})$ called the *tree of components* in [14]. It is obtained by first adding an extra vertex into each inner face (or *loop*) of $\mathbf{l}$, and then connecting this vertex by an edge to all the vertices of the corresponding face (the edges of $\mathbf{l}$ are discarded). Our definition of looptree differs from [15, 14] (since we allow several loops to be glued at the same vertex). For conciseness, we do not detail rooting conventions throughout the paper.

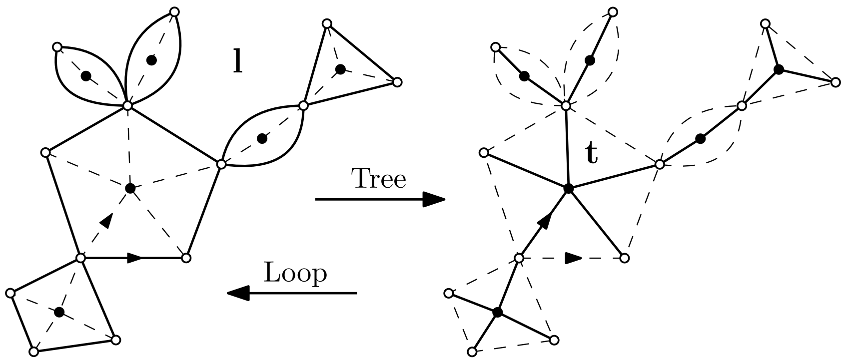

Figure 3: The mappings Tree and Loop.

We now extend our definition to infinite looptrees. Formally, an infinite looptree is a looptree whose root face is the unique infinite face. The set of finite and infinite looptrees is denoted by $\mathcal{L}$. The application Loop extends to any $\mathbf{t} \in \mathcal{T}_{\mathrm{loc}}$ by using the consistent sequence of planar maps $\{\mathrm{Loop}(\mathbf{B}_{2R}(\mathbf{t})) : R \in \mathbb{Z}_+\}$. When $\mathbf{t}$ is infinite and one-ended, $\mathrm{Loop}(\mathbf{t})$ is an infinite looptree. The inverse mapping Tree also extends to any infinite looptree $\mathbf{l} \in \mathcal{L}$ by using the consistent sequence of planar maps $\{\mathrm{Tree}(\mathbf{B}'_R(\mathbf{l})) : R \in \mathbb{Z}_+\}$, where $\mathbf{B}'_R(\mathbf{l})$ is the looptree made of all the internal faces of $\mathbf{l}$ having a vertex at distance less than R from the origin. Note that Tree and Loop are both continuous with respect to the local topology.

We now use these definitions to describe planar maps with a boundary. By splitting $\mathbf{m} \in \mathcal{M}$ at the cut-vertices (or pinch-points) of its boundary, we obtain a collection of connected components called the *irreducible components* of $\mathbf{m}$ in [12, 17]. We also define the so-called *scooped-out map* $\mathrm{Scoop}(\mathbf{m})$, which is the map obtained from the boundary of $\mathbf{m}$ by duplicating the edges whose sides both belong to the root face.

Let us assume that $\mathbf{m} \in \mathcal{M}$ has no infinite irreducible component. Then, $\mathrm{Scoop}(\mathbf{m})$ is a looptree and we call *tree of components* of $\mathbf{m}$ the locally finite plane tree $\mathbf{Tree}(\mathbf{m}) := \mathrm{Tree}(\mathrm{Scoop}(\mathbf{m}))$ (see Figure 4). By construction, to every vertex v at odd height in $\mathbf{Tree}(\mathbf{m})$ with degree $k \geq 2$ corresponds an internal face of $\mathrm{Scoop}(\mathbf{m})$ with the same degree, and a map $\mathbf{m}_v \in \widehat{\mathcal{M}}_k$ with a simple boundary (which is the irreducible component of $\mathbf{m}$ delimited by the face). Thus, $\mathbf{m}$ is recovered from $\mathrm{Scoop}(\mathbf{m})$ by gluing $\mathbf{m}_v$ in the associated face of $\mathrm{Scoop}(\mathbf{m})$ (meaning that we identify their boundaries). This results in an application

$$\Phi : \mathbf{m} \mapsto (\mathbf{t} = \mathbf{Tree}(\mathbf{m}), \{\mathbf{m}_v : v \in \mathbf{t}_\bullet\})$$

that associates to the planar map with a boundary $\mathbf{m} \in \mathcal{M}$ its tree of components $\mathbf{t} = \mathbf{Tree}(\mathbf{m})$ together with a collection $\{\mathbf{m}_v : v \in \mathbf{t}_\bullet\}$ of maps with a simple boundary having respective perimeter $\deg(v)$ attached to vertices at odd height of $\mathbf{t}$. Note that the inverse mapping Φ^{-1} that consists in filling in the loops of $\mathrm{Loop}(\mathbf{t})$ with the collection $\{\mathbf{m}_v : v \in \mathbf{t}_\bullet\}$ is continuous with respect to the local topology.

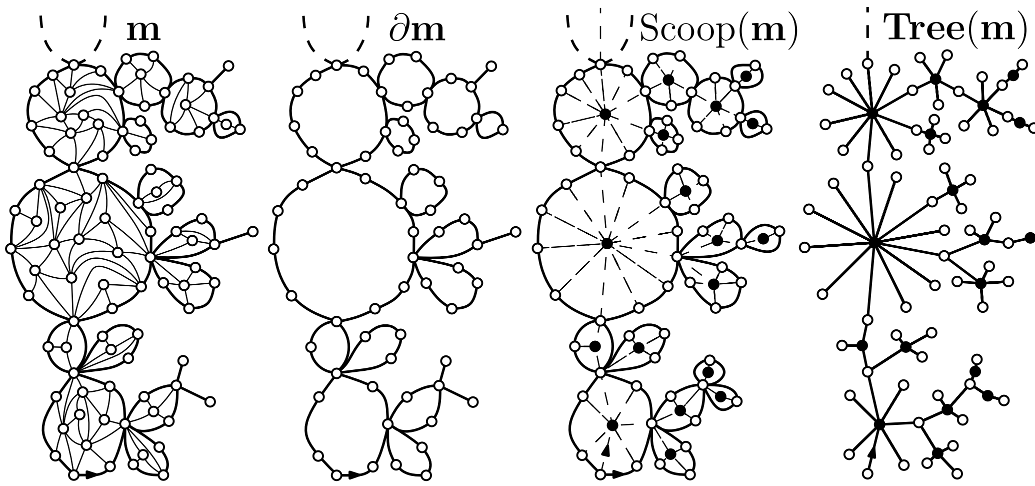

Figure 4: A triangulation with a boundary $\mathbf{m}$, its scooped-out map $\mathrm{Scoop}(\mathbf{m})$ (dashed edges and black vertices excluded) and the associated tree of components $\mathbf{Tree}(\mathbf{m})$.

Multi-type Galton-Watson trees. Let $\nu_\circ$ and $\nu_\bullet$ be probability measures on $\mathbb{Z}_+$. A random plane tree is an (alternated two-type) Galton-Watson tree with offspring distribution $(\nu_\circ, \nu_\bullet)$ if all the vertices at even (resp. odd) height have offspring distribution $\nu_\circ$ (resp. $\nu_\bullet$) all independently of each other. From now on, we assume that the pair $(\nu_\circ, \nu_\bullet)$ is critical (i.e., its mean vector $(m_\circ, m_\bullet)$ satisfies $m_\circ m_\bullet = 1$). Then, the law $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$ of such a tree is characterized by

$$\mathsf{GW}_{\nu_\circ,\nu_\bullet}(\mathbf{t}) = \prod_{v\in\mathbf{t}_\circ} \nu_\circ\left(k_v(\mathbf{t})\right) \prod_{v\in\mathbf{t}_\bullet} \nu_\bullet\left(k_v(\mathbf{t})\right), \quad \mathbf{t} \in \mathcal{T}_f.$$

The construction of *Kesten's tree* [24, 26] has been generalized in [34, Theorem 3.1] to multi-type Galton-Watson trees conditioned to survive as follows. Assume that the critical pair $(\nu_\circ, \nu_\bullet)$ satisfies $\mathsf{GW}_{\nu_\circ,\nu_\bullet}(\{|\mathbf{t}| = n\}) > 0$ for every $n \in \mathbb{Z}_+$. Let T_n be a plane tree with distribution $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$ conditioned to have n vertices. Then, in the sense of weak convergence, for the local topology

$$T_n \xrightarrow[n\to\infty]{(d)} \mathbf{T}_\infty.$$

The random infinite plane tree $\mathbf{T}_\infty = \mathbf{T}_\infty(\nu_\circ, \nu_\bullet)$ is a multi-type version of Kesten's tree, whose law is denoted by $\mathsf{GW}^{(\infty)}_{\nu_\circ,\nu_\bullet}$. Let us describe the alternative construction of $\mathbf{T}_\infty$ as explained in [34]. For every probability measure ν on $\mathbb{Z}_+$ with mean $m \in (0, \infty)$, the size-biased distribution $\bar{\nu}$ reads

$$\bar{\nu}(k) := \frac{k\nu(k)}{m}, \quad k \in \mathbb{Z}_+.$$

The tree $\mathbf{T}_\infty$ has a.s. a unique spine, in which white vertices have offspring distribution $\bar{\nu}_\circ$ while black vertices have offspring distribution $\bar{\nu}_\bullet$. Each vertex of the spine has a unique child in the spine, chosen uniformly at random among the offspring. Out of the spine, white and black vertices have offspring distribution $\nu_\circ$ and $\nu_\bullet$ respectively, and the number of offspring are all independent. We will use two variants of $\mathbf{T}_\infty$, which are obtained by discarding all the vertices and edges on the left (resp. right) of the spine, excluding the children of black vertices of the spine. Their distributions are denoted by $\mathsf{GW}^{(\infty,l)}_{\nu_\circ,\nu_\bullet}$ and $\mathsf{GW}^{(\infty,r)}_{\nu_\circ,\nu_\bullet}$ respectively.

2.3 Statement of the results

Uniform infinite necklace. A necklace is a map that was first introduced in [11], see also [14]. Formally, an infinite necklace is a triangulation of the upper half-plane $\mathbb{H}$ with no inner vertex.

Consider the graph of $\mathbb{Z}$ embedded in the plane. Let $\{z_i : i \in \mathbb{N}\}$ be a sequence of independent random variables with Bernoulli distribution of parameter $1/2$, and define the simple random walk $S_k := \sum_{i=1}^k z_i$ for $k \in \mathbb{N}$. The uniform infinite necklace is the random map obtained from $\mathbb{Z}$ by adding the set of edges $\{(-S_k, k+1-S_k) : k \in \mathbb{N}\}$ in a non-crossing manner. It is a.s. an infinite necklace in the aforementioned sense, and can also be interpreted as a gluing of triangles along their sides, with the tip oriented to the left or to the right equiprobably and independently. Its distribution is denoted by $\mathsf{UN}(\infty,\infty)$.

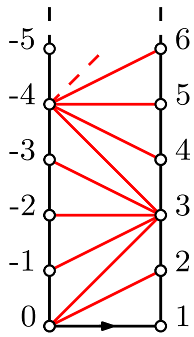

Figure 5: The uniform infinite necklace.

In the next part, we will perform gluing operations of planar maps with a boundary along infinite necklaces. Let $\mathbf{m}, \mathbf{m}' \in \mathcal{M}_\infty$ be maps with an infinite boundary. Let $\{e_i : i \in \mathbb{N}\}$ be the sequence of edges of the root face of $\mathbf{m}$ on the right of the origin vertex, listed in contour order. Similarly, the left boundary of $\mathbf{m}'$ defines the sequence of edges $\{e'_i : i \in \mathbb{N}\}$. Let $\mathbf{n}$ be an infinite necklace, with a boundary identified to $\mathbb{Z}$. The gluing $\Psi_{\mathbf{n}}(\mathbf{m}, \mathbf{m}')$ of $\mathbf{m}$ and $\mathbf{m}'$ along $\mathbf{n}$ is the infinite map defined as follows. For every $i \in \mathbb{N}$, we identify the edge $(i, i+1)$ of $\mathbf{n}$ with e'_i, and the edge $(-i, -i+1)$ of $\mathbf{n}$ with e_i (see Figure 7).

Note that the gluing $\Psi_{(\mathbf{n},\mathbf{n}')}(\mathbf{m}, \mathbf{m}', \mathbf{m}'')$ of three rooted maps with an infinite boundary $\mathbf{m}$, $\mathbf{m}'$ and $\mathbf{m}''$ along the pair of infinite necklaces $(\mathbf{n}, \mathbf{n}')$ can be defined as the map $\Psi_{(\mathbf{n},\mathbf{n}')}(\mathbf{m}, \mathbf{m}', \mathbf{m}'') := \Psi_{\mathbf{n}'}(\mathbf{m}^*, \mathbf{m}'')$, with $\mathbf{m}^* := \Psi_{\mathbf{n}}(\mathbf{m}, \mathbf{m}')$ (see Figure 9 for an example). These gluing operations are continuous with respect to the local topology.

Decomposition of the UIHPT**.** We consider the UIHPT decorated with a critical percolation model, and work conditionally on the "Black-White" boundary condition of Figure 6. We let $\mathcal{H}_b$ and $\mathcal{H}_w$ be the hulls of the percolation clusters of the origin and the target of the root. We denote by $\mathbf{Tree}(\mathcal{H}_b)$ and $\mathbf{Tree}(\mathcal{H}_w)$ their respective tree of components, and by

$$\left\{M^b_v : v \in \mathbf{Tree}(\mathcal{H}_b)_\bullet\right\} \quad \text{and} \quad \left\{M^w_v : v \in \mathbf{Tree}(\mathcal{H}_w)_\bullet\right\}$$

their irreducible components (i.e., the second components of $\Phi(\mathcal{H}_b)$ and $\Phi(\mathcal{H}_w)$). The boundary conditions of the irreducible components are determined by the hull. We define the probability measures $\mu_\circ$ and $\mu_\bullet$ by

$$\mu_\circ(k) := \frac{2}{3}\left(\frac{1}{3}\right)^k \quad \text{and} \quad \mu_\bullet(k) := 6q_k, \quad k \in \mathbb{Z}_+. \tag{2.3}$$

By [4, Proposition 3], we have $\sum_{k\in\mathbb{N}} kq_k = 1/3$ so that the pair $(\mu_\circ, \mu_\bullet)$ is critical.

Figure 6: The "Black-White" boundary condition.

Theorem 2.1. *In the critical Bernoulli percolation model on the* UIHPT *with "Black-White" boundary condition:*

- *The trees of components* $\mathbf{Tree}(\mathcal{H}_b)$ *and* $\mathbf{Tree}(\mathcal{H}_w)$ *are independent with respective distribution* $\mathsf{GW}^{(\infty,l)}_{\mu_\circ,\mu_\bullet}$ *and* $\mathsf{GW}^{(\infty,r)}_{\mu_\circ,\mu_\bullet}$.
- *Conditionally on* $\mathbf{Tree}(\mathcal{H}_b)$ *and* $\mathbf{Tree}(\mathcal{H}_w)$*, the associated irreducible components* $\{M_v^b : v \in \mathbf{Tree}(\mathcal{H}_b)_\bullet\}$ *and* $\{M_v^w : v \in \mathbf{Tree}(\mathcal{H}_w)_\bullet\}$ *are independent critically percolated Boltzmann triangulations with a simple boundary and respective distribution* $\mathbf{W}_{\deg(v)}$.

Finally, the UIHPT *is recovered as the gluing* $\Psi_{\mathbf{N}}(\mathcal{H}_b, \mathcal{H}_w)$ *of* $\mathcal{H}_b$ *and* $\mathcal{H}_w$ *along a uniform infinite necklace* $\mathbf{N}$ *with distribution* $\mathsf{UN}(\infty,\infty)$ *independent of* $(\mathcal{H}_b, \mathcal{H}_w)$.

Remark 2.2. The result of Theorem 2.1 can be seen as a discrete counterpart to [18, Theorem 1.16-1.17], as we will discuss in Section 6. It could also be stated without reference to percolation: By discarding the colouring of the vertices, we obtain a decomposition of the UIHPT into two independent looptrees filled in with Boltzmann triangulations and glued along a uniform necklace. An illustration is provided in Figure 7.

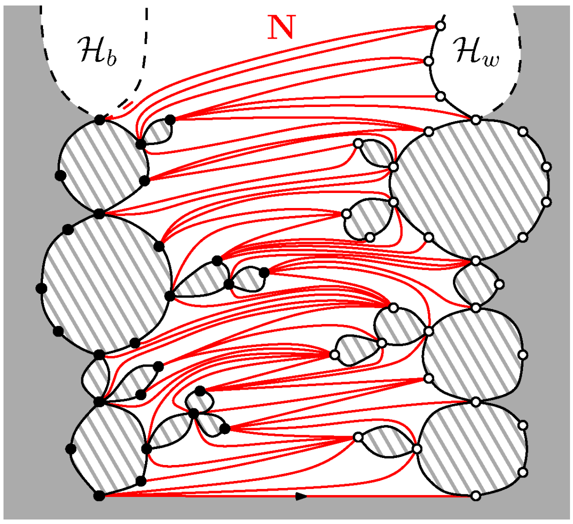

Figure 7: The decomposition of the UIHPT into its percolation hulls and the uniform infinite necklace. The dashed areas are filled-in with independent critically percolated Boltzmann triangulations of the given boundary length.

The incipient infinite cluster. We now consider the UIHPT decorated with a Bernoulli percolation model with parameter p conditionally on the "White-Black-White" boundary condition of Figure 8. In a map with such a boundary condition, we let $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$ be the hulls of the clusters of the origin and its left and right neighbours on the boundary. We let $\mathbf{Tree}(\mathcal{H})$, $\mathbf{Tree}(\mathcal{H}_l)$ and $\mathbf{Tree}(\mathcal{H}_r)$ be their respective tree of components, and denote by

$$\{M_v : v \in \mathbf{Tree}(\mathcal{H})_\bullet\}, \quad \{M_v^l : v \in \mathbf{Tree}(\mathcal{H}_l)_\bullet\} \quad \text{and} \quad \{M_v^r : v \in \mathbf{Tree}(\mathcal{H}_r)_\bullet\}$$

their irreducible components (i.e., the second components of $\Phi(\mathcal{H})$, $\Phi(\mathcal{H}_l)$ and $\Phi(\mathcal{H}_r)$). Again, the boundary conditions of these components are forced by the hulls.

The *height* $h(\mathcal{C})$ of the open percolation cluster of the origin $\mathcal{C}$ will be defined in Section 5.1. It corresponds to the maximal length of the open segment revealed when exploring the percolation interface between the origin and its left neighbour on the boundary.

Figure 8: The "White-Black-White" boundary condition.

Theorem 2.3. *Let $\mathbf{P}_p$ be the law of the* UIHPT *equipped with a Bernoulli percolation model with parameter p and "White-Black-White" boundary condition. Then, there exists a probability measure $\mathbf{P}_{\mathsf{IIC}}$ such that in the sense of weak convergence for the local topology,*

$$\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty) \underset{p \downarrow p_c}{\Longrightarrow} \mathbf{P}_{\mathsf{IIC}} \quad \textit{and} \quad \mathbf{P}_{p_c}(\cdot \mid h(\mathcal{C}) \geq n) \underset{n \to \infty}{\Longrightarrow} \mathbf{P}_{\mathsf{IIC}}.$$

The probability measure $\mathbf{P}_{\mathsf{IIC}}$ is called (the law of) the Incipient Infinite Cluster of the UIHPT *or* IIC*. The* IIC *is a.s. a percolated triangulation of the half-plane with "White-Black-White" boundary condition. Moreover, in the* IIC*:*

- *The trees of components $\mathbf{Tree}(\mathcal{H})$, $\mathbf{Tree}(\mathcal{H}_l)$ and $\mathbf{Tree}(\mathcal{H}_r)$ are independent with respective distribution $\mathsf{GW}^{(\infty)}_{\mu_\circ,\mu_\bullet}$, $\mathsf{GW}^{(\infty,l)}_{\mu_\circ,\mu_\bullet}$ and $\mathsf{GW}^{(\infty,r)}_{\mu_\circ,\mu_\bullet}$.*
- *Conditionally on $\mathbf{Tree}(\mathcal{H})$, $\mathbf{Tree}(\mathcal{H}_l)$ and $\mathbf{Tree}(\mathcal{H}_r)$, the irreducible components $\{M_v : v \in \mathbf{Tree}(\mathcal{H})_\bullet\}$, $\{M_v^l : v \in \mathbf{Tree}(\mathcal{H}_l)_\bullet\}$ and $\{M_v^r : v \in \mathbf{Tree}(\mathcal{H}_r)_\bullet\}$ are independent critically percolated Boltzmann triangulations with a simple boundary and respective distribution $\mathbf{W}_{\deg(v)}$.*

Finally, the IIC *is recovered as the gluing $\Psi_{(\mathbf{N}_l,\mathbf{N}_r)}(\mathcal{H}_l,\mathcal{H},\mathcal{H}_r)$ of $\mathcal{H}_l$, $\mathcal{H}$ and $\mathcal{H}_r$ along a pair of independent uniform infinite necklaces $(\mathbf{N}_l,\mathbf{N}_r)$ with distribution $\mathsf{UN}(\infty,\infty)$, also independent of $(\mathcal{H}_l,\mathcal{H},\mathcal{H}_r)$.*

Remark 2.4. In the work of Kesten [23], the IIC is defined for bond percolation on $\mathbb{Z}^2$ by considering a supercritical open percolation cluster, and letting p decrease towards the critical point p_c. Equivalently, the IIC arises directly in the critical model when conditioning the open cluster of the origin to reach the boundary of $[-n,n]^2$, and letting n go to infinity. Theorem 2.3 is the analogous result for site percolation on the UIHPT; however, we use a slightly different conditioning in the critical setting, which is more adapted to the use of the peeling techniques.

Remark 2.5. Theorem 2.3 should be seen as a counterpart to Theorem 2.1. Indeed, the decomposition of the IIC shows that when conditioning the open cluster of the origin to be infinite, one adds ex-nihilo an infinite looptree in the UIHPT, as shown in Figure 9. This describes how the zero measure event we condition on twists the geometry of the initial random half-planar triangulation.

Strategy of the proof. Let us give a rough idea of the proofs of Theorems 2.1 and 2.3. We first define suitable exploration processes (Sections 4.1 and 5.1) that follow the percolation interfaces in the UIHPT. Then, we identify the distribution of the maps revealed

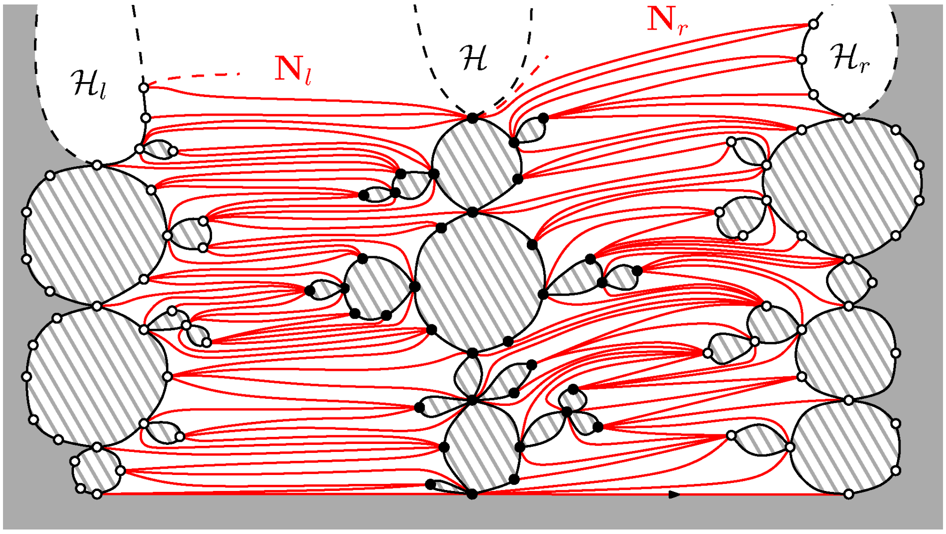

Figure 9: The decomposition of the IIC into its percolation hulls and the uniform necklaces. The dashed areas are filled-in with independent critically percolated Boltzmann triangulations of the given boundary length.

by these explorations (Sections 4.2 and 5.2) by decomposing them into percolation hulls and necklaces. This is noticeably harder in the IIC case, due to the conditioning. The last part of the argument (Sections 4.3 and 5.4) consists in showing that if the explorations last long enough, then the revealed maps look like the constructions of Theorems 2.1 and 2.3 around their root edge. Finally, Section 3 introduces an encoding of looptrees and preliminary results that will be useful in the next parts of the paper.

3 Coding of looptrees

3.1 The contour function

We first describe the encoding of looptrees via an analogue of the contour function for trees [1, 25]. This bears similarities with the coding of continuum random looptrees of [14]. In order to shorten the presentation, rooting conventions are not detailed.

Finite looptrees. Let $n \in \mathbb{Z}_+$ and $C = \{C_k : 0 \le k \le n\}$ a discrete *excursion* with no positive jumps (larger than 1) and no constant step, that is $C_0 = C_n = 0$, and for every $0 \le k < n$, $C_k \in \mathbb{Z}_+$, $C_{k+1} - C_k \le 1$ and $C_k \ne C_{k+1}$. The equivalence relation $\sim$ on $\{0, \dots, n\}$ is defined by

$$i \sim j \quad \text{iff} \quad C_i = C_j = \inf_{i \wedge j \le k \le i \vee j} C_k. \tag{3.1}$$

The quotient space $\{0, \dots, n\}/\sim$ inherits the graph structure of the chain $\{0, \dots, n\}$, and can be embedded in the plane as follows. Consider the graph of C (with linear interpolation), together with the set of edges E containing all the pairs $\{(i, C_i), (j, C_j)\}$ such that $i \sim j$. This defines a planar map $\mathbf{m}_C$, whose vertices are identified to $\{0, \dots, n\}$. The embedding of $\{0, \dots, n\}/\sim$ is obtained by contracting the edges of E as in Figure 10. We obtain a looptree denoted by $\mathbf{L}_C$. Let us describe the tree of components $\mathbf{T}_C := \mathrm{Tree}(\mathbf{L}_C)$. The black vertices of $\mathbf{T}_C$ are the internal faces of $\mathbf{m}_C$, and the white vertices of $\mathbf{T}_C$ are the equivalence classes of $\sim$. For every black vertex f, the white vertices incident to f are the classes that have a representative j incident to the face f

in $\mathbf{m}_C$, with the natural cyclic order. This construction extends to the case $C_n > 0$, but the resulting map in not a looptree in general.

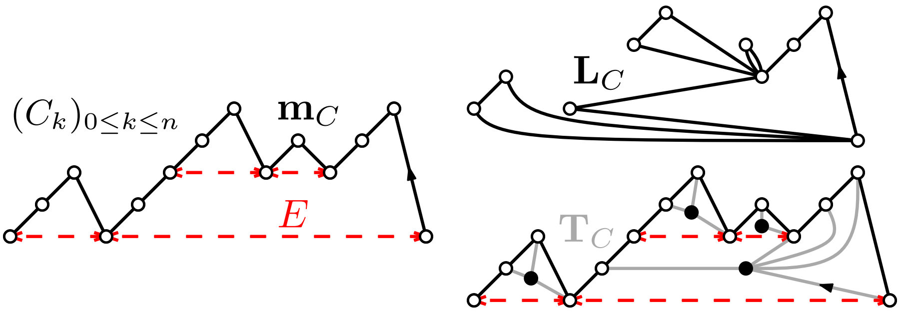

Figure 10: The construction of $\mathbf{L}_C$ and $\mathbf{T}_C$ from the excursion C.

Infinite looptrees. Let us extend the construction to infinite looptrees. Let $C : \mathbb{Z}_+ \to \mathbb{Z}$ be a function such that $C_0 = 0$, with no positive jump, no constant step and such that

$$\liminf_{k\to\infty} C_k = -\infty. \tag{3.2}$$

The function C is extended to $\mathbb{Z}$ by setting $C_k = k$ for $k \in \mathbb{Z}_-$. We define an equivalence relation $\sim$ on $\mathbb{Z}$ by applying (3.1) with the function C. The graph of $-C$ and the set of edges E containing all the pairs $\{(i, -C_i), (j, -C_j)\}$ such that $i \sim j$ define an infinite map $\mathbf{m}_C$, whose vertices are identified to $\mathbb{Z}$ (the root connects 0 to 1). By contracting the edges of E, we obtain the infinite looptree $\mathbf{L}_C$ (which is an embedding of $\mathbb{Z}/\sim$). The tree $\mathbf{T}_C := \mathrm{Tree}(\mathbf{L}_C)$ is defined as in the finite setting. By the assumption (3.2), internal faces of $\mathbf{m}_C$ (the black vertices) and equivalence classes of $\mathbb{Z}/\sim$ (the white vertices) are finite. Thus, $\mathbf{T}_C$ is locally finite. We let

$$\tau_0 = 0 \quad \text{and} \quad \tau_{k+1} := \inf\{i \geq \tau_k : C_i < C_{\tau_k}\}, \quad k \in \mathbb{Z}_+. \tag{3.3}$$

For every $k \in \mathbb{N}$, the white vertex of $\mathbf{T}_C$ associated to τ_k disconnects the root from infinity, as well as its (black) parent in $\mathbf{T}_C$. This exhibits the unique spine of $\mathbf{T}_C$ (and a spine of faces in $\mathbf{L}_C$). Since $C_k = k$ for negative k, there is no vertex on the left of the spine of $\mathbf{L}_C$.

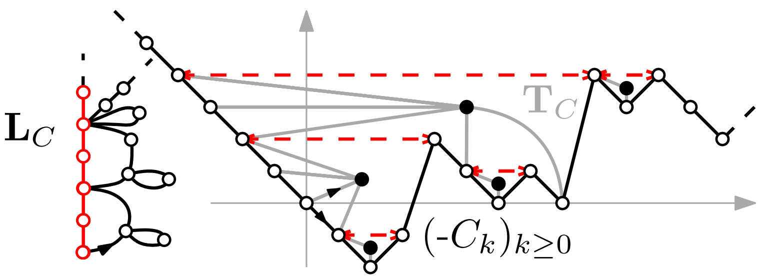

Figure 11: The construction of $\mathbf{L}_C$ and $\mathbf{T}_C$ from C.

We finally define a looptree out of a pair of functions $C, C' : \mathbb{Z}_+ \to \mathbb{Z}$, with $C_0 = C'_0 = 0$, no positive jumps, no constant steps and so that C satisfies (3.2) and C' is nonnegative. First define a looptree $\mathbf{L}_C$ as above, and let $\{e_i : i \in \mathbb{Z}_+\}$ be the edges of the left boundary of $\mathbf{L}_C$ in contour order. Then, we define an equivalence relation $\sim$ on $\mathbb{Z}_+$ by applying (3.1) with the function C'. Let $R_0 = -1$ and for every $k \in \mathbb{N}$, $R_k := \sup\{i \in \mathbb{Z}_+ : C'_i = k - 1\}$.

For every $k \in \mathbb{Z}_+$, the excursion $\{C'_{R_k+i+1} - k : 0 \le i \le R_{k+1} - R_k - 1\}$ of C' above its future infimum defines a looptree $\mathbf{L}_k$. We now consider the graph of $\mathbb{Z}_+$ embedded in the plane and for every $k \in \mathbb{Z}_+$, attach the looptree $\mathbf{L}_k$ on the left of the vertex $k \in \mathbb{Z}_+$. We obtain a *forest* of looptrees $\mathbf{F}_{C'}$, isomorphic to $\mathbb{Z}_+/\sim$. The infinite looptree $\mathbf{L}_{C,C'}$ is obtained by gluing the left boundary of $\mathbf{L}_C$ to the right boundary of $\mathbf{F}_{C'}$ (i.e., by identifying the edge e_i with the edge $(i+1,i)$ of $\mathbb{N}$ for every $i \in \mathbb{N}$). The tree of components $\mathbf{T}_{C,C'} := \mathrm{Tree}(\mathbf{L}_{C,C'})$ has a unique spine inherited from $\mathbf{T}_C$. We can also define a function $C^* : \mathbb{Z} \to \mathbb{Z}$ by

$$C^*_k = \begin{cases} -C_k & \text{if} \quad k \in \mathbb{Z}_+ \\ C'_{-k} & \text{if} \quad k \in \mathbb{Z}_- \end{cases}. \tag{3.4}$$

and an equivalence relation $\sim$ on $\mathbb{Z}$ by

$$i \sim j \quad \text{iff} \quad \begin{cases} i \vee j > 0 \quad \text{and} \quad \inf_{k \le i \wedge j} C^*_k \ge C^*_i = C^*_j \ge \sup_{(i\wedge j)\vee 0 \le k \le i \vee j} C^*_k \\ \qquad \text{or} \\ i \vee j \le 0 \quad \text{and} \qquad\qquad \inf_{i \wedge j \le k \le i \vee j} C^*_k \ge C^*_i = C^*_j \end{cases} \tag{3.5}$$

(with $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$). Then, $\mathbf{L}_{C,C'}$ is isomorphic to $\mathbb{Z}/\sim$ (see Figure 12). In the next part, we let p_C and $p_{C,C'}$ denote the canonical projection on $\mathbf{L}_C$ and $\mathbf{L}_{C,C'}$.

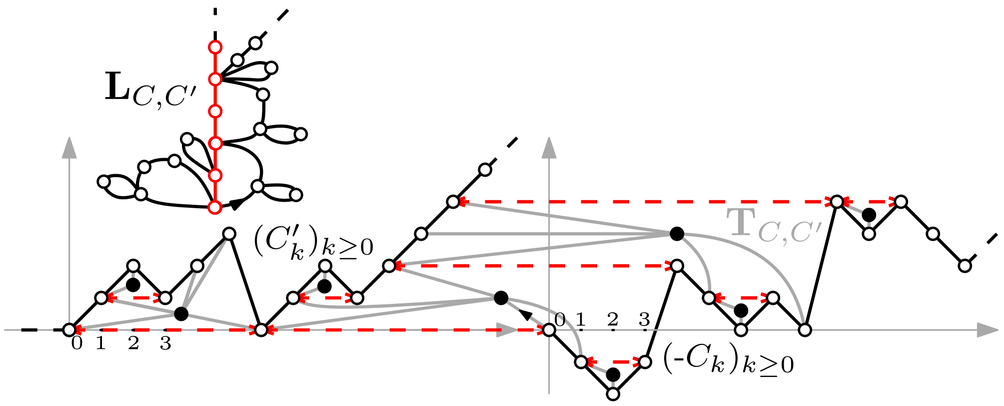

Figure 12: The construction of $\mathbf{L}_{C,C'}$ and $\mathbf{T}_{C,C'}$ from C and C'.

3.2 Random walks

We now gather results on random walks. For every probability measure ν on $\mathbb{Z}$ and every $x \in \mathbb{Z}$, let P^ν_x be the law of the simple random walk started at x with step distribution ν (we may omit the exponent ν). Let $Z = \{Z_k : k \in \mathbb{Z}_+\}$ be the canonical process and $T := \inf\{k \in \mathbb{Z}_+ : Z_k < 0\}$. We assume that ν is centered and $\nu((1,+\infty)) = 0$ (the random walk is called upwards-skip-free or with no positive jumps). For every $k \in \mathbb{Z}$, we let $\widehat{\nu}(k) = \nu(-k)$.

Overshoot. We start with a result on the *overshoot* at the first entrance in $(-\infty, 0)$.

Lemma 3.1. *We have*

$$P^\nu_0(Z_{T-1} - Z_T = k) = \frac{k\nu(-k)}{\nu(1)}, \quad k \in \mathbb{Z}_+.$$

Conditionally on $Z_{T-1} - Z_T$, $-Z_T$ is uniform on $\{1, \dots, Z_{T-1} - Z_T\}$. Moreover, under P^ν_0 and conditionally on Z_{T-1}, the reversed process $\{\widehat{Z}_k : 0 \le k < T\} := \{Z_{T-1-k} : 0 \le k < T\}$ has the same law as $\{Z_k : 0 \le k < T\}$ under $P^{\widehat{\nu}}_{Z_{T-1}}$.

Proof. Let $n \in \mathbb{N}$ and $x_0, \dots, x_{n-1} \geq 0$. On the one hand

$$\begin{aligned}
&P_0^{\nu}(T = n, \widehat{Z}_0 = x_0, \dots, \widehat{Z}_{n-1} = x_{n-1}) \\
&\quad = P_0^{\nu}\left(Z_0 = x_{n-1}, \dots, Z_{n-1} = x_0, Z_n < 0\right) \\
&\quad = \mathbf{1}_{\{x_{n-1}=0\}} \nu(x_{n-2} - x_{n-1}) \cdots \nu(x_0 - x_1) \nu((-\infty, -x_0))
\end{aligned}$$

and on the other hand since $\nu((1, +\infty)) = 0$,

$$\begin{aligned}
&P_{x_0}^{\widehat{\nu}}\left(T = n, Z_0 = x_0, \dots, Z_{n-1} = x_{n-1}\right) \\
&\quad = \mathbf{1}_{\{x_{n-1}=0\}} \nu(-(x_1 - x_0)) \cdots \nu(-(x_{n-1} - x_{n-2})) \nu(1).
\end{aligned}$$

Now, while computing the probability $P_0^{\nu}\left(Z_{T-1} = x_0\right)$ one gets

$$P_0^{\nu}\left(Z_{T-1} = x_0\right) = \nu((-\infty, -x_0)) \sum_{n \in \mathbb{N}} \sum_{x_1, \dots, x_{n-1} \geq 0} \mathbf{1}_{\{x_{n-1}=0\}} \nu(x_{n-2} - x_{n-1}) \cdots \nu(x_0 - x_1),$$

and still using $\nu((1, +\infty)) = 0$, we have

$$1 = P_{x_0}^{\widehat{\nu}}\left(Z_{T-1} = 0\right) = \nu(1) \sum_{n \in \mathbb{N}} \sum_{x_1, \dots, x_{n-1} \geq 0} \mathbf{1}_{\{x_{n-1}=0\}} \nu(x_{n-2} - x_{n-1}) \cdots \nu(x_0 - x_1).$$

The last assertion follows, as well as $P_0^{\nu}\left(Z_{T-1} = i\right) = \nu((-\infty, -i))/\nu(1)$ for every $i \in \mathbb{Z}_+$. By a direct computation, for every $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_- \backslash \{0\}$,

$$P_0^{\nu}\left(Z_T = j \mid Z_{T-1} = i\right) = \frac{\nu(j - i)}{\nu((-\infty, -i))} \quad \text{and} \quad P_0^{\nu}\left(Z_{T-1} = i, Z_T = j\right) = \frac{\nu(j - i)}{\nu(1)}.$$

Then, for every $k \in \mathbb{Z}_+$ and $l \in \{1, \dots, k\}$,

$$P_0^{\nu}\left(Z_{T-1} - Z_T = k\right) = \frac{k\nu(-k)}{\nu(1)} \quad \text{and} \quad P_0^{\nu}\left(-Z_T = l \mid Z_{T-1} - Z_T = k\right) = \frac{1}{k},$$

which ends the proof. □

Remark 3.2. Since $\nu((1, +\infty)) = 0$, by putting $\widehat{Z}_T := \widehat{Z}_{T-1} - 1$ we have that under P_0^{ν} and conditionally on Z_{T-1}, $\{\widehat{Z}_k : 0 \leq k \leq T\}$ is distributed as $\{Z_k : 0 \leq k \leq T\}$ under $P_{Z_{T-1}}^{\widehat{\nu}}$.

Random walk conditioned to stay nonnegative. We now recall the construction of the so-called *random walk conditioned to stay nonnegative* of [7] (see also [19, 33]). We let $T_n := \inf\{k \in \mathbb{N} : Z_k \geq n\}$ for every $n \in \mathbb{Z}_+$. Let $H := \{H_k : k \in \mathbb{Z}_+\}$ be the strict ascending ladder height process of $-Z$. Namely, let $L_0 = 0$ and

$$H_k = -Z_{L_k}, \quad L_{k+1} = \inf\{j > L_k : -Z_j > H_k\}, \quad k \in \mathbb{Z}_+.$$

Then, the *renewal function* associated with H is defined by

$$V(x) := \sum_{k=0}^{\infty} P_0(H_k \leq x) = E_0\left(\sum_{k=0}^{T_0 - 1} \mathbf{1}_{\{Z_k \geq -x\}}\right), \quad x \geq 0, \tag{3.6}$$

where the equality follows from the duality lemma ([19, Chapter XII.2]). For every $x \geq 0$, we denote by $P_x^{\uparrow}$ the (Doob) h-transform of P_x by V. That is, for $k \in \mathbb{Z}_+$ and $F : \mathbb{Z}^{k+1} \to \mathbb{R}$,

$$E_x^{\uparrow}(F(Z_0, \dots, Z_k)) = \frac{1}{V(x)} E_x\left(V(Z_k) F(Z_0, \dots, Z_k) \mathbf{1}_{k < T}\right). \tag{3.7}$$

Theorem 3.3. [7, Theorem 1] *For every $x \geq 0$, in the sense of weak convergence of finite-dimensional distributions,*

$$P_x\left(\cdot \mid T_n < T\right) \underset{n\to+\infty}{\Longrightarrow} P_x^{\uparrow} \quad \text{and} \quad P_x\left(\cdot \mid T \geq n\right) \underset{n\to+\infty}{\Longrightarrow} P_x^{\uparrow}.$$

The probability measure $P_x^{\uparrow}$ is the law of the random walk conditioned to stay nonnegative.

We now recall Tanaka's pathwise construction, and let $T_+ := \inf\{k \in \mathbb{Z}_+ : Z_k > 0\}$.

Theorem 3.4. [35, Theorem 1] *Let $\{w_k : k \in \mathbb{Z}_+\}$ be independent copies of the reversed excursion*

$$\left(0, Z_{T_+} - Z_{T_+-1}, \ldots, Z_{T_+} - Z_1, Z_{T_+}\right)$$

under P_0, with $w_k = (w_k(0), \ldots, w_k(s_k))$. Let for every $k \in \mathbb{Z}_+$

$$Y'_k := \sum_{j=0}^{i-1} w_j(s_j) + w_i\left(k - \sum_{j=0}^{i-1} w_j(s_j)\right) \quad \text{for} \quad \sum_{j=0}^{i-1} w_j(s_j) < k \leq \sum_{j=0}^{i} w_j(s_j),$$

and $Y_k := Y'_{k+1} - 1$. Then, the process $\{Y_k : k \in \mathbb{Z}_+\}$ has law $P_0^{\uparrow}$.

Remark 3.5. In [35], $\{Y'_k : k \in \mathbb{Z}_+\}$ is the h-transform of P_0 by a suitable renewal function V'. This function differs from the function V of (3.6) and rather defines a random walk conditioned to stay positive. However, when the random walk is upwards-skip-free and we remove its first step (which gives $\{Y_k : k \in \mathbb{Z}_+\}$), the associated renewal function equals V up to a multiplicative constant. This ensures that $\{Y_k : k \in \mathbb{Z}_+\}$ has law $P_0^{\uparrow}$.

Let us rephrase this theorem. Let $R_0 = -1$ and $R_k := \sup\{i \in \mathbb{Z}_+ : Z_i \leq k-1\}$ for $k \in \mathbb{N}$.

Corollary 3.6. *Under $P_0^{\uparrow}$, the reversed excursions*

$$\left\{Z_i^{(k)},\ 0 \leq i < R_{k+1} - R_k\right\} := \left\{Z_{R_{k+1}-i} - k,\ 0 \leq i < R_{k+1} - R_k\right\}, \quad k \in \mathbb{Z}_+$$

are independent and distributed as $\{Z_k : 0 \leq k < T\}$ under $P_0^{\widehat{\nu}}$.

The law of the conditioned random walk stopped at a first hitting time is explicit.

Lemma 3.7. *Let $n \in \mathbb{Z}_+$. Under $P_0^{\uparrow}$, $\{Z_k : 0 \leq k \leq T_n\}$ has distribution $P_0\left(\cdot \mid T_n < T\right)$.*

Proof. Let $k \in \mathbb{Z}_+$ and $x_0, \ldots, x_k \geq 0$. By [7, Theorem 1],

$$P_0\left(T_n = k, Z_0 = x_0, \ldots, Z_k = x_k \mid T_m < T\right) \underset{m\to\infty}{\longrightarrow} P_0^{\uparrow}\left(T_n = k, Z_0 = x_0, \ldots, Z_k = x_k\right).$$

It is thus sufficient to prove that for m large enough,

$$P_0\left(T_n = k, Z_0 = x_0, \ldots, Z_k = x_k \mid T_m < T\right) = P_0\left(T_n = k, Z_0 = x_0, \ldots, Z_k = x_k \mid T_n < T\right).$$

We have $T_n < T_m$ whenever $m > n$, and $Z_{T_n} = n$ P_0-a.s.. The strong Markov property gives

$$P_0\left(T_n = k, Z_0 = x_0, \ldots, T_m < T\right) = P_0\left(T_n = k, Z_0 = x_0, \ldots, T_n < T\right) P_n\left(T_m < T\right).$$

We conclude the proof by using the identity $P_0\left(T_m < T\right) = P_0\left(T_n < T\right) P_n\left(T_m < T\right)$. □

We now deal with the conditioned random walk started at large values. For every $x \geq 0$, let ϕ_x be defined for every $y \in \mathbb{R}^{\mathbb{N}}$ by $\phi_x(y) = (y_i - x : i \in \mathbb{N})$. We use the notation f_*P for the pushforward measure of P by the function f.

Lemma 3.8. *In the sense of weak convergence of finite-dimensional distributions,*

$$(\phi_x)_* P_x^{\uparrow} \underset{x\to\infty}{\Longrightarrow} P_0.$$

Proof. Let $k \in \mathbb{Z}_+$ and $x_0, \cdots, x_k \in \mathbb{Z}$. From (3.7), we have

$$(\phi_x)_* P_x^{\uparrow}(Z_0 = x_0, \dots, Z_k = x_k) = \frac{V(x+x_k)}{V(x)} P_x(k > T, Z_0 = x + x_0, \dots, Z_k = x + x_k).$$

Up to choosing x large enough, we can assume that $x_i + x \geq 0$ for $i \in \{0, \dots, k\}$ and get

$$(\phi_x)_* P_x^{\uparrow}(Z_0 = x_0, \dots, Z_k = x_k) = \frac{V(x+x_k)}{V(x)} P_0(Z_0 = x_0, \dots, Z_k = x_k).$$

We now assume that $x_k \in \mathbb{Z}_+$ (the case $x_k \in \mathbb{Z}_-$ can be treated similarly). We have

$$\frac{V(x+x_k)}{V(x)} = 1 + \frac{1}{V(x)} E_0 \left(\sum_{k=0}^{T_0 - 1} \mathbf{1}_{\{-x-x_k \leq Z_k < -x\}} \right).$$

By monotone convergence, since $T_0 < \infty$ P_0-a.s.,

$$E_0 \left(\sum_{k=0}^{T_0 - 1} \mathbf{1}_{\{-x-x_k \leq Z_k < -x\}} \right) \leq E_0 \left(\sum_{k=0}^{T_0 - 1} \mathbf{1}_{\{Z_k < -x\}} \right) \underset{x\to\infty}{\longrightarrow} 0.$$

By (3.6) and monotone convergence once again, since $E_0(T_0) = \infty$, $V(x) \to \infty$ as x goes to infinity, which concludes the proof. $\square$

Random walk with positive drift. We consider random walks with positive drift conditioned to stay nonnegative. Let ν and $\{\nu_p : p \in \mathbb{Z}_+\}$ be upwards-skip-free probability measure, such that ν is centered, $m_p := E_0^{\nu_p}(Z_1) > 0$ and $\nu_p \Rightarrow \nu$ weakly as $p \to \infty$. The random walk conditioned to stay nonnegative is well-defined in the usual sense, since $P_0^{\nu_p}(T = \infty) > 0$ for $p \in \mathbb{Z}_+$. We denote its law by $P_x^{\nu_p\uparrow}$. It is also the h-transform of $P_x^{\nu_p}$ by the renewal function V_p associated to the strict ascending ladder height process of $-Z$, which satisfies

$$V_p(x) = \frac{P_x^{\nu_p}(T = \infty)}{P_0^{\nu_p}(T = \infty)}, \quad x \geq 0,\ p \in \mathbb{Z}_+.$$

We let $T_{-n} := \inf\{k \in \mathbb{N} : Z_k \leq -n\}$ for every $n \in \mathbb{N}$.

Lemma 3.9. *For $x \in \mathbb{Z}_+$, in the sense of weak convergence of finite-dimensional distributions,*

$$P_x^{\nu_p\uparrow} \underset{p\to+\infty}{\Longrightarrow} P_x^{\nu\uparrow}.$$

Proof. Let $x, k \in \mathbb{Z}_+$ and $x_0, \cdots, x_k \in \mathbb{Z}_+$. By (3.7),

$$P_x^{\nu_p\uparrow}(Z_0 = x_0, \dots, Z_k = x_k) = \frac{V_p(x_k)}{V_p(x)} P_x^{\nu_p}(k > T, Z_0 = x_0, \dots, Z_k = x_k). \tag{3.8}$$

Since $\nu_p \Rightarrow \nu$ weakly, we have

$$P_x^{\nu_p}(k > T, Z_0 = x_0, \dots, Z_k = x_k) \underset{p\to\infty}{\longrightarrow} P_x^{\nu}(k > T, Z_0 = x_0, \dots, Z_k = x_k).$$

By (3.6), for every $x, k \in \mathbb{Z}_+$,

$$\begin{aligned} V_p(x) &= E_0^{\nu_p} \left(\sum_{k=0}^{\infty} \mathbf{1}_{\{H_k \leq x\}} \right) \\ &= E_0^{\nu_p} \left(\mathbf{1}_{\{T_{-x} \leq K\}} \sum_{k=0}^{\infty} \mathbf{1}_{\{H_k \leq x\}} \right) + E_0^{\nu_p} \left(\mathbf{1}_{\{T_{-x} > K\}} \sum_{k=0}^{\infty} \mathbf{1}_{\{H_k \leq x\}} \right). \end{aligned}$$

The first variable being measurable with respect to the K first steps of Z, we get

$$E_0^{\nu_p}\left(\mathbf{1}_{\{T_{-x}\le K\}}\sum_{k=0}^{\infty}\mathbf{1}_{\{H_k\le x\}}\right)\underset{p\to\infty}{\longrightarrow}E_0^{\nu}\left(\mathbf{1}_{\{T_{-x}\le K\}}\sum_{k=0}^{\infty}\mathbf{1}_{\{H_k\le x\}}\right).$$

Since H is the *strict* ascending ladder height process of $-Z$ and Z takes only integer values,

$$E_0^{\nu_p}\left(\mathbf{1}_{\{T_{-x}> K\}}\sum_{k=0}^{\infty}\mathbf{1}_{\{H_k\le x\}}\right)\le (x+1)P_0^{\nu_p}\left(T_{-x}>K\right)\underset{p\to\infty}{\longrightarrow}(x+1)P_0^{\nu}\left(T_{-x}>K\right).$$

As a consequence, $\limsup_{p\to\infty}|V_p(x)-V(x)|\le 2(x+1)P_0^{\nu}\left(T_{-x}>K\right)$. Furthermore, T_{-x} is finite P_0^{ν}-a.s., so that $V_p(x)\to V(x)$ as $p\to\infty$ for every $x\ge 0$. Applying this to (3.8) together with (3.7) yields the expected result. □

3.3 Contour functions of random looptrees.

Let ν be a centered upwards-skip-free probability measure on $\mathbb{Z}$ such that $\nu(0)=0$. We define the probability measures $\nu_\circ$ and $\nu_\bullet$ (with means $m_\circ$ and $m_\bullet$) by

$$\nu_\circ(k):=\nu(1)\left(1-\nu(1)\right)^k\quad\text{and}\quad\nu_\bullet(k):=\frac{\nu(-k)}{1-\nu(1)},\quad k\in\mathbb{Z}_+.\tag{3.9}$$

The fact that ν is centered entails $m_\circ m_\bullet=1$, i.e., the pair $(\nu_\circ,\nu_\bullet)$ is critical.

Finite looptrees. We consider a random walk $\{\widehat{C}_k : k\in\mathbb{Z}_+\}$ with law $P_0^{\widehat{\nu}}$. We let $T:=\inf\{k\in\mathbb{Z}_+ : \widehat{C}_k<0\}$, and $C=\{C_k : 0\le k<T\}:=\{\widehat{C}_{T-1-k} : 0\le k<T\}$.

Lemma 3.10. *The tree of components* $\mathbf{T}_C=\mathrm{Tree}(\mathbf{L}_C)$ *has law* $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$.

Proof. The excursion C satisfies a.s. the assumptions of Section 3.1 and defines a looptree $\mathbf{L}_C$. By construction, the number of offspring of the root vertex in $\mathbf{T}_C$ is the number of excursions of $\widehat{C}$ above zero before T: $k_\emptyset(\mathbf{T}_C)=\inf\{k\in\mathbb{Z}_+ : \widehat{C}_{\sigma_k+1}<0\}$, where $\sigma_0=0$ and for every $k\in\mathbb{Z}_+$, $\sigma_{k+1}=\inf\{i>\sigma_k : \widehat{C}_i\le 0\}$. By the strong Markov property, $k_\emptyset(\mathbf{T}_C)$ has geometric distribution with parameter $\widehat{\nu}(-1)$, which is exactly $\nu_\circ$. The descendants of the children of the root are coded by the excursions $\{\widehat{C}_{\sigma_{k+1}-i} : 0\le i\le\sigma_{k+1}-\sigma_k\}$ for $0\le k<k_\emptyset(\mathbf{T}_C)$ and are i.i.d.. Thus, we focus on the child of the root v coded by the first of these excursions (i.e., the first child if $k_\emptyset(\mathbf{T}_C)=1$, the second otherwise).

The number of offspring of v is $k_v(\mathbf{T}_C)=\widehat{C}_1$ (conditionally on $\{T>1\}$, i.e., the root vertex has at least one child). Its law is $\widehat{\nu}$ conditioned to take positive values, which is $\nu_\bullet$. The descendants of the children of v are coded by the excursions $\{\widehat{C}_{\sigma'_{k+1}-i-1} : 0\le i<\sigma'_{k+1}-\sigma'_k\}$ for $0\le k<k_v(\mathbf{T}_C)$ (where $\sigma'_0=1$ and $\sigma'_{k+1}=\inf\{i\ge\sigma'_k : \widehat{C}_i<\widehat{C}_{\sigma'_k}\}$). These excursions are independent with the same law as $\{C_k : 0\le k<T\}$, which concludes the argument. □

Remark 3.11. We can extend Lemma 3.10 to $C=\{\widehat{C}_{T-1-k} : 0\le k<T\}$ where $\widehat{C}$ has distribution $P_x^{\widehat{\nu}}$ with $x>0$ (using the construction of Section 3.1). By decomposing $\widehat{C}$ into its excursions above its infimum (as in (3.3)), we get a forest $\mathbf{F}_C$ of $x+1$ independent looptrees, whose trees of components have distribution $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$ (see Figure 13 for an illustration).

Infinite looptrees. We first consider a random walk $C=\{C_k : k\in\mathbb{Z}_+\}$ with law P_0^{ν}.

Proposition 3.12. *The tree of components* $\mathbf{T}_C=\mathrm{Tree}(\mathbf{L}_C)$ *has law* $\mathsf{GW}_{\nu_\circ,\nu_\bullet}^{(\infty,l)}$.

Proof. The function C satisfies a.s. the assumptions of Section 3.1. We denote by $\{s_k : k\in\mathbb{Z}_+\}$ the a.s. unique spine of $\mathbf{T}_C$. Recall from (3.3) the definition of the

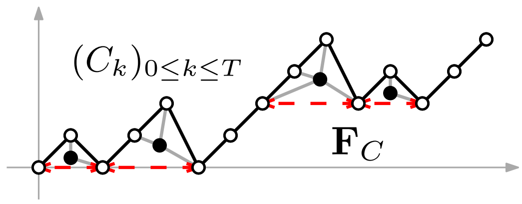

Figure 13: The construction of the forest $\mathbf{F}_C$ from the excursion C with $C_{T-1} = 4$.

excursions endpoints $\{\tau_k : k \in \mathbb{Z}_+\}$. The (white) vertices of the spine $\{s_{2k} : k \in \mathbb{Z}_+\}$ are also $\{p_C(\tau_k) : k \in \mathbb{Z}_+\}$. For every $k \in \mathbb{Z}_+$, let $\mathbf{T}_k$ be the sub-tree of $\mathbf{T}_C$ containing all the offspring of s_{2k} that are not offspring of s_{2k+2} (with the convention that s_{2k} and s_{2k+2} belong to $\mathbf{T}_k$). The tree $\mathbf{T}_k$ is coded by $\{C_{\tau_k+i} - C_{\tau_k} : 0 \le i \le \tau_{k+1} - \tau_k\}$, see Figure 14. By the strong Markov property, the trees $\{\mathbf{T}_k : k \in \mathbb{Z}_+\}$ are i.i.d. and it suffices to determine the law of $\mathbf{T}_0$.

The vertex s_1 is the unique black vertex of $\mathbf{T}_0$ that belongs to the spine of $\mathbf{T}_C$. Its number of offspring read $k_{s_1} = k_{s_1}(\mathbf{T}_C) = k_{s_1}(\mathbf{T}_0) = C_{\tau_1-1} - C_{\tau_1}$. Moreover, if $k_{s_1}^{(l)}$ (resp. $k_{s_1}^{(r)}$) is the number of offspring of s_1 on the left (resp. right) of the spine, we have $k_{s_1}^{(l)} = -C_{\tau_1} - 1$ and $k_{s_1}^{(r)} = C_{\tau_1-1}$. Thus, the position of the child $s_2 = p_C(\tau_1)$ of s_1 that belongs to the spine among its $C_{\tau_1-1} - C_{\tau_1}$ children is $-C_{\tau_1}$. By Lemma 3.1,

$$P(k_{s_1} = j) = \frac{j\nu(-j)}{\nu(1)} = \bar{\nu}_\bullet(j), \quad j \in \mathbb{N},$$

and conditionally on k_{s_1}, the rank of s_2 is uniform among $\{1, \dots, k_{s_1}\}$. We now work conditionally on $(C_{\tau_1} - 1, C_{\tau_1})$, and let $\{v_j : 0 \le j \le k_{s_1}\}$ be the neighbours of s_1 in counterclockwise order, v_0 being the root vertex of $\mathbf{T}_0$. Then, the descendants of $\{v_j : 0 \le j \le k_{s_1}^{(r)}\}$ are the trees of components of the finite forest coded by $\{C_i : 0 \le i < \tau_1\}$ (see Figure 14). By Lemma 3.1, conditionally on C_{τ_1-1}, the reversed excursion $\{C_{\tau_1-i-1} : 0 \le i < \tau_1\}$ has distribution $P^{\widehat{\nu}}_{C_{\tau_1-1}}$ and by Remark 3.11, the trees of components form a forest of $C_{\tau_1-1} + 1$ independent trees with distribution $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$, that are grafted on the right of the vertices $\{v_j : 0 \le j \le k_{s_1}^{(r)}\}$. By construction, children of s_1 on the left of the spine have no offspring. It remains to identify the offspring distribution of the white vertex of the spine in $\mathbf{T}_0$, i.e., the root vertex. It has one (black) child s_1 on the spine, which is its leftmost offspring, and a tree with distribution $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$ grafted on the right of it. As a consequence,

$$P(k_\emptyset(\mathbf{T}_C) = k) = \nu_\circ(k-1), \quad k \in \mathbb{N}.$$

By standard properties of the geometric distribution, this is the law of a uniform variable on $\{1, \dots, X\}$, with X of law $\bar{\nu}_\circ$. We get Kesten's multi-type tree with pruning on the left. □

Lastly, we consider a random walk $C = \{C_k : k \in \mathbb{Z}_+\}$ with law P^ν_0, together with an independent process $C' = \{C'_k : k \in \mathbb{Z}_+\}$ with law $P^{\nu\uparrow}_0$.

Proposition 3.13. *The tree of components $\mathbf{T}_{C,C'} = \mathrm{Tree}(\mathbf{L}_{C,C'})$ has law $\mathsf{GW}^{(\infty)}_{\nu_\circ,\nu_\bullet}$.*

Proof. The processes C and C' satisfy a.s. the assumptions of Section 3.1. There, we defined $\mathbf{L}_{C,C'}$ as the gluing along their boundaries of $\mathbf{L}_C$ and the infinite forest $\mathbf{F}_{C'}$. By Proposition 3.12, $\mathbf{T}_C = \mathrm{Tree}(\mathbf{L}_C)$ has distribution $\mathsf{GW}^{(\infty,l)}_{\nu_\circ,\nu_\bullet}$. Recall from Section 3.1 that the looptrees $\{\mathbf{L}_k : k \in \mathbb{Z}_+\}$ defining $\mathbf{F}_{C'}$ are coded by the excursions $\{C'_{R_k+i+1} - k : 0 \le i \le R_{k+1} - R_k - 1\}$. By Lemma 3.6, the time-reverse of these excursions are

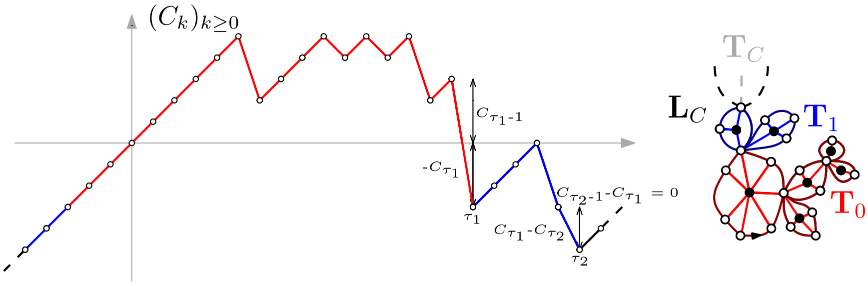

Figure 14: The decomposition of the tree $\mathbf{T}_C$.

independent and distributed as $(Z_k)_{0\leq k\leq T-1}$ under $P_0^{\widehat{\nu}}$. By Lemma 3.10, $\mathbf{T}_k := \mathrm{Tree}(\mathbf{L}_k)$ has distribution $\mathsf{GW}_{\nu_\circ,\nu_\bullet}$. So vertices of $\mathbf{T}_{C,C'}$ have independent number of offspring, and offspring distribution $\nu_\circ$ and $\nu_\bullet$ out of the spine.

By Proposition 3.12, black vertices of the spine have offspring distribution $\bar{\nu}_\bullet$, and a unique child in the spine with uniform rank conditionally on the number of offspring. It remains to identify the offspring distribution of white vertices of the spine, and thus of the root vertex. By construction, we have $k_\emptyset\left(\mathbf{T}_{C,C'}\right) = k_\emptyset\left(\mathbf{T}_0\right) + k_\emptyset\left(\mathbf{T}_C\right)$, where the variables on the right hand side are independent with respective distribution $\nu_\circ(\cdot - 1)$ and $\nu_\circ$. Then,

$$P\left(k_\emptyset\left(\mathbf{T}_{C,C'}\right) = k\right) = \sum_{i=1}^{k} \nu_\circ(i-1)\nu_\circ(k-i) = \bar{\nu}_\circ(k), \quad k \in \mathbb{N}.$$

The child of the root that belongs to the spine of $\mathbf{T}_{C,C'}$ is the leftmost child of the root in $\mathbf{T}_C$. Its rank among the $k_\emptyset\left(\mathbf{T}_{C,C'}\right)$ children of the root in $\mathbf{T}_{C,C'}$ is $k_\emptyset\left(\mathbf{T}_0\right) + 1$. Furthermore,

$$P\left(k_\emptyset\left(\mathbf{T}_0\right) + 1 = i \mid k_\emptyset\left(\mathbf{T}_{C,C'}\right) = k\right) = \nu_\circ(i-1)\nu_\circ(k-i)\frac{m_\circ}{k\nu_\circ(k)} = \frac{1}{k}, \quad k \in \mathbb{N},\ 1 \leq i \leq k,$$

so that this rank is uniform among the offspring. We obtain Kesten's multi-type tree. $\square$

Note that in the definition of $\mathbf{L}_{C,C'}$, the process C encodes the spine together with vertices and edges on its right, while C' only encodes vertices and edges on the left of the spine.

4 Decomposition of the UIHPT

In this section, we introduce a decomposition of the UIHPT along a percolation interface and prove Theorem 2.1. The idea of this decomposition first appears in [18, Section 1.7.3], where it served as a discrete intuition for a continuous model (see Section 6 for details).

4.1 Exploration process

We consider a Bernoulli percolation model with parameter p on the UIHPT, conditionally on the "Black-White" boundary condition of Figure 6. The decomposition arises from the exploration of the percolation interface between the open and closed clusters of the boundary. Our approach is based on a peeling process introduced in [3], notably to compute the critical threshold. Although we will only use this peeling process at criticality in the remainder of this section, we define it for any $p \in (0,1)$ in view of forthcoming applications.

Algorithm 4.1. ([3]) Let $p \in (0,1)$, and consider a percolated UIHPT with distribution $\mathbf{P}_p$ and a "Black-White" boundary condition.

- Reveal the face incident to the edge of the boundary whose endpoints have different colour (the "Black-White" edge).
- Repeat the algorithm on the UIHPT given by the unique infinite connected component of the map without the revealed face.

This peeling process is well-defined in the sense that the pattern of the boundary is preserved, and the spatial Markov property implies that its steps are i.i.d.. We now introduce a sequence of random variables describing the evolution of the peeling process. Let $\mathcal{E}_i$, $\mathcal{R}_{l,i}$, $\mathcal{R}_{r,i}$ and c_i be the number of exposed edges, swallowed edges on the left and right, and colour of the revealed vertex (if any, or a cemetery state otherwise) at step i of the peeling process.

Definition 4.2. *For every $i \in \mathbb{N}$, let $Y_i = (Y_i^{(1)}, Y_i^{(2)}) := \left(\mathbf{1}_{\{c_i=1\}} - \mathcal{R}_{l,i}, \mathbf{1}_{\{c_i=0\}} - \mathcal{R}_{r,i}\right)$. The exploration process $X = \{X_k = (X_k^{(1)}, X_k^{(2)}) : k \in \mathbb{Z}_+\}$ is defined by*

$$X_0 = (0,0) \quad \textit{and} \quad X_k := \sum_{i=1}^{k} Y_i, \quad k \in \mathbb{N}.$$

By the properties established in Section 2.1, the variables $\{Y_i : i \in \mathbb{N}\}$ are independent and distributed as $Y = \left(Y^{(1)}, Y^{(2)}\right)$ such that

$$Y = \begin{cases} (1,0) & \text{with probability} \quad 2p/3 \\ (0,1) & \text{with probability} \quad 2(1-p)/3 \\ (-k,0) & \text{with probability} \quad q_k \quad (k \in \mathbb{N}) \\ (0,-k) & \text{with probability} \quad q_k \quad (k \in \mathbb{N}) \end{cases}. \tag{4.1}$$

Let μ_p^0 be the law of $Y^{(1)}$, so that $Y^{(2)}$ has law μ_{1-p}^0. (This defines a probability measure since $\sum_{k \in \mathbb{N}} q_k = 1/6$.) Note that $Y^{(1)} = 0$ or $Y^{(2)} = 0$ but not both a.s., and $\mu_{1/2}^0$ is centered (since $\sum_{k \in \mathbb{N}} k q_k = 1/3$). We call the random walk X the *exploration process*, as it fully describes Algorithm 4.1. We now extract from X information on the percolation clusters. Let $\sigma_0^B = \sigma_0^W = 0$ and for every $k \in \mathbb{N}$,

$$\sigma_k^B := \inf\left\{ j \geq \sigma_{k-1}^B : X_j^{(1)} \neq X_{\sigma_{k-1}^B}^{(1)} \right\} \quad \text{and} \quad \sigma_k^W := \inf\left\{ j \geq \sigma_{k-1}^W : X_j^{(2)} \neq X_{\sigma_{k-1}^W}^{(2)} \right\}. \tag{4.2}$$

These stopping times are a.s. finite. We also let

$$z_i := \mathbf{1}_{\left\{Y_i^{(1)} \neq 0\right\}} = \mathbf{1}_{\left\{Y_i^{(2)} = 0\right\}}, \quad k \in \mathbb{N}, \tag{4.3}$$

In a word, $\{z_i : i \in \mathbb{N}\}$ is the sequence of colours of the third vertex of the faces revealed by the exploration. The processes $B = \{B_k : k \in \mathbb{Z}_+\}$ and $W = \{W_k : k \in \mathbb{Z}_+\}$ are defined by

$$B_k := X_{\sigma_k^B}^{(1)} \quad \text{and} \quad W_k := X_{\sigma_k^W}^{(2)}, \quad k \in \mathbb{Z}_+.$$

Lemma 4.3. *Let $p \in (0,1)$. Under $\mathbf{P}_p$, B and W are independent random walks started at 0 with step distribution*

$$\mu_p := \frac{\mu_p^0(\cdot \cap \mathbb{Z}\backslash\{0\})}{\mu_p^0(\mathbb{Z}\backslash\{0\})}.$$

Moreover, $\{z_i : i \in \mathbb{N}\}$ are independent with Bernoulli distribution of parameter $g(p) := 2p/3 + 1/6$, and independent of B and W.

Proof. As we noticed, for every $i \in \mathbb{Z}_+$, $Y_i^{(1)} = 0$ or $Y_i^{(2)} = 0$ a.s.. Thus, the sequences $\{\sigma_k^B : k \in \mathbb{N}\}$ and $\{\sigma_k^W : k \in \mathbb{N}\}$ induce a partition of $\mathbb{N}$. Moreover, we have

$$B_k = \sum_{j=0}^{k} Y^{(1)}_{\sigma_j^B} \quad \text{and} \quad W_k = \sum_{j=0}^{k} Y^{(2)}_{\sigma_j^W}, \quad k \in \mathbb{Z}_+. \tag{4.4}$$

The variables $\{Y_i : i \in \mathbb{N}\}$ being independent, B and W are independent. We also have

$$\sigma_k^B := \inf\left\{ i \geq \sigma_{k-1}^B : Y_i^{(1)} \neq 0 \right\} \quad \text{and} \quad \sigma_k^W := \inf\left\{ i \geq \sigma_{k-1}^W : Y_i^{(2)} \neq 0 \right\}, \quad k \in \mathbb{N}.$$

The strong Markov property applied at these times entails the first assertion.

We now turn to the sequence $\{z_i : i \in \mathbb{N}\}$. First, the distribution of z_i follows from the definition of the exploration process and the identity $\mu_p^0(0) = 2p/3 + 1/6$. Then, observe that

$$z_i = \mathbf{1}_{\{i \in \{\sigma_k^B :\, k \in \mathbb{N}\}\}} = \mathbf{1}_{\{i \notin \{\sigma_k^W :\, k \in \mathbb{N}\}\}}, \quad i \in \mathbb{N}.$$

Thanks to the strong Markov property, for every $k \in \mathbb{N}$, σ_k^B (resp. σ_k^W) is independent of $Y_{\sigma_k^B}$ (resp. $Y_{\sigma_k^W}$) so that $\{z_i : i \in \mathbb{N}\}$ is independent of B and W using (4.4). □

Note that for $p = p_c = 1/2$, $\mu := \mu_{p_c}$ is centered and $g(p_c) = 1/2$, while μ_p has positive mean if $p > p_c$. In the remainder of this section, we assume that $p = p_c$ and work under $\mathbf{P}$.

4.2 Percolation hulls and necklace

We now describe the percolation hulls $\mathcal{H}_b$ and $\mathcal{H}_w$ of the origin and the target of the root at criticality. Recall from (2.3) the definition of $(\mu_\circ, \mu_\bullet)$, and the boundary condition of Figure 6.

Proposition 4.4. *The trees of components* $\mathbf{Tree}(\mathcal{H}_b)$ *and* $\mathbf{Tree}(\mathcal{H}_w)$ *are independent with respective distribution* $\mathsf{GW}^{(\infty,l)}_{\mu_\circ,\mu_\bullet}$ *and* $\mathsf{GW}^{(\infty,r)}_{\mu_\circ,\mu_\bullet}$.

Moreover, conditionally on $\mathbf{Tree}(\mathcal{H}_b)$ *and* $\mathbf{Tree}(\mathcal{H}_w)$*, the irreducible components* $\{M_v^b : v \in \mathbf{Tree}(\mathcal{H}_b)_\bullet\}$ *and* $\{M_v^w : v \in \mathbf{Tree}(\mathcal{H}_w)_\bullet\}$ *are independent critically percolated Boltzmann triangulations with a simple boundary and respective distribution* $\mathbf{W}_{\deg(u)}$.

Proof. We first deal with the open hull $\mathcal{H}_b$. The equivalence relation $\sim$ is defined by applying (3.1) with the process $B = \{B_k : k \in \mathbb{Z}_+\}$. For every $k \in \mathbb{Z}_+$, let $\mathbf{L}_k$ be the quotient space of $(-\infty, k] \cap \mathbb{Z}$ by the restriction of $\sim$ to this set (with the embedding convention of Section 3.1). We also let $\mathbf{S}_k$ be the part of $\mathrm{Scoop}(\mathcal{H}_b)$ discovered at step σ_k^B of the peeling process.

We now prove by induction that $\mathbf{S}_k$ is a map isomorphic to $\mathbf{L}_k$ for every $k \in \mathbb{Z}_+$. This is clear for $k = 0$, since the initial open cluster is isomorphic to $\mathbb{N}$. We assume that this holds for $k \in \mathbb{N}$, and denote by p_k the canonical projection on $\mathbf{L}_k$. The exploration steps between σ_k^B and σ_{k+1}^B reveal the face incident to $p_k(k)$ and the leftmost white vertex. They leave invariant the open cluster, so we restrict our attention to the step σ_{k+1}^B at which two cases are likely.

1. An inner open vertex is discovered ($Y^{(1)}_{\sigma_{k+1}^B} = 1$). Then, $\mathbf{S}_k$ is isomorphic to $\mathbf{L}_k$ plus an extra vertex in its external face connected only to $p_k(k)$. On the other hand, $B_{k+1} = B_k + 1$ so that $\mathbf{L}_{k+1}$ is isomorphic to $\mathbf{S}_k$.
2. The third vertex of the revealed triangle is on the (left) boundary and $l \in \mathbb{N}$ edges are swallowed ($Y^{(1)}_{\sigma_{k+1}^B} = -l$). Then, $\mathbf{S}_k$ is isomorphic to $\mathbf{L}_k$ plus an edge between $p_k(k)$ and l-th vertex after $p_k(k)$ in left contour order on the boundary of $\mathbf{L}_k$. Since $B_{k+1} = B_k - l$, $\mathbf{S}_k$ is isomorphic to $\mathbf{L}_{k+1}$.

In the second case, by the spatial Markov property, the loop of perimeter $l+1$ added to $\mathbf{L}_k$ is filled in with an independent percolated triangulation with a simple boundary having distribution $\mathbf{W}_{l+1}$. This is the irreducible component $M_v^\bullet$ associated to this loop in $\mathrm{Scoop}(\mathcal{H}_b)$. The peeling process follows the right boundary of $\mathcal{H}_b$. Since $\liminf_{k\to+\infty} B_k = -\infty$ a.s., the left and right boundaries of the hull intersect infinitely many times during the exploration. This ensures that the whole hull $\mathcal{H}_b$ is revealed by the peeling process (i.e., that $\mathbf{B}_R(\mathcal{H}_b)$ is eventually revealed for every $R \geq 0$). Moreover, the sequence $\{\mathbf{L}_k : k \in \mathbb{N}\}$ is a consistent exhaustion of the looptree $\mathbf{L}_B$. Thus, the scooped-out boundary $\mathrm{Scoop}(\mathcal{H}_b)$ is isomorphic to $\mathbf{L}_B$. By Proposition 3.12, the tree of components $\mathbf{T}_B = \mathbf{Tree}(\mathcal{H}_b)$ has distribution $\mathsf{GW}^{(\infty,l)}_{\mu_\circ,\mu_\bullet}$.

The same argument shows that $\mathrm{Scoop}(\mathcal{H}_w)$ is isomorphic to $\mathbf{L}_W$ (up to a reflection and a suitable rooting convention), and the independence of B and W concludes the proof. $\square$

Let us explain how the hulls are connected in the UIHPT. We define a planar map with an infinite simple boundary $\mathbf{N}$ as follows. Let $\{c_i : i \in \mathbb{N}\}$ be the corners of the right boundary of $\mathcal{H}_b$ listed in contour order, and similarly for $\{c_i' : i \in \mathbb{N}\}$ with the left boundary of $\mathcal{H}_w$. Then, let $\mathbf{N}$ be the map with vertex set $\{c_i, c_i' : i \in \mathbb{N}\}$, such that two vertices are neighbours iff the associated corners are connected by an edge in the UIHPT. (Loosely speaking, we consider the sub-map of the UIHPT generated by the right boundary of $\mathcal{H}_b$ and the left boundary of $\mathcal{H}_w$, but we split the pinch-points of these boundaries.)

Proposition 4.5. *The infinite planar map $\mathbf{N}$ has distribution $\mathsf{UN}(\infty,\infty)$. Otherwise said, $\mathcal{H}_b$ and $\mathcal{H}_w$ are glued along an independent uniform infinite necklace.*

Remark 4.6. Due to the "Black-White" boundary condition, Proposition 4.5 ensures that the map revealed by the peeling process is $\Psi_{\mathbf{N}}(\mathcal{H}_b, \mathcal{H}_w)$.

Proof. For every $i \in \mathbb{Z}_+$ such that $Y_i^{(1)} \neq 0$, the third vertex of the revealed face at step i of the peeling process is open, and defines the i-th corner of $\partial\mathcal{H}_b$ in right contour order. In such a situation, there is an edge between this corner and the last (closed) corner of $\partial\mathcal{H}_w$ that has been discovered. The converse occurs when $Y_i^{(2)} \neq 0$. By Lemma 4.3, the variables

$$z_i := \mathbf{1}_{\left\{Y_i^{(2)}=0\right\}} = \mathbf{1}_{\left\{Y_i^{(1)}\neq 0\right\}}, \quad i \in \mathbb{N},$$

are independent with Bernoulli distribution of parameter $1/2$, and independent of B and W. We obtain the uniform infinite necklace. $\square$

4.3 Proof of the decomposition result

Proof of Theorem 2.1. The proof is based on Propositions 4.4 and 4.5. However, it remains to show that the percolation hulls $\mathcal{H}_b$ and $\mathcal{H}_w$, and the infinite necklace $\mathbf{N}$ cover the entire map, or in other words that the peeling process discovers the whole UIHPT. Indeed, by following the percolation interface, the peeling algorithm could *a priori* avoid some vertices. Let us show that this is a.s. not the case.

For every $n \in \mathbb{N}$, let M_n be the map revealed at step n of the peeling process, and denote by M_∞ the underlying UIHPT (with origin vertex ρ). We denote by $\{\tau_k^\bullet : k \in \mathbb{Z}_+\}$ and $\{\tau_k^\circ : k \in \mathbb{Z}_+\}$ the endpoints of the excursions intervals of B and W above their infimums, defined as in (3.3), and set

$$\phi(n) := \sigma^B_{\tau_n^\bullet} \vee \sigma^W_{\tau_n^\circ}, \quad n \in \mathbb{N}.$$

We have $\phi(n) < \infty$ a.s. for every $n \in \mathbb{N}$. We consider the sequence of sub-maps of M_∞ given by $\{M_{\phi(n)} : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, we let d_n stand for the graph distance on

$M_{\phi(n)}$ and denote by $\partial M_{\phi(n)}$ the boundary of $M_{\phi(n)}$ as a sub-map of M_∞. More precisely,

$$\partial M_{\phi(n)} := \left\{ v \in M_{\phi(n)} : \exists\, u \in M_\infty \backslash M_{\phi(n)} \text{ s.t. } u \sim v \right\}. \tag{4.5}$$

Let $\{v_k^\bullet : k \in \mathbb{Z}_+\}$ and $\{v_k^\circ : k \in \mathbb{Z}_+\}$ be the white vertices of the spine in $\mathbf{T}_B$ and $\mathbf{T}_W$, seen as vertices of $\mathbf{L}_B = \mathrm{Scoop}(\mathcal{H}_b)$ and $\mathbf{L}_W = \mathrm{Scoop}(\mathcal{H}_w)$ (and thus of $\mathcal{H}_b$ and $\mathcal{H}_w$). Namely,

$$v_k^\bullet := p_B\left(B_{\tau_k^\bullet}\right) \quad \text{and} \quad v_k^\circ := p_W\left(W_{\tau_k^\circ}\right), \quad k \in \mathbb{Z}_+.$$

Note that the vertices $v_0^\bullet, \dots, v_n^\bullet$ and $v_0^\circ, \dots, v_n^\circ$ can be identified in $M_{\phi(n)}$, since $\{\tau_k^\bullet : k \in \mathbb{Z}_+\}$ and $\{\tau_k^\circ : k \in \mathbb{Z}_+\}$ are stopping times in the filtration of the exploration process. Moreover, these vertices are cut-points: they disconnect the origin from infinity in $\mathbf{L}_B$ (resp. $\mathbf{L}_W$). We now define an equivalence relation $\approx$ on the set $V(M_{\phi(n)})$ of vertices of $M_{\phi(n)}$ as follows:

For every $k \in \{0, \dots, n\}$ and every $v \in V(\mathcal{H}_b \cap M_{\phi(n)})$, $v \approx v_k^\bullet$ iff there exists a geodesic path from v to the origin of $\mathcal{H}_b$ that contains $v_k^\bullet$, but does not contain $v_{k+1}^\bullet$ if $k < n$.

We define symmetric identifications on $V(\mathcal{H}_w \cap M_{\phi(n)})$. Roughly speaking, for every $k \in \{0, \dots, n-1\}$, the vertices of $\mathcal{H}_b$ between $v_k^\bullet$ and $v_{k+1}^\bullet$ (excluded) are identified to $v_k^\bullet$, and all the vertices of $\mathcal{H}_b$ above $v_n^\bullet$ are identified to $v_n^\bullet$ (and similarly in $\mathcal{H}_w$). We denote the quotient map $M_{\phi(n)}/\approx$ by $M'_{\phi(n)}$ (the root edge of $M'_{\phi(n)}$ is the root edge of M_∞). The graph distance on $M'_{\phi(n)}$ is denoted by d'_n. The family $\{M'_{\phi(n)} : n \in \mathbb{N}\}$ is a consistent sequence of locally finite maps with origin $v_0^\bullet = \rho$. Moreover, for every $n \in \mathbb{N}$, the boundary of $M'_{\phi(n)}$ in $M'_{\phi(n+1)}$ is $\{v_n^\bullet, v_n^\circ\}$. Thus, the sequences $\{d'_n(\rho, v_n^\circ) : n \in \mathbb{N}\}$ and $\{d'_n(\rho, v_n^\bullet) : n \in \mathbb{N}\}$ are non-decreasing and diverge a.s.. By definition of $\approx$ and since we discover the finite regions swallowed by the peeling process, the representatives of $\partial M_{\phi(n)}$ in $M'_{\phi(n)}$ are $v_n^\bullet$ and v_n°. As a consequence,

$$d_n\left(\rho, \partial M_{\phi(n)}\right) \underset{n\to\infty}{\longrightarrow} \infty \quad \text{a.s.}. \tag{4.6}$$

This implies that a.s. for every $R \in \mathbb{Z}_+$, the ball of radius R of the UIHPT $\mathbf{B}_R(M_\infty)$ is contained in $M_{\phi(n)}$ for n large enough, and concludes the argument. □

5 The incipient infinite cluster of the UIHPT

The goal of this section is to introduce the IIC probability measure, and prove the convergence and decomposition result of Theorem 2.3.

5.1 Exploration process

We consider a Bernoulli percolation model with parameter p on the UIHPT, conditionally on the "White-Black-White" boundary condition of Figure 8 (the consecutive open vertices on the boundary then form what we call an *open segment*). From now on, we assume that $p \in [p_c, 1)$. As in Section 4 we use a peeling process of the UIHPT, which combines two versions of Algorithm 4.1.

Algorithm 5.1. Let $p \in [p_c, 1)$ and consider a percolated UIHPT with distribution $\mathbf{P}_p$ and "White-Black-White" boundary condition. Let $n, m \in \mathbb{N}$ such that $n \geq m$. The first part of the algorithm is called the *left peeling*.

1. **Left peeling.** While the finite open segment on the boundary has size less than $n+1$:
   - Reveal the face incident to the (leftmost) "White-Black" edge on the boundary. If there is no such edge, reveal the face incident to the leftmost exposed edge at the previous step.

- Repeat the operation on the unique infinite connected component of the map without the revealed face.

When the finite open segment on the boundary has size larger than $n+1$, we start a second part which we call the *right peeling*.

2. **Right peeling.** While the finite open segment on the boundary of the map has size greater than $n+1-m$:
   - Reveal the face incident to the (rightmost) “Black-White” edge on the boundary. If there is no such edge, reveal the face incident to the leftmost exposed edge at the previous step.
   - Repeat the operation on the unique infinite connected component of the map without the revealed face.

   The algorithm ends when the left and right peelings are completed.

Remark 5.2. By definition, the left and right peelings stop when the length of the open segment reaches a given value. However, it is convenient to define both peeling processes *continued forever*. We systematically consider such processes, and use the terminology *stopped peeling process* otherwise. Nevertheless, the right peeling is defined on the event that the left peeling ends, i.e., that the open segment on the boundary reaches size $n+1$.

Intuitively, the left and right peelings explore the percolation interface between the open cluster of the origin and the closed clusters of its left and right neighbours on the boundary. Note that the peeling processes are still defined when the open segment on the boundary is swallowed, although they do not follow any percolation interface in such a situation.

As for Algorithm 4.1, by the spatial Markov property, Algorithm 5.1 is well-defined and has i.i.d. steps. We use the notation of Section 4.1 for the number of exposed edges, swallowed edges on the left and right, and colour of the revealed vertex (if any) at step k of the left and right peeling processes. We use the exponents l and r to distinguish the quantities concerning the left and right peelings.

The peeling processes are fully described by the associated exploration processes $X^{(l)} = \{X_k^{(l)} = (X_k^{(l,1)}, X_k^{(l,2)}) : k \in \mathbb{Z}_+\}$ and $X^{(r)} = \{X_k^{(r)} = (X_k^{(r,1)}, X_k^{(r,2)}) : k \in \mathbb{Z}_+\}$, defined by

$$X_0^{(l)} = X_0^{(r)} = (0,0) \quad \text{and} \quad X_k^{(l)} := \sum_{i=1}^{k} Y_i^{(l)},\ X_k^{(r)} := \sum_{i=1}^{k} Y_i^{(r)}, \quad k \in \mathbb{N},$$

where for every $i \in \mathbb{N}$,

$$Y_i^{(l)} := \left(\mathbf{1}_{\left\{c_i^{(l)}=1\right\}} - \mathcal{R}_{l,i}^{(l)}, \mathbf{1}_{\left\{c_i^{(l)}=0\right\}} - \mathcal{R}_{r,i}^{(l)}\right), Y_i^{(r)} := \left(\mathbf{1}_{\left\{c_i^{(r)}=1\right\}} - \mathcal{R}_{l,i}^{(r)}, \mathbf{1}_{\left\{c_i^{(r)}=0\right\}} - \mathcal{R}_{r,i}^{(r)}\right).$$

The exploration processes $X^{(l)}$ and $X^{(r)}$ have respective lifetimes

$$\sigma_n^{(l)} := \inf\left\{k \in \mathbb{Z}_+ : X_k^{(l,1)} \geq n\right\} \quad \text{and} \quad \sigma_m^{(r)} := \inf\left\{k \in \mathbb{Z}_+ : X_k^{(r,1)} \leq -m\right\}. \tag{5.1}$$

The right peeling process is defined on the event $\{\sigma_n^{(l)} < \infty\}$. However, the assumption $p \geq p_c$ guarantees that a.s. for every $n \in \mathbb{N}$, $\sigma_n^{(l)} < \infty$ and the left peeling ends. On the contrary, when $p > p_c$, with positive probability $\sigma_m^{(r)} = \infty$ and the right peeling does not end.

We now extract information on the percolation clusters, and introduce the processes $B^{(l)} = \{B_k^{(l)} : k \in \mathbb{Z}_+\}$, $W^{(l)} = \{W_k^{(l)} : k \in \mathbb{Z}_+\}$, $B^{(r)} = \{B_k^{(r)} : k \in \mathbb{Z}_+\}$ and $W^{(r)} = \{W_k^{(r)} : k \in \mathbb{Z}_+\}$ defined by

$$\left(B_k^{(l)}, W_k^{(l)}\right) := \left(X^{(l,1)}_{\sigma_k^{(B,l)}}, X^{(l,2)}_{\sigma_k^{(W,l)}}\right) \quad \text{and} \quad \left(B_k^{(r)}, W_k^{(r)}\right) := \left(X^{(r,1)}_{\sigma_k^{(B,r)}}, X^{(r,2)}_{\sigma_k^{(W,r)}}\right), \quad k \in \mathbb{Z}_+,$$

where we use the same definitions as in (4.2) for the stopping times. We define as in (4.3) the random variables $\{z_i^{(l)} : i \in \mathbb{N}\}$ and $\{z_i^{(r)} : i \in \mathbb{N}\}$. The exploration process $X^{(l)}$ is measurable with respect to $B^{(l)}$, $W^{(l)}$ and $z^{(l)}$ (and the same holds when replacing l by r). The lifetimes of the processes $B^{(l)}$ and $B^{(r)}$ are defined by

$$T_n^{(B,l)} := \inf\left\{k \in \mathbb{Z}_+ : B_k^{(l)} \geq n\right\} \quad \text{and} \quad T_m^{(B,r)} := \inf\left\{k \in \mathbb{Z}_+ : B_k^{(r)} \leq -m\right\}, \tag{5.2}$$

while the lifetimes of $W^{(l)}$ and $W^{(r)}$ read

$$T_n^{(W,l)} := \sigma_n^{(l)} - T_n^{(B,l)} \quad \text{and} \quad T_m^{(W,r)} := \sigma_m^{(r)} - T_m^{(B,r)}.$$

When $p > p_c$, $\sigma_m^{(r)} = \infty$ with positive probability, in which case $T_m^{(B,r)} = \infty$ (and by convention, $T_m^{(W,r)} = \infty$). Thus, $\{\sigma_m^{(r)} < \infty\}$ is measurable with respect to $T_m^{(B,r)}$ and $T_m^{(W,r)}$. For every $k \in \mathbb{N} \cup \{\infty\}$ and every $p \in (0,1)$, we denote by $\mathsf{NB}(k,p)$ the negative binomial distribution with parameters k and p (where $\mathsf{NB}(\infty,p)$ is a Dirac mass at infinity).

Lemma 5.3. *Let $p \in [p_c, 1)$ and $n, m \in \mathbb{N}$ such that $n \geq m$. The following hold under $\mathbf{P}_p$.*

- ***Left peeling.*** *$\{B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)}\}$ is a random walk with step distribution μ_p, killed at the first entrance in $[n, +\infty)$. Conditionally on $T_n^{(B,l)}$, $T_n^{(W,l)}$ has distribution $\mathsf{NB}(T_n^{(B,l)}, g(p))$. Then, conditionally on $T_n^{(W,l)}$, $\{W_k^{(l)} : 0 \leq k \leq T_n^{(W,l)}\}$ is a random walk with step distribution μ_{1-p}, independent of $B^{(l)}$. Finally, conditionally on $T_n^{(B,l)}$ and $T_n^{(W,l)}$, $\{z_i^{(l)} : 1 \leq i \leq \sigma_n^{(l)} - 1\}$ is uniformly distributed among the set of binary sequences with $T_n^{(W,l)}$ zeros and $T_n^{(B,l)} - 1$ ones.*
- ***Right peeling.*** *The above results hold when replacing l by r, except that $\{B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)}\}$ is killed at the first entrance in $(-\infty, -m]$. Moreover, $\sigma_m^{(r)}$ is possibly infinite: conditionally on $T_m^{(B,r)}$ and $T_m^{(W,r)}$, $\{z_i^{(r)} : 1 \leq i \leq \sigma_m^{(r)} - 1\}$ is uniformly distributed among the set of binary sequences with $T_m^{(W,r)}$ zeros and $T_m^{(B,r)} - 1$ ones on the event $\{\sigma_m^{(r)} < \infty\}$, and distributed as a sequence of i.i.d. variables with Bernoulli distribution of parameter $g(p)$ on the event $\{\sigma_m^{(r)} = \infty\}$.*

Lastly, $X^{(r)}$ is independent of $\{X_k^{(l)} : 0 \leq k \leq \sigma_n^{(l)}\}$. (However, it is not independent of the whole process $X^{(l)}$.)

Proof. By the spatial Markov property, $X^{(r)}$ is independent the map revealed by the left peeling process, i.e., of $\{X_k^{(l)} : 0 \leq k \leq \sigma_n^{(l)}\}$. Moreover, $X^{(l)}$ and $X^{(r)}$ are distributed as the process X of Definition 4.2. We restrict our attention to the left peeling. By Lemma 4.3, $B^{(l)}$ and $W^{(l)}$ are independent random walks with step distribution μ_p and μ_{1-p}, while $z^{(l)}$ is an independent sequence of i.i.d. variables with Bernoulli distribution of parameter $g(p)$. The first assertion follows from the definition of $T_n^{(B,l)}$. The random time $\sigma_n^{(l)}$ being measurable with respect to $X^{(l,1)}$, it is independent of $W^{(l)}$. Thus, conditionally on $T_n^{(B,l)}$ the lifetime of $W^{(l)}$ is exactly the number of failures before the $T_n^{(B,l)}$-th success in a sequence of independent Bernoulli trials with parameter $g(p)$. This is the second assertion. By definition, $Y^{(1,l)}_{\sigma_n^{(l)}} \neq 0$ so that $z^{(l)}_{\sigma_n^{(l)}} = 1$. Conditionally on $T_n^{(B,l)}$ and $T_n^{(W,l)}$, the binary sequence $\{z_i^{(l)} : 1 \leq i \leq \sigma_n^{(l)} - 1\}$ is uniform among all possibilities by definition of the negative binomial distribution. The properties of the right peeling follow from a slight adaptation of these arguments. □

We now describe the events we condition on to force an infinite critical open percolation cluster and define the IIC. As in [23], we use two distinct conditionings. Firstly, $B^{(l)}$ describes the length of the open segment on the boundary of the unexplored map (before the left peeling stops). This segment represents the revealed part of the left boundary of $\mathcal{C}$ (the open cluster of the origin). We define the height $h(\mathcal{C})$ of $\mathcal{C}$ by

$$h(\mathcal{C}) := \sup\left\{B_k^{(l)} : 0 \leq k \leq T\right\}, \quad \text{where} \quad T := \inf\left\{k \in \mathbb{Z}_+ : B_k^{(l)} < 0\right\}. \tag{5.3}$$

This is also the length of the loop-erasure of the open path revealed by the left peeling. Note that both the perimeter of the hull $\mathcal{H}$ of $\mathcal{C}$ and its size $|\mathcal{C}|$ are larger than $h(\mathcal{C})$. Consistently, we want to condition the height of the cluster to be larger than n. In terms of the exploration process, this exactly means that $B^{(l)}$ reaches $[n, +\infty)$ before $(-\infty, 0)$:

$$\{h(\mathcal{C}) \geq n\} = \left\{T_n^{(B,l)} < T\right\}.$$

We thus work under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and let n go to infinity. Secondly, we let $p > p_c = 1/2$ and condition $\mathcal{C}$ to be infinite (which is an event of positive probability under $\mathbf{P}_p$). In other words, we work under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$ and let p decrease towards p_c.

We now describe the law of the exploration process under these conditional probability measures, starting with $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$. In the next part, we voluntarily choose the same parameter n for the definition of the left peeling process and the conditioning $\{h(\mathcal{C}) \geq n\}$. This will make our argument simpler. The event $\{h(\mathcal{C}) \geq n\} = \{T_n^{(B,l)} < T\}$ is measurable with respect to $\{B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)}\}$. By Lemma 5.3, it is then independent of the other variables involved in the stopped left peeling process.

Lemma 5.4. *Let $n, m \in \mathbb{N}$ such that $n \geq m$. Under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$, the statements of Lemma 5.3 (under $\mathbf{P}$) hold except that $\{B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)}\}$ is a random walk conditioned to reach $[n, +\infty)$ before $(-\infty, 0)$ (and killed at that entrance time).*

We now describe the exploration process under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, for $p > p_c$.

Lemma 5.5. *Let $p > p_c$ and $n, m \in \mathbb{N}$ such that $n \geq m$. Under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, the statements of Lemma 5.3 (under $\mathbf{P}_p$) hold except that $\{B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)}\}$ is a random walk conditioned to stay nonnegative killed at the first entrance in $[n, +\infty)$, and $\{B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)}\}$ is a random walk conditioned to stay larger than $-n$ killed at the first entrance in $(-\infty, -m]$.*

Proof. We consider the left peeling first. From the proof of $p_c = 1/2$ in [3], we get that

$$\{|\mathcal{C}| = \infty\} = \{T = \infty\} \quad \mathbf{P}_p\text{-a.s.}.$$

Indeed, with probability one, if T is finite the open segment on the boundary is swallowed by the exploration and $\mathcal{C}$ is confined in a finite region, while if T is infinite the exploration reveals infinitely many open vertices connected to the origin by an open path. In particular, the event $\{|\mathcal{C}| = \infty\}$ is measurable with respect to $B^{(l)}$, and thus independent of $W^{(l)}$ and $z^{(l)}$ by Lemma 4.3. The first assertion follows.

We now focus on the right peeling process and denote by M_∞ the underlying UIHPT. The event $\{T_n^{(B,l)} < \infty\} \cap \{T_n^{(B,l)} < T\}$ has probability one under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$. Thus, the right exploration is performed in a half-planar triangulation M'_∞ with distribution $\mathbf{P}_p$ and a "White-Black-White" boundary condition (the open segment on the boundary has size $n+1$ and its rightmost vertex is the origin of M_∞). Since the stopped left peeling reveals a.s. finitely many vertices, the open cluster of the origin $\mathcal{C}$ in M_∞ is infinite iff the open cluster of the origin $\mathcal{C}'$ in M'_∞ is infinite. In other words, the right exploration is distributed as the exploration process of the UIHPT with the above boundary condition under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, and is independent of $\{X_k^{(l)} : 0 \leq k \leq \sigma_n^{(l)}\}$. By the same argument as above,

$$\{|\mathcal{C}| = \infty\} = \{T'_{-n} = \infty\} \quad \mathbf{P}_p\text{-a.s.,} \quad \text{where } T'_{-n} := \inf\left\{k \in \mathbb{Z}_+ : B_k^{(r)} < -n\right\}.$$

Again, $\{|\mathcal{C}| = \infty\}$ is measurable with respect to $B^{(r)}$, and thus independent of $W^{(r)}$ and $z^{(r)}$ by Lemma 4.3. This ends the proof. □

5.2 Distribution of the revealed map

We consider the peeling process of Algorithm 5.1 under the probability measures $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$. We let M_∞ be the underlying infinite triangulation of the half-plane.

We denote by $M_{n,m}$ the planar map that is revealed by the peeling process of Algorithm 5.1 with parameters $n, m \in \mathbb{N}$. Namely, the vertices and edges of $M_{n,m}$ are those discovered by the stopped peeling process (including the swallowed regions). However, by convention, we exclude the edges and vertices discovered at the last step of both the left and right (stopped) peeling processes. This will make our description simpler. The root edge of $M_{n,m}$ is the root edge of M_∞. By definition, $M_{n,m}$ is infinite on the event $\{\sigma_m^{(r)} = \infty\}$. The goal of this section is to provide a decomposition of the map $M_{n,m}$.

The percolation hulls. We now extend our definition of the percolation hulls to the possibly finite map $M_{n,m}$. The origin ρ of M_∞ and its left and right neighbours ρ_l and ρ_r on the boundary belong to $M_{n,m}$. The percolation hull $\mathcal{H}$ of ρ in $M_{n,m}$ is the cluster of ρ together with the connected components of its complement in $M_{n,m}$ that do not contain ρ_l or ρ_r. This extends to the hulls of ρ_l and ρ_r, for which we use the notation $\mathcal{H}_l$ and $\mathcal{H}_r$.

We let

$$\underline{W}^{(l)} := \min_{0 \leq k \leq T_n^{(W,l)}} W_k^{(l)} \quad \text{and} \quad \underline{W}^{(r)} := \min_{0 \leq k \leq T_m^{(W,r)}} W_k^{(r)},$$

and define the finite sequences $w^{(l)}$ and $w^{(r)}$ by

$$w_k^{(l)} = \begin{cases} k & \text{if} \quad -\underline{W}^{(l)} \leq k < 0 \\ W_k^{(l)} & \text{if} \quad 0 \leq k \leq T_n^{(W,l)} \end{cases} \quad \text{and} \quad w_k^{(r)} = \begin{cases} k & \text{if} \quad -\underline{W}^{(r)} \leq k < 0 \\ W_{-k}^{(r)} & \text{if} \quad 0 \leq k \leq T_m^{(W,r)} \end{cases}. \tag{5.4}$$

We define equivalence relations on $\{-T_n^{(W,l)}, \ldots, -\underline{W}^{(l)}\}$ and $\{-T_m^{(W,r)}, \ldots, -\underline{W}^{(r)}\}$ by applying (3.1) with $w^{(l)}$ and $w^{(r)}$. By the construction of Section 3.1, this defines two planar maps which are not looptrees in general. Nonetheless, we denote them by $\mathbf{L}_{w^{(l)}}$ and $\mathbf{L}_{w^{(r)}}$ to keep the notation simple. We replace $\mathbf{L}_{w^{(l)}}$ by its image under a reflection. The left (resp. right) boundary of $\mathbf{L}_{w^{(l)}}$ is the projection of nonpositive (resp. nonnegative) integers of $\{-T_n^{(W,l)}, \ldots, -\underline{W}^{(l)}\}$ on $\mathbf{L}_{w^{(l)}}$. The same holds when replacing l by r, up to inverting left and right boundaries. We choose the root edges consistently. The sequence $w^{(l)}$ and $\mathbf{L}_{w^{(l)}}$ are a.s. finite, while $w^{(r)}$ and thus $\mathbf{L}_{w^{(r)}}$ are infinite on the event $\{\sigma_m^{(r)} = \infty\}$. We introduce the finite sequences $b^{(l)}$ and $b^{(r)}$ defined by

$$b_k^{(l)} = B_k^{(l)}, \quad 0 \leq k < T_n^{(B,l)} \quad \text{and} \quad b_k^{(r)} = B_k^{(r)}, \quad 0 \leq k < T_m^{(B,r)}. \tag{5.5}$$

We define an equivalence relation on $\{-T_n^{(B,l)} + 1, \ldots, T_m^{(B,r)} - 1\}$ by applying (3.5), with $b^{(l)}$ (resp. $b^{(r)}$) playing the role of C' (resp. C). By the construction of Section 3.1, this defines a planar map that we denote by $\mathbf{L}_{b^{(l)},b^{(r)}}$, which is not a looptree either. (It is also obtained by defining $\mathbf{L}_{b^{(r)}}$ and the finite forest of looptrees $\mathbf{F}_{b^{(l)}}$ as above, and gluing them along their boundaries as in Section 3.1.) The left (resp. right) boundary of $\mathbf{L}_{b^{(l)},b^{(r)}}$ is the projection of nonpositive (resp. nonnegative) integers of $\{-T_n^{(B,l)}+1,\dots,T_m^{(B,r)}-1\}$ (and we choose the root edge consistently). Again, $\mathbf{L}_{b^{(l)},b^{(r)}}$ is infinite on the event $\{\sigma_m^{(r)}=\infty\}$.

Proposition 5.6. *Let $n,m\in\mathbb{N}$ such that $n\geq m$. Under $\mathbf{P}(\cdot\mid h(\mathcal{C})\geq n)$ and $\mathbf{P}_p(\cdot\mid|\mathcal{C}|=\infty)$, in the map $M_{n,m}$, the percolation hulls $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$ are the independent random maps $\mathbf{L}_{b^{(l)},b^{(r)}}$, $\mathbf{L}_{w^{(l)}}$ and $\mathbf{L}_{w^{(r)}}$ in which each internal face of degree $l\geq 2$ is filled in with an independent triangulation with distribution $\mathbb{W}_l$ equipped with a Bernoulli percolation model with parameter p_c (resp. p).*

Proof. The proof follows the same lines as Proposition 4.4, to which we refer for more details. We begin with the left peeling, which follows the percolation interface between $\mathcal{H}_l$ and $\mathcal{H}$. On the one hand, the right contour of $\mathcal{H}_l$ is encoded by $\{W_k^{(l)}:0\leq k\leq T_n^{(W,l)}\}$ and $-\underline{W}^{(l)}$ vertices are discovered on the left boundary of M_∞. We obtain the map $\mathbf{L}_{w^{(l)}}$. On the other hand, the left contour of $\mathcal{H}$ is encoded by $b^{(l)}$, which defines the forest of finite looptrees $\mathbf{F}_{b^{(l)}}$ (by applying (3.1)). We now deal with the right peeling, that starts a.s. in a triangulation of the half-plane with an open segment of size $n+1$ on the boundary. This segment corresponds (up to the last vertex) to the right boundary of $\mathbf{F}_{b^{(l)}}$, i.e., to the set

$$\mathsf{V}_l:=\left\{0\leq k<T_n^{(B,l)}:B_k^{(l)}=\inf_{k\leq i<T_n^{(B,l)}}B_i^{(l)}\right\}. \tag{5.6}$$

The right peeling follows the percolation interface between $\mathcal{H}$ and $\mathcal{H}_r$. In particular, the right contour of $\mathcal{H}$ is encoded by $b^{(r)}$. By construction, vertices associated to the set

$$\mathsf{V}_r:=\left\{0\leq k<T_m^{(B,r)}:B_k^{(r)}=\inf_{0\leq i\leq k}B_i^{(r)}\right\} \tag{5.7}$$

are identified with the right boundary of $\mathbf{F}_{b^{(l)}}$. Precisely, every $k\in\mathsf{V}_r$ such that $B_k^{(r)}=-j$ is matched to the $(j+1)$-th element of V_l (note that $-\inf(b^{(r)})\leq m\leq n$). We obtain the map $\mathbf{L}_{b^{(l)},b^{(r)}}$. Finally, $\{W_k^{(r)}:0\leq k\leq T_m^{(W,r)}\}$ encodes the left contour of $\mathcal{H}_r$ and $-\underline{W}^{(r)}$ vertices are discovered on the right boundary of M_∞, which gives the map $\mathbf{L}_{w^{(r)}}$.

The spatial Markov property shows that the finite faces of $\mathbf{L}_{b^{(l)},b^{(r)}}$, $\mathbf{L}_{w^{(l)}}$ and $\mathbf{L}_{w^{(r)}}$ are filled in with independent percolated Boltzmann triangulations with a simple boundary (the boundary conditions are fixed by the hulls and the percolation parameter by the underlying model). By definition of $M_{n,m}$, we then recover the whole percolation hulls $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$. □

We now focus on the connection between the percolation hulls in $M_{n,m}$. In order to make the next statement simpler, we generalize the definition of the uniform necklace.

Definition 5.7. *Let $x,y\in\mathbb{N}$ and $\{z_i:1\leq i\leq x+y\}$ uniform among the set of binary sequence with x ones and y zeros. Define*

$$S_k:=\sum_{i=1}^k z_i,\quad 1\leq k\leq x+y.$$

The uniform necklace of size (x,y) is obtained from the graph of $[-x,y+1]\cap\mathbb{Z}$ by adding the set of edges $\{(-S_k,k+1-S_k):1\leq k\leq x+y\}$. Its distribution is denoted by $\mathsf{UN}(x,y)$.

By convention, for $x=y=\infty$, $\mathsf{UN}(\infty,\infty)$ is the uniform infinite necklace of Section 2.3.

We now use a construction similar to that preceding Proposition 4.5. Let $V_l:=\{c_i:1\leq i\leq T_n^{(W,l)}+1\}$ be the corners of the right boundary of $\mathcal{H}_l$ listed in contour order, and similarly for $V_r := \{c'_i : 1 \leq i \leq T_n^{(B,l)}\}$ with the left boundary of $\mathcal{H}$. Then, let N_l be the planar map with vertex set $V_l \cup V_r$, such that two vertices are neighbours iff the associated corners are connected by an edge in the UIHPT. The planar map N_r is defined symmetrically with the right boundary of $\mathcal{H}$ and the left boundary of $\mathcal{H}_r$.

Proposition 5.8. *Let $n, m \in \mathbb{N}$ such that $n \geq m$. Under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, conditionally on $T_n^{(B,l)}$, $T_m^{(B,r)}$, $T_n^{(W,l)}$ and $T_m^{(W,r)}$, the following holds: N_l and N_r are independent uniform necklaces with distribution* $\mathsf{UN}(T_n^{(W,l)}, T_n^{(B,l)} - 1)$ *and* $\mathsf{UN}(T_m^{(B,r)} - 1, T_m^{(W,r)})$. *Otherwise said, in the map $M_{n,m}$, $\mathcal{H}_l$, $\mathcal{H}$ and $\mathcal{H}_r$ are glued along a pair of independent uniform necklaces with respective size $(T_n^{(W,l)}, T_n^{(B,l)} - 1)$ and $(T_m^{(B,r)} - 1, T_m^{(W,r)})$.*

Proof. We follow the arguments of Proposition 4.5. For every $k \in \mathbb{Z}_+$ such that $Y_k^{(l,1)} \neq 0$, there is an edge between the revealed open corner of the left boundary of $\mathcal{H}$ and the last revealed closed corner of the right boundary of $\mathcal{H}_l$. The converse occurs when $Y_k^{(l,2)} \neq 0$. Then, N_l is the uniform necklace generated by $\{z_i^{(l)} : 1 \leq i < \sigma_n^{(l)}\}$. Similarly, N_r is the uniform necklace generated by $\{z_i^{(r)} : 1 \leq i < \sigma_m^{(r)}\}$. We conclude by Lemmas 5.4 and 5.5. □

Since $M_{n,m}$ is the map revealed by the peeling process, we obtain a decomposition of this map illustrated in Figure 15 (in the finite case). The map $M_{n,m}$ is measurable with respect to the processes $w^{(l)}$, $w^{(r)}$, $b^{(l)}$ and $b^{(r)}$, the variables $z^{(l)}$ and $z^{(r)}$ defining the uniform necklaces and the percolated triangulations with a simple boundary that fill in the internal faces.

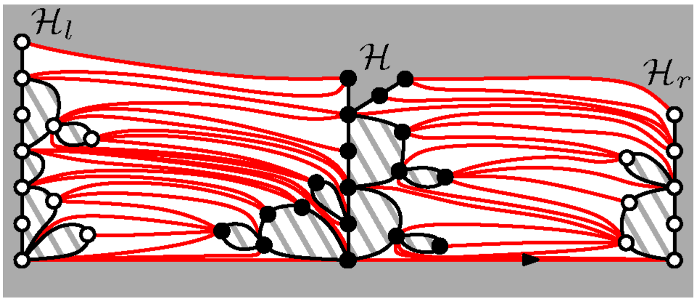

Figure 15: The decomposition of the map $M_{n,m}$ into percolation hulls and necklaces.

5.3 The IIC probability measure

In this section, we define a triangulation of the half-plane with distribution $\mathbf{P}_{\mathsf{IIC}}$. We use infinite looptrees, uniform necklaces and Boltzmann triangulations as building blocks.

Definition of the IIC. Let P be a probability measure and under P, let $A^{(l)} := \{A_k^{(l)} : k \in \mathbb{Z}_+\}$ be a random walk with law $P_0^{\mu\uparrow}$. Let also $A^{(r)} := \{A_k^{(r)} : k \in \mathbb{Z}_+\}$, $V^{(l)} := \{V_k^{(l)} : k \in \mathbb{Z}_+\}$ and $V^{(r)} := \{V_k^{(r)} : k \in \mathbb{Z}_+\}$ be random walks with law P_0^{μ}. Finally, let $y^{(l)} = \{y_i^{(l)} : i \in \mathbb{N}\}$ and $y^{(r)} = \{y_i^{(r)} : i \in \mathbb{N}\}$ be sequences of i.i.d. variables with Bernoulli distribution of parameter $1/2$. We assume that these processes are all independent under P.

Following Section 3.3, we define the looptrees $\mathbf{L}_{A^{(l)},A^{(r)}}$, $\mathbf{L}_{V^{(l)}}$ and $\mathbf{L}_{V^{(r)}}$. We replace $\mathbf{L}_{V^{(r)}}$ by its image under a reflection (with the root edge going from the vertex 0 to -1). By Propositions 3.12 and 3.13, the associated trees of components $\mathbf{T}_{A^{(l)},A^{(r)}}$, $\mathbf{T}_{V^{(l)}}$ and

$\mathbf{T}_{V^{(r)}}$ have respective distribution $\mathsf{GW}^{\infty}_{\mu_\circ,\mu_\bullet}$, $\mathsf{GW}^{(\infty,l)}_{\mu_\circ,\mu_\bullet}$ and $\mathsf{GW}^{(\infty,r)}_{\mu_\circ,\mu_\bullet}$. We agree that the vertices of $\mathbf{L}_{A^{(l)},A^{(r)}}$ are open, while vertices of $\mathbf{L}_{V^{(l)}}$ and $\mathbf{L}_{V^{(r)}}$ are closed. For every vertex $v \in (\mathbf{T}_{A^{(l)},A^{(r)}})_\bullet$ (resp. $(\mathbf{T}_{V^{(l)}})_\bullet$, $(\mathbf{T}_{V^{(r)}})_\bullet$) we let M_v (resp. M_v^l, M_v^r) be a Boltzmann triangulation with distribution $\mathbf{W}_{\deg(u)}$ (independent of all the other variables). Then, $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$ are defined by

$$\mathcal{H} := \Phi^{-1}\left(\mathbf{T}_{A^{(l)},A^{(r)}}, \left\{M_v : v \in \left(\mathbf{T}_{A^{(l)},A^{(r)}}\right)_\bullet\right\}\right),\ \mathcal{H}_l := \Phi^{-1}\left(\mathbf{T}_{V^{(l)}}, \left\{M_v^l : v \in \left(\mathbf{T}_{V^{(l)}}\right)_\bullet\right\}\right),$$

and similarly for $\mathcal{H}_r$ replacing l by r. Finally, we let $\mathbf{N}_l$ and $\mathbf{N}_r$ be the uniform infinite necklaces with distribution $\mathsf{UN}(\infty,\infty)$ generated by $y^{(l)}$ and $y^{(r)}$. The infinite planar map M_∞ is defined by gluing $(\mathcal{H}_l,\mathcal{H},\mathcal{H}_r)$ along $(\mathbf{N}_l,\mathbf{N}_r)$, i.e.,

$$M_\infty := \Psi_{(\mathbf{N}_l,\mathbf{N}_r)}(\mathcal{H}_l,\mathcal{H},\mathcal{H}_r).$$

In particular, the root edge of M_∞ connects the origin of $\mathbf{L}_{A^{(l)},A^{(r)}}$ to that of $\mathbf{L}_{V^{(r)}}$. The probability measure $\mathbf{P}_{\mathsf{IIC}}$ is the distribution of M_∞ under P.

The infinite planar map M_∞ is by construction one-ended, since infinite looptrees are one-ended themselves. It is also a (type 2) triangulation with an infinite simple boundary, and thus a triangulation of the upper half-plane. By definition, vertices of M_∞ are coloured and M_∞ has the "White-Black-White" boundary condition of Figure 8. Moreover, the planar maps $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$ are the percolation hulls of the origin vertex and its neighbours on the boundary in M_∞, which justifies the choice of notation.

Exploration of the IIC. For every $n,m\in\mathbb{N}$, we define a finite map $M_{n,m}$ under P, by replicating the construction of Section 5.2. Let

$$T_n^{(A,l)} := \inf\left\{k \in \mathbb{Z}_+ : A_k^{(l)} \geq n\right\} \quad \text{and} \quad T_m^{(A,r)} := \inf\left\{k \in \mathbb{Z}_+ : A_k^{(r)} \leq -m\right\}, \tag{5.8}$$

and conditionally on $T_n^{(A,l)}$ and $T_m^{(A,r)}$, let $T_n^{(V,l)}$ and $T_m^{(V,r)}$ be independent random variables with distribution $\mathsf{NB}(T_n^{(A,l)},1/2)$ and $\mathsf{NB}(T_m^{(A,r)},1/2)$. We let

$$\rho_n^{(l)} := T_n^{(A,l)} + T_n^{(V,l)} \quad \text{and} \quad \rho_m^{(r)} := T_m^{(A,r)} + T_m^{(V,r)}.$$

We define $v^{(l)}$ and $v^{(r)}$ as in (5.4) (replacing W by V), and $a^{(l)}$ and $a^{(r)}$ as in (5.5) (replacing B by A). Note that all the random times considered here are finite P-a.s.. Let us consider the finite planar maps $\mathbf{L}_{a^{(l)},a^{(r)}}$, $\mathbf{L}_{v^{(l)}}$ and $\mathbf{L}_{v^{(r)}}$, defined according to the construction of Section 5.2. As we will see in Proposition 5.9, these are possibly (though not always) sub-maps of the infinite looptrees $\mathbf{L}_{A^{(l)},A^{(r)}}$, $\mathbf{L}_{V^{(l)}}$ and $\mathbf{L}_{V^{(r)}}$. We now fill in each internal face of degree $l \geq 2$ of the finite maps with an independent percolated Boltzmann triangulation with distribution $\mathbf{W}_l$. We agree that we use the same triangulations to fill in corresponding faces in the finite maps and their infinite counterparts when this is possible. Finally, we glue the right boundary of $\mathbf{L}_{v^{(l)}}$ and the left boundary of $\mathbf{L}_{a^{(l)},a^{(r)}}$ along the uniform necklace with size $(T_n^{(V,l)}, T_n^{(A,l)}-1)$ generated by $\{y_i^{(l)} : 1 \leq i < \rho_n^{(l)}\}$. Similarly, we glue the left boundary of $\mathbf{L}_{v^{(r)}}$ and the right boundary of $\mathbf{L}_{a^{(l)},a^{(r)}}$ along the uniform necklace with size $(T_m^{(V,r)}, T_m^{(A,r)}-1)$ generated by $\{y_i^{(r)} : 1 \leq i < \rho_m^{(r)}\}$. These gluing operations are defined as in Section 2.3 provided minor adaptations to the finite setting. The resulting planar map is denoted by $M_{n,m}$.

Proposition 5.9. *A.s., for every $n\in\mathbb{N}$ and every $m\in\mathbb{N}$ such that $m \leq \inf\{A_k^{(l)} : k \geq T_n^{(A,l)}\}$, $M_{n,m}$ is a sub-map of M_∞.*

Proof. Let $n,m\in\mathbb{N}$. We first consider the equivalence relation $\sim$ defined by applying (3.1) to $V^{(l)}$. The relation $i \sim j$ is determined by the values $\{V_k^{(l)} : i \wedge j \leq k \leq i \vee j\}$. Thus, the restriction of $\sim$ to $[-T_n^{(V,l)}, -\underline{V}^{(l)}] \cap \mathbb{Z}$ is isomorphic to $\mathbf{L}_{v^{(l)}}$, and $\mathbf{L}_{v^{(l)}}$ is a sub-map of $\mathbf{L}_{V^{(l)}}$. The same argument proves that $\mathbf{L}_{v^{(r)}}$ is a sub-map of $\mathbf{L}_{V^{(r)}}$, and that $\mathbf{F}_{a^{(l)}}$ (resp. $\mathbf{L}_{a^{(r)}}$) is a sub-map of the forest $\mathbf{F}_{A^{(l)}}$ (resp. the looptree $\mathbf{L}_{A^{(r)}}$). Moreover, the finite necklaces generated by $\{y_i^{(l)} : 1 \leq i < \rho_n^{(l)}\}$ and $\{y_i^{(r)} : 1 \leq i < \rho_m^{(r)}\}$ are sub-maps of the infinite uniform necklaces $\mathbf{N}_l$ and $\mathbf{N}_r$ generated by $y^{(l)}$ and $y^{(r)}$ in M_∞.

The only thing that remains to check is the gluing of $\mathbf{F}_{a^{(l)}}$ and $\mathbf{L}_{a^{(r)}}$ into $\mathbf{L}_{a^{(l)},a^{(r)}}$. Let $a := \{a_k : -T_n^{(A,l)} + 1 \leq k \leq T_m^{(A,r)} - 1\}$ and $A := \{A_k : k \in \mathbb{Z}\}$ be defined from $(a^{(l)}, a^{(r)})$ and $(A^{(l)}, A^{(r)})$ as in (3.4). We let $\sim_a$ and $\sim_A$ be the associated equivalence relations, defined from (3.5). When m is too large compared to n, $\sim_a$ is finer than the restriction of $\sim_A$ to $[-T_n^{(A,l)}+1, T_m^{(A,r)}-1] \cap \mathbb{Z}$ (see Figure 16 for an example). In other words, $\mathbf{L}_{a^{(l)},a^{(r)}}$ cannot be realized as a subset of $\mathbf{L}_{A^{(l)},A^{(r)}}$. By definition of $\sim_A$, such a situation occurs only if

$$-\inf_{0 \leq k \leq T_m^{(A,r)}-1} A_k^{(r)} > \inf_{k \geq T_n^{(A,l)}-1} A_k^{(l)}, \tag{5.9}$$

which is avoided for $m \leq \inf\{A_k^{(l)} : k \geq T_n^{(A,l)}\}$. This concludes the proof. $\square$

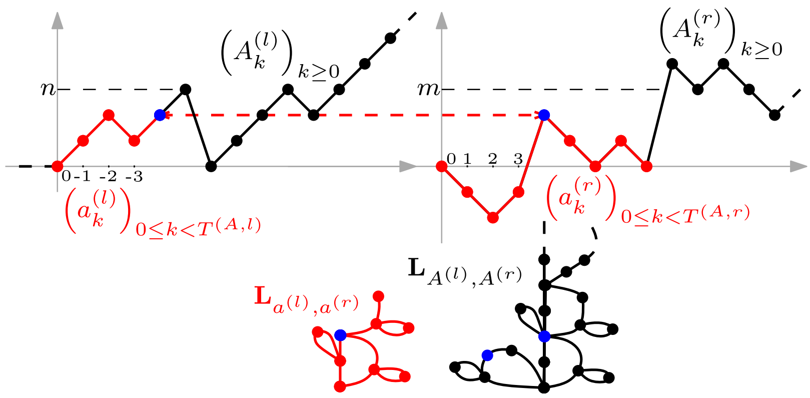

Figure 16: A configuration where $M_{n,m}$ cannot be realized as a subset of M_∞.

Convergence of the exploration processes. The goal of this paragraph is to prove that the distributions of the maps $M_{n,m}$ are close in total variation distance under P, $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$ for a suitable choice of n, m, and p. In the next part, $\mathcal{F}$ denotes the set of Borel functions from $\mathbb{Z}^{\mathbb{N}}$ to $[0,1]$. For every process Z and every stopping time T, $Z' := \{Z_k : 0 \leq k \leq T\}$ is interpreted as an element of $\mathbb{Z}^{\mathbb{N}}$ by putting $Z'_k = 0$ for $k > T$. We use the notation of Section 3.2 and start with two preliminary lemmas.

Lemma 5.10. *For every $n, m \in \mathbb{N}$ such that $n \geq m$,*

$$\limsup_{p \downarrow p_c} \sup_{F \in \mathcal{F}} \left| \mathbb{E}_p \left(F\left(B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)} \right) \,\middle|\, |\mathcal{C}| = \infty \right) - E_0^{\mu\uparrow}\left(F\left(Z_k : 0 \leq k \leq T_n \right) \right) \right| = 0.$$

Proof. Let $\varepsilon > 0$, $n, m \in \mathbb{N}$ such that $n \geq m$ and $F \in \mathcal{F}$. By Lemma 5.5,

$$\mathbb{E}_p \left(F\left(B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)} \right) \,\middle|\, |\mathcal{C}| = \infty \right) = E_0^{\mu_p\uparrow}\left(F\left(Z_k : 0 \leq k \leq T_n \right) \right).$$

Furthermore, by Lemma 3.9, for every $K \in \mathbb{N}$,

$$P_0^{\mu_p\uparrow}(T_n > K) \underset{p \downarrow p_c}{\longrightarrow} P_0^{\mu\uparrow}(T_n > K).$$

Since T_n is finite $P_0^{\mu\uparrow}$-a.s., up to choosing K large enough and then p close enough to p_c,

$$\max\left(P_0^{\mu_p\uparrow}(T_n > K), P_0^{\mu\uparrow}(T_n > K)\right) \leq \varepsilon.$$

Then,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}\left|\mathbb{E}_p\left(F\left(B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)}\right) \,\middle|\, |\mathcal{C}| = \infty\right) - E_0^{\mu\uparrow}\left(F\left(Z_k : 0 \leq k \leq T_n\right)\right)\right| \\
&\qquad \leq 2\varepsilon + \sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_0^{\mu_p\uparrow}\left(F\left(Z_0,\dots,Z_K\right)\right) - E_0^{\mu\uparrow}\left(F\left(Z_0,\dots,Z_K\right)\right)\right|
\end{aligned}$$

and by Lemma 3.9,

$$\sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_0^{\mu_p\uparrow}\left(F\left(Z_0,\dots,Z_K\right)\right) - E_0^{\mu\uparrow}\left(F\left(Z_0,\dots,Z_K\right)\right)\right| \underset{p\downarrow p_c}{\longrightarrow} 0.$$

This concludes the proof since ε is arbitrary. □

Lemma 5.11. *For every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$*

$$\limsup_{p\downarrow p_c}\sup_{F\in\mathcal{F}}\left|\mathbb{E}_p\left(F\left(B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)}\right) \,\middle|\, |\mathcal{C}| = \infty\right) - E_0^{\mu}\left(F\left(Z_k : 0 \leq k \leq T_{-m}\right)\right)\right| \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ such that $n \geq m$ and $F \in \mathcal{F}$. By Lemma 5.5 and translation invariance,

$$\begin{aligned}
\mathbb{E}_p\left(F\left(B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)}\right) \,\middle|\, |\mathcal{C}| = \infty\right) &= E_0^{\mu_p}\left(F\left(Z_k : 0 \leq k \leq T_{-m}\right) \mid T_{-n} = \infty\right) \\
&= E_n^{\mu_p}\left(F\left(Z_k - n : 0 \leq k \leq T_{n-m}\right) \mid T = \infty\right) \\
&= E_n^{\mu_p\uparrow}\left(F\left(Z_k - n : 0 \leq k \leq T_{n-m}\right)\right).
\end{aligned}$$

Furthermore, by Lemma 3.9 and then Lemma 3.8, for every $K \in \mathbb{N}$,

$$P_n^{\mu_p\uparrow}(T_{n-m} > K) \underset{p\downarrow p_c}{\longrightarrow} P_n^{\mu\uparrow}(T_{n-m} > K) \underset{n\to\infty}{\longrightarrow} P_0^{\mu}(T_{-m} > K).$$

Since T_{-m} is finite P_0^{μ}-a.s., up to choosing K large enough, then n large enough and finally p close enough to p_c,

$$\max\left(P_n^{\mu_p\uparrow}(T_{n-m} > K), P_0^{\mu}(T_{-m} > K)\right) \leq \varepsilon.$$

Then,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}\left|\mathbb{E}_p\left(F\left(B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)}\right) \,\middle|\, |\mathcal{C}| = \infty\right) - E_0^{\mu}\left(F\left(Z_k : 0 \leq k \leq T_{-m}\right)\right)\right| \\
&\leq 2\varepsilon + \sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_n^{\mu_p\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right) - E_0^{\mu}\left(F\left(Z_0,\dots,Z_K\right)\right)\right| \\
&\leq 2\varepsilon + \sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_n^{\mu_p\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right) - E_n^{\mu\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right)\right| \\
&\qquad + \sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_n^{\mu\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right) - E_0^{\mu}\left(F\left(Z_0,\dots,Z_K\right)\right)\right|
\end{aligned}$$

By Lemma 3.8, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_n^{\mu\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right) - E_0^{\mu}\left(F\left(Z_0,\dots,Z_K\right)\right)\right| \leq \varepsilon,$$

and by Lemma 3.9, for every $n \in \mathbb{N}$,

$$\sup_{F:\mathbb{Z}^{K+1}\to[0,1]}\left|E_n^{\mu_p\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right) - E_n^{\mu\uparrow}\left(F\left(Z_0 - n,\dots,Z_K - n\right)\right)\right| \underset{p\downarrow p_c}{\longrightarrow} 0.$$

This concludes the proof. □

In what follows, $\mathcal{A}$ denotes the Borel σ-field of the local topology.

Proposition 5.12. *For every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\limsup_{p \downarrow p_c} \sup_{A \in \mathcal{A}} |\mathbf{P}_p(M_{n,m} \in A \mid |\mathcal{C}| = \infty) - \mathsf{P}(M_{n,m} \in A)| \leq \varepsilon.$$

Moreover, for every $n, m \in \mathbb{N}$ such that $n \geq m$, the random planar map $M_{n,m}$ has the same law under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and P.

Proof. We start with the first assertion. Throughout this proof, we use $\mathbf{P}_p^\infty := \mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$ in order to shorten the notation. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. By Lemma 5.11 and the definition of the IIC, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\limsup_{p \downarrow p_c} \sup_{F \in \mathcal{F}} \left| \mathbb{E}_p^\infty \left(F \left(B_k^{(r)} : 0 \leq k \leq T_m^{(B,r)} \right) \right) - \mathsf{E} \left(F \left(A_k^{(r)} : 0 \leq k \leq T_m^{(A,r)} \right) \right) \right| \leq \varepsilon. \tag{5.10}$$

We now fix $n \geq N$, and by Lemma 5.10

$$\lim_{p \downarrow p_c} \sup_{F \in \mathcal{F}} \left| \mathbb{E}_p^\infty \left(F \left(B_k^{(l)} : 0 \leq k \leq T_n^{(B,l)} \right) \right) - \mathsf{E} \left(F \left(A_k^{(l)} : 0 \leq k \leq T_n^{(A,l)} \right) \right) \right| = 0. \tag{5.11}$$

Furthermore, by Lemma 5.5 and the definition of the IIC, for every $K \in \mathbb{N}$ and $F \in \mathcal{F}$,

$$\mathbb{E}_p^\infty \left(F \left(W_k^{(l)} : 0 \leq k \leq T_n^{(W,l)} \right) \;\middle|\; T_n^{(B,l)} = K \right) = \mathsf{E} \left(F \left(V_k^{(l)} : 0 \leq k \leq T_n^{(V,l)} \right) \;\middle|\; T_n^{(A,l)} = K \right). \tag{5.12}$$

For the same reason, for every $K, K' \in \mathbb{Z}_+$ and $F \in \mathcal{F}$,

$$\begin{aligned} &\mathbb{E}_p^\infty \left(F \left(z_i^{(l)} : 0 \leq i \leq \sigma_n^{(l)} \right) \;\middle|\; \left(T_n^{(B,l)}, T_n^{(W,l)} \right) = (K, K') \right) \\ &\qquad = \mathsf{E} \left(F \left(y_i^{(l)} : 0 \leq i \leq \rho_n^{(l)} \right) \;\middle|\; \left(T_n^{(A,l)}, T_n^{(V,l)} \right) = (K, K') \right). \end{aligned} \tag{5.13}$$

The assertions (5.12) and (5.13) hold when replacing l by r. Let $M'_{n,m}$ be the planar map obtained from the construction of $M_{n,m}$ in Sections 5.2 and 5.3 without filling the internal faces with Boltzmann triangulations with a simple boundary. Using (5.10), (5.11), (5.12) and (5.13), we get that

$$\limsup_{p \downarrow p_c} \sup_{A \in \mathcal{A}} \left| \mathbf{P}_p^\infty (M'_{n,m} \in A) - \mathsf{P}(M'_{n,m} \in A) \right| \leq \varepsilon. \tag{5.14}$$

Since the colouring of the vertices in the Boltzmann triangulations filling in the faces of $M'_{n,m}$ is a Bernoulli percolation with parameter p (resp. p_c), for every finite map $\mathbf{m} \in \mathcal{M}_f$ we have

$$\limsup_{p \downarrow p_c} \sup_{A \in \mathcal{A}} \left| \mathbf{P}_p^\infty (M_{n,m} \in A \mid M'_{n,m} = \mathbf{m}) - \mathsf{P}(M_{n,m} \in A \mid M'_{n,m} = \mathbf{m}) \right| = 0.$$

Since $M'_{n,m}$ is P-a.s. finite, together with (5.14) this proves the first assertion.

For the second assertion, by Lemma 3.7, the above coding processes have the same law under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and P. The same arguments apply and conclude the proof. □

5.4 Proof of the IIC results

Proof of Theorem 2.3. Let $R \in \mathbb{Z}_+$ and $\varepsilon > 0$. We first prove that under P, $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, $M_{n,m}$ contains $\mathbf{B}_R(M_\infty)$ with high probability for a good choice of the parameters, where M_∞ is the underlying infinite half-planar triangulation (with origin vertex ρ). This closely follows the proof of Theorem 2.1, to which we refer for more details.

We first restrict our attention to P, and define the random maps

$$M_N := M_{N,\xi_N}, \quad \xi_N := \inf\left\{A_k^{(l)} : k \geq T_N^{(A,l)}\right\}, \quad N \in \mathbb{N}.$$

We denote by d_N the graph distance on M_N. By Proposition 5.9, M_N is P-a.s. a sub-map of M_∞ and we denote by ∂M_N its boundary as such (as in (4.5)). Recall that by Tanaka's theorem, we have $\xi_N \to \infty$ P-a.s. as $N \to \infty$. Let $\{\tau_k : k \in \mathbb{Z}_+\}$, $\{\tau_k^l : k \in \mathbb{Z}_+\}$ and $\{\tau_k^r : k \in \mathbb{Z}_+\}$ be the endpoints of the excursion intervals of $A^{(r)}$, $V^{(l)}$ and $V^{(r)}$ above their infimum processes, as in (3.3). They define cut-points that disconnect the origin from infinity in $\mathbf{L}_{A^{(l)},A^{(r)}}$, $\mathbf{L}_{V^{(l)}}$ and $\mathbf{L}_{V^{(r)}}$ (and thus in $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$) respectively, by the identities

$$v_k := p_{A^{(r)}}\left(A^{(r)}_{\tau_k}\right), \; v_k^l := p_{V^{(l)}}\left(V^{(l)}_{\tau_k^l}\right) \text{ and } v_k^r := p_{V^{(r)}}\left(V^{(r)}_{\tau_k^r}\right), \quad k \in \mathbb{Z}_+.$$

The numbers of cut-points of $\mathcal{H}$ identified in M_N read

$$K(N) := \#\left\{k \in \mathbb{Z}_+ : \tau_k < T_{\xi_N}^{(A,r)}\right\},$$

and similarly for $\mathcal{H}_l$ and $\mathcal{H}_r$ with

$$K_l(N) := \#\left\{k \in \mathbb{Z}_+ : \tau_k^l < T_N^{(V,l)}\right\} \quad \text{and} \quad K_r(N) := \#\left\{k \in \mathbb{Z}_+ : \tau_k^r < T_{\xi_N}^{(V,r)}\right\}.$$

Since the processes $A^{(r)}$, $V^{(l)}$ and $V^{(r)}$ are centered random walks and $\xi_N \to \infty$ P-a.s., we get

$$K(N),\ K_l(N),\ K_r(N) \underset{N\to\infty}{\longrightarrow} \infty \quad \text{P-a.s..}$$

We define an equivalence relation $\approx$ as in Theorem 2.1, by identifying vertices between consecutive cut-points in $\mathcal{H}$, $\mathcal{H}_l$ and $\mathcal{H}_r$. We denote the quotient map $M_N/\approx$ by M_N' (the root edge of M_N' is the root edge of M_∞) and the graph distance on M_N' by d_N'. The family $\{M_N' : n \in \mathbb{N}\}$ is a consistent sequence of locally finite maps with origin $v_0 = \rho$. Moreover, for every $N \in \mathbb{N}$, the boundary of M_N' in M_{N+1}' is $\{v_{K(N)}, v^l_{K_l(N)}, v^r_{K_r(N)}\}$. Thus, the sequences

$$\left\{d_N'\left(\rho, v_{K(N)}\right) : N \in \mathbb{N}\right\}, \; \left\{d_N'\left(\rho, v^l_{K_l(N)}\right) : N \in \mathbb{N}\right\} \text{ and } \left\{d_N'\left(\rho, v^r_{K_r(N)}\right) : N \in \mathbb{N}\right\}$$

are non-decreasing and diverge P-a.s.. By definition of $\approx$, since we discover the finite regions that are swallowed during the exploration, the representatives of ∂M_N in M_N' are $v_{K(N)}$, $v^l_{K_l(N)}$ and $v^r_{K_r(N)}$. As a consequence,

$$d_N\left(\rho, \partial M_N\right) \underset{N\to\infty}{\longrightarrow} \infty \quad \text{P-a.s..} \tag{5.15}$$

Let us choose $N \in \mathbb{N}$ such that $\mathsf{P}(d_N(\rho, \partial M_N) < R) \leq \varepsilon$. By the proof of Proposition 5.9, we have that P-a.s., for every $n \geq N$ and $m \geq \xi_N$, M_N is a sub-map of $M_{n,m}$. Since ξ_N is P-a.s. finite, we can fix $m \in \mathbb{N}$ such that $\mathsf{P}(\xi_N > m) \leq \varepsilon$, and thus for every $n \geq N$,

$$\mathsf{P}(M_N \subseteq M_{n,m}) \geq 1 - \varepsilon \tag{5.16}$$

Since $\xi_n \to \infty$ P-a.s., there exists $N_1 \geq N$ such that for every $n \geq N_1$, $\mathsf{P}(\xi_n < m) \leq \varepsilon$. By Proposition 5.9, it follows that for every $n \geq N_1$,

$$\mathsf{P}(M_{n,m} \subseteq M_\infty) \geq 1 - \varepsilon. \tag{5.17}$$

By construction, when $M_N \subseteq M_{n,m} \subseteq M_\infty$, we have $d_N\left(\rho, \partial M_N\right) \leq d\left(\rho, \partial M_{n,m}\right)$ (where d is the graph distance on $M_{n,m}$ and $\partial M_{n,m}$ its boundary in M_∞). Thus, for every $n \geq N_1$,

$$\mathsf{P}(M_{n,m} \subseteq M_\infty, d(\rho, \partial M_{n,m}) \geq R) \geq 1 - 3\varepsilon. \tag{5.18}$$

By Proposition 5.12, we find $N_2 \geq N_1$ such that for every $n \geq N_2$,

$$\limsup_{p \downarrow p_c} \sup_{A \in \mathcal{A}} |\mathbf{P}_p(M_{n,m} \in A \mid |\mathcal{C}| = \infty) - \mathsf{P}(M_{n,m} \in A)| \leq \varepsilon, \tag{5.19}$$

while $M_{n,m}$ has the same distribution under P and $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$. Under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$ and $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$, $M_{n,m}$ is a.s. a sub-map of M_∞. By the construction of $M_{n,m}$, (5.18) and (5.19) we have for every $n \geq N_2$,

$$\limsup_{p \downarrow p_c} \mathbf{P}_p(d(\rho, \partial M_{n,m}) < R \mid |\mathcal{C}| = \infty) \leq 4\varepsilon \quad \text{and} \quad \mathbf{P}(d(\rho, \partial M_{n,m}) < R \mid h(\mathcal{C}) \geq n) \leq 3\varepsilon. \tag{5.20}$$

Finally, on the event $\{M_{n,m} \subseteq M_\infty\}$, $d(\rho, \partial M_{n,m}) \geq R$ enforces that $\mathbf{B}_R(M_{n,m}) = \mathbf{B}_R(M_\infty)$, which concludes the first part of the proof.

Now, let $A \in \mathcal{A}$ be a Borel set for the local topology. By (5.20), for every $n \geq N_2$,

$$\begin{aligned}
&\limsup_{p \downarrow p_c} |\mathbf{P}_p(\mathbf{B}_R(M_\infty) \in A \mid |\mathcal{C}| = \infty) - \mathbf{P}_p(\mathbf{B}_R(M_{n,m}) \in A \mid |\mathcal{C}| = \infty)| \\
&\leq 2 \limsup_{p \downarrow p_c} \mathbf{P}_p(d(\rho, \partial M_{n,m}) < R \mid |\mathcal{C}| = \infty) \leq 8\varepsilon.
\end{aligned}$$

Then, by (5.19), for every $n \geq N_2$,

$$\limsup_{p \downarrow p_c} |\mathbf{P}_p(\mathbf{B}_R(M_{n,m}) \in A \mid |\mathcal{C}| = \infty) - \mathsf{P}(\mathbf{B}_R(M_{n,m}) \in A)| \leq \varepsilon.$$

Finally, by (5.18), for every $n \geq N_1$,

$$|\mathsf{P}(\mathbf{B}_R(M_{n,m}) \in A) - \mathsf{P}(\mathbf{B}_R(M_\infty) \in A)| \leq 2\left(1 - \mathsf{P}(M_{n,m} \subseteq M_\infty, d(\rho, \partial M_{n,m}) \geq R)\right) \leq 6\varepsilon.$$

This concludes the proof under $\mathbf{P}_p(\cdot \mid |\mathcal{C}| = \infty)$ since R and ε are arbitrary.

Under $\mathbf{P}(\cdot \mid h(\mathcal{C}) \geq n)$, the proof is simpler. By (5.20) once again, for every $n \geq N_2$,

$$\begin{aligned}
&|\mathbf{P}(\mathbf{B}_R(M_\infty) \in A \mid h(\mathcal{C}) \geq n) - \mathbf{P}(\mathbf{B}_R(M_{n,m}) \in A \mid h(\mathcal{C}) \geq n)| \\
&\qquad \leq 2\mathbf{P}(d_{\mathsf{gr}}(\rho, \partial M_{n,m}) < R \mid h(\mathcal{C}) \geq n) \leq 6\varepsilon,
\end{aligned}$$

and by Proposition 5.12,

$$\mathbf{P}(\mathbf{B}_R(M_{n,m}) \in A \mid h(\mathcal{C}) \geq n) = \mathsf{P}(\mathbf{B}_R(M_{n,m}) \in A).$$

This concludes the proof. □

Remark 5.13. In view of [7, Theorem 1], we believe that this proof can be adapted to show that in the sense of weak convergence, for the local topology

$$\mathbf{P}_{p_c}(\cdot \mid T \geq n) \underset{n \to \infty}{\Longrightarrow} \mathbf{P}_{\mathsf{IIC}}.$$

Otherwise said, the IIC arises as a local limit of a critically percolated UIHPT when conditioning the exploration process to survive a long time. It is also natural to conjecture that conditioning the open cluster of the origin to have large hull perimeter (as in [14]) or to reach the boundary of a ball of large radius (as in [23]) yields the same local limit. However, our techniques do not seem to allow to tackle this problem.

6 Scaling limits and perspectives

In the recent work [6] (see also [21]), Baur, Miermont and Ray introduced the scaling limit of the quadrangular analogous of the UIHPT, called the Uniform Infinite Half-Planar Quadrangulation (UIHPQ). Precisely, they consider a map $\mathbf{Q}_\infty^\infty$ having the law of the UIHPQ as a metric space equipped with its graph distance d_{gr}, and multiply the distances by a scaling factor λ that goes to zero, proving that [6, Theorem 3.6]

$$(\mathbf{Q}_\infty^\infty, \lambda d_{\mathrm{gr}}) \xrightarrow[\lambda\to 0]{(d)} \mathsf{BHP},$$

in the local Gromov-Hausdorff sense (see [13, Chapter 8] for more on this topology). The limiting object is called the *Brownian Half-Plane* and is a half-planar analog of the Brownian Plane of [16]. Such a convergence is believed to hold also in the triangular case.

We now discuss the conjectural continuous counterpart of Theorem 2.1, and the connection with the BHP. On the one hand, the processes B and W introduced in Section 4.2 have a scaling limit. Namely, using the asymptotics of Z_k (see [4, Section 2.2]) and standard results of convergence of random walks with steps in the domain of attraction of a stable law [10, Chapter 8] one has

$$\left(\lambda^2 B_{\lfloor t/\lambda^3\rfloor}\right)_{t\geq 0} \xrightarrow[\lambda\to 0]{(d)} (X_t)_{t\geq 0},$$

in distribution for Skorokhod's topology, where X is (up to a multiplicative constant) the spectrally negative $3/2$-stable process. This suggests that the looptrees $\mathbf{L}_B$ and $\mathbf{L}_W$ converge when distances are rescaled by the factor λ^2 towards a non-compact version of the random stable looptrees of [15] (with parameter $3/2$), in the local Gromov-Hausdorff sense. This object is supposed to be coded by the process X extended to $\mathbb{R}$ by the relation $X_t = -t$ for every $t \leq 0$ and equivalence relation

$$s \sim t \quad \text{iff} \quad X_t = X_s = \inf_{s\wedge t\leq u\leq s\vee t} X_u. \tag{6.1}$$

On the other hand, one can associate to each negative jump of size ℓ of X (which codes a loop of the same length in the infinite stable looptree, see [15]) a sequence of jumps of B (equivalently, of loops of $\mathbf{L}_B$) with sizes $\{\ell_\lambda : \lambda > 0\}$ satisfying

$$\lambda^2 \ell_\lambda \xrightarrow[\lambda\to 0]{} \ell.$$

With each negative jump of B is associated a Boltzmann triangulation M_λ with a simple boundary of size ℓ_λ and graph distance d_λ, that fills in the corresponding loop in the decomposition of the UIHPT. Inspired by [9, Theorem 8], we expect that there exists a constant $c > 0$ such that

$$\left(M_\lambda, (c\ell_\lambda)^{-1/2} d_\lambda\right) \xrightarrow[\lambda\to 0]{(d)} \mathsf{FBD}_1,$$

in the Gromov-Hausdorff sense, where FBD_1 is a compact metric space called the Free Brownian Disk with perimeter 1, originally introduced in [8] (this result has been proved for Boltzmann bipartite maps with a general boundary). By a scaling argument, we obtain

$$(M_\lambda, \lambda d_\lambda) \xrightarrow[\lambda\to 0]{(d)} \mathsf{FBD}_{c\ell},$$

where $\mathsf{FBD}_{c\ell}$ is the FBD with perimeter $c\ell$. From these observations, it is natural to conjecture that the looptrees $\mathbf{L}_B$ and $\mathbf{L}_W$ filled in with independent Boltzmann triangulations converge when rescaled by a factor that goes to zero, towards a collection of independent FBD with perimeters given by the jumps of a Lévy $3/2$-stable process, and glued together according to the looptree structure induced by this process. Based on Theorem 2.1, we believe that there is a way to glue two independent copies of the above looptrees of brownian disks along their boundaries so that the resulting object has the law of the BHP. Similarly, one may give a rigorous sense to the IIC embedded in the BHP, by passing to the scaling limit in Theorem 2.3. However, there are several possible metrics on the topological quotient, and it is not clear which metric should be chosen.

This question is connected to [18, 31], where gluings of quantum surfaces are discussed. As we already mentioned, Theorem 2.1 can be seen as a discrete counterpart to [18, Theorem 1.16-1.17]. Applying these results with the choice of parameter $\gamma = \sqrt{8/3}$, we obtain that the $\theta = \pi$ quantum wedge (which is believed to be a conformal version of the BHP) is obtain as the gluing of two independent forested wedges with parameter $\alpha = 3/2$ (an infinite counterpart to the $3/2$-stable looptree, where loops are filled in with disks equipped with a metric defined in terms of the Gaussian Free Field - a conformal version of the brownian disk). Moreover, the counterflow line that separates the forested wedges is a Schramm-Loewner Evolution with parameter 6, which is also the scaling limit of percolation interfaces in the triangular lattice [30, 32]. In the recent work [22], Gwynne and Miller also unveiled such a decomposition and proved the convergence of the percolation interface for face percolation on the UIHPQ towards (chordal) Schramm-Loewner Evolution with parameter 6 on the BHP.

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