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Dirichlet form associated with the Φ_3^4 model*

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Abstract

We construct the Dirichlet form associated with the dynamical Φ_3^4 model obtained in [23, 7] and [37]. This Dirichlet form on cylinder functions is identified as a classical gradient bilinear form. As a consequence, this classical gradient bilinear form is closable and then by a well-known result its closure is also a quasi-regular Dirichlet form, which means that there exists another (Markov) diffusion process, which also admits the Φ_3^4 field measure as an invariant (even symmetrizing) measure.

Keywords: Φ_3^4 model; Dirichlet form; paracontrolled distributions; regularity structures; space-time white noise; renormalisation.
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1 Introduction

Recall that the usual continuum Euclidean Φ_d^4-quantum field theory on a torus is heuristically described by the following probability measure:

$$\mu(dx) = N^{-1}\Pi_{\xi\in\mathbb{T}^d} dx(\xi) \exp\bigg(-\int_{\mathbb{T}^d}(|\nabla x(\xi)|^2 + mx^2(\xi) + \frac{\lambda}{2}x^4(\xi))d\xi\bigg), \tag{1.1}$$

where N is the normalization constant, m is a real constant, $\lambda \geq 0$ is the coupling constant and x is the real-valued field and $\mathbb{T}^d$ is the d-dimensional torus. There have been many approaches to the problem of giving a meaning to the above heuristic measure for $d = 2$ and $d = 3$ (see [21, 15] and references therein). The construction of this Φ_3^4 field measure μ has been achieved in [12] for λ small enough, which was one of the major achievements of the programme of constructive quantum field theory. In [40]

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Parisi and Wu proposed a program for Euclidean quantum field theory of getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study the properties of the Gibbs states. This procedure is called stochastic field quantization (see [29]). The Φ^4_d model is the simplest non-trivial Euclidean quantum field (see [15] and the references therein). The issue of the stochastic quantization of the Φ^4_d model on the d-torus is to solve the following equation:

$$d\Phi =(\Delta\Phi - \lambda\Phi^3 - m\Phi)dt + dW(t) \quad \Phi(0) = \Phi_0. \tag{1.2}$$

where W is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$. In the following we take λ small enough (weak coupling) as in [6] and in the following when we analyze (1.2) we omit λ for simplicity if there is no confusion. The solution Φ is also called dynamical Φ^4_d model. The main difficulty in this case is that W and hence the solutions Φ are so singular that the non-linear term is not well-defined in the classical sense.

In two spatial dimensions, the dynamical Φ^4_2 model was first treated in [2] by using the Dirichlet form approach: The authors considered the following bilinear form on $L^2(E;\mu)$ with E being a separable Banach space and $\mu(E) = 1$:

$$\mathcal{E}(u,v) := \frac{1}{2}\int \langle Du, Dv\rangle_{L^2} d\mu,$$

where Du means L^2-derivative, which is defined in Section 4. By the corresponding integration by parts formula for μ they obtained that the bilinear form is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form. Then according to a general result in [34] (see Theorem D.4), we know that there exists a (Markov) diffusion process $M = (\Omega, \mathcal{F}, X(t), (P^x)_{x\in E})$ on E *properly associated with* $(\mathcal{E}, D(\mathcal{E}))$. The sample paths of the associated process satisfy (1.2) in the (probabilistically) weak sense for quasi-surely every Φ_0.

Later in [10] and [36], the authors split Φ as $\Phi = \Phi_1 + v$, where

$$d\Phi_1 = \Delta\Phi_1 dt + dW,$$

$$\partial_t v = \Delta v - (v^3 + 3v^2\Phi_1 + 3v :\Phi_1^2: + :\Phi_1^3:) - m(\Phi_1 + v), \tag{1.3}$$

where $:\Phi_1^2:, :\Phi_1^3:$ are defined as Wick products. Then the nonlinear terms are well defined in the classical sense and they obtained a (probabilistically) strong solution to (1.3).

In three spatial dimensions both techniques break down. For the Dirichlet form approach we cannot directly obtain that the bilinear form:

$$\mathcal{E}(u,v) := \frac{1}{2}\int_E \langle Du, Dv\rangle_{L^2} d\mu, \quad u,v \in \mathcal{F}C_b^\infty,$$

is closable since the measure μ is more singular and may be not quasi-invariant along smooth direction (see [1]). Here for the definition of $\mathcal{F}C_b^\infty$ we refer to section 4. Nobody has constructed the Dirichlet form associated with Φ^4_3 model successfully and the closability of the corresponding bilinear form has been a long-standing open problem for more than 25 years ([2]). For the second approach (1.3) is also not well defined in the classical sense since the noise is more rough. It was a long-standing open problem to give a meaning to the equation (1.2) in the three dimensional case. A breakthrough result was achieved recently by Martin Hairer in [23], where he introduced a theory of regularity structures and gave a meaning to equation (1.2) successfully. Also by using the paracontrolled distributions proposed by Gubinelli, Imkeller and Perkowski in [17] existence and uniqueness of local solutions to (1.2) have been obtained in [7]. Recently, these two approaches have been successful in giving a meaning to a lot of ill-posed stochastic PDEs like the Kardar-Parisi-Zhang (KPZ) equation ([30, 4, 22]), the stochastic 3D-Navier-Stokes equation driven by space-time white noise ([49, 51]), the dynamical sine-Gordon equation ([27]) and so on (see [26] for more other interesting examples). These two approaches are inspired by the theory of rough paths [33]. In [31] the author also uses renormalization group techniques to make sense of the dynamical Φ_3^4 model. Recently in [37] the authors obtained global well-posedness of the solution to (1.2) in the three dimensional case based on the paracontrolled distribution method.

The aim of this paper is to construct the Dirichlet form associated to the Φ_3^4 model. Dirichlet form techniques have developed into a powerful method to combine analytic and functional analysis, as well as potential theoretic and probabilistic methods to study the properties of stochastic processes. In [44, 45] M. Röckner and the authors of this paper combine the Dirichlet form approach and the SPDE approach to obtain new properties in the two dimensional case (such as restricted Markov uniqueness and the characterization of the Φ_2^4 field). We hope this paper is a start to study the dynamical Φ_3^4 model combining Dirichlet form techniques and the theory of regularity structures as well as the paracontrolled distributions approach.

Different from [2], our idea is to construct the Dirichlet form from the global solution $\Phi(t)$ obtained in [37]. It has been proved in [24] that $\Phi(t)$ satisfies the Markov property. Moreover, it is easy to obtain that $\Phi(t)$ satisfies the Feller property (see Lemma 4.1), which implies that $\Phi(t)$ satisfies the strong Markov property. Then we prove $\Phi(t)$ is reversible with respect to μ by the lattice approximations obtained in [50] (see Lemma 4.2). Hence we obtain our first main result of this paper:

Theorem 1.1. *There exists a quasi-regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ associated with $\Phi(t)$. Moreover, Φ is* properly associated with *$(\mathcal{E}, D(\mathcal{E}))$ in the sense that the Markovian transition semigroup for Φ is a quasi-continuous version of the semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$. Furthermore, $\mathcal{F}C_b^\infty \subset D(\mathcal{E})$ and $\langle l, \cdot\rangle \in D(\mathcal{E})$ for any $l \in E^*$.*

For definitions of quasi-regular Dirichlet form we refer to Appendix D. Here $\mathcal{F}C_b^\infty$ denotes all the smooth with all derivatives bounded cylinder functions on the state space E, E^* is the dual space of E and $\langle \cdot, \cdot \rangle$ is the dualization between E and E^*. For the explicit definition we refer to Section 4. Moreover, we can identify the Dirichlet form on the cylinder functions as a gradient Dirichlet form:

Theorem 1.2. *For $f, g \in \mathcal{F}C_b^\infty$, $\mathcal{E}(f, g) = \frac{1}{2}\int \langle Df, Dg\rangle d\mu$ with $\langle \cdot, \cdot \rangle$ being the inner product of $L^2(\mathbb{T}^3)$ and Df is L^2-derivative defined in Section 4.*

As a byproduct of Theorem 1.2 we can also deduce that Φ is an energy solution in the stationary case (see Remark 5.2). Energy solution is a notion of weak solutions for KPZ equation to describe the large scale fluctuations of a wide class of weakly asymmetric particle systems (see [16, 18, 20]). For the dynamical Φ_3^4 case we can also introduce the notion of energy solution.

As a consequence of Theorem 1.2, we obtain that the bilinear form is closable, which we cannot directly obtain as we mentioned before:

Theorem 1.3. *The bilinear form $\bar{\mathcal{E}}(f, g) = \frac{1}{2}\int \langle Df, Dg\rangle d\mu, f, g \in \mathcal{F}C_b^\infty$, is closable and its closure $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ is a quasi-regular Dirichlet form. Then there exists a (Markov) diffusion process* properly associated with *$(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$, which admits μ as an invariant measure.*

By using Theorem 1.3 and Dirichlet form theory (see [14, Theorem 1.6.3]) we obtain the following result easily:

Corollary 1.4. *$(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ and $(\mathcal{E}, D(\mathcal{E}))$ are recurrent in the sense that their associated* *semigroups* $(T^i_t)_{t>0}, i=1,2,$ *satisfy for* $i=1,2$

$$\int_0^\infty T^i_t f dt = 0 \text{ or } \infty \text{ a.e. for any } f\in L^1(E;\mu) \text{ with } f\geq 0.$$

Here we use $(T^i_t)_{t>0}$ *to denote the semigroup associated with the above Dirichlet forms respectively.*

Recently a new uniform estimate for the solution Φ has been obtained in [38], which combined with the strong Feller property for Φ obtained in [25] and a support theorem in [28] for Φ, may imply the exponential convergence to equilibrium in this case. By this result we can deduce the following estimate by using Dirichlet form constructed above.

Corollary 1.5. *Suppose that the exponential convergence in the* L^2*-sense hold for the semigroup* $\bar{P}_t$ *associated with the solution* Φ*. Then the following Poincaré inequality holds:*

$$\mu(f^2)\leq C\mathcal{E}(f,f)+\mu(f)^2, \quad f\in D(\mathcal{E})$$

for some $C>0$*. Moreover, there exists* $c_0>0$ *such that*

$$\int e^{c_0\|x\|_E}\mu(dx)<\infty, \tag{1.4}$$

where E *is the state space we introduced in Section 4.*

Remark 1.6. (i) In fact (1.4) can be obtained by using [6] and further calculations. However, the construction of Dirichlet form only need an uniform moment estimate of the lattice field measures. Here we use Dirichlet form theory to give a new proof of (1.4).

(ii) We recall the following result from Dirichlet form theory: Poincaré inequality implies the irreducibility of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Then by Corollary 1.4 and [14, Theorem 4.7.1], for any nearly Borel non-exceptional set B,

$$P^x(\sigma_B\circ\theta_n<\infty, \forall n\geq 0)=1, \quad \text{for q.e. } x\in E.$$

Here $\sigma_B=\inf\{t>0:\Phi_t\in B\}$, θ is the shift operator for the Markov process Φ, and for the definition of any nearly Borel non-exceptional set we refer to [14]. Moreover by [14, Theorem 4.7.3] we obtain the following strong law of large numbers: for $f\in L^1(E,\mu)$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\Phi_s)ds=\int f d\mu, \quad P^x-a.s.,$$

for q.e. $x\in E$.

Remark 1.7. From Theorem 1.3 we know that there exists another Markov process which admits μ as an invariant measure. Is this Markov process the same as the solution Φ to (1.2) obtained in [37]? In Dirichlet form theory it corresponds to the problem of the relations between the domains of the Dirichlet forms $D(\mathcal{E})$ and $D(\bar{\mathcal{E}})$. In the two dimensional case, they are the same (corresponding to restricted Markov uniqueness, see [44]). In the three dimensional case we do not know the answer until now, since the measure is more singular and we do not know along which vector fields the integration by parts formula holds. This is also a major problem in Dirichlet form theory, which is related to the long-standing open problem whether Markov uniqueness holds for the associated generator.

The structure of this paper is as follows. In Section 2 we prove some useful estimates for the solutions to (1.2). In Section 3 we recall the lattice approximations, which is required to prove Φ is reversible w.r.t. μ. In Section 4 we give the proof of our first main result. In Section 5 we identify the Dirichlet form on the cylinder functions. In

Appendix A, we recall some basic notions and results for the paracontrolled distribution method. In Appendix B, we calculate the convergence of the stochastic terms. We recall the paracontrolled analysis for the solutions to the lattice approximations in Appendix C. We also recall the definitions of Markov processes and quasi-regular Dirichlet forms in Appendix D.

Notations: Let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d=[-1,1]^d$. For $\alpha\in\mathbb{R}$, the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha=B^\alpha_{\infty,\infty}(\mathbb{T}^d)$ and for $p>1$ we use the notation $B^\alpha_p:=B^\alpha_{p,\infty}$. For the definition of the general Besov spaces $B^\alpha_{p,q}$ and the paraproduct see Appendix A. For $\beta>0,\alpha\in\mathbb{R}$ we write $\|\cdot\|_\alpha$, $C_T\mathcal{C}^\alpha$ and $C^\beta_T\mathcal{C}^\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$, $C([0,T];\mathcal{C}^\alpha)$ and $C^\beta([0,T];\mathcal{C}^\alpha)$, respectively in the following for simplicity. For a Banach space E, $\mathcal{B}(E)$ denotes the Borel-algebra on E and $C_b(E)$ and $\mathcal{B}_b(E)$ denote the bounded continuous function and the bounded measurable functions on E, respectively. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$. The heat semigroup is denoted by $P_t:=e^{t\Delta}$. To simplify the arguments below, when we analyze the equations, we assume that $\mathcal{F}W(0)=0$ and this can be easily removed by adding a linear term on the right hand side of equation.

For $f\in\mathcal{S}'(\mathbb{T}^3)$ we write $\rho_\varepsilon * f:=\sum_k g(\varepsilon k)\langle f,e_k\rangle e_k$ with g being a smooth radical function with compact support and $g(0)=1$, $g(\varepsilon k)=\mathcal{F}\rho_\varepsilon(k)$. Here and in the following $\langle\cdot,\cdot\rangle$ denotes $L^2(\mathbb{T}^3)$-inner product and $e_k(\xi)=2^{-3/2}e^{\iota\pi k\cdot\xi}$ for $k=(k^1,k^2,k^3)\in\mathbb{Z}^3,\xi=(\xi^1,\xi^2,\xi^3)\in\mathbb{T}^3$. We also use $|k|_\infty=\max(|k^1|,|k^2|,|k^3|)$ and $\delta_{st}f:=f(t)-f(s)$. To make our paper better readable we summarize the graph notation used in the paper in the following table. The definition of them will be introduced below.

Φ_1	$\bar\Phi_1^\varepsilon$	$-\Phi_2$	$-\bar\Phi_2^\varepsilon$	$-\rho_\varepsilon*\Phi_2$	$(\Phi_1)^{\diamond,2}$	$(\bar\Phi_1^\varepsilon)^{\diamond,2}$
K	$\bar K^\varepsilon$	$\rho_\varepsilon * K$	$(\rho_\varepsilon*\Phi_1)^{\diamond,3}$	$\Phi_1\diamond\Phi_2$	$(\Phi_1)^{\diamond,2}\diamond\Phi_2$	
				$-$	$-$	

2 A uniform estimate

In this section we give an uniform estimate of the solution to (1.2). In the following we assume that $\Phi_0\in\mathcal{C}^{-z}$ and $z\in(\frac12,\frac23)$. We fix $\kappa,\gamma>0$ satisfying

$$z-\frac{1}{2}>2\kappa,\quad 6\kappa<\gamma,\quad 10\kappa+3\gamma<2-3z.$$

Parameters κ,γ satisfying the above conditions can always be found. Indeed, we first choose $\gamma<\frac{2-3z}{3}$. Then the conditions are satisfied if we choose $\kappa>0$ small enough satisfying $\kappa<\frac{\gamma}{6}\wedge\frac{2z-1}{4}\wedge\frac{2-3z-3\gamma}{10}$.

Now we recall that the solution obtained by [7] and [37]: (1.2) can be split as follows: $\Phi=\Phi_1+\Phi_2+\Phi_3$ and

$$\Phi_1(t)=\int_{-\infty}^t P_{t-s}dW = ,$$

$$\Phi_2(t)=-\lim_{\varepsilon\to0}\int_0^t P_{t-s}\ ds:=- ,$$

and

$$\Phi_3(t) = P_t(\Phi_0 - \Phi_1(0)) - \int_0^t P_{t-s}\Big[\Phi_3^3 + 3\Phi_3^2(\vert - \Psi) + \Phi_3(3(\Psi)^2 - 6\,{\setminus}\!\Psi) + 3\vee \diamond \Phi_3 \\ + 3(\vert \diamond (\Psi)^2 - \vee\!\!\!\Psi) - (\Psi)^3 - (9\varphi - m)\Phi\Big] ds. \tag{2.1}$$

Here Φ_2 does not depend on ε (see [7]) and we use $\vert$ to denote Φ_1 and $\vdots$ to denote $\rho_\varepsilon * \Phi_1$ and introduce $\vee$, Ψ to represent $\Phi_1^{\diamond 2}, -\Phi_2$, respectively.

$$\vee,\ \curlyvee,\ {\setminus}\!\Psi,\ \vert \diamond (\Psi)^2,\ \vee\!\!\!\Psi,\ \varphi$$

involve a renormalization procedure and are defined in Appendix B. Throughout this paper we do not use the explicit formulation of these stochastic terms, but only use their regularity. We will introduce their regularity in (2.2) below. The most difficult part for renormalization is $\vee \diamond \Phi_3$. For this term we define

$$K(t) := \int_0^t P_{t-s}(\Phi_1)^{\diamond,2} ds := Y.$$

We have the following paracontrolled ansatz

$$\Phi_3 = -3\pi_<(-\Psi + \Phi_3, Y) + \Phi^\sharp$$

with $\Phi^\sharp(t) \in \mathcal{C}^{1+3\kappa}$ for $t > 0$. Here $\Phi^\sharp$ is the regular term in the paracontrolled ansatz. Then

$$\begin{aligned} \vee \diamond \Phi_3 :=& \pi_0(\Phi^\sharp, \vee) - 3C(-\Psi + \Phi_3, Y, \vee) \\ &- 3(-\Psi + \Phi_3)\pi_{0,\diamond}(Y, \vee) + \pi_{<,>}(\Phi_3, \vee), \end{aligned}$$

where $C(-\Psi + \Phi_3, Y, \vee)$ is defined in Lemma A.3 and $\pi_{0,\diamond}(Y, \vee)$ is defined in Appendix B. Now we introduce the following notations:

$$\begin{aligned} C_W(T) := \sup_{t\in[0,T]} &[\|\vert\|_{-\frac{1}{2}-2\kappa} + \|\vee\|_{-1-2\kappa} + \|\Psi\|_{\frac{1}{2}-2\kappa} + \|\pi_{0,\diamond}(Y, \vert)\|_{-2\kappa} \\ &+ \|\pi_{0,\diamond}(\Psi, \vee)\|_{-\frac{1}{2}-2\kappa} + \|\pi_{0,\diamond}(Y, \vee)\|_{-2\kappa}] + \|\Psi\|_{C_T^{\frac{1}{8}}\mathcal{C}^{\frac{1}{4}-2\kappa}}, \end{aligned} \tag{2.2}$$

and

$$\rho_L := \inf\{t \geq 0 : C_W(t) \geq L\}.$$

By [7] $P(C_W(T) < \infty, \forall T > 0) = 1$ and by [7] on this set there exists a unique local solution Φ_3 to (2.1). Recently in [37] the authors proved that the solution to (2.1) does not blow up in finite time. In fact we can check that the solution obtained in [37] satisfies (2.1) by smooth approximation. In the following we consider the solution Φ obtained in [7] and [37].

Then we have the following estimate for Φ:

Proposition 2.1. *For any $T > 0$ there exist $C_0, \bar{m} > 0$ depending on L, T such that on the set $\{\rho_L > T\}$*

$$\sup_{t\in[0,T]}[\|\Phi\|_{-z} + t^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3\|_{\frac{1}{2}+4\kappa}] \leq C_0(\|\Phi_0\|^{\bar{m}}_{-z} + 1).$$

Remark 2.2. Here we obtain the estimate on the set $\{\rho_L > T\}$, since on this set we can choose t^* below and the bound is independent of ω.

Proof. Set

$$Q(t) := t^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3\|_{\frac{1}{2}+4\kappa} + t^{\frac{3(\gamma+z+\kappa)}{2}}\|\Phi^\sharp\|_{1+3\kappa} + 1.$$

By similar calculations as in [50, Section 4] there exists $q > 1$ such that for $t \leq \rho_L \wedge T$

$$Q(t)^q \leq \bar{C}(\|\Phi_0\|^q_{-z} + 1) + \bar{C}\int_0^t Q(s)^{3q}ds,$$

where the constant $\bar{C}$ depends on L, T, q. Then Bihari's inequality implies that on the set $\{\rho_L > T\}$ for $t^* := \bar{C}^{-1}[2\bar{C}(\|\Phi_0\|^q_{-z} + 1)]^{-2} \wedge T$

$$\sup_{t\in[0,t^*]} Q(t)^q \leq C(\|\Phi_0\|^q_{-z} + 1).$$

Here and in the following the constant C depends on L, T, q. Then we obtain that

$$\sup_{t\in[0,t^*]}[t^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3\|_{\frac{1}{2}+4\kappa}] \leq C(\|\Phi_0\|_{-z} + 1). \tag{2.3}$$

Moreover, by similar calculations as in [50, Section 4] there exists $m_0 > 0$ such that

$$\sup_{t\in[0,t^*]}\|\Phi_3(t)\|_{-z} \leq C(\|\Phi_0\|^{m_0}_{-z} + 1).$$

By the main result in [38] we have on the set $\{\rho_L > T\}$ for $t^* < t \leq T$

$$\|\Phi_3(t)\|_{-z} \leq Ct^{-1/2} \leq C(t^*)^{-1/2} \leq C(\|\Phi_0\|^q_{-z} + 1).$$

Combining the above estimates we obtain that on the set $\{\rho_L > T\}$ there exists $m_1 > 0$ such that

$$\sup_{t\in[0,T]}\|\Phi_3(t)\|_{-z} \leq C(\|\Phi_0\|^{m_1}_{-z} + 1). \tag{2.4}$$

Let $\bar{t}^* = \bar{C}^{-1}[2\bar{C}(\|\Phi_0\|^{m_1 q}_{-z} + 1)]^{-2} \wedge t^*$. Consider the solution to (2.1) at time $t > \bar{t}^*$, then it can be viewed as a solution starting from $t - \frac{\bar{t}^*}{2}$. A similar argument as above implies that on the set $\{\rho_L > T\}$ for $\bar{t}^* < t \leq T$

$$\begin{aligned}&(\frac{\bar{t}^*}{2})^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3(t)\|_\gamma + (\frac{\bar{t}^*}{2})^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3(t)\|_{\frac{1}{2}+4\kappa}\\ \leq& C(\|\Phi_3(t - \frac{\bar{t}^*}{2})\|_{-z} + 1)\\ \leq& C(\|\Phi_0\|^{m_1}_{-z} + 1).\end{aligned}$$

Then we have for $\bar{t}^* < t \leq \rho_L \wedge T$

$$\begin{aligned}&t^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3(t)\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3(t)\|_{\frac{1}{2}+4\kappa}\\ \leq& C\|\Phi_3(t)\|_{\frac{1}{2}+4\kappa}\\ \leq& (\bar{t}^*)^{-\frac{\frac{1}{2}+z+5\kappa}{2}}C(\|\Phi_0\|^{m_1}_{-z} + 1) \leq C(\|\Phi_0\|^{m}_{-z} + 1),\end{aligned} \tag{2.5}$$

where $m > 0$ is a constant.

Thus the result follows from (2.3) (2.4) and (2.5). ☐

3 Lattice approximation

In this section we will recall the lattice approximation in [50] for later use. For $N \geq 1$, let $\Lambda^N = \{-N, -(N-1), ..., N\}^3$. Set $\varepsilon = \frac{2}{2N+1}$. Every point $k \in \Lambda^N$ can be identified with $\xi = \varepsilon k \in \Lambda_\varepsilon = \{\xi = (\xi^1, \xi^2, \xi^3) \in \varepsilon\mathbb{Z}^3 : -1 < \xi^1, \xi^2, \xi^3 < 1\}$. We view Λ_ε as a discretisation of the continuous three-dimensional torus $\mathbb{T}^3$ identified with $[-1, 1]^3$. Then for $n \geq 1$ we set $L^{2n}(\Lambda^\varepsilon) := \{\|f\|^{2n}_{L^{2n}(\Lambda^\varepsilon)} := \sum_{x\in\Lambda^\varepsilon} \varepsilon^3|f(x)|^{2n} < \infty\}$. (1.1) can be approximated by the following lattice Φ^4_3-field measure $\mu^\varepsilon(dx)$:

$$
\begin{aligned}
N_\varepsilon^{-1}\Pi_{\xi\in\Lambda_\varepsilon} dx_\xi \exp\Bigg(-\varepsilon \sum_{|\xi_1-\xi_2|=\varepsilon, \xi_1,\xi_2\in\Lambda_\varepsilon} (x(\xi_1) - x(\xi_2))^2 + (3C_0^\varepsilon - 9C_1^\varepsilon - m) \sum_{\xi\in\Lambda_\varepsilon} \varepsilon^3 x^2(\xi) \\
- \frac{1}{2} \sum_{\xi\in\Lambda_\varepsilon} \varepsilon^3 x^4(\xi)\Bigg),
\end{aligned}
$$

where N_ε is a normalization constant and we choose $C_0^\varepsilon, C_1^\varepsilon$ as in [50, Section 1]. The following stochastic PDEs on Λ_ε are the stochastic quantizations associated with the lattice Φ^4_3-field measure:

$$
\begin{aligned}
d\Phi^\varepsilon(t) =&(\Delta_\varepsilon\Phi^\varepsilon(t) - (\Phi^\varepsilon)^3(t) + (3C_0^\varepsilon - 9C_1^\varepsilon - m)\Phi^\varepsilon(t))dt \\
&+ dW_N(t) \\
\Phi^\varepsilon(0) =&\Phi_0^\varepsilon,
\end{aligned} \tag{3.1}
$$

where we fix a cylindrical Wiener process in (1.2) on $L^2(\mathbb{T}^3)$ given by $\sum_k \beta_k e_k(\xi)$ for $\xi \in \mathbb{T}^3$ and restrict it to $L^2(\Lambda_\varepsilon)$ as $W_N(\xi) = \sum_{|k|_\infty\leq N} \beta_k e_k(\xi)$ for $\xi \in \Lambda_\varepsilon$, which is also a cylindrical Wiener process on $L^2(\Lambda_\varepsilon)$. Here $\{\beta_k\}$ is a family of independent Brownian motions on $(\Omega, \mathcal{F}, P)$. Also we take Φ_0^ε independent of W. For $\xi \in \Lambda_\varepsilon$ define

$$
\Delta_\varepsilon f(\xi) := \varepsilon^{-2} \sum_{y\in\Lambda_\varepsilon, y\sim\xi} (f(y) - f(\xi)),
$$

where the nearest neighbor relation $\xi \sim y$ is to be understood with periodic boundary conditions on Λ_ε. For Φ_0^ε satisfying $E\|\Phi_0^\varepsilon\|^2_{L^2(\Lambda_\varepsilon)} < \infty$ by [43, Theorem 3.1.1] there exists a unique solution Φ^ε to (3.1).

Following [35, 50] we define a suitable extension of functions defined on Λ_ε onto all of the torus $\mathbb{T}^3$ (which we identify with the interval $[-1, 1]^3$) in the following way:

$$
\mathrm{Ext}Y(\xi) := \frac{1}{2^3} \sum_{k\in\{-N,...,N\}^3} \sum_{y\in\Lambda_\varepsilon} \varepsilon^3 e^{\imath\pi k\cdot(\xi-y)} Y(y). \tag{3.2}
$$

Now we extend the solutions of (3.1) to all of $\mathbb{T}^3$. Let $u^\varepsilon = \mathrm{Ext}\Phi^\varepsilon$ for simplicity. We have the following equation:

$$
u^\varepsilon(t) = P_t^\varepsilon \mathrm{Ext}\Phi_0^\varepsilon - \int_0^t P_{t-s}^\varepsilon Q_N[(u^\varepsilon)^3 - (3C_0^\varepsilon - 9C_1^\varepsilon - m)u^\varepsilon]ds + \int_0^t P_{t-s}^\varepsilon P_N dW. \tag{3.3}
$$

where $P_t^\varepsilon = \mathrm{Ext}e^{t\Delta_\varepsilon}$ and $Q_N u(x) = P_N u(x) + \Pi_N u(x)$ with

$$
P_N = \mathcal{F}^{-1} 1_{|k|_\infty\leq N}\mathcal{F},
$$

and Π_N is defined for u satisfying $\mathrm{supp}\mathcal{F}u \subset \{k : |k|_\infty \leq 3N\}$

$$
\begin{aligned}
\Pi_N u(x) =& \sum_{i_1,i_2,i_3\in\{-1,0,1\}, \sum_{j=1}^3 i_j^2\neq 0} e_N^{i_1i_2i_3}\mathcal{F}^{-1}1_{k\in P^{i_1i_2i_3}}\mathcal{F}u(x) \\
=& \sum_{i_1,i_2,i_3\in\{-1,0,1\}, \sum_{j=1}^3 i_j^2\neq 0} P_N[e_N^{i_1i_2i_3}u]
\end{aligned}
$$

with $P^{i_1i_2i_3} = \{k : k^j i_j > N \text{ if } i_j = -1, 1; |k^j| \leq N, \text{ if } i_j = 0\}$ is a rectangular division of $\mathbb{Z}^3 \backslash \{k \in \mathbb{Z}^3, |k|_\infty \leq N\}$, $e_N^{i_1i_2i_3}(\xi) = \Pi_{j=1}^3 e^{-\imath\pi(2N+1)i_j\xi^j}$.

As in [50] we split (3.3) into the following three equations:

$$u_1^\varepsilon(t) = \int_{-\infty}^t P_{t-s}^\varepsilon P_N dW,$$

$$u_2^\varepsilon(t) = -\int_0^t P_{t-s}^\varepsilon Q_N[(u_1^\varepsilon)^{\diamond,3}]ds$$

and

$$\begin{aligned} u_3^\varepsilon(t) =& P_t^\varepsilon(\mathrm{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)) - \int_0^t P_{t-s}^\varepsilon \Big[Q_N[6u_1^\varepsilon u_2^\varepsilon u_3^\varepsilon + 3u_1^\varepsilon(u_3^\varepsilon)^2 + 3u_1^\varepsilon(u_2^\varepsilon)^2 + (u_2^\varepsilon + u_3^\varepsilon)^3] \\ & + P_N[3(u_1^\varepsilon)^{\diamond,2} \diamond (u_2^\varepsilon + u_3^\varepsilon) + 3e_N^{i_1i_2i_3}(u_1^\varepsilon)^{\diamond,2} \diamond (u_2^\varepsilon + u_3^\varepsilon) - (9\varphi^\varepsilon - m)u^\varepsilon] \Big] ds. \end{aligned} \tag{3.4}$$

Here the terms containing $\diamond$ are defined as in [50, Section 4]. For (3.4) we can do paracontrolled analysis as in [50, Section 4] and define the corresponding regular term $u^{\varepsilon,\sharp}$ in the paracontrolled ansatz. Also we define

$$C_W^\varepsilon(T), E_W^\varepsilon(T), A_N(T), D_N(T), \delta C_W^\varepsilon(T)$$

similarly as the corresponding stochastic terms in [50]. Here for the completeness of the paper we include the definition of all these terms in Appendix C. Now we introduce the following definition:

$$\rho_L^\varepsilon := \inf\{t \geq 0 : C_W^\varepsilon(t) + E_W^\varepsilon(t) + A_N(t) + D_N(t) \geq L\}, \tag{3.5}$$

and

$$\tau_{C_0}^\varepsilon := \inf\{t \geq 0 : t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} \|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa} \geq C_0(\|\Phi_0\|_{-z}^{\bar{m}} + 1) + 1\}, \tag{3.6}$$

with $C_0, \bar{m}$ obtained in Proposition 2.1.

Now we obtain the following estimate for the lattice approximations:

Proposition 3.1. *We have for any $T > 0$ on the set $\{\rho_L > T\}$ there exists $C_1, m > 0$ such that*

$$\begin{aligned} &\sup_{t\in[0,T\wedge\rho_L^\varepsilon\wedge\tau_{C_0}^\varepsilon]} [\|u^\varepsilon - \Phi\|_{-z} + t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon - \Phi_3\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} \|u_3^\varepsilon - \Phi_3\|_{\frac{1}{2}+4\kappa}] \\ \leq& C_1(\varepsilon^{\frac{\kappa}{2}} + \delta C_W^\varepsilon(T) + E_W^\varepsilon(T) + A_N(T) + D_N(T) + \|\mathit{Ext}\Phi_0^\varepsilon - \Phi_0\|_{-z}) e^{C_1(\|\Phi_0\|_{-z}^m + 1)}, \end{aligned}$$

where the constant C_1 depends on L, T.

Proof. Let

$$L^\varepsilon(t) := t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon - \Phi_3\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} \|u_3^\varepsilon - \Phi_3\|_{\frac{1}{2}+4\kappa} + t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon,\sharp} - \Phi^\sharp\|_{1+3\kappa}.$$

Since the nonlinear terms are given by polynomials, by similar calculations as in [50] and Proposition 2.1 we have that on the set $\{\rho_L > T\}$ there exists $q, m > 1$ such that for $t \in [0, T \wedge \rho_L \wedge \rho_L^\varepsilon \wedge \tau_{C_0}^\varepsilon]$

$$\begin{aligned} L^\varepsilon(t)^q \leq& C(\|\Phi_0\|_{-z}^m + 1)(\varepsilon^{\kappa/2} + \delta C_W^\varepsilon(T) + E_W^\varepsilon(T) + A_N(T) + D_N(T) + \|\mathrm{Ext}\Phi_0^\varepsilon - \Phi_0\|_{-z})^q \\ & + C(\|\Phi_0\|_{-z}^m + 1) \int_0^t L^\varepsilon(s)^q ds, \end{aligned}$$

which by Gronwall's inequality implies that for $t \in [0, T \wedge \rho_L \wedge \rho^\varepsilon_L \wedge \tau^\varepsilon_{C_0}]$

$$L^\varepsilon(t) \le (\varepsilon^{\kappa/2} + \delta C^\varepsilon_W(T) + E^\varepsilon_W(T) + A_N(T) + D_N(T) + \|\mathrm{Ext}\Phi^\varepsilon_0 - \Phi_0\|_{-z})e^{C(\|\Phi_0\|^m_{-z}+1)},$$

on $\{\rho_L > T\}$. Moreover, by similar calculations as in [50] we obtain that on $\{\rho_L > T\}$ for $t \in [0, T \wedge \rho_L \wedge \rho^\varepsilon_L \wedge \tau^\varepsilon_{C_0}]$

$$\begin{aligned}\|u^\varepsilon(t) - \Phi(t)\|_{-z} \le (&\varepsilon^{\kappa/2} + \delta C^\varepsilon_W(T) + E^\varepsilon_W(T) + A_N(T) + D_N(T)\\ &+ \|\mathrm{Ext}\Phi^\varepsilon_0 - \Phi_0\|_{-z})e^{C(\|\Phi_0\|^m_{-z}+1)}.\end{aligned}$$

□

Similarly as in the proof of [24, Proposition 7.7] we obtain the following estimate for the measure $\bar\mu^\varepsilon := \mu^\varepsilon \circ \mathrm{Ext}^{-1}$. Since μ^ε is a measure on $L^2(\Lambda^\varepsilon)$ and Ext is an isometry from $L^2(\Lambda_\varepsilon)$ to $P_N L^2(\mathbb{T}^3)$, $\bar\mu^\varepsilon$ has full support on $P_N L^2(\mathbb{T}^3)$:

Lemma 3.2. *Let $n \in \mathbb{N}$. Then there exists a constant C independent of ε such that*

$$\int \|x\|^{2n}_{-z}\bar\mu^\varepsilon(dx) \le C.$$

Moreover, $\bar\mu^\varepsilon$ weakly converges to μ on $\mathcal{C}^{-z}$.

Remark 3.3. The proof of this lemma depends on the results in [6]. In fact the results in [6] are established for the whole space based on the calculations for the correlation functions, which can be extended to the torus case (see the argument in [6, Section 2] and [42, Theorem 2.1]). Therefore, we use these results for the torus as was done in [24, Proposition 7.7].

Proof. The following calculations on Λ_ε essentially follow [35, Lemma 8.4]. Suppose $supp\theta \subset \{a \le |k| \le b\}$ for θ as in Appendix A and $a, b > 0$. If $2^j a > \sqrt{3}N$, then $\int \|\Delta_j x\|^{2n}_{L^{2n}(\mathbb{T}^3)}\bar\mu^\varepsilon(dx) = 0$. For $x \in \mathrm{supp}\bar\mu^\varepsilon$ we have

$$\Delta_j x = \sum_{|k|_\infty \le N} \theta_j(k)\langle x, e_k\rangle e_k = \sum_{|k|_\infty \le N} \theta_j(k)\langle \mathrm{Ext}^{-1}x, e_k\rangle_\varepsilon e_k,$$

where $\theta_j(\cdot) := \theta(2^{-j}\cdot)$ and $\langle\cdot,\cdot\rangle, \langle\cdot,\cdot\rangle_\varepsilon$ denote the inner products in $L^2(\mathbb{T}^3)$ and $L^2(\Lambda_\varepsilon)$, respectively. Here we can take Ext^{-1} since Ext is an isometry from $L^2(\Lambda_\varepsilon)$ to $P_N L^2(\mathbb{T}^3)$. If $2^j b < N - 1$, then by changing variables we have

$$\begin{aligned}
&\int \|\Delta_j x\|^{2n}_{L^{2n}(\mathbb{T}^3)}\bar\mu^\varepsilon(dx)\\
=&\int \|\sum_{|k|_\infty \le N} \theta_j(k)\langle x, e_k\rangle_\varepsilon e_k\|^{2n}_{L^{2n}(\mathbb{T}^3)}\mu^\varepsilon(dx)\\
=&2^{-3n}\int \sum_{y_i\in\Lambda_\varepsilon, i=1,\dots,2n} \varepsilon^{6n} \sum_{|k_i|_\infty\le N, i=1,\dots,2n} \big(\Pi^{2n}_{i=1}\theta_j(k_i)e_{k_i}(\xi - y_i)\big)S^\varepsilon_{2n}(y_1,\dots,y_{2n})d\xi\\
=&C\int \sum_{y_i\in 2^j\Lambda_\varepsilon, i=1,\dots,2n} \varepsilon^{6n}2^{6nj} \sum_{|k_i|_\infty\le 2^{-j}N, k_i\in 2^{-j}\mathbb{Z}^3, i=1,\dots,2n} 2^{-6jn}\big(\Pi^{2n}_{i=1}\theta(k_i)e^{\pi\iota k_i(2^j\xi - y_i)}\big)\\
&S^\varepsilon_{2n}(\frac{y_1}{2^j},\dots,\frac{y_{2n}}{2^j})d\xi\\
=&C\int \sum_{y_i\in 2^j\Lambda_\varepsilon, i=1,\dots,2n} \varepsilon^{6n}2^{6jn}\left(\Pi^{2n}_{i=1}\frac{1}{[1 + 2^{2j}\sum^3_{l=1}2(1 - \cos(\pi 2^{-j}(2^j\xi^l - y^l_i)))]^2}\right)\\
&S^\varepsilon_{2n}(\frac{y_1}{2^j},\dots,\frac{y_{2n}}{2^j}) \sum_{|k_i|_\infty\le 2^{-j}N, k_i\in 2^{-j}\mathbb{Z}^3, i=1,\dots,2n} 2^{-6jn}\big(\Pi^{2n}_{i=1}\theta(k_i)(1 - \underline{\Delta}_j)^2 e^{\pi\iota k_i(2^j\xi - y_i)}\big)d\xi
\end{aligned}$$

$$
\begin{aligned}
=&C\int\sum_{y_i\in 2^j\Lambda_\varepsilon,i=1,...,2n}\varepsilon^{6n}2^{6jn}\Big(\Pi_{i=1}^{2n}\frac{1}{[1+2^{2j}\sum_{l=1}^3 2(1-\cos(\pi 2^{-j}(2^j\xi^l-y_i^l)))]^2}\Big)\\
&S^\varepsilon_{2n}(\frac{y_1}{2^j},...,\frac{y_{2n}}{2^j})\sum_{|k_i|_\infty\leq 2^{-j}N,k_i\in 2^{-j}\mathbb{Z}^3,i=1,...,2n}2^{-6jn}\big(\Pi_{i=1}^{2n}(1-\underline{\Delta}_j)^2\theta(k_i)e^{\pi\iota k_i(2^j\xi-y_i)}\big)d\xi\\
\leq&C\int(\sum_{y_1,y_2\in 2^j\Lambda_\varepsilon}\varepsilon^6 2^{6j}\frac{1}{(1+|2^j\xi-y_1|^2)^2}\frac{1}{(1+|2^j\xi-y_2|^2)^2}(C^\varepsilon(\frac{y_1}{2^j},\frac{y_2}{2^j})+\lambda^2))^n d\xi\\
\lesssim&2^{jn},
\end{aligned}
$$

where $S^\varepsilon_{2n}(y_1,...,y_{2n})$ is the $2n$ point function for μ^ε from [6] and C^ε is the covariance for the corresponding Gaussian measure on the lattice and

$$\underline{\Delta}_j f(k)=2^{2j}\sum_{k'\in 2^{-j}\mathbb{Z}^3,k\sim k'}(f(k')-f(k)).$$

Here in the last equality we use the integration by parts formula, since on the boundary θ vanishes and in the first inequality we used that the support of θ is contained in an annulus to count the number of non-zero terms and deduce

$$\Big|\sum_{|k_i|_\infty\leq 2^{-j}N,k_i\in 2^{-j}\mathbb{Z}^3,i=1,...,2n}2^{-6jn}\big(\Pi_{i=1}^{2n}(1-\underline{\Delta}_j)^2\theta(k_i)e^{\pi\iota k_i(2^j\xi-y_i)}\big)\Big|\lesssim 1.$$

In addition, we use (8.2) and Theorem 6.1 in [6] to control S^ε_{2n} and the following: when $\xi^1\in[-1,1]$, $\frac{1}{1-\cos(\pi\xi^1)}\leq\frac{C}{(\xi^1)^2}$ and when $\xi^1\in[1,2]$, $\frac{1}{1-\cos(\pi\xi^1)}=\frac{1}{1-\cos(\pi(\xi^1-2))}\leq\frac{C}{(\xi^1-2)^2}$ and when $\xi^1\in[-2,-1]$, $\frac{1}{1-\cos(\pi\xi^1)}=\frac{1}{1-\cos(\pi(\xi^1+2))}\leq\frac{C}{(\xi^1+2)^2}$. Furthermore, in the last step we use that the covariance $C^\varepsilon(y_1,y_2)$ of the Gaussian measure is of order $|y_1-y_2|^{-1}$.

If $\frac{2^ja}{\sqrt{3}}\leq N\leq 2^jb+1$, we choose a smooth function χ which equals 1 on $\{\frac{a}{2}\leq|k|\leq 4b\}$ and vanishes outside the annulus $\{\frac{a}{3}\leq|k|\leq 5b\}$. Let $\chi_j=\chi(2^{-j}\cdot)$. We have

$$
\begin{aligned}
&\int\|\Delta_j x\|^{2n}_{L^{2n}(\mathbb{T}^3)}\bar{\mu}^\varepsilon(dx)\\
=&\int\|\sum_k\theta_j(k)\chi_j(k)\langle x,e_k\rangle e_k\|^{2n}_{L^{2n}(\mathbb{T}^3)}\bar{\mu}^\varepsilon(dx)\leq C\int\|\sum_k\chi_j(k)\langle x,e_k\rangle e_k\|^{2n}_{L^{2n}(\mathbb{T}^3)}\bar{\mu}^\varepsilon(dx)\\
\lesssim&N^3\int\|\sum_k\chi_j(k)\langle x,e_k\rangle_\varepsilon e_k\|^{2n}_{L^{2n}(\Lambda_\varepsilon)}\mu^\varepsilon(dx)\\
\lesssim&2^{3j}\sum_{\xi\in\Lambda_\varepsilon}\varepsilon^3\sum_{y_i\in\Lambda_\varepsilon,i=1,...,2n}\sum_{|k_i|_\infty\leq N,i=1,...,2n}\varepsilon^{6n}\big(\Pi_{i=1}^{2n}\chi_j(k_i)e_{k_i}(\xi-y_i)\big)S^\varepsilon_{2n}(y_1,...,y_{2n})\\
\lesssim&2^{3j+jn}.
\end{aligned}
$$

Here in the second inequality we used Lemma C.2 and the estimate in the last inequality can be obtained by a similar argument as above and the integration by parts formula holds for the periodic boundary conditions. Thus, the first result holds by choosing n large enough and because of Lemma A.1. In fact, for any $\alpha<-\frac{1}{2}$, $\int\|x\|^{2n}_\alpha\bar{\mu}^\varepsilon(dx)\leq C$. The second result follows from the tightness of the $\bar{\mu}^\varepsilon$ and from the fact that the corresponding Schwinger functions converge (see [41] and [24, Prop. 7.7]). □

4 Existence of the Dirichlet form

Consider the normal filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W. As we mentioned in Section 2, by [23, 7, 37] for every $x\in\mathcal{C}^{-z}$ there exists a unique solution $\Phi(x)$ to (1.2) starting from x. By [24] we have that Φ satisfies the Markov property on $\mathcal{C}^{-z}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Define

$$P^x(A):=P(\Phi(x)\in A).$$

P^x is a measure on $\Omega' := C([0,\infty);\mathcal{C}^{-z})$ and we use E^x to denote the expectation under P^x. We use X to denote the canonical process on Ω' and equip Ω' by the natural filtration $(\mathcal{M}_t)_{t\geq 0}$ generated by X (cf. [34, Chapter IV, (1.7)]). We know X has the same distribution as Φ. By the Markov property of Φ we know $(\Omega', \mathcal{M} := \vee_{t\geq 0}\mathcal{M}_t, (\mathcal{M}_t)_{t\geq 0}, X, P^x)_{x\in\mathcal{C}^{-z}}$ is also a Markov process (cf. Definition D.2). Here iii) in Definition D.2 follows from the measurablity of $x\mapsto \Phi(x)$. Now we prove the following:

Lemma 4.1. *$(\Omega', \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, X, P^x)_{x\in\mathcal{C}^{-z}}$ is a Feller process on $\mathcal{C}^{-z}$.*

Proof. It suffices to check that $E^x f(X(t))$ is a continuous function on $\mathcal{C}^{-z}$ for $f\in C_b(\mathcal{C}^{-z})$. We have

$$\begin{aligned}&|E^{x_1}f(X(t)) - E^{x_2}f(X(t))| = |Ef(\Phi(t,x_1)) - Ef(\Phi(t,x_2))|\\ \leq& E|f(\Phi(t,x_1)) - f(\Phi(t,x_2))|1_{t\leq\rho_L} + CP(t>\rho_L).\end{aligned}$$

Here $\Phi(x)$ denotes the solution to (1.2) starting from x and ρ_L is defined as in Section 2. The first term goes to zero as x_1 goes to x_2 in $\mathcal{C}^{-z}$ by [23] and the second term goes to zero as L goes to infinity since $EC_W(t)\leq C$ with C_W defined in (2.2). □

By $P^x(X\in C([0,\infty);\mathcal{C}^{-z})) = 1$ for $x\in\mathcal{C}^{-z}$ and by [9, Section 2.3 Theorem 1] we know that the Feller process $(\Omega', \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, X, P^x)_{x\in\mathcal{C}^{-z}}$ satisfies the corresponding strong Markov property (cf. iii) in Definition D.3).

To construct the Dirichlet form associated with X, we first extend the Markov process to starting points from a larger space, which contains $L^2(\mathbb{T}^3)$ as a subspace. Choose $E = H^{-z-\epsilon} := B^{-z-\epsilon}_{2,2}$ with $\epsilon>0$ and $H = L^2(\mathbb{T}^3)$. By Lemma A.1 we have $\mathcal{C}^{-z}\subset E$ and the following relation holds:

$$E^*\subset H^*\simeq H\subset E.$$

In the following we use $\langle\cdot,\cdot\rangle, |\cdot|$ to denote the inner product and norm on H respectively and $\langle\cdot,\cdot\rangle$ also denotes the dual relation between E^* and E if there is no confusion. Now we would like to extend X to a process X' with state space E in such a way that each $x\in E\backslash\mathcal{C}^{-z}$ is a trap for X' (see [34, page 118]). For notation's simplicity we still use $(\Omega', \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, X, P^x)_{x\in E}$ to denote X'. In the following $(\Omega', \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, X, P^x)_{x\in E}$ is a continuous strong Markov process with state space E. Define the associated semigroup for $f\in\mathcal{B}_b(E), x\in E$

$$\bar{P}_t f(x) := E^x f(X(t)).$$

We also introduce the following cylinder functions

$$\mathcal{F}C_b^\infty = \{f_1(\langle l_1,\cdot\rangle, ..., \langle l_m,\cdot\rangle)|m\in\mathbb{N}, f_1\in C_b^\infty(\mathbb{R}^m), l_1,...,l_m\in E^*\}.$$

Define for $f\in\mathcal{F}C_b^\infty$ and $l\in H$,

$$\frac{\partial f}{\partial l}(z) := \frac{d}{ds}f(z+sl)|_{s=0}, z\in E,$$

that is, by the chain rule,

$$\frac{\partial f}{\partial l}(z) = \sum_{j=1}^m \partial_j f_1(\langle l_1,z\rangle, \langle l_2,z\rangle, ..., \langle l_m,z\rangle)\langle l_j,l\rangle_H.$$

Let Df denote the H-derivative of $f\in\mathcal{F}C_b^\infty$, i.e. the map from E to H such that

$$\langle Df(z), l\rangle = \frac{\partial f}{\partial l}(z) \text{ for all } l\in H, z\in E.$$

In the following we prove that $\bar{P}_t$ is a symmetric semigroup with respect to μ. For this we use lattice approximation in Section 3 and let $\Phi^\varepsilon(x)$ be the solution to (3.1) obtained in Section 3 starting from $x \in L^2(\Lambda_\varepsilon)$. By existence and uniqueness of the solutions to (3.1) and similar arguments as in [43, 32, Section 4.3] we obtain that Φ^ε satisfies the Markov property w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$. We define the semigroup of the lattice approximation: for $f \in C_b(L^2(\Lambda_\varepsilon)), x \in L^2(\Lambda_\varepsilon)$,

$$\tilde{P}^\varepsilon_t f(x) = E(f(\Phi^\varepsilon(t,x))).$$

Since (3.1) is a gradient system, by [11, Theorem 12.3.2] we have for $f, g \in C_b(L^2(\Lambda_\varepsilon))$

$$\int \tilde{P}^\varepsilon_t f(x) g(x) \mu^\varepsilon(dx) = \int f(x) \tilde{P}^\varepsilon_t g(x) \mu^\varepsilon(dx). \tag{4.1}$$

We also define the semigroup for the extension of the lattice approximation on $P_N E$: for $f \in C_b(P_N E)$, $x \in P_N E$,

$$\bar{P}^\varepsilon_t f(x) = E(f(u^\varepsilon(t,x))),$$

where P_N is as introduced in Section 3 and $u^\varepsilon(x)$ is the solution to (3.3) starting from x. Then we prove that $\bar{P}^\varepsilon_t$ is symmetric with respect to $\bar{\mu}^\varepsilon$. Since the extension operator Ext defined in (3.2) is an isometry from $L^2(\Lambda_\varepsilon)$ to $P_N E$, we view $\bar{\mu}^\varepsilon$ as a measure on $P_N E$.

Lemma 4.2. *For $\varepsilon = \frac{2}{2N+1}$ and $f, g \in \mathcal{F}C^\infty_b$ we have*

$$\int \bar{P}^\varepsilon_t (f|_{P_N E})(x) g|_{P_N E}(x) \bar{\mu}^\varepsilon(dx) = \int f|_{P_N E}(x) \bar{P}^\varepsilon_t (g|_{P_N E})(x) \bar{\mu}^\varepsilon(dx),$$

where we used that $P_N E \subset E$.

Proof. Without loss of generality we assume that $f(x) = f_1(\langle x, l\rangle), g(x) = g_1(\langle x, h\rangle)$ with $f_1, g_1 \in C^\infty_b$. Then we have that for $l_1 = \sum_{|k|_\infty \leq N} \langle l, e_k\rangle e_k, h_1 = \sum_{|k|_\infty \leq N} \langle h, e_k\rangle e_k$,

$$\begin{aligned}
&\int \bar{P}^\varepsilon_t (f|_{P_N E})(x) g|_{P_N E}(x) \bar{\mu}^\varepsilon(dx) = \int E(f_1(\langle u^\varepsilon(t,x), l_1\rangle)) g_1(\langle x, h_1\rangle) \bar{\mu}^\varepsilon(dx) \\
=& \int E(f_1(\langle \Phi^\varepsilon(t, \mathrm{Ext}^{-1} x), l_1\rangle_\varepsilon)) g_1(\langle \mathrm{Ext}^{-1} x, h_1\rangle_\varepsilon) \bar{\mu}^\varepsilon(dx) \\
=& \int E(f_1(\langle \Phi^\varepsilon(t,x), l_1\rangle_\varepsilon)) g_1(\langle x, h_1\rangle_\varepsilon) \mu^\varepsilon(dx) \\
=& \int E(g_1(\langle \Phi^\varepsilon(t,x), h_1\rangle_\varepsilon)) f_1(\langle x, l_1\rangle_\varepsilon) \mu^\varepsilon(dx) = \int E(g_1(\langle u^\varepsilon(t,x), h_1\rangle)) f_1(\langle x, l_1\rangle) \bar{\mu}^\varepsilon(dx) \\
=& \int \bar{P}^\varepsilon_t (g|_{P_N E})(x) f|_{P_N E}(x) \bar{\mu}^\varepsilon(dx).
\end{aligned}$$

Here in the second equality we used $\langle x, l_1\rangle = \langle \mathrm{Ext}^{-1} x, l_1\rangle_\varepsilon$ for $x \in P_N E$ to deduce $\langle \Phi^\varepsilon_t, l_1\rangle_\varepsilon = \langle u^\varepsilon_t, l_1\rangle$ and in the forth equality we used (4.1). □

By Lemma 4.2 and [34, Chapter II Prop. 4.3] we know that $(\bar{P}^\varepsilon_t)_{t>0}$ can be extended as a strongly continuous Markovian semigroup of contractions on $L^2(P_N E; \bar{\mu}^\varepsilon)$. By [34, Chap I] there exists a corresponding Dirichlet form for $(\bar{P}^\varepsilon_t)_{t>0}$. In Proposition 4.4 we will give the explicit formula for this Dirichlet form. Now we prove that $\bar{P}_t$ is symmetric with respect to μ.

Proposition 4.3. *For $f, g \in \mathcal{F}C^\infty_b$ we have for $t \geq 0$*

$$\int \bar{P}_t f(x) g(x) \mu(dx) = \int f(x) \bar{P}_t g(x) \mu(dx).$$

Proof. By Lemma 4.2 it suffices to prove that for $f, g \in \mathcal{F}C_b^\infty$

$$\lim_{\varepsilon\to 0}\int \bar{P}_t^\varepsilon(f|_{P_N E})(x)g|_{P_N E}(x)\bar{\mu}^\varepsilon(dx) = \int \bar{P}_t f(x)g(x)\mu(dx). \tag{4.2}$$

Lemmas 3.2 and 4.1 imply that

$$\lim_{\varepsilon\to 0}\int \bar{P}_t f(x)g(x)\bar{\mu}^\varepsilon(dx) = \int \bar{P}_t f(x)g(x)\mu(dx).$$

We also have

$$\begin{aligned}&\int |\bar{P}_t^\varepsilon(f|_{P_N E})(x) - \bar{P}_t f(x)||g(x)|\bar{\mu}^\varepsilon(dx)\\ \le& C\int E\big(|f(u^\varepsilon(t,x)) - f(\Phi(t,x))|1_{\{t<\rho_L^\varepsilon\wedge\rho_L\}}\big)\bar{\mu}^\varepsilon(dx) + CP(t\ge\rho_L\wedge\rho_L^\varepsilon),\end{aligned} \tag{4.3}$$

where $\rho_L, \rho_L^\varepsilon$ are as introduced in Section 2 and (3.5), respectively. The second term in (4.3) is bounded by a constant times

$$\begin{aligned}&P((C_W^\varepsilon + E_W^\varepsilon + A_N + D_N)(t) > L) + P(C_W(t) > L)\\ \le& C/L,\end{aligned}$$

which uniformly goes to zero as L goes to ∞. For some $\delta_0 > 0$ the first term in (4.3) is bounded by

$$\begin{aligned}&\varepsilon^{\delta_0}C\int P(\|u^\varepsilon(t,x) - \Phi(t,x)\|_{-z} < \varepsilon^{\delta_0})\bar{\mu}^\varepsilon(dx)\\ &+ C\int P(t<\rho_L^\varepsilon\wedge\rho_L, \|u^\varepsilon(t,x) - \Phi(t,x)\|_{-z} > \varepsilon^{\delta_0})\bar{\mu}^\varepsilon(dx).\end{aligned} \tag{4.4}$$

Then the first term is bounded by $C\varepsilon^{\delta_0}$ and the second integral in (4.4) is bounded by

$$\begin{aligned}&\int[P(t<\rho_L^\varepsilon\wedge\rho_L, t<\tau_{C_0}^\varepsilon, \|u^\varepsilon(t,x) - \Phi(t,x)\|_{-z} > \varepsilon^{\delta_0}) + P(t<\rho_L^\varepsilon\wedge\rho_L, t\ge\tau_{C_0}^\varepsilon)]\bar{\mu}^\varepsilon(dx)\\ \le& 2\int P\Big(\sup_{s\in[0,\rho_L^\varepsilon\wedge\rho_L\wedge t\wedge\tau_{C_0}^\varepsilon]}[\|u^\varepsilon(x) - \Phi(x)\|_{-z} + s^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon - \Phi_3\|_\gamma\\ &+ s^{\frac{\frac12+z+5\kappa}{2}}\|u_3^\varepsilon - \Phi_3\|_{\frac12+4\kappa}] > \varepsilon^{\delta_0}\Big)\bar{\mu}^\varepsilon(dx)\\ \le& 2\int P(2C_1\varepsilon^{\kappa_0}e^{C_1(\|x\|_{-z}^m+1)} > \varepsilon^{\delta_0})\bar{\mu}^\varepsilon(dx)\\ &+ 2\int P(\delta C_W^\varepsilon(t) + A_N(t) + E_W^\varepsilon(t) + D_N(t) > \varepsilon^{\kappa_0})\bar{\mu}^\varepsilon(dx)\\ \le& 2\int 1_{\{\|x\|_{-z}^m > \frac{1}{C_1}\ln\frac{\varepsilon^{\delta_0-\kappa_0}}{2C_1} - 1\}}\bar{\mu}^\varepsilon(dx) + 2C\varepsilon^{\kappa_1-\kappa_0}\\ \le& 2\int\frac{1}{\frac{1}{C_1}\ln\frac{\varepsilon^{\delta_0-\kappa_0}}{2C_1} - 1}\|x\|_{-z}^m\bar{\mu}^\varepsilon(dx) + 2C\varepsilon^{\kappa_1-\kappa_0} \to 0, \quad \text{as } \varepsilon\to 0,\end{aligned}$$

where u_3^ε, Φ_3 correspond to $u^\varepsilon(x), \Phi(x)$ respectively and $\tau_{C_0}^\varepsilon$ is defined in (3.6) and in the first inequality we used Proposition 2.1 and the definition of $\tau_{C_0}^\varepsilon$ to deduce

$$\sup_{s\in[0,\rho_L^\varepsilon\wedge\rho_L\wedge t\wedge\tau_{C_0}^\varepsilon]}[\|u^\varepsilon - \Phi\|_{-z} + s^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon - \Phi_3\|_\gamma + s^{\frac{1/2+z+5\kappa}{2}}\|u_3^\varepsilon - \Phi_3\|_{\frac12+4\kappa}] > \varepsilon^{\delta_0}.$$

In the second inequality we used Proposition 3.1 and in the third inequality we used Proposition C.1 and in the last step we used Lemma 3.2. Here we choose $0 < \delta_0 < \kappa_0 < \kappa_1 \wedge \frac{\kappa}{2}$ for $\frac{\kappa}{2}, \kappa_1$ coming from Proposition 3.1 and Proposition C.1, respectively. Summarizing, we obtain the result. □

Now we identify the Dirichlet form associated with $(\bar{P}^\varepsilon_t)_{t>0}$ on $L^2(P_NE,\bar{\mu}^\varepsilon)$.

Proposition 4.4. *The Dirichlet form associated with $(\bar{P}^\varepsilon_t)_{t>0}$ can be written as the closure of the following bilinear form*

$$\mathcal{E}^\varepsilon(f,g)=\frac{1}{2}\sum_{|k|_\infty\leq N}\int_{P_NE}\frac{\partial f}{\partial e_k}\frac{\partial g}{\partial e_k}d\bar{\mu}^\varepsilon,\quad f,g\in C_b^\infty(P_NE),$$

where $C_b^\infty(P_NE)$ means smooth functions on P_NE with bounded derivatives.

Proof. It is standard to obtain that the closure of $(\mathcal{E}^\varepsilon, C_b^\infty(P_NE))$ is a quasi-regular Dirichlet form (cf. Definition D.1, [34, Chap IV Section 4]), which is denoted by $(\mathcal{E}^\varepsilon, D(\mathcal{E}^\varepsilon))$. By Theorem D.4 there exists a Markov process with continuous sample paths properly associated with $(\mathcal{E}^\varepsilon, D(\mathcal{E}^\varepsilon))$. Now we want to prove that the associated Markov process has the same distribution as u^ε.

We can easily conclude that the log-derivative of μ^ε along e_k for $|k|_\infty\leq N$ is given by

$$b_k(x)=2\langle x,\Delta_\varepsilon e_k\rangle_\varepsilon-2\langle x^3-(3C_0^\varepsilon-9C_1^\varepsilon-m)x,e_k\rangle_\varepsilon \text{ for } x\in L^2(\Lambda_\varepsilon),$$

which implies that for $f\in C_b^\infty(P_NE)$ and $|k|_\infty\leq N$

$$\int\frac{\partial f}{\partial e_k}(x)d\bar{\mu}^\varepsilon=\int\frac{\partial}{\partial e_k}(f\circ\mathrm{Ext})(x)d\mu^\varepsilon=-\int f(\mathrm{Ext}x)b_k(x)d\mu^\varepsilon=-\int f(x)b_k(\mathrm{Ext}^{-1}x)d\bar{\mu}^\varepsilon,$$

we obtain that the log-derivative of $\bar{\mu}^\varepsilon$ is

$$\beta_k(x)=b_k(\mathrm{Ext}^{-1}x)=2\langle x,\Delta_\varepsilon e_k\rangle_{L^2(\mathbb{T}^3)}-2\langle Q_N(x^3-(3C_0^\varepsilon-9C_1^\varepsilon-m)x),e_k\rangle_{L^2(\mathbb{T}^3)},$$

for $x\in P_NE, |k|_\infty\leq N$, where we used that $\mathrm{Ext}(\mathrm{Ext}^{-1}x)^3=Q_N(x^3)$ for $x\in P_NE$. This implies that the associated Markov process is a probabilistically weak solution to the equation (3.3). On the other hand, the equation (3.3) is a finite dimensional stochastic differential equation and we can easily obtain the pathwise uniqueness of the solutions to the equation (3.3), which implies the uniqueness in law of the solutions to (3.3). This implies that u^ε has the same distribution as the Markov process given by the Dirichlet form $(\mathcal{E}^\varepsilon, D(\mathcal{E}^\varepsilon))$, since u^ε is also a solution to (3.3). By Theorem D.4 we know that the semigroup $(\bar{P}^\varepsilon_t)_{t>0}$ of u^ε is properly associated with $(\mathcal{E}^\varepsilon, D(\mathcal{E}^\varepsilon))$. ☐

Proof of Theorem 1.1. By Proposition 4.3 we have that $\int\bar{P}_tfd\mu=\int fd\mu$ for $f\in\mathcal{F}C_b^\infty$. Since $\sigma(\mathcal{F}C_b^\infty)=\mathcal{B}(E)$, we deduce that μ is an invariant measure for the semigroup $\bar{P}_t$, which implies that

$$\int\bar{P}_tfd\mu=\int fd\mu \text{ for } f\in\mathcal{B}_b(E). \tag{4.5}$$

By Proposition 4.3 and using (4.5) and the fact that $\mathcal{F}C_b^\infty$ is dense in $L^2(E;\mu)$, we have that for $f,g\in\mathcal{B}_b(E)$

$$\int\bar{P}_tf(x)g(x)\mu(dx)=\int f(x)\bar{P}_tg(x)\mu(dx).$$

Since $(\bar{P}_t)_{t>0}$ is sub-Markovian, by [34, Chapter II Proposition 4.1] it can be extended to $L^2(E,\mu)$. This extension is still denoted by $(\bar{P}_t)_{t>0}$. On the other hand, since Φ has continuous path in E, we can deduce that $\bar{P}_tf\to_{t\to0}f$ in μ-measure for $f\in\mathcal{F}C_b^\infty$. Then by [34, Chapter II Proposition 4.3] $(\bar{P}_t)_{t>0}$ is a strongly continuous contraction semigroup on $L^2(E;\mu)$. Then there exists a corresponding Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ associated with $(\bar{P}_t)_{t>0}$.

We know that $(\Omega',\mathcal{M},(\mathcal{M}_t)_{t>0},X,P^z)_{z\in E}$ is a right process in the sense of Definition D.3, which implies that $(\mathcal{E},D(\mathcal{E}))$ is a quasi-regular Dirichlet form by Theorem D.4.

In the following we prove that $\mathcal{F}C_b^\infty \subset D(\mathcal{E})$. By (4.2) and since $\bar{\mu}^\varepsilon$ converges weakly to μ we know that for $f \in \mathcal{F}C_b^\infty$,

$$\begin{aligned}\sup_{t>0}\frac{1}{t}\int(\bar{P}_t f - f)f d\mu &= \sup_{t>0}\lim_{\varepsilon\to 0}\frac{1}{t}\int(\bar{P}_t^\varepsilon(f|_{P_N E}) - f|_{P_N E})f|_{P_N E}d\bar{\mu}^\varepsilon \\ &\leq \liminf_{\varepsilon\to 0}\sup_{t>0}\frac{1}{t}\int(\bar{P}_t^\varepsilon(f|_{P_N E}) - f|_{P_N E})f|_{P_N E}d\bar{\mu}^\varepsilon \\ &= \liminf_{\varepsilon\to 0}\mathcal{E}^\varepsilon(f|_{P_N E}, f|_{P_N E}) < \infty,\end{aligned}$$

where in the last inequality we used Proposition 4.4. This implies that $\mathcal{F}C_b^\infty \subset D(\mathcal{E})$ and for $f \in \mathcal{F}C_b^\infty$,

$$\mathcal{E}(f,f) \leq \frac{1}{2}\int |Df|^2 d\mu. \tag{4.6}$$

For $l \in E^*$ by (4.6) we can easily find $f_n \in \mathcal{F}C_b^\infty$ such that $f_n \to \langle l, \cdot\rangle$ in $L^2(E,\mu)$ and f_n is a Cauchy sequence in $D(\mathcal{E})$, which implies $\langle l,\cdot\rangle \in D(\mathcal{E})$ since $(\mathcal{E}, D(\mathcal{E}))$ is a closed form. $\square$

5 Identification of the Dirichlet form

In this section we identify the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{F}C_b^\infty$. To complete this, we first write the nonlinear term as an additive functional of the solution. Here we use paracontrolled analysis to prove the solution Φ to (1.2) satisfies the following equation in the analytic weak sense:

$$\Phi(t) = \Phi_0 + \int_0^t \Delta\Phi ds - \lim_{\varepsilon\to 0}\int_0^t[(\rho_\varepsilon * \Phi)^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \Phi - 9\tilde{C}_1^\varepsilon\Phi - m\Phi)]ds + W(t), \tag{5.1}$$

where $\bar{C}_0^\varepsilon$ and $\tilde{C}_1^\varepsilon$ are defined below. For this we consider the following approximation: Let $\bar{\Phi}^\varepsilon$ be the solutions to the following equation:

$$d\bar{\Phi}^\varepsilon = \Delta\bar{\Phi}^\varepsilon dt + \rho_\varepsilon * dW - (\bar{\Phi}^\varepsilon)^3 dt + (3\bar{C}_0^\varepsilon - 9\bar{C}_1^\varepsilon - m)\bar{\Phi}^\varepsilon dt, \tag{5.2}$$

$$\bar{\Phi}^\varepsilon(0) = \Phi_0.$$

Here $\bar{C}_0^\varepsilon$ and $\bar{C}_1^\varepsilon$ are the corresponding constants defined in Appendix B. For this equation we can also write $\bar{\Phi}^\varepsilon = \bar{\Phi}_1^\varepsilon + \bar{\Phi}_2^\varepsilon + \bar{\Phi}_3^\varepsilon$ and define $\bar{\Phi}_1^\varepsilon$, $\bar{\Phi}_2^\varepsilon$, $\bar{\Phi}_3^\varepsilon$, $\bar{K}^\varepsilon$, $\bar{\Phi}^{\varepsilon,\sharp}$ similarly as in Section 2. Here we also introduce graph notations for them. We use to denote $\bar{\Phi}_1^\varepsilon$ and to denote $-\bar{\Phi}_2^\varepsilon$. Moreover, is used to denote $\bar{K}^\varepsilon$. The corresponding renormalized terms , , $\pi_{0,\diamond}($, $)$, $\pi_{0,\diamond}($, $)$ are defined as in Appendix B. Furthermore, we use and to denote $-\rho_\varepsilon * \Phi_2$ and $\rho_\varepsilon * K$, respectively. We summarise the graph notations after the introduction. We also introduce the following:

$$\pi_{0,\diamond}(\ ,\ ) := \pi_0(\ ,\ ) - 3(\tilde{C}_1^\varepsilon + \tilde{\varphi}^\varepsilon)\ ,$$

$$\pi_{0,\diamond}(\ ,\ ) := \pi_0(\ ,\ ) - (\tilde{C}_1^\varepsilon + \tilde{\varphi}^\varepsilon),$$

with

$$\tilde{C}_1^\varepsilon = 2^{-7}\int\int \frac{g(\varepsilon k_1)g(\varepsilon k_2)g(\varepsilon k_{[12]})}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{[12]}|^2)\pi^6}dk_1 dk_2,$$

and

$$\tilde{\varphi}^\varepsilon(t) = -2^{-7}\int\int \frac{e^{-t\pi^2(|k_1|^2+|k_2|^2+|k_{[12]}|^2)}g(\varepsilon k_1)g(\varepsilon k_2)g(\varepsilon k_{[12]})}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|k_{[12]}|^2)\pi^6}dk_1dk_2.$$

Here $k_{[12]} = k_1 + k_2$ and the integral is on the set $\mathbb{Z}^3\backslash\{0\}$.

We also define

$$\begin{aligned}
\delta\bar{C}^\varepsilon_W(T) := \sup_{t\in[0,T]}\big[&\|\pi_0(\ ,\ ) - \pi_0(\ ,\ )\|_{-2\kappa}\\
&+\|\pi_{0,\diamond}(\ ,\ ) - \pi_{0,\diamond}(\ ,\ )\|_{-\frac{1}{2}-2\kappa}\\
&+\|\pi_{0,\diamond}(\ ,\ ) - \pi_{0,\diamond}(\ ,\ )\|_{-2\kappa} + \|\ -\ \|_{-\frac{1}{2}-2\kappa}\\
&+\|\ -\ \|_{-1-2\kappa} + \|\ -\ \|_{\frac{1}{2}-2\kappa} + \|\pi_0(\ ,\ ) - \pi_0(\ ,\ )\|_{-2\kappa}\\
&+\|\pi_{0,\diamond}(\ ,\ ) - \pi_{0,\diamond}(\ ,\ )\|_{-\frac{1}{2}-2\kappa}\\
&+\|\pi_{0,\diamond}(\ ,\ ) - \pi_{0,\diamond}(\ ,\ )\|_{-2\kappa}\big]\\
&+\|\ -\ \|_{C_T^{\frac{1}{8}}\mathcal{C}^{\frac{1}{4}-2\kappa}}.
\end{aligned}$$

By Appendix B we can find a subsequence of ε going to zero such that for any $T > 0$ $\lim_{\varepsilon\to0}\delta\bar{C}^\varepsilon_W(T) = 0, \lim_{\varepsilon\to0}\int_0^T \ ds$ exists P-a.s.. Here and in the following for simplicity we still use the notation ε to denote this subsequence. Set

$$\Omega_0 = \{\lim_{\varepsilon\to0}\delta\bar{C}^\varepsilon_W(T) = 0, C_W(T) < \infty, \lim_{\varepsilon\to0}\int_0^T \ ds \text{ exists, for any } T > 0\}.$$

Then $P(\Omega_0) = 1$.

Lemma 5.1. Φ *satisfies (5.1) in the analytically weak sense on* Ω_0.

Proof. First we prove the following:

$$\lim_{\varepsilon\to0}\int_0^t[(\rho_\varepsilon*\Phi)^3 - (3\bar{C}^\varepsilon_0\rho_\varepsilon*\Phi - 9\tilde{C}^\varepsilon_1\Phi - m\Phi)]ds = \lim_{\varepsilon\to0}\int_0^t[(\bar{\Phi}^\varepsilon)^3 - (3\bar{C}^\varepsilon_0 - 9\bar{C}^\varepsilon_1 - m)\bar{\Phi}^\varepsilon]ds. \tag{5.3}$$

In fact,

$$\begin{aligned}
&\int_0^t[(\bar{\Phi}^\varepsilon)^3 - (3\bar{C}^\varepsilon_0 - 9\bar{C}^\varepsilon_1 - m)\bar{\Phi}^\varepsilon]ds\\
=&\int_0^t[(\bar{\Phi}^\varepsilon_3)^3 + 3(\ -\ )(\bar{\Phi}^\varepsilon_3)^2 + (3(\ )^2 - 6\ \ )\bar{\Phi}^\varepsilon_3 + 3\ \diamond\bar{\Phi}^\varepsilon_3 \qquad (5.4)\\
&+\ + 3\ (\ )^2 - (\ )^3 - 3\ \diamond\ - (9\bar{\varphi}^\varepsilon - m)\bar{\Phi}^\varepsilon]ds,
\end{aligned}$$

and

$$\int_0^t [(\rho_\varepsilon * \Phi)^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \Phi - 9\tilde{C}_1^\varepsilon \Phi - m\Phi)]ds$$
$$= \int_0^t [(\rho_\varepsilon * \Phi_3)^3 + 3(\quad - \quad)(\rho_\varepsilon * \Phi_3)^2 + (3(\quad)^2 - 6 \quad \quad)\rho_\varepsilon * \Phi_3 + 3 \quad \diamond \rho_\varepsilon * \Phi_3$$
$$+ \quad + 3 \quad (\quad)^2 - (\quad)^3 - 3 \quad \diamond \quad - (9\tilde{\varphi}^\varepsilon - m)\Phi]ds, \tag{5.5}$$

where

$$\quad \diamond \quad := \quad \quad - 3(\tilde{C}_1^\varepsilon + \tilde{\varphi}^\varepsilon) \quad ,$$
$$\quad \diamond \rho_\varepsilon * \Phi_3 := \rho_\varepsilon * \Phi_3 \quad + 3(\tilde{C}_1^\varepsilon + \tilde{\varphi}^\varepsilon)(- \quad + \Phi_3),$$

and the other terms containing $\diamond$ and $\bar{\varphi}^\varepsilon$ are defined in Appendix B and Φ_3 satisfies equation (2.1). Now we only need to prove that each term converges. First we check the relations between , $\rho_\varepsilon * \Phi_3$ and , $\bar{\Phi}_3^\varepsilon$. We have that on Ω_0 for any $T > 0$ and $\epsilon > 0$ small enough

$$\sup_{t\in[0,T]} \| \quad - \quad \|_{\frac{1}{2}-2\kappa-\epsilon} \leq \sup_{t\in[0,T]} (\| \quad - \quad \|_{\frac{1}{2}-2\kappa} + \| \quad - \quad \|_{\frac{1}{2}-2\kappa-\epsilon}) \to 0.$$

Now we consider $\rho_\varepsilon * \Phi_3 - \bar{\Phi}_3$. We define $\bar{C}_W^\varepsilon(T,\omega)$ for (5.2) similarly as $C_W(T,\omega)$ in (2.2) and we have that for $\omega \in \Omega_0$, there exists a constant $C_1(T,\omega)$ such that $\bar{C}_W^\varepsilon(T,\omega) \leq C_1(T,\omega)$ for the subsequence of ε. Since $\bar{\Phi}_3^\varepsilon$ satisfies a similar equation as Φ_3, by a similar argument as in Proposition 2.1 we obtain that

$$\sup_{t\in[0,T]} [t^{\frac{\gamma+z+\kappa}{2}}(\|\Phi_3\|_\gamma + \|\bar{\Phi}_3^\varepsilon\|_\gamma) + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}(\|\Phi_3\|_{\frac{1}{2}+4\kappa} + \|\bar{\Phi}_3^\varepsilon\|_{\frac{1}{2}+4\kappa})] \leq C(T,\omega,\|\Phi_0\|_{-z}).$$

Then a similar argument as in Proposition 3.1 yields that on Ω_0

$$\sup_{t\in[0,T]} [t^{\frac{\gamma+z+\kappa}{2}}\|\Phi_3 - \bar{\Phi}_3^\varepsilon\|_\gamma + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\Phi_3 - \bar{\Phi}_3^\varepsilon\|_{\frac{1}{2}+4\kappa} + t^{\frac{3(\gamma+z+\kappa)}{2}}\|\Phi^\sharp - \bar{\Phi}^{\varepsilon,\sharp}\|_{1+3\kappa}] \to 0,$$

which combined with the fact the $\|\rho_\varepsilon * \Phi_3 - \Phi_3\|_{\beta-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}}\|\Phi_3\|_\beta$ implies that on Ω_0 for $\epsilon > 0$ small enough

$$\begin{aligned} \sup_{t\in[0,T]} [& t^{\frac{\gamma+z+\kappa}{2}}\|\rho_\varepsilon * \Phi_3 - \bar{\Phi}_3^\varepsilon\|_{\gamma-\epsilon} + t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|\rho_\varepsilon * \Phi_3 - \bar{\Phi}_3^\varepsilon\|_{\frac{1}{2}+4\kappa-\epsilon} \\ &+ t^{\frac{3(\gamma+z+\kappa)}{2}}\|\rho_\varepsilon * \Phi^\sharp - \bar{\Phi}^{\varepsilon,\sharp}\|_{1+3\kappa-\epsilon}] \to 0. \end{aligned} \tag{5.6}$$

Hence by Lemma A.2 we obtain that the terms which do not need to be renormalized in (5.4) and (5.5) converge. Now we concentrate on the renormalization terms. For the renormalized terms , $\diamond$, $(\quad)^2$ by Lemmas A.2, A.3 it is sufficient to consider the following terms: Since $\delta\bar{C}_W^\varepsilon \to 0$ on Ω_0, we have on Ω_0

$$\pi_0(\quad , \quad) - \pi_0(\quad , \quad) \to 0 \quad \text{in } C_T\mathcal{C}^{-2\kappa},$$

and

$$\pi_{0,\diamond}(\quad , \quad) - \pi_{0,\diamond}(\quad , \quad) \to 0 \quad \text{in } C_T\mathcal{C}^{-\frac{1}{2}-2\kappa}.$$

Now we focus on the convergence of $\diamond\, \rho_\varepsilon * \Phi_3$. It is sufficient to consider $\pi_{0,\diamond}(\rho_\varepsilon * \Phi_3, \;) := \pi_0(\rho_\varepsilon * \Phi_3, \;) + 3(- \; + \Phi_3)\pi_{0,\diamond}(\; , \;)$. We have

$$\Phi_3 = -3\pi_<(- \; + \Phi_3, \;) + \Phi^\sharp.$$

Then we obtain that

$$\pi_0(\rho_\varepsilon * \Phi_3, \;) = -3\pi_0(\rho_\varepsilon * \pi_<(- \; + \Phi_3, \;), \;) + \pi_0(\rho_\varepsilon * \Phi^\sharp, \;).$$

For the second term we can easily obtain the convergence by (5.6). For the first term we have

$$\begin{aligned}
&\pi_0(\rho_\varepsilon * \pi_<(- \; + \Phi_3, \;), \;) \\
=&\pi_0(\rho_\varepsilon * \pi_<(- \; + \Phi_3, \;), \;) - \pi_0(\pi_<(- \; + \Phi_3, \;), \;) \\
&+ C(- \; + \Phi_3, \; , \;) + (- \; + \Phi_3)\pi_0(\; , \;),
\end{aligned}$$

where the first two terms converge to zero as $\varepsilon \to 0$ by Lemma A.5 and the third term converges to the corresponding term by Lemma A.3 and the last term should be renormalized and converges to the corresponding term on Ω_0. Since $\lim_{\varepsilon\to 0}\int_0^t \; ds$ exists, combining the above arguments (5.3) follows. Moreover, on Ω_0 we know that for any $t > 0$,

$$\bar\Phi^\varepsilon(t) = \Phi_0 + \int_0^t \Delta\bar\Phi^\varepsilon ds - \int_0^t [(\bar\Phi^\varepsilon)^3 - (3\bar C_0^\varepsilon - 9\bar C_1^\varepsilon - m)\bar\Phi^\varepsilon] ds + \rho_\varepsilon * W(t).$$

Then taking the limit on both sides we obtain the result. □

Proof of Theorem 1.2. The idea is to prove that the drift term in (5.1) is the zero-energy part in the Fukushima decomposition (cf. [14, Theorem 5.2.2]). In the proof we take the space of continuous paths $C([0,\infty); E)$ as the sample paths $\bar\Omega$ and we denote the t-th coordinate of the path ω by $\bar X_t(\omega)$. For $t \in [0,\infty)$ let $(\bar{\mathcal{F}}_t)$ be the natural filtration for $\bar X$ given in [34, Chapter IV, (1.7)]. Set $\bar{\mathcal{F}} := \cup_{t\geq 0}\bar{\mathcal{F}}_t$ and define on $\bar\Omega$

$$P^x(\bar X \in A) := P(\Phi(x) \in A),$$

for $A \in \mathcal{B}(\bar\Omega)$. Here for $x \in \mathcal{C}^{-z}$, Φ on the right hand side is the solution from Section 2 starting from x. Then for $x \in \mathcal{C}^{-z}$ under P^x, $\bar X$ is the solution to (1.2) starting from x. For $x \in E\backslash\mathcal{C}^{-z}$ the process $\bar X$ is a trap under P^x as in Section 4. Let θ be the associated shift operator. By Theorem D.4 the (Markov) diffusion process $(\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t>0}, \theta_t, \bar X, P^x)_{x\in E}$ is properly associated with $(\mathcal{E}, D(\mathcal{E}))$. Define

$$\Omega_1 := \{\omega : \lim_{\varepsilon\to 0}\int_0^\cdot \langle(\rho_\varepsilon * \bar X)^3 - (3\bar C_0^\varepsilon\rho_\varepsilon * \bar X - 9\tilde C_1^\varepsilon\bar X - m\bar X), \varphi\rangle dr \text{ exists in } C([0,\infty);\mathbb{R}), \forall\varphi\in\mathcal{D}\}.$$

and for $\varphi \in C^\infty(\mathbb{T}^3)$,

$$H_t^\varphi := \begin{cases} \lim_{\varepsilon\to 0}\int_0^t \langle(\rho_\varepsilon * \bar X)^3 - (3\bar C_0^\varepsilon\rho_\varepsilon * \bar X - 9\tilde C_1^\varepsilon\bar X - m\bar X), \varphi\rangle ds, & \text{for } \omega\in\Omega_1 \\ 0 & \text{otherwise.}\end{cases}$$

Now we would like to check that H_t^φ is an additive functional (AF) in the sense of [14, Section 5.1]:

i) It's obvious that H_t^φ is $\bar{\mathcal{F}}_t$-measurable;

ii) For $\omega \in \Omega$, $H_\cdot^\varphi(\omega)$ is continuous, $H_0(\omega) = 0$. Since $P^x(\bar{X} \in C([0,\infty); \mathcal{C}^{-z})) = 1$ for $x \in \mathcal{C}^{-z}$ and $\mu(\mathcal{C}^{-z}) = 1$, it is sufficient to check that for $x \in \mathcal{C}^{-z}$ $P^x(\Omega_1) = 1$, $\theta_t\Omega_1 \subset \Omega_1$, and for $\omega \in \Omega_1$

$$H_{t+s}^\varphi(\omega) = H_t^\varphi(\omega) + H_s^\varphi(\theta_t\omega). \tag{5.7}$$

$P(\Omega_0) = 1$ implies that $P^x(\Omega_1) = 1$ by Lemma 5.1. Since $\bar{X}(t+s) = \bar{X}(s) \circ \theta_t$, we can easily deduce that $\theta_t\Omega_1 \subset \Omega_1$ and that

$$\begin{aligned}
&\int_0^{t+s} \langle (\rho_\varepsilon * \bar{X})^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \bar{X} - 9\tilde{C}_1^\varepsilon \bar{X} - m\bar{X}), \varphi\rangle dr \\
=& \int_0^{t} \langle (\rho_\varepsilon * \bar{X})^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \bar{X} - 9\tilde{C}_1^\varepsilon \bar{X} - m\bar{X}), \varphi\rangle dr \\
+& \int_0^{s} \langle (\rho_\varepsilon * \bar{X})^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \bar{X} - 9\tilde{C}_1^\varepsilon \bar{X} - m\bar{X}), \varphi\rangle dr \circ \theta_t,
\end{aligned}$$

which implies that (5.7) holds for $\omega \in \Omega_1$.

Now we know that H_t^φ is an AF. Define

$$M_t^\varphi := \langle \bar{X}(t) - \bar{X}(0), \varphi\rangle - \int_0^t \langle \bar{X}, \Delta\varphi\rangle ds + H_t^\varphi.$$

We know that M^φ is also an AF, since $\langle \bar{X}(t) - \bar{X}(0), \varphi\rangle$ and $\int_0^t \langle \bar{X}, \Delta\varphi\rangle ds$ are AFs (see [14, Section 5.2]. Moreover, by Lemma 5.1 we have

$$E^x M_t^\varphi = 0, \quad E^x(M_t^\varphi)^2 = |\varphi|^2 t < \infty,$$

which implies that M^φ is also a martingale additive functional (MAF) in the sense of [14, Chapter V]. Here $|\cdot|$ denotes the L^2-norm.

Let us fix an arbitrary $T > 0$ and consider the space Ω_T of all continuous paths from $[0,T]$ to E. We also use $(\bar{\mathcal{F}}_t)$ to denote the natural filtration generated by the canonical process. We introduce the time reversal operator r_T on Ω_T defined by

$$r_T\omega(t) = \omega(T-t), \quad 0 \le t \le T, \omega \in \Omega_T.$$

By [14, Lemma 5.7.1] and the symmetry of the semigroup $\bar{P}_t$ we have that for any $\bar{\mathcal{F}}_T$-measurable set A on Ω_T

$$P^\mu(r_T\omega \in A) = P^\mu(A), \tag{5.8}$$

where $P^\mu = \int P^x \mu(dx)$. Now we have

$$\langle \bar{X}(t) - \bar{X}(0), \varphi\rangle = M_t^\varphi + \bar{H}_t^\varphi \quad P^\mu - a.s.,$$

with $\bar{H}_t^\varphi = \int_0^t \langle \bar{X}, \Delta\varphi\rangle ds - H_t^\varphi$. By (5.8) we have for $0 \le t \le T$

$$\langle \bar{X}(T-t) - \bar{X}(T), \varphi\rangle = M_t^\varphi(r_T) + \bar{H}_t^\varphi(r_T) \quad P^\mu - a.s.. \tag{5.9}$$

Moreover, under P^μ,

$$\begin{aligned}
&\bar{H}_t^\varphi(r_T) \\
=& \int_0^t \langle \bar{X} \circ r_T, \Delta\varphi\rangle ds - \lim_{\varepsilon\to 0} \int_0^t \langle (((\rho_\varepsilon * \bar{X}) \circ r_T)^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \bar{X} - 9\tilde{C}_1^\varepsilon \bar{X} - m\bar{X}) \circ r_T, \varphi\rangle ds
\end{aligned}$$

$$=\int_{T-t}^{T}\langle \bar{X},\Delta\varphi\rangle ds-\lim_{\varepsilon\to 0}\int_{T-t}^{T}\langle(\rho_\varepsilon*\bar{X})^3-(3\bar{C}_0^\varepsilon\rho_\varepsilon*\bar{X}-9\tilde{C}_1^\varepsilon\bar{X}-m\bar{X}),\varphi\rangle ds$$
$$=\bar{H}_T^\varphi-\bar{H}_{T-t}^\varphi. \tag{5.10}$$

By (5.9), (5.10) we have

$$M_t^\varphi(r_T)=\langle\bar{X}(T-t)-\bar{X}(T),\varphi\rangle-\bar{H}_T^\varphi+\bar{H}_{T-t}^\varphi,$$

which implies that

$$\begin{aligned}&M_{T-t}^\varphi(r_T)-M_T^\varphi(r_T)\\ =&\langle\bar{X}(t)-\bar{X}(T),\varphi\rangle-\bar{H}_T^\varphi+\bar{H}_t^\varphi-\langle\bar{X}(0)-\bar{X}(T),\varphi\rangle+\bar{H}_T^\varphi\\ =&2\langle\bar{X}(t)-\bar{X}(0),\varphi\rangle-M_t^\varphi.\end{aligned}$$

Now we know that

$$\langle\bar{X}(t)-\bar{X}(0),\varphi\rangle=\frac{1}{2}(M_t^\varphi-M_t^\varphi\circ r_t)\quad P^\mu-a.s.\forall t>0.$$

By [13, Theorem 2.2] we have that $M^\varphi\equiv M^{[\varphi]}$, where $M^{[\varphi]}$ is the MAF from the Fukushima decomposition for $\langle\cdot,\varphi\rangle$ (see [14, Section 5.2]. Hence, we have that $\bar{H}_t^\varphi=N_t^{[\varphi]}$ is the associated zero-energy additive functional (NAF), which implies that Φ is a Dirichlet process.

Now for $f=f_1(\langle\cdot,l_1\rangle,\langle\cdot,l_2\rangle,...,\langle\cdot,l_k\rangle)$ with l_i,f_1 smooth, denote the MAF in the Fukushima decomposition associated with $\langle\cdot,l_i\rangle$ by M^{l_i}. By Itô's formula for Dirichlet process in [8, Theorem 4.7] and [39, Theorem 4.1], we have

$$\begin{aligned}f(\bar{X}(t))-f(\bar{X}(0))=&\sum_{i=1}^k\int_0^t\partial_i f(\bar{X}(s))dM_t^{l_i}+\sum_{i=1}^k\int_0^t\partial_i f(\bar{X}(s))d\bar{H}_t^{l_i}\\ &+\frac{1}{2}\sum_{i,j=1}^k\int_0^t\partial_{ij}f(\bar{X}(s))\langle l_i,l_j\rangle ds\\ :=&\sum_{i=1}^k\int_0^t\partial_i f(\bar{X}(s))dM_t^{l_i}+\bar{H}_t^f,\end{aligned}$$

where $\partial_i f:=\partial_i f_1(\langle\cdot,l_1\rangle,\langle\cdot,l_2\rangle,...,\langle\cdot,l_k\rangle)$ and the stochastic integral $\int_0^t\partial_i f(\bar{X}(s))d\bar{H}_t^{l_i}$ w.r.t. NAF is defined in [8]. We know that $\sum_{i=1}^k\int_0^t\partial_i f(\bar{X}(s))dM_t^{l_i}$ is an MAF and $\bar{H}_t^f$ is an NAF, which implies that

$$\sum_{i=1}^k\int_0^t\partial_i f(\bar{X}(s))dM_t^{l_i}\equiv M_t^{[f]}, \tag{5.11}$$

where $M_t^{[f]}$ is the MAF obtained in the Fukushima decomposition.

By (5.11) we know that

$$\mathcal{E}(f,f)=e(M_t^{[f]}):=\lim_{t\downarrow 0}\frac{1}{2t}E^\mu(M_t^{[f]})^2=\frac{1}{2}\int|Df|^2d\mu.$$

Then for $g\in\mathcal{F}C_b^\infty$ we can use the above f's to approximate it and obtain $\mathcal{E}(g,g)=\frac{1}{2}\int|Dg|^2d\mu$. □

Remark 5.2. From the above proof we can check that Φ starting from μ is an energy solution in the sense that $(\Phi,N)_{0\le t\le T}$ has continuous paths in E such that

i) the law of Φ is μ for all $t\in[0,T]$;

ii) for any test function $\varphi \in C^\infty(\mathbb{T}^3)$ the process $t \to N_t$ is a.s. of zero quadratic variation, $N_0(\varphi) = 0$ and the pair $(\Phi, N)_{0\le t\le T}$ satisfies the equation

$$\langle \Phi_t, \varphi\rangle = \langle \Phi_0, \varphi\rangle + \int_0^t \langle \Phi_s, \Delta\varphi\rangle ds + \langle N_t, \varphi\rangle + \langle M_t, \varphi\rangle,$$

where $(\langle M_t, \varphi\rangle)_{0\le t\le T}$ is a martingale with respect to the filtration generated by $(\Phi, N)_{0\le t\le T}$ with quadratic variation $|\varphi|^2 t$.

iii) the reversed processes $\hat{\Phi}_t = \Phi_{T-t}, \hat{N}_t = N_T - N_{T-t}$ satisfies the same equation with the associated martingale $\hat{M}_t$ with respect to its own filtration and the quadratic variation of $\langle \hat{M}, \varphi\rangle$ is also $|\varphi|^2 t$.

iv) $N_t = -\lim_{\varepsilon\to 0}\int_0^t [(\rho_\varepsilon * \Phi)^3 - (3\bar{C}_0^\varepsilon \rho_\varepsilon * \Phi - 9\tilde{C}_1^\varepsilon \Phi - m\Phi)]ds$ a.s. with $\bar{C}_0^\varepsilon, \tilde{C}_1^\varepsilon$ introduced at the beginning of Section 5.

The concept of energy solutions has been introduced in [16] and has been extended in [18] for the stochastic Burger's equation, which is equivalent to the KPZ equation. In their case, the energy solutions satisfy the corresponding i)-iv) with the invariant distribution given by the Gaussian white noise and N_t related to the nonlinear terms in the Burger's equation. The main difference in the definition is that $\hat{N}_t$ should be given by $-(N_T - N_{T-t})$, since their case are non-reversible while in our case the process is reversible.

Proof of Theorem 1.3. By Theorem 1.2 we know that for $f, g \in \mathcal{F}C_b^\infty$ as elements in $L^2(E, \mu)$, $\int \langle Df, Dg\rangle d\mu$ is well defined, which implies that $(\bar{\mathcal{E}}, \mathcal{F}C_b^\infty)$ is a well-defined symmetric bilinear form. Since the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is an extension of $(\bar{\mathcal{E}}, \mathcal{F}C_b^\infty)$, it is obvious that $(\bar{\mathcal{E}}, \mathcal{F}C_b^\infty)$ is closable. We denote its closure by $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$. Then by similar arguments as in [34, Chapter II Proposition 3.5] we obtain that for $u \in D(\bar{\mathcal{E}})$, $v = u \vee 0 \wedge 1 \in D(\bar{\mathcal{E}})$ and $\bar{\mathcal{E}}(v, v) \le \bar{\mathcal{E}}(u, u)$. Moreover, by similar arguments as in the proof of [34, Chapter IV Proposition 4.2] i) in Definition D.1 follows, which implies that $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ is a quasi-regular Dirichlet form (cf. Definition D.1). Then existence of the Markov process follows from Theorem D.4. □

Proof of Corollary 1.5. By general theory of Markov semigroup and Dirichlet form (cf. [48]) we know the following Poincaré inequality holds:

$$\mu(f^2) \le C\mathcal{E}(f, f) + \mu(f)^2, \quad f \in D(\mathcal{E}) \tag{5.12}$$

for some $C > 0$. In the following we follows essentially the same argument from [48, Section 1.2] to deduce (1.4). Since

$$\|x\|_E^2 = \sum_k \lambda_k \langle x, \hat{e}_k\rangle^2,$$

where $\lambda_k \in \mathbb{R}$ satisfies $\lambda_k \to 0, k \to \infty$ and $\{\hat{e}_k\}$ is a real smooth eigenbasis on $L^2(\mathbb{T}^3)$. We first prove that for $r \ge 0, n \in \mathbb{N}$, $f_n(\cdot) := e^{\frac{r}{2}((\sum_k \lambda_k \langle\cdot, \hat{e}_k\rangle^2 + 1)^{\frac{1}{2}} \wedge n)} \in D(\mathcal{E})$. By approximation we can easily check that $f_{n,N} := e^{\frac{r}{2}((\sum_{|k|_\infty \le N} \lambda_k \langle\cdot, \hat{e}_k\rangle^2 + 1)^{\frac{1}{2}} \wedge n)} \in D(\mathcal{E})$. Moreover, by direct computation we know that

$$\mathcal{E}_1(f_{n,N} - f_n, f_{n,N} - f_n) \to 0, \quad N \to \infty,$$

with $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_{L^2(E;\mu)}$. We also have

$$\mathcal{E}(f_{n,N}, f_{n,N}) \le \frac{r^2}{4}\int f_{n,N}^2 d\mu,$$

which implies the following by letting $N \to \infty$

$$\mathcal{E}(f_n, f_n) \le \frac{r^2}{4} \int f_n^2 d\mu.$$

Let $h_n(r) := \mu(f_n^2)$. By (5.12) we know that

$$h_n(r) \le \frac{Cr^2}{4} h_n(r) + h_n(r/2)^2.$$

Thus, for any $r \in (0, 2/\sqrt{C})$ we have

$$h_n(r) \le \frac{4}{4 - Cr^2} h_n(r/2)^2. \tag{5.13}$$

Next, for any $m > 0$, let $p_m = \mu(x : (\sum_k \lambda_k \langle x, \hat{e}_k \rangle^2 + 1)^{1/2} \ge m)$. We have

$$h_n(r/2)^2 \le \left[e^{mr/2} + \mu(1_{\{(\sum_k \lambda_k \langle x, \hat{e}_k \rangle^2 + 1)^{1/2} \ge m\}} f_n) \right]^2 \le 2e^{mr} + 2p_m h_n(r).$$

Substituting this into (5.13) we have

$$h_n(r) \le \frac{8}{4 - Cr^2} e^{mr} + \frac{8}{4 - Cr^2} p_m h_n(r), \quad 0 < r < 2/\sqrt{C}.$$

By Lemma 3.2 we know that $p_m \to 0$ as $m \to \infty$, which implies that there exists $m_0 > 0$ such that $\frac{8p_{m_0}}{4 - Cr^2} \le \frac{1}{2}$. Therefore,

$$h_n(r) \le \frac{16}{4 - Cr^2} e^{m_0 r}.$$

Letting $n \to \infty$ we arrive at

$$\int e^{r\|x\|_E} \mu(dx) \le \int e^{r(\sum_k \lambda_k \langle x, \hat{e}_k \rangle^2 + 1)^{\frac{1}{2}}} \mu(dx) < \infty, \quad r \in (0, 2/\sqrt{C}). \qquad \square$$

A Besov spaces and paraproduct

In this appendix we recall the definitions and some properties of Besov spaces and paraproducts. For a general introduction to these theories we refer to [3, 17]. First we introduce the following notations. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

i. the support of χ is contained in a ball and the support of θ is contained in an annulus;

ii. $\chi(z) + \sum_{j \ge 0} \theta(2^{-j} z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j} \cdot)) = \emptyset$ for $j \ge 1$ and $\mathrm{supp}(\theta(2^{-i} \cdot)) \cap \mathrm{supp}(\theta(2^{-j} \cdot)) = \emptyset$ for $|i - j| > 1$.

We call such a pair (χ, θ) a dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [3, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j} \cdot) \mathcal{F} u).$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [46]). More precisely, Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}} := \Big(\sum_{j \ge -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q\Big)^{1/q}.$$

We will need the following Besov embedding theorem on the torus (c.f. [17, Lemma 41]):

Lemma A.1. *i) Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$, and let $\alpha \in \mathbb{R}$. Then $B^{\alpha}_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.*

ii) (Besov embedding [47, Chapter 6]) Let $\alpha_1 < \alpha_2$, $1 \le p_1 \le p_2 \le \infty$, and $1 \le q_1 \le q_2 \le \infty$. Then

$$B^{\alpha_2}_{p_1,q_2}(\mathbb{T}^d) \subset B^{\alpha_1}_{p_1,q_1}(\mathbb{T}^d); \quad B^{\alpha_1}_{p_1,q_1}(\mathbb{T}^d) \subset B^{\alpha_1}_{p_1,q_2}(\mathbb{T}^d), \quad B^{\alpha_1}_{p_2,q_1}(\mathbb{T}^d) \subset B^{\alpha_1}_{p_1,q_1}(\mathbb{T}^d).$$

iii) ([36, Remarks 3.5, 3.6]) For $p > 1$

$$B^0_{p,1}(\mathbb{T}^d) \subset L^p \subset B^0_{p,\infty}(\mathbb{T}^d).$$

Now we recall the following paraproduct introduced by Bony (see [5]). In general, the product fg of two distributions $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley blocks, the product fg can be formally decomposed as

$$fg = \sum_{j\ge -1}\sum_{i\ge -1} \Delta_i f \Delta_j g = \pi_<(f,g) + \pi_0(f,g) + \pi_>(f,g),$$

with

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{j\ge -1}\sum_{i<j-1} \Delta_i f \Delta_j g, \quad \pi_0(f,g) = \sum_{|i-j|\le 1} \Delta_i f \Delta_j g.$$

The basic result about these bilinear operations is given by the following estimates:

Lemma A.2 (Paraproduct estimates, [5, 37, Proposition A.7])**.** *Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then we have

$$\|\pi_<(f,g)\|_{B^\beta_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^\beta_{p_2,q}} \quad f \in L^{p_1}, g \in B^\beta_{p_2,q},$$

and for $\alpha < 0$, furthermore,

$$\|\pi_<(f,g)\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^\alpha_{p_1,q}} \|g\|_{B^\beta_{p_2,q}} \quad f \in B^\alpha_{p_1,q}, g \in B^\beta_{p_2,q}.$$

For $\alpha + \beta > 0$ we have

$$\|\pi_0(f,g)\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^\alpha_{p_1,q}} \|g\|_{B^\beta_{p_2,q}} \quad f \in B^\alpha_{p_1,q}, g \in B^\beta_{p_2,q}.$$

The following basic commutator lemma is important for our use:

Lemma A.3 ([17, Lemma 5], [37, Proposition A.9])**.** *Assume that $\alpha \in (0,1), \beta, \gamma \in \mathbb{R}$ and $p, p_1, p_2, p_3 \in [1, \infty]$ are such that*

$$\alpha + \beta + \gamma > 0, \quad \beta + \gamma < 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Then for smooth f, g, h, the trilinear operator

$$C(f,g,h) = \pi_0(\pi_<(f,g),h) - f\pi_0(g,h)$$

satisfies the bound

$$\|C(f,g,h)\|_{B^{\alpha+\beta+\gamma}_{p,\infty}} \lesssim \|f\|_{B^\alpha_{p_1,\infty}} \|g\|_{B^\beta_{p_2,\infty}} \|h\|_{B^\gamma_{p_3,\infty}}.$$

Thus, C can be uniquely extended to a bounded trilinear operator from $B^\alpha_{p_1,\infty} \times B^\beta_{p_2,\infty} \times B^\gamma_{p_3,\infty}$ to $B^{\alpha+\beta+\gamma}_{p,\infty}$.

Now we recall the following estimate for the heat semigroup $P_t := e^{t\Delta}$.

Lemma A.4 ([17, Lemma 47], [37, Proposition A.13]). *Let $u \in B^{\alpha}_{p,q}$ for some $\alpha \in \mathbb{R}, p, q \in [1,\infty]$. Then for every $\delta \geq 0$*

$$\|P_t u\|_{B^{\alpha+\delta}_{p,q}} \lesssim t^{-\delta/2}\|u\|_{B^{\alpha}_{p,q}}.$$

Lemma A.5 ([7, Lemma A.1]). *Let $\alpha < 1$ and $\beta \in \mathbb{R}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, let $u \in \mathcal{C}^{\alpha}$, and $v \in \mathcal{C}^{\beta}$. Then for every $\varepsilon > 0$ and every $\delta \geq -1$ we have*

$$\|\varphi(\varepsilon D)\pi_<(u,v) - \pi_<(u,\varphi(\varepsilon D)v)\|_{\alpha+\beta+\delta} \lesssim \varepsilon^{-\delta}\|u\|_{\alpha}\|v\|_{\beta}.$$

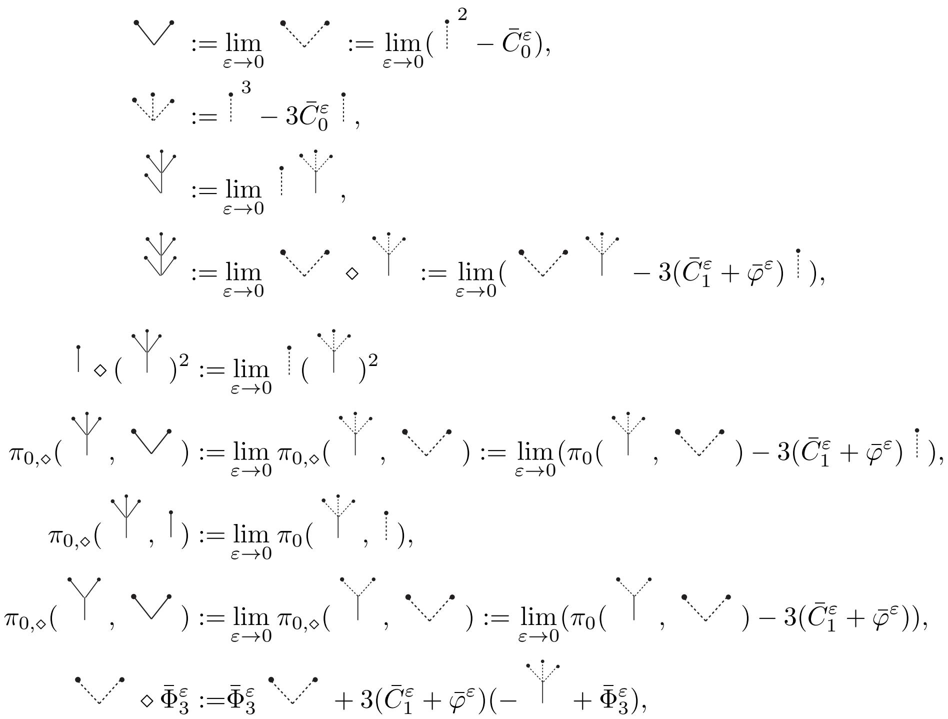
where $\varphi(D)u = \mathcal{F}^{-1}(\varphi\mathcal{F}u)$.

Lemma A.6 ([7, Lemma 2.5], [37, Proposition A.13]). *Let $u \in B^{\alpha+\delta}_{p,q}$ for some $\alpha \in \mathbb{R}, 0 \leq \delta \leq 2, p, q \in [1,\infty]$. Then for every $t \geq 0$*

$$\|(P_t - I)u\|_{B^{\alpha}_{p,q}} \lesssim t^{\delta/2}\|u\|_{B^{\alpha+\delta}_{p,q}}.$$

B Convergence of the stochastic terms

We first recall the definition of the stochastic terms from [7] we use in the paper:



where $\bar{C}^{\varepsilon}_0, \bar{C}^{\varepsilon}_1, \bar{\varphi}^{\varepsilon}$ are terms for renormalization and are defined in [7]. Here the notations $\diamond$ are not the usual Wick products and we used them for the reromalization terms. We do not recall the explicit formula of them since this is not used in our paper. The convergence above is in the corresponding space (see (2.2)). The convergence of $\delta\bar{C}^{\varepsilon}_W \to 0$ can be obtained partially from [7] and a similar argument as in [7]. In this part we consider the convergence of $\int_0^T$  ds. We follow the notations from [19, Section 9]. We represent the white noise in terms of its spatial Fourier transform. More precisely, let $E_0 = \mathbb{Z}^3\backslash\{0\}$ and let $W(s,k) = \langle W(s), e_k\rangle$ and we view $W(s,k)$ as a Gaussian process on $\mathbb{R} \times E$ with covariance given by

$$E\left[\int_{\mathbb{R}\times E_0} f(\eta)W(d\eta)\int_{\mathbb{R}\times E_0} g(\eta')W(d\eta')\right] = \int_{R\times E_0} g(\eta_1)f(\eta_{-1})d\eta_1,$$

where $\eta_a = (s_a, k_a)$, $s_{-a} = s_a, k_{-a} = -k_a$ and the measure $d\eta_a = ds_a dk_a$ is the product of the Lebesgue measure ds_a on $\mathbb{R}$ and of the counting measure dk_a on E_0. Denote by

$$\int_{(\mathbb{R}\times E_0)^n} f(\eta_{1\dots n}) W(d\eta_{1\dots n})$$

a generic element of the n-th chaos of W on $\mathbb{R} \times E_0$. Recall that

$$\int_0^t \quad d\sigma = 2^{-3} \int_{(\mathbb{R}\times E)^3} e_{k_{[123]}} \int_0^t P^\varepsilon_{\sigma-s_1}(k_1) P^\varepsilon_{\sigma-s_2}(k_2) P^\varepsilon_{\sigma-s_3}(k_3) d\sigma W(d\eta_{123}).$$

Here $P^\varepsilon_t(k) = e^{-|k|^2 t\pi^2} 1_{\{t\geq 0\}} g(\varepsilon k)$ and $k_{[123]} = k_1 + k_2 + k_3$. By a straightforward calculation we obtain that

$$\begin{aligned}
& E|\Delta_q(\int_s^t (\bar{\Phi}^{\varepsilon_1}_1)^{\diamond,3} d\sigma - \int_s^t (\bar{\Phi}^{\varepsilon_2}_1)^{\diamond,3} d\sigma)|^2 \\
\lesssim & \int_{(\mathbb{R}\times E)^2} \theta(2^{-q} k_{[123]})^2 \left| \int_s^t [\Pi_{i=1}^3 P^{\varepsilon_1}_{\sigma-s_i}(k_i) - \Pi_{i=1}^3 P^{\varepsilon_2}_{\sigma-s_i}(k_i)] d\sigma \right|^2 d\eta_{123} \\
\lesssim & (\varepsilon_1^\kappa + \varepsilon_2^\kappa) \int \theta(2^{-q} k_{[123]})^2 \int_s^t \int_s^t \frac{e^{-\pi^2(|k_1|^2+|k_2|^2+|k_3|^2)|\sigma-\bar{\sigma}|} \sum_{i=1}^3 |k_i|^\kappa}{|k_1|^2 |k_2|^2 |k_3|^2} d\sigma d\bar{\sigma} dk_{123} \\
& \lesssim (\varepsilon_1^\kappa + \varepsilon_2^\kappa) \int \theta(2^{-q} k_{[123]}) \frac{|t-s| \sum_{i=1}^3 |k_i|^\kappa}{|k_1|^2 |k_2|^2 |k_3|^2 [|k_1|^2 + |k_2|^2 + |k_3|^2]} dk_{123} \\
& \lesssim (\varepsilon_1^\kappa + \varepsilon_2^\kappa) \int_E \theta(2^{-q} k) \frac{|t-s|}{|k|^{2-\kappa}} dk \lesssim (\varepsilon_1^\kappa + \varepsilon_2^\kappa) 2^{q(1+\kappa)} |t-s|.
\end{aligned}$$

Then by Gaussian hypercontractivity and Lemma A.1 we obtain that for any $\delta > 0, p > 1$, $\int_0^t \quad ds$ converges in $L^p(\Omega; C_T \mathcal{C}^{-\frac{1+\delta}{2}})$.

C Paracontrolled analysis for the solution to the lattice approximation

In this appendix we recall paracontrolled analysis for the solution to (3.4) in [50]. To avoid confusion we do not use the graph notation for the lattice approximation in this paper. For the graph notation for u^ε we refer to [50]. We define

$$K^\varepsilon(t) := \int_0^t P^\varepsilon_{t-s} (u^\varepsilon_1)^{\diamond,2} ds, \quad \tilde{K}^\varepsilon(t) := \int_0^t \tilde{P}^\varepsilon_{t-s} (u^\varepsilon_1)^{\diamond,2} ds,$$

and

$$K^\varepsilon_1(t) := \int_0^t P^\varepsilon_{t-s} [e_N^{i_1 i_2 i_3} (u^\varepsilon_1)^{\diamond,2}] ds, \quad \tilde{K}^\varepsilon_1(t) := \int_0^t \tilde{P}^\varepsilon_{t-s} [e_N^{i_1 i_2 i_3} (u^\varepsilon_1)^{\diamond,2}] ds,$$

with

$$\tilde{P}^\varepsilon_t := \mathcal{F}^{-1} e^{-t|k|^2 f(\varepsilon k)} \varphi_0(\varepsilon k) \mathcal{F},$$

where φ_0 is a smooth function and equals to 1 on $\{|x|_\infty \leq 1\}$ with supp$\varphi_0 \subset \{|x| \leq 1.8\}$ and for $k = (k^1, k^2, k^3) \in \mathbb{R}^3$

$$f(k) = \frac{4}{|k|^2} (\sin^2 \frac{k^1 \pi}{2} + \sin^2 \frac{k^2 \pi}{2} + \sin^2 \frac{k^3 \pi}{2}).$$

Then we write the paracontrolled ansatz for the solution to (3.4) as follows:

$$u^\varepsilon_3 = -3 P_N [\pi_<(u^\varepsilon_2 + u^\varepsilon_3, \tilde{K}^\varepsilon + \tilde{K}^\varepsilon_1)] + u^{\varepsilon,\sharp}$$

with $u^{\varepsilon,\sharp}(t) \in \mathcal{C}^{1+3\kappa}$. Now we introduce the stochastic terms for the lattice approximation: for $T > 0$

$$\begin{aligned} C^\varepsilon_W(T) := \sup_{t\in[0,T]} \big[&\|u^\varepsilon_1\|_{-\frac12-2\kappa} + \|(u^\varepsilon_1)^{\diamond,2}\|_{-1-2\kappa} + \|u^\varepsilon_2\|_{\frac12-2\kappa} + \|\pi_0(u^\varepsilon_2,u^\varepsilon_1)\|_{-2\kappa} \\ &+ \|\pi_{0,\diamond}(u^\varepsilon_2,(u^\varepsilon_1)^{\diamond,2})\|_{-\frac12-2\kappa} + \|\pi_{0,\diamond}(K^\varepsilon,(u^\varepsilon_1)^{\diamond,2})\|_{-2\kappa}\big] + \|u^\varepsilon_2\|_{C_T^{\frac18}\mathcal{C}^{\frac14-2\kappa}}, \end{aligned}$$

$$\begin{aligned} &E^\varepsilon_W(T) \\ &:= \sup_{t\in[0,T]} \big[\|(u^\varepsilon_1)^{\diamond,2} e^{i_1i_2i_3}_N\|_{-1-2\kappa} + \|\pi_0(u^\varepsilon_2, e^{i_1i_2i_3}_N u^\varepsilon_1)\|_{-2\kappa} + \|\pi_{0,\diamond}(u^\varepsilon_2, e^{i_1i_2i_3}_N (u^\varepsilon_1)^{\diamond,2})\|_{-\frac12-2\kappa} \\ &\quad + \|\pi_0(K^\varepsilon, e^{i_1i_2i_3}_N (u^\varepsilon_1)^{\diamond,2})\|_{-2\kappa} + \|\pi_0(K^\varepsilon_1,(u^\varepsilon_1)^{\diamond,2})\|_{-2\kappa} + \|\pi_{0,\diamond}(K^\varepsilon_1, e^{i_1i_2i_3}_N (u^\varepsilon_1)^{\diamond,2})\|_{-2\kappa}\big], \end{aligned}$$

and

$$\begin{aligned} \delta C^\varepsilon_W(T) := \sup_{t\in[0,T]} \big[&\|u^\varepsilon_1 - \ \|_{-\frac12-2\kappa} + \|(u^\varepsilon_1)^{\diamond,2} - \ \|_{-1-2\kappa} + \|u^\varepsilon_2 + \ \|_{\frac12-2\kappa} \\ &+ \|\pi_0(u^\varepsilon_2,u^\varepsilon_1) + \pi_{0,\diamond}(\ ,\ )\|_{-2\kappa} + \|\pi_{0,\diamond}(u^\varepsilon_2,(u^\varepsilon_1)^{\diamond,2}) + \pi_{0,\diamond}(\ ,\ )\|_{-\frac12-2\kappa} \\ &+ \|\pi_{0,\diamond}(K^\varepsilon,(u^\varepsilon_1)^{\diamond,2}) - \pi_{0,\diamond}(\ ,\ )\|_{-2\kappa}\big] + \|u^\varepsilon_2 + \ \|_{C_T^{\frac18}\mathcal{C}^{\frac14-2\kappa}}. \end{aligned}$$

Here the terms containing $\diamond$ are renormlized terms defined in [50, Section 4]. Moreover, we introduce the following operators

$$A^1_N(g,h)(f) := -\pi_0((I-P_N)\pi_<(f,P_N g),h),$$

and

$$A^2_N(g,h)(f) := \pi_0(P_N\pi_<(f,(P_{3N}-P_N)g),h).$$

Then we define

$$A_N(T) := \|(A^1_N + A^2_N)(\tilde K^\varepsilon + \tilde K^\varepsilon_1, (u^\varepsilon_1)^{\diamond,2} + e^{i_1i_2i_3}_N (u^\varepsilon_1)^{\diamond,2})\|_{C_T L(\mathcal{C}^{1-3\kappa},\mathcal{C}^{-\frac12-5\kappa})}$$

and

$$\begin{aligned} D_N(T) := \sup_{t\in[0,T]} (\| &-\pi_0((I-P_N)\pi_<(u^\varepsilon_2, K^\varepsilon + K^\varepsilon_1),(u^\varepsilon_1)^{\diamond,2} + e^{i_1i_2i_3}_N(u^\varepsilon_1)^{\diamond,2}) \\ &+ \pi_0(P_N\pi_<(u^\varepsilon_2,(P_{3N}-P_N)(\tilde K^\varepsilon + \tilde K^\varepsilon_1)),(u^\varepsilon_1)^{\diamond,2} + e^{i_1i_2i_3}_N (u^\varepsilon_1)^{\diamond,2})\|_{-\kappa}). \end{aligned}$$

By the calculations in [50] we obtain the following result.

Proposition C.1. *There exists $\kappa_1, C > 0$ such that*

$$E[\delta C^\varepsilon_W(T) + A_N(T) + E^\varepsilon_W(T) + D_N(T)] \le C\varepsilon^{\kappa_1}.$$

Moreover, by a similar argument as in [36, Lemma A.6] we obtain the following estimate on the extension operator defined in (3.2):

Lemma C.2. *Let f be a function on Λ_ε. Then we have*

$$\|\mathit{Ext}f\|_{L^{2n}(\mathbb{T}^3)} \lesssim N^{\frac{3}{2n}}\|f\|_{L^{2n}(\Lambda_\varepsilon)},$$

where the implicit constant depends on n.

Proof. By (3.2) we have

$$\mathrm{Ext}f(x)=\sum_{z\in\Lambda_\varepsilon}\frac{\varepsilon^3}{8}f(z)\Pi_{j=1}^3\frac{\sin\frac{\pi}{2}(2N+1)(x^j-z^j)}{\sin\frac{\pi}{2}(x^j-z^j)}.$$

Then we have

$$|\mathrm{Ext}f(x)|^{2n}\lesssim\sum_{z\in\Lambda_\varepsilon}\frac{\varepsilon^3}{8}|f(z)|^{2n}[\sum_{z\in\Lambda_\varepsilon}\frac{\varepsilon^3}{8}\Pi_{j=1}^3|\frac{\sin\frac{\pi}{2}(2N+1)(x^j-z^j)}{\sin\frac{\pi}{2}(x^j-z^j)}|^{\frac{2n}{2n-1}}]^{2n-1}.$$

By the proof of [36, Lemma A.6] we obtain that

$$[\sum_{z\in\Lambda_\varepsilon}\frac{\varepsilon^3}{8}\Pi_{j=1}^3|\frac{\sin\frac{\pi}{2}(2N+1)(x^j-z^j)}{\sin\frac{\pi}{2}(x^j-z^j)}|^{\frac{2n}{2n-1}}]^{2n-1}\lesssim N^3,$$

where the implicit constant does not depend on x, which implies the result. □

D Symmetric quasi regular Dirichlet forms and Markov processes

In this section we recall some general Dirichlet form results from [34]. Let E be a Hausdorff topological space, m a σ-finite measure on E, and let $\mathcal{B}$ the smallest σ-algebra of subsets of E with respect to which all continuous functions on E are measurable. Let $\mathcal{E}$ be a symmetric Dirichlet form acting in the real $L^2(m)$-space, i.e. $\mathcal{E}$ is a positive, symmetric, bilinear, closed form with domain $D(\mathcal{E})$ dense in $L^2(m)$, and such that $\mathcal{E}(\Phi(u),\Phi(u))\leq\mathcal{E}(u,u)$, for any $u\in D(\mathcal{E})$, where $\Phi(t)=(0\vee t)\wedge 1, t\in\mathbb{R}$. The latter condition is known to be equivalent with the condition that the associated C_0-contraction semigroup $T_t, t\geq 0$, is submarkovian (i.e. $0\leq u\leq 1$ m-a.e. implies $0\leq T_tu\leq 1$ m-a.e., for all $u\in L^2(m)$); association means that $\lim_{t\downarrow 0}\frac{1}{t}\langle u-T_tu,v\rangle_{L^2(m)}=\mathcal{E}(u,v),\forall u,v\in D(\mathcal{E})$.

Definition D.1 (cf. [34, Chap. IV, Defi. 3.1])**.** *A symmetric Dirichlet form is called quasi-regular if the following holds:*

i) There exists a sequence $(F_k)_{k\in\mathbb{N}}$ of compact subsets of E such that $\cup_k D(\mathcal{E})_{F_k}$ is $\mathcal{E}_1^{1/2}$-dense in $D(\mathcal{E})$ (where $D(\mathcal{E})_{F_k}:=\{u\in D(\mathcal{E})|u=0$ m-a.e. on $E-F_k\}$; $\mathcal{E}_1^{1/2}$ is the norm given by the scalar product in $L^2(m)$ defined by $\mathcal{E}_1$, where $\mathcal{E}_1(u,v):=\mathcal{E}(u,v)+\langle u,v\rangle$, $\langle,\rangle$ being the scalar product in $L^2(m)$. Such a sequence $(F_k)_{k\in\mathbb{N}}$ is called an $\mathcal{E}$-nest.

ii) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi continuous m-versions. A real function u on E is called quasi continuous when there exists an $\mathcal{E}$-nest (F_k) s.t. u restricted to F_k is continuous.

iii) There exists $u_n\in D(\mathcal{E}), n\in\mathbb{N}$, with $\mathcal{E}$-quasi continuous m-versions $\tilde{u}_n$ and there exists an $\mathcal{E}$-exceptional subset N of E s.t. $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ separates the points of $E-N$. An $\mathcal{E}$-exceptional subset of E is a subset $N\subset\cap_k(E-F_k)$ for some $\mathcal{E}$-nest (F_k).

To recall the main results in [34] we recall the definitions of a Markov process and a right process. Here we consider only Markov processes with life time ∞.

Definition D.2 (cf. [34, Chap. IV Defi. 1.5])**.** *A collection $\mathrm{M}:=(\Omega,\mathcal{M},(X_t)_{t\geq 0},(P^z)_{z\in E})$ is called a Markov process (with state space E) if it has the following properties.*

i) There exists a filtration $(\mathcal{M}_t)$ on $(\Omega,\mathcal{M})$ such that $(X_t)_{t\geq 0}$ is an $(\mathcal{M}_t)_{t\geq 0}$ adapted stochastic process with state space E.

ii) For each $t\geq 0$ there exists a shift operator $\theta_t:\Omega\to\Omega$ such that $X_s\circ\theta_t=X_{s+t}$ for all $s,t\geq 0$

iii) $P^z, z\in E$, are probability measures on $(\Omega,\mathcal{M})$ such that $z\mapsto P^z(\Gamma)$ is $\mathcal{B}(E)^*$-measurable for each $\Gamma\in\mathcal{M}$ resp. $\mathcal{B}(E)$-measurable if $\Gamma\in\sigma\{X_s|s\in[0,\infty)\}$, where $\mathcal{B}(E)^*:=\cap_{P\in\mathcal{P}(E)}\mathcal{B}^P(E)$ for $\mathcal{P}(E)$ denoting the family of all probability measures on $(E,\mathcal{B}(E))$ and $\mathcal{B}^P(E)$ denotes the completion of the σ-algebra $\mathcal{B}(E)$ w.r.t. a probability P.

iv) (Markov property) For all $A \in \mathcal{B}(E)$ and any $t, s \geq 0$

$$P^z[X_{s+t} \in A|\mathcal{M}_s] = P^{X_s}[X_t \in A] \quad P^z - a.s., z \in E.$$

Definition D.3 (cf. [34, Chap. IV Defi. 1.8]). *Let $\mathrm{M} := (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P^z)_{z\in E})$ be a Markov process with state space E and corresponding filtration $(\mathcal{M}_t)$. M is called a right process if it has the following additional properties.*

i) (Normal property) $P^z(X_0 = z) = 1$ for all $z \in E$.

ii) (Right continuity) For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$.

iii) (Strong Markov property) $(\mathcal{M}_t)$ is right continuous and for every $(\mathcal{M}_t)$-stopping time σ and every $\nu \in \mathcal{P}(E)$

$$P^\nu[X_{\sigma+t} \in A|\mathcal{M}_\sigma] = P^{X_\sigma}[X_t \in A] \quad P^\nu - a.s.$$

for all $A \in \mathcal{B}(E)$, $t \geq 0$.

Theorem D.4 (cf. [34, Chap. IV Thm 6.7]). *Let E be a metrizable Lusin space. Then a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, m)$ is quasi-regular if and only if there exists a right process M associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. the semigroup of M is an m-version of the semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$. In this case M is always properly associated with $(\mathcal{E}, D(\mathcal{E}))$.*

Remark D.5. The results in [34, Chap. IV] are more general and can be applied for general Hausdorff topological spaces and more general Markov processes. Lusin spaces are enough for our use in this paper.

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