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On the Liouville heat kernel for k-coarse MBRW*

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Abstract

We study the Liouville heat kernel (in the L^2 phase) associated with a class of logarithmically correlated Gaussian fields on the two dimensional torus. We show that for each $\varepsilon > 0$ there exists such a field, whose covariance is a bounded perturbation of that of the two dimensional Gaussian free field, and such that the associated Liouville heat kernel satisfies the short time estimates,

$$\exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}-\varepsilon}\right) \le p_t^\gamma(x,y) \le \exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\varepsilon}\right),$$

for $\gamma < 1/2$. In particular, these are different from predictions, due to Watabiki, concerning the Liouville heat kernel for the two dimensional Gaussian free field.

Keywords: Liouville quantum gravity; Liouville Brownian motion; Liouville heat kernel.
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1 Introduction

In recent years, there has been much interest and progress in the understanding of two dimensional *Liouville quantum gravity*, and associated processes. We do not provide an extensive bibliography and refer instead to the original (mathematical) articles and surveys [10, 11, 5] for background. A starting point for this study is the measure which is the exponential of the Gaussian free field and is constructed rigorously using Kahane's theory of Gaussian multiplicative chaos [19].[1]

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[1] With some abuse, we refer in the sequel to this measure as the LQG measure. Thus, in our terminology, the LQG measure is the Gaussian Multiplicative Chaos (GMC) built from a logarithmically correlated Gaussian field. As pointed out to us by Remi Rhodes and by an anonymous referee, in the physics literature the LQG measure is often meant to represent a modification of this measure, e.g. by normalizition with respect to the

One aspect that has received attention is the construction of Liouville Brownian motion using the Liouville measure and the theory of Dirichlet forms. Mathematically, this has been achieved in [13] (see also [4]), and properties of the associated Liouville heat kernel have been discussed in [14, 17, 2]. One important motivation behind the study of the Liouville heat kernel is that it can be used to study the geometry (and critical exponents) of Liouville quantum gravity. Indeed, a particularly nice application of the construction of the Liouville heat kernel is that it allows for a clean derivation of the so-called KPZ relations [3]. Another important motivation, discussed in [17], are the predictions of Watabiki [20] concerning the short time behavior of the Liouville heat kernel. (We emphasize that the paper [20] is a physics paper, and its results are not formulated as precise theorems and in particular do not refer explicitly to heat kernel exponents. What we refer to as “Watabiki’s predictions” are reasonable extrapolations from the formulae in [20], under the common mapping of exponents from dimensions to heat kernels. This point is discussed in greater detail in [17].) See [17, 2] for existing (weak) estimates on the diffusivity exponents of the Liouville heat kernel.

An important aspect of the class of logarithmically correlated Gaussian fields, that is fields whose covariance is a bounded perturbation of the logarithmic covariance, of which the 2D Gaussian free field is arguably the prominent example, is the universality of many quantitites. We mention explictly Hausdorff dimensions, statistics of the maximum, etc., see [19, 7]. One could naively expect that for Gaussian fields in this class, the predicted exponents of the Liouville heat kernel would be universal. If that is indeed the case, we would say that the heat kernel exponents are universal.

Our goal in this paper is to provide an example where the explicit predictions on Liouville heat-kernel exponents (which can be extrapolated from [20] and discussed in [17, 2]) do not hold for some two dimensional logarithmically correlated Gaussian fields which are bounded perturbations of the Gaussian free field. Namely, we study in this paper the heat kernel for Liouville Brownian motion constructed with respect to a particular logarithmically correlated field, introduced in [6] under the name *k-coarse modified branching random walk* (MBRW for short). Given $k > 0$ integer, this is the centered Gaussian field on the torus $\mathbb{T} = \mathbb{R}^2/(4\mathbb{Z})^2$, denoted $h = \{h(x)\}_{x\in\mathbb{T}}$, with covariance

$$G(x,y) = k\log 2\sum_{j=0}^{\infty} A(x,y;2^{-kj}),$$

where $A(x,y;R) = |B(x,R)\cap B(y,R)|/|B(x,R)|$, $B(z,R)$ is the (open) ball centered at z with radius R with respect to the natural metric on the torus, and $|B|$ is the Lebesgue measure of a set B. The particular choice of the scaling of the torus $\mathbb{T} = \mathbb{R}^2/(4\mathbb{Z})^2$ (rather than $\mathbb{R}^2/\mathbb{Z}^2$) is not important and only done for convenience.

We will show in Section 2.1 that for all k,

$$G(x,y) = \log\frac{1}{|x-y|} + \lambda(|x-y|), \tag{1.1}$$

where λ is continuous in $(0,2]$ and $|\lambda| \le 6k$. Fixing $\gamma\in(0,2)$, we introduce in Section 2.3, following [13], the Liouville measure μ^γ, Liouville Brownian motion (LBM) $\{Y_t\}$, and Liouville heat kernel (LHK) $p_t^\gamma(x,y)$, associated with (γ,h). Formally, the Liouville measure on $\mathbb{T}$ is defined as $\mu^\gamma(dx) := e^{\gamma h(x)-\frac{1}{2}\gamma^2\mathbb{E}h(x)^2}dx$; one then introduces the positive

total mass of the GMC, adding point singularities, etc. In this paper we follow the terminology established in [10], and only note that global, absolutely continuous modifications such as a normalization by the area would not change the main results. For more on this issue from the mathematical perspective, see [12] and references therein.

continuous additive functional (PCAF) with respect to μ^γ as

$$F(v) := \int_0^v e^{\gamma h(X_u) - \frac{\gamma^2}{2}\mathbb{E}h(X_u)^2} du,$$

where $\{X_t\}$ denotes a standard Brownian motion (SBM) on $\mathbb{T}$. The LBM is then defined formally as $Y_t := X_{F^{-1}(t)}$, and the LHK $p_t^\gamma(x,y)$ is then the density of the Liouville semigroup with respect to μ^γ, i.e.

$$E^x f(Y_t) = \int p_t^\gamma(x,y) f(y) \mu^\gamma(dy),$$

where the superscript x is to recall that $Y_0 = X_0 = x$.

Let $\mathbb{P}$ denote the Gaussian law of h. The main result of this paper is as follows.

Theorem 1.1. *Suppose $0 \le \gamma < \frac{1}{2}$, and $x, y \in \mathbb{T}$ with $x \neq y$. For any $\varepsilon > 0$, there exist $k(\varepsilon, x, y)$ and a random variable T_0 depending on $(x, y, \gamma, k, \varepsilon, h)$ only so that for any $k \ge k(\varepsilon, x, y)$ and $t < T_0$,*

$$\exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}-\varepsilon}\right) \le p_t^\gamma(x,y) \le \exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\varepsilon}\right), \quad \mathbb{P}\text{-a.s.}. \tag{1.2}$$

Remark 1.2. The results of this paper shows that the exponent of the LHK with respect to the k-coarse MBRW is for large k and small γ, roughly $(1 + o_k(1))/(1 + \gamma^2/2)$. In particular, it does not match values one could guess from Watabiki's formula, see [20, 17], based on which one would predict that for γ small, the exponent is $(1 + o(\gamma))/(1 + 7\gamma^2/4)$.

In a forthcoming paper [9], the present authors relate the heat kernel exponent to the exponents of distances derived from the LQG itself. Together with [8] and the results of this paper, this shows that the heat kernel estimate for the LQG built from the standard GFF is not the same as the one for the k-coarse MBRW, and both differ from Watabiki's prediction. This is yet another manifestation of the expected non-universality of exponents related to Liouville quantum gravity, across the class of logarithmically correlated Gaussian fields. See [6, 8] for other examples. We do not know what are the different universality classes for the exponents. In particular, as pointed out by a referee, it is possible that fields with a certain regularity (i.e., such that $f(x,y) = G(x,y) + \log|x-y|$ is a bounded *continuous* function, also on the diagonal) all belong to the same universality class.

Heuristics. We describe the strategy behind the proof of the lower bound, and the upper bound is similar. First, represent hierarchically the k-coarse MBRW as follows. Let h_j be independent centered Gaussian fields on $\mathbb{T}$ with covariance

$$\mathbb{E}h_j(x)h_j(y) = k \log 2 \times A(x, y; 2^{-kj}) =: g_j(x,y). \tag{1.3}$$

Formally, $h = \sum_{j=0}^\infty h_j$. For given t, choose r such that $t = 2^{-kr(1+\frac{1}{2}\gamma^2 - o(1))}$, and decompose the field h into a coarse field φ_r and a fine field ψ_r, with

$$\varphi_r := \sum_{j=0}^{r-1} h_j, \quad \psi_r := \sum_{j=r}^{\infty} h_j, \tag{1.4}$$

with respective covariances

$$G_r^{(1)}(x,y) = k\log 2 \sum_{j=0}^{r-1} A(x,y;2^{-kj}), \quad G_r^{(2)}(x,y) = k \log 2 \sum_{j=r}^{\infty} A(x,y;2^{-kj}). \tag{1.5}$$

Note that much like the MBRW, the fine field is not defined pointwise but only in the sense of distributions.

With k, r fixed, we partition $\mathbb{T}$ into $2^{2(kr+2)}$ boxes of side length $s = 2^{-kr}$, elements of

$$\mathcal{BD}_r = \{[a2^{-kr}, (a+1)2^{-kr}) \times [b2^{-kr}, (b+1)2^{-kr})\}_{a,b \in [0, 2^{kr+2}) \cap \mathbb{Z}}.$$

We call the elements of $\mathcal{BD}_r$ *s-boxes*. Similarly to [6], we will find a sequence of neighboring s-boxes B_i, $1 \le i \le I$ (with $I \le 2^{kr(1+\delta)}$, δ chosen below) connecting x to y, so that the following properties (of the B_i's) hold. The coarse field φ_r throughout each B_i is bounded above by $\delta kr \log 2$, where $\delta > 0$ is small and will be chosen according to ε in Theorem 1.1. With probability at least s^δ, the LBM associated with the fine field ψ_r crosses each B_i within time $s^{2-\delta}$. Forcing the original LBM to pass through this sequence of boxes, we will then conclude that it spends time at most $2^{kr(1+\delta)} \times 2^{\delta\gamma kr - \frac{1}{2}\gamma^2 kr} s^{2-\delta} = 2^{-kr(1+\frac{1}{2}\gamma^2 - (2+\gamma)\delta)} = t^{1+O(\varepsilon)}$ crossing from x to the s-box containing y. This happens with probability at least $(s^\delta)^{2^{kr(1+\delta)}} \ge \exp(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\varepsilon})$, and, modulo a localization argument, completes the proof of the lower bound.

Structure of the paper. The preliminaries Section 2 is devoted to the study of the covariance of the k-coarse MBRW h, and in particular to verifying that its covariance is a bounded perturbation of that of the Gaussian free field. We also discuss the power law spectrum of μ^γ and the construction of the LBM with its corresponding PCAF. In addition, Section 2.2 is devoted to a study of the coarse field φ_r, and results in estimates on its fluctuations and maximum in a box. Section 3 is devoted to a study of the fine field; we introduce the notions of slow and fast points/boxes and estimate related probabilities. (The property of being fast is used in the proof of the lower bound, and that of being slow is used in the upper bound.) Finally, the proof of lower bound is contained in Section 4, and that of upper bound is contained in Section 5. Both these sections borrow crucial arguments from [6].

Notation convention. Throughout the paper, we restrict attention to $0 \le \gamma < 1/2$. $\mathbb{T}$ is equipped with the natural metric inherited from the Euclidean distance. We choose $\delta > 0$ small and k large integer (as functions of ε) and keep them fixed throughout. We let C_i, $i = 0, 1, \dots$ be universal positive constants, independent of all other parameters. With r as described above, we let $BD_r(x)$ denote the unique element of $\mathcal{BD}_r$ containing x. For $\ell > 0$, an ℓ-box means a box of side length ℓ. Let $B_\ell(x)$ denote the ℓ-box centered at x, and let $B(x, \ell)$ denote the ball centered at x with radius ℓ. For any box B, let c_B denote the center of B. If B is an ℓ-box, denote by B^* the (5ℓ)-box centered at c_B. We use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability and expectation related to the Gaussian field h. Let P^x and E^x be the probability and expectation related to the SBM starting at x. We let F^x and F_r^x be the PCAFs for the LBM and ψ_r-LBM started at x, respectively. When the starting point x needs not be emphasized, we drop the superscript x.

2 Preliminaries

Section 2.1 is devoted to the proof of (1.1). In Section 2.2, we study the coarse field φ_r and bound its maximum on small boxes as well as the fluctuation across such boxes. Section 2.3 is devoted to a quick review of the construction and existence of the LBM and the LHK.

2.1 Proof of (1.1)

Let d denote the $\mathbb{T}$ distance between x, y, and fix $r_0 := r_0(d) \ge 0$ integer so that

$$2^{-k(r_0+1)} < \frac{d}{2} \le 2^{-kr_0}.$$

Denote

$$\theta_{j,d} := \arcsin(2^{kj} d/2), \quad j = 0, 1, \dots, r_0.$$

We compute the covariance $g_j(x,y)$, c.f. (1.3). For $j \le r_0$, note that $R := 2^{-kj} \ge \frac{d}{2}$; set $\theta = \theta_{j,d}$. Then $|B(x,R) \cap B(y,R)| = (\pi - 2\theta)R^2 - 2R^2 \sin(\theta)\cos(\theta) = \pi R^2 - R^2(2\theta + \sin(2\theta))$, which implies that $A(x,y;R) = 1 - \frac{1}{\pi}(2\theta + \sin(2\theta))$. It follows that with $j \in \mathbb{Z}_+$,

$$g_j(x,y) = \begin{cases} k\log 2 - \frac{k\log 2}{\pi}\big(2\theta_{j,d} + \sin(2\theta_{j,d})\big), & \text{if } j \le r_0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

We now write

$$G(x,y) = \sum_{j=0}^{\infty} g_j(x,y) = \sum_{j=0}^{r_0} g_j(x,y) = k\log 2 \left((r_0+1) - \frac{1}{\pi}\sum_{j=0}^{r_0}\big(2\theta_{j,d} + \sin(2\theta_{j,d})\big) \right). \tag{2.2}$$

Since $r_0 = r_0(d)$, we obtain that

$$G(x,y) = g(d) \text{ for some function } g : (0,2] \to \mathbb{R}_+. \tag{2.3}$$

We now show that g is continuous. Indeed, note that for any fixed j, $d \mapsto \theta_{j,d}$ is continuous (in $d \in [0, 2^{1-kj}]$). Thus the only possible discontinuities of g on $(0,2]$ are whenever $-\log_2(d/2)/k$ is an integer (*i.e.* equals $r_0(d)$); however, for such d we obtain that $\theta_{r_0(d),d} = \pi/2$, which together with the continuity of $d \mapsto \theta_{j,d}$, yields the continuity of g.

To estimate $g(d)$, note that for all $\theta \in [0, \frac{\pi}{2}]$, $0 \le \sin(2\theta) \le 2\sin(\theta)$ and $\theta \le 2\sin(\theta)$, and therefore

$$0 \le 2\theta + \sin(2\theta) \le 6\sin(\theta). \tag{2.4}$$

In particular,

$$\frac{1}{\pi}\Big|\sum_{j=0}^{r_0}(2\theta_{j,d} + \sin(2\theta_{j,d}))\Big| \le \frac{6}{\pi}\sum_{j=0}^{r_0} 2^{-k(r_0-j)} \le \frac{6}{\pi}\sum_{i=0}^{\infty} 2^{-ki} \le \frac{12}{\pi} \le 4.$$

On the other hand, $|k(r_0+1)\log 2 + \log d| \le (k+1)\log 2 \le 2k$. Combining the last two displays with (2.2) shows that

$$|g(d) + \log d| \le 6k\,,$$

yielding (1.1).

2.2 The coarse field

Note that $g_j(x,y)$ is a positive definite kernel on $L^2(\mathbb{T})$, since, with $R = R_j = 2^{-kj}$,

$$\hat{g}_j(x,y) = |B(0,R)| g_j(x,y) = \int_{\mathbb{T}} dz\, \mathbf{1}_{\{|z-x| \le R\}} \mathbf{1}_{\{|z-y| \le R\}}$$

and therefore, for any $f \in L^2(\mathbb{T})$,

$$\int_{(\mathbb{T})^2} f(x)f(y)\hat{g}_j(x,y)dxdy = \int_{\mathbb{T}} dz \left(\int_{\mathbb{T}} dx\; f(x)\mathbf{1}_{\{|x-z| \le R\}} \right)^2 \ge 0\,.$$

Since $g_j(x,y)$ is Lipshitz continuous, Kolmogorov's criterion implies that the associated Gaussian field $x \mapsto h_j(x)$ is continuous almost surely (more precisely, there exists a version of the field which is continuous almost surely). Consequently, the coarse field φ_r is also smooth. In this section, we estimate the maximum value as well as the fluctuations of φ_r in a box.

We begin by recalling an easy consequence of Dudley's criterion.

Lemma 2.1. *([1, Theorem 4.1]) Let $B \subset \mathbb{Z}^2$ be a box of side length ℓ and $\{\eta_w : w \in B\}$ be a mean zero Gaussian field satisfying*

$$\mathbb{E}(\eta_z - \eta_w)^2 \le |z-w|_\infty/\ell \quad \text{for all } z, w \in B.$$

Then $\mathbb{E}\max_{w\in B}\eta_w \le C_0$, where C_0 is a universal constant.

The next lemma is usually referred to as the Borell, or Ibragimov-Sudakov-Tsirelson, inequality. See, e.g., [16, (7.4), (2.26)] as well as discussions in [16, Page 61].

Lemma 2.2. *Let $\{\eta_z : z \in B\}$ be a Gaussian field on a finite index set B. Set $\sigma^2 = \max_{z\in B}\mathrm{Var}(\eta_z)$. Then for all $\lambda, a > 0$,*

$$\mathbb{E}[\exp\{\lambda(\max_{z\in B}\eta_z - \mathbb{E}\max_{z\in B}\eta_z)\}] \le e^{\frac{\lambda^2\sigma^2}{2}}, \quad \text{and} \quad \mathbb{P}(|\max_{z\in B}\eta_z - \mathbb{E}\max_{z\in B}\eta_z| \ge a) \le 2e^{-\frac{a^2}{2\sigma^2}}.$$

Proposition 2.3. *Suppose k is large. For all $r \ge 1$,*

$$\mathbb{E}(\varphi_r(x) - \varphi_r(y))^2 \le 2^{kr}|x-y|, \quad \forall x, y \in \mathbb{T}.$$

Proof. Use the notation in Section 2.1. Let $d = |x-y|$, $r_0 = r_0(d)$. By (2.1) and (2.4),

$$\mathbb{E}(h_j(x) - h_j(y))^2 \le \begin{cases} \frac{2k\log 2}{\pi}\big(2\theta_{j,d} + \sin(2\theta_{j,d})\big) \le 2kd2^{kj}, & \forall j \le r_0, \\ 2k, & \forall j > r_0, \end{cases}$$

where we use $\sin(\theta_{j,d}) = 2^{kj}d/2$ in the case $j \le r_0$.

If $r_0 \ge r-1$,

$$\mathbb{E}(\varphi_r(x) - \varphi_r(y))^2 = \sum_{j=0}^{r-1}\mathbb{E}(h_j(x) - h_j(y))^2 \le 2kd\sum_{j=0}^{r-1}2^{kj} \le 2^{kr}d.$$

Otherwise, $r_0 \le r-2$.

$$\mathbb{E}(\varphi_r(x) - \varphi_r(y))^2 = 2k(r-r_0-1) + \sum_{j=0}^{r_0}2kd2^{kj} \le 2k(r-r_0-1) + 4kd2^{kr_0}.$$

Note $2^{kr}d \ge 2^{k(r-r_0-1)+1}$ and $r - r_0 - 1 \ge 1$. It follows that

$$\mathbb{E}(\varphi_r(x) - \varphi_r(y))^2 \le \frac{k(r-r_0-1)}{2^{k(r-r_0-1)}}2^{kr}d + \frac{4k}{2^{k(r-r_0)}}2^{kr}d \le 2^{kr}d,$$

since k is large enough. ☐

Corollary 2.4. *Suppose k is large. Let B denote a box of side length ℓ, and set $M := \max_{z\in B}\varphi_r(z)$. Then, $\mathbb{E}M \le \sqrt{2}C_0\sqrt{2^{kr}\ell}$.*

Proof. We discretize B by dividing B into 2^{2n} identical boxes $\tilde{B}$'s and identifying the lower left corner $\tilde{c}$ of each $\tilde{B}$ as a point in $\mathbb{Z}^2$. Denote by M_n the maximum value of φ_r over these $\tilde{c}$'s. By the continuity of the coarse field, M_n increases to M as $n \to \infty$. By Proposition 2.3, we can apply Lemma 2.1 to $\varphi_r/\sqrt{2^{kr}2\ell}$ and conclude that $\mathbb{E}M_n \le \sqrt{2}C_0\sqrt{2^{kr}\ell}$. The monotone convergence theorem yields the result. ☐

Corollary 2.5. *There exist $r_0 = r_0(k,\delta)$ such that the following holds for k large and $r \ge r_0$. Enumerate the boxes in $\mathcal{BD}_r$ arbitrarily as B_i, $i = 1, \dots, 2^{2(kr+2)}$. Denote $M_i = \max_{x\in B_i^*}\varphi_r(x)$, $M_i^f = \sup_{x\in B_i^*}|\varphi_r(x) - \varphi_r(c_{B_i})|$, and $M^f = \max_{1\le i\le 2^{2(kr+2)}}M_i^f$. Then*

$$\mathbb{P}(M_i \ge \delta kr\log 2) \le 2e^{-\frac{1}{8}\delta^2 kr\log 2}, \quad \mathbb{P}(M^f \ge \delta kr\log 2) \le e^{-r}.$$

Proof. Note that, for all x, $\mathbb{E}\varphi_r(x)^2 = kr\log 2$. By Corollary 2.4, $\mathbb{E}M_i \le \sqrt{2}C_0\sqrt{5} \le \frac{1}{2}\delta kr\log 2$ for $r \ge r_0(k,\delta)$. By Lemma 2.2,

$$\mathbb{P}(M_i \ge \delta kr\log 2) \le \mathbb{P}(M_i - \mathbb{E}M_i \ge \frac{1}{2}\delta kr\log 2) \le 2e^{-(\frac{1}{2}\delta kr\log 2)^2/(2kr\log 2)} = 2e^{-\frac{1}{8}\delta^2 kr\log 2}.$$

Denote $\hat{M}_i^f := \sup_{x\in B_i^*}(\varphi_r(x) - \varphi_r(c_{B_i}))$. Similarly, we have $\mathbb{P}(\hat{M}_i^f \ge \delta kr\log 2) \le 2e^{-\frac{1}{32}(\delta kr\log 2)^2}$, noting $\mathbb{E}\hat{M}_i^f = \mathbb{E}M_i$ and by Proposition 2.3, $\mathbb{E}(\varphi_r(x) - \varphi_r(c_{B_i}))^2 \le 2^{kr}|x - c_{B_i}| \le 4$ for all $x \in B_i^*$. Furthermore, by a union bound and symmetry,

$$\mathbb{P}(M^f \ge \delta kr\log 2) \le \sum_{i=1}^{2^{2(kr+2)}} 2\mathbb{P}(\hat{M}_i^f \ge \delta kr\log 2) \le 64 \times 2^{2kr} e^{-\frac{(\delta k\log 2)^2}{32} r^2} \le e^{-r},$$

where in the last inequality we use $r \ge r_0(k,\delta)$. □

2.3 Construction of the LBM and LHK

There are several ways to construct the Liouville measure μ^γ with respect to h, say, via the method of Gaussian multiplicative chaos [15]. In our case, since we deal with $\gamma < 1/2$, it is particulaly simple by applying L^2 methods. So, in the rest of this section we concentrate on the construction of the LBM and LHK.

Suppose $\varepsilon = 2^{-kr}$. Noting (2.3) and $d \le 2$, we can assume $x, y \in [-1,1]^2$, which is regarded as a subset of $\mathbb{T}$. Then,

$$G(x,y) = G_r^{(2)}(\varepsilon x, \varepsilon y), \quad \textit{i.e.}\ G(\varepsilon x, \varepsilon y) = G(x,y) + G_r^{(1)}(\varepsilon x, \varepsilon y) \tag{2.5}$$

since $A(\varepsilon x, \varepsilon y; 2^{-k(r+j)}) = A(x,y;2^{-kj})$. By (2.1),

$$G_r^{(1)}(\varepsilon x, \varepsilon y) \le G_r^{(1)}(\varepsilon x, \varepsilon x) = kr\log 2 = \log\frac{1}{\varepsilon}\ .$$

It follows that

$$G(\varepsilon x, \varepsilon y) \le G(x,y) + \log\frac{1}{\varepsilon}\ . \tag{2.6}$$

By a standard reasoning (see [19, Theorem 2.14] for example), one has

$$\mathbb{E}\mu^\gamma(B(0,\varepsilon))^q \le \hat{C}(q)\varepsilon^{\xi(q)},$$

where $\hat{C}(q)$ is a constant depending on q (as well as γ), and

$$\xi(q) = (2 + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2.$$

For any $2^{-k(r+1)} < \varepsilon \le 2^{-kr}$, we take $C(q) = \hat{C}(q)2^{-k\xi(q)}$ and conclude that

$$\mathbb{E}\mu^\gamma(B(0,\varepsilon))^q \le \mathbb{E}\mu^\gamma(B(0,2^{-kr}))^q \le \hat{C}(q)2^{-kr\xi(q)} \le C(q)\varepsilon^{\xi(q)}. \tag{2.7}$$

Recall that the coarse field φ_r is smooth, so

$$H_r(u) := \int_0^u e^{\gamma\varphi_r(X_v) - \frac{1}{2}\gamma^2\mathbb{E}\varphi_r(X_v)^2} dv$$

is well-defined.

With (2.6) and (2.7), one can follow the arguments in [13, Section 2] and obtain the following conclusions. Let F denote the PCAF associated with μ^γ. Then, $\mathbb{P}$-a.s., the limit of H_r in P^x-probability exists and it is the PCAF F; that is, $P^x(\sup_{0\le t\le T}|F(u) - H_r(u)| > a) \to_{r\to\infty} 0$, for all $a > 0$ and $T > 0$. Further, the process $Y_t := X_{F^{-1}(t)}$ is a strong Markov process, which is called the LBM with respect to μ^γ. The LHK $p_t^\gamma(x,y)$ exists and satisfies $E^x f(Y_t) = \int f(y)p_t(x,y)\mu^\gamma(dy)$. Furthermore, by [14, Theorem 0.1] and parallel arguments in [17], $p_t^\gamma(x,y)$ is continuous in (t,x,y).

3 Fast/slow points/boxes of the fine field

This section is devoted to the study of properties of the fine field. For the lower bound on the LHK, we need to construct regions which are fast to cross for the LBM, while for the upper bound we will need to create obstacles, i.e. regions which force the LBM to be slow. Toward this end, we introduce in Definitions 3.1 and 3.2 the notions of fast/slow points and boxes, and estimate, in Lemmas 3.3 and 3.4, the probability that a point/box is fast/slow.

Throughout, we fix $s = 2^{-kr}$ for an appropriate integer $r \geq 1$ (as explained in the introduction, r, and hence s, are chosen so that $t = s^{1+\frac{1}{2}\gamma^2+o(1)}$). This choice determines the fine field ψ_r, see (1.4). With this choice, one can construct the PCAF F_r based on ψ_r in the same way as F was constructed, by replacing the measure μ^γ with the truncated measure μ_r^γ written formally as $\mu_r^\gamma(dx) = e^{\gamma\psi_r(x) - \frac{\gamma^2}{2}\mathbb{E}\psi_r(x)^2}dx$ (as before, the actual construction involves the smooth cutoff $\psi_{r,w} := \sum_{j=r}^{w} h_j$ and taking the limit as $w \to \infty$). Formally, we write

$$F_r(v) = \int_0^v e^{\gamma\psi_r(X_u) - \frac{1}{2}\gamma^2\mathbb{E}\psi_r(X_u)^2}du.$$

We note also that the sequence of approximating PCAF

$$F_{r,w}(v) := \int_0^v e^{\gamma\psi_{r,w}(X_u) - \frac{1}{2}\gamma^2\mathbb{E}\psi_{r,w}(X_u)^2}du$$

converges as $w \to \infty$, in the sense described at the end of Section 2, to F_r.

Fix $\delta_1, \delta_2, \delta_3, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ small, possibly depending on k, γ and s. Fix $z \in \mathbb{T}$ and recall that $B_\ell(z)$ denotes the ℓ-box centered at z. Let $\sigma_{z,\ell}$ denote the time that the SBM (starting from z) hits $\partial B_\ell(z)$.

Definition 3.1 (Fast points and boxes)**.** *A point z is said to be* fast *if*

$$P^z(F_r(s^2 \wedge \sigma_{z,6s}) \leq s^2/\delta_1) \geq 1 - \delta_2.$$

The set of fast points is denoted by $\mathcal{F}$. An s-box B is said to be fast *if $|B \cap \mathcal{F}| \geq \delta_3 s^2$.*

Definition 3.2 (Slow points and boxes)**.** *A point z is said to be* slow *if*

$$P^z(F_r(\sigma_{z,s}) \geq \varepsilon_1 s^2) \geq \varepsilon_2. \tag{3.1}$$

The set of slow points is denoted by $\mathcal{S}$. An s-box B is said to be slow *if $|B \cap \mathcal{S}| \geq \varepsilon_3 s^2$.*

We emphasize that the notions of fast/slow points and boxes depend on the fine field ψ_r only. Further, a point (or box) may be fast and slow simultaneously.

Our fundamental estimate concerning fast/slow points is contained in the next lemma.

Lemma 3.3. *There exist universal positive constants C_2, C_3 such that the following hold.*
(i) $\mathbb{P}(z \in \mathcal{F}) \geq 1 - \frac{\delta_1}{\delta_2}$.
(ii) *For $\varepsilon_1 \leq C_2$ and $\varepsilon_2 \leq C_3 e^{-6k\gamma^2}$, we have $\mathbb{P}(z \in \mathcal{S}) \geq 120 C_3 e^{-6k\gamma^2}$.*

Proof. (i) Set $\xi = F_r^z(s^2 \wedge \sigma_{z,6s})$ and $\eta = P^z(\xi > s^2/\delta_1)$. By definition,

$$\mathbb{P}(z \notin \mathcal{F}) = \mathbb{P}(\eta > \delta_2) \leq \mathbb{E}\eta/\delta_2.$$

Note that

$$\mathbb{E}\eta = E^z\mathbb{P}(\xi > s^2/\delta_1) \leq \frac{\delta_1}{s^2}E^z\mathbb{E}\xi = \frac{\delta_1}{s^2}E^z(s^2 \wedge \sigma_{z,6s}) \leq \delta_1.$$

Combining the last two displays, one obtains $\mathbb{P}(z \notin \mathcal{F}) \leq \delta_1/\delta_2$, completing the proof.

(ii) We use the abbreviation $\sigma = \sigma_{z,s}$ and set now $\xi = F_r^z(\sigma)$ and $\eta = P^z(\xi \geq \varepsilon_1 s^2)$. Without loss of generality, we suppose $z = (0,0)$ and consistently drop z from the notation, writing $B_s = B_s(z)$. Since $\eta \leq 1$, we have $\mathbb{E}\eta = \mathbb{E}\eta \mathbf{1}_{\{\eta \geq \varepsilon_2\}} + \mathbb{E}\eta \mathbf{1}_{\{\eta < \varepsilon_2\}} \leq \mathbb{P}(\eta \geq \varepsilon_2) + \varepsilon_2$. By definition,

$$\mathbb{P}((0,0) \in \mathcal{S}) = \mathbb{P}(\eta \geq \varepsilon_2) \geq \mathbb{E}\eta - \varepsilon_2 = E\mathbb{P}(\xi \geq \varepsilon_1 s^2) - \varepsilon_2. \tag{3.2}$$

We are going to estimate $\mathbb{P}(\xi \geq \varepsilon_1 s^2)$ via the second moment method. Recall that $\mathbb{E}\xi = \sigma$, which is of order s^2. To compute the second moment, note that since $\gamma < 1/2$, the sequence of squares of approximating PCAFs $(F_{r,w})^2$ are uniformly (in w) integrable (see the argument just after (3.3) below) and therefore

$$\begin{aligned}\mathbb{E}\xi^2 &= \mathbb{E}F_r(\sigma)^2 = \int_0^\sigma \int_0^\sigma \mathbb{E}e^{\gamma\psi_r(X_u) - \frac{1}{2}\gamma^2 \mathbb{E}\psi_r(X_u)^2 + \gamma\psi_r(X_v) - \frac{1}{2}\gamma^2\mathbb{E}\psi_r(X_v)^2} du dv \\ &= \int_0^\sigma \int_0^\sigma e^{\gamma^2 G_r^{(2)}(X_u, X_v)} du dv = \int_{w,w' \in B_s} e^{\gamma^2 G_r^{(2)}(w,w')} \nu(dw)\nu(dw') =: I_{\gamma^2},\end{aligned}$$

where $\{X_u\}$ is the SBM starting from $(0,0)$, $G_r^{(2)}$ is defined in (1.5), and ν denotes the occupation measure of $\{X_u\}$ before exiting B_s, i.e.

$$\int_{w \in B_s} f(w)\nu(dw) = \int_0^\sigma f(X_u) du.$$

Let $\hat{w} = 2^{kr} w$ and $\hat{w}' = 2^{kr} w'$, with $\hat{w}, \hat{w}' \in \mathbb{T}$. By (1.1) and (2.5),

$$G_r^{(2)}(w,w') = G(\hat{w}, \hat{w}') \leq \log \frac{1}{|\hat{w} - \hat{w}'|} + 6k = \log \frac{s}{|w - w'|} + 6k.$$

Consequently,

$$I_{\gamma^2} \leq e^{6k\gamma^2} s^{\gamma^2} \int_{w,w' \in B_s} \frac{1}{|w - w'|^{\gamma^2}} \nu(dw)\nu(dw') = e^{6k\gamma^2} s^{\gamma^2} \int_0^\sigma \int_0^\sigma \frac{1}{|X_u - X_v|^{\gamma^2}} du dv.$$

Let $\hat{X}_u = \frac{1}{s} X_{s^2 u}$, and let $\hat{\sigma} = \sigma / s^2$ be the time that the SBM $\{\hat{X}_u\}$ started at $(0,0)$ exits $[-1/2, 1/2]^2$. Then

$$I_{\gamma^2} \leq e^{6k\gamma^2} s^4 \int_0^{\hat{\sigma}} \int_0^{\hat{\sigma}} \frac{1}{|\hat{X}_u - \hat{X}_v|^{\gamma^2}} du dv.$$

Note $|\frac{\hat{X}_u - \hat{X}_v}{\sqrt{2}}|^{\gamma^2} \geq |\frac{\hat{X}_u - \hat{X}_v}{\sqrt{2}}|^{1/4}$, since $|\hat{X}_u - \hat{X}_v| \leq \sqrt{2}$ and $\gamma^2 \leq 1/4$. Thus,

$$|\hat{X}_u - \hat{X}_v|^{\gamma^2} \geq \frac{1}{2} |\hat{X}_u - \hat{X}_v|^{1/4}.$$

It follows that

$$I_{\gamma^2} \leq 2e^{6k\gamma^2} s^4 \hat{I}, \quad \text{where } \hat{I} = \int_0^{\hat{\sigma}} \int_0^{\hat{\sigma}} \frac{1}{|\hat{X}_u - \hat{X}_v|^{1/4}} du dv. \tag{3.3}$$

Note that $\hat{I}$ is a random variable depending only on the SBM $\{\hat{X}_u\}$. By [18, Theorem 4.33], $E\hat{I} < \infty$. Consequently, there exists a universal constant C_1 such that $P(\hat{I} \leq \frac{1}{2} C_1) \geq 3/4$. Hence, the event $E_1 := \{\mathbb{E}\xi^2 \leq C_1 e^{6k\gamma^2} s^4\}$ has probability $P(E_1) \geq 3/4$. By the scaling invariance of the SBM, there exists a universal positive constant C_2 such that the event $E_2 = \{\sigma \geq 2C_2 s^2\}$ has probability $\geq 3/4$. Thus, $P(E_1 \cap E_2) \geq 1/4$.

Assume $E_1 \cap E_2$ happens. On the one hand, on E_1,

$$\mathbb{P}(\xi \geq \varepsilon_1 s^2) \geq \frac{\left(\mathbb{E}\xi \mathbf{1}_{\{\xi \geq \varepsilon_1 s^2\}}\right)^2}{\mathbb{E}\xi^2} \geq \frac{1}{C_1 e^{6k\gamma^2} s^4} \left(\mathbb{E}\xi \mathbf{1}_{\{\xi \geq \varepsilon_1 s^2\}}\right)^2.$$

On the other hand, on E_2, $\xi = F_r(\sigma) \geq F_r(2C_2 s^2) =: \zeta$. Note that $2C_2 s^2 = \mathbb{E}\zeta \leq \mathbb{E}\zeta \mathbf{1}_{\{\zeta \geq \varepsilon_1 s^2\}} + \varepsilon_1 s^2$. We have $\mathbb{E}\xi \mathbf{1}_{\{\xi \geq \varepsilon_1 s^2\}} \geq \mathbb{E}\zeta \mathbf{1}_{\{\zeta \geq \varepsilon_1 s^2\}} \geq (2C_2 - \varepsilon_1)s^2 \geq C_2 s^2$, where we use the assumption $\varepsilon_1 \leq C_2$. Thus,

$$\mathbb{P}(\xi \geq \varepsilon_1 s^2) \geq \frac{\left(C_2 s^2\right)^2}{C_1 e^{6k\gamma^2} s^4} = \frac{C_2^2}{C_1} e^{-6k\gamma^2}, \quad \text{on } E_1 \cap E_2.$$

Consequently,

$$E\mathbb{P}(\xi \geq \varepsilon_1 s^2) \geq E\left(\mathbb{P}(\xi \geq \varepsilon_1 s^2)\mathbf{1}_{\{E_1 \cap E_2\}}\right) \geq \frac{C_2^2}{C_1} e^{-6k\gamma^2} \times P(E_1 \cap E_2) \geq \frac{C_2^2}{4C_1} e^{-6k\gamma^2}.$$

Take $C_3 := C_2^2/(484C_1)$. Then $E\mathbb{P}(\xi \geq \varepsilon_1 s^2) \geq 121 C_3 e^{-6k\gamma^2}$. This, together with (3.2) and the assumption $\varepsilon_2 \leq C_3 e^{-6k\gamma^2}$, implies the result. □

The next lemma estimates the probability that an s-box B is fast/slow.

Lemma 3.4. (i) $\mathbb{P}(B$ *is fast*$) \geq 1 - \frac{\delta_1}{\delta_2} - \delta_3$.

(ii) *Suppose* $\varepsilon_2 \leq C_3 e^{-6k\gamma^2}$ *and* $\varepsilon_3 \leq C_3^2 e^{-12k\gamma^2}$. *Then,* $\mathbb{P}(B$ *is slow*$) \geq 1 - \varepsilon_1^{C_3 e^{-6k\gamma^2} 2^{-2k}}$ *if* ε_1 *is less than some constant* $\varepsilon_1(\gamma, k)$.

Proof. (i) By Lemma 3.3(i) and the translation invariance of the fine field ψ_r, $\mathbb{E}|B \cap \mathcal{F}| \geq (1 - \frac{\delta_1}{\delta_2})s^2$. Since $|B \cap \mathcal{F}| \leq |B| \leq s^2$, $|B \cap \mathcal{F}| \leq |B \cap \mathcal{F}|\mathbf{1}_{\{|B\cap\mathcal{F}| < \delta_3 s^2\}} + |B \cap \mathcal{F}|\mathbf{1}_{\{|B\cap\mathcal{F}| \geq \delta_3 s^2\}} \leq \delta_3 s^2 + s^2 \mathbf{1}_{\{|B\cap\mathcal{F}| \geq \delta_3 s^2\}}$. Hence, $\mathbb{E}|B \cap \mathcal{F}| - \delta_3 s^2 \leq s^2 \mathbb{P}(|B \cap \mathcal{F}| \geq \delta_3 s^2) = s^2 \mathbb{P}(B \text{ is fast})$. Therefore, $\mathbb{P}(B \text{ is fast}) \geq \frac{1}{s^2}\left(\mathbb{E}|B \cap \mathcal{F}| - \delta_3 s^2\right) \geq 1 - \frac{\delta_1}{\delta_2} - \delta_3$.

(ii) Our strategy is as follows. We will divide B into n^2 identical boxes $\tilde{B}$ of side length $\tilde{s} = s/n$, where n is to be chosen properly to support the following arguments. In each box $\tilde{B}$, one can find $O(s^2/n^2)$ slow points in average, by Lemma 3.3(ii). Then, we would like to use large deviations to show that, with high probability, there are at least $\delta_3 s^2$ slow points in B, i.e. B is slow. Unfortunately, the random variables $|\tilde{B} \cap \mathcal{S}|$'s, measuring the size of the cluster of slow points in the smaller boxes $\tilde{B}$, are heavily dependent. To obtain the appropriate large deviation estimates by independence, we will replace $\sigma_{z,s}$ in (3.1) by $\sigma_{z,\tilde{s}}$, and use a new parameters $\tilde{\varepsilon}_1$ to define the property of a point to be $\widetilde{slow}$. Let $\tilde{\mathcal{S}}$ consist of $\widetilde{\text{slow}}$ points. Then, the random variables $|B_i \cap \tilde{\mathcal{S}}|$'s are almost independent, and good large deviation estimates for their sums can be obtained. Finally, we will show that by choosing $\tilde{\varepsilon}_1$ properly, $B \cap \tilde{\mathcal{S}} \subseteq B \cap \mathcal{S}$ with high probability, completing the proof.

The actual proof is in four steps. In the first step, we set the parameters n and $\tilde{\varepsilon}_1$, and give the definition of being $\widetilde{\text{slow}}$. In the second step, we will show $|B \cap \tilde{\mathcal{S}}| \geq \delta_3 s^2$ with high probability. In the third step, we will show $B \cap \tilde{\mathcal{S}} \subseteq B \cap \mathcal{S}$ with high probability. In the last step, we collect the results obtained and show (ii).

Step 1. Let

$$\kappa := \sqrt{-\log \varepsilon_1}, \quad r_0 := \lfloor \frac{1}{k} \log_2 \kappa \rfloor, \quad n := 2^{kr_0}. \tag{3.4}$$

Equivalently, we write ε_1 in the form of $e^{-\kappa^2}$, pick r_0 such that $2^{kr_0} \leq \kappa < 2^{k(r_0+1)}$, and set $n = 2^{kr_0}$. Take

$$\tilde{\varepsilon}_1 = n^{2\gamma n + \frac{\gamma^2}{2} + 2} \varepsilon_1. \tag{3.5}$$

The parameters n and $\tilde{\varepsilon}_1$ depend only on ε_1 (and k,γ). As $\varepsilon_1 \to 0$, we have $\kappa \to \infty$, and $r_0 \to \infty$ as well as $n \to \infty$. Furthermore, $\tilde{\varepsilon}_1 \to 0$, since $\tilde{\varepsilon}_1 \leq e^{(2\gamma n + \gamma^2/2 + 2)\log n} e^{-\kappa^2} \leq e^{(2\gamma n + \gamma^2/2+2)\log n - n^2}$ and $n \to \infty$. Therefore, there exists a constant $\varepsilon_1(\gamma, k)$ such that $\tilde{\varepsilon}_1 \leq C_2$ if $\varepsilon_1 \leq \varepsilon_1(\gamma, k)$. Furthermore, we pick $\varepsilon_1(\gamma, k)$ such that

$$2e^{-\frac{(2n\log n - 2C_0\sqrt{n})^2}{2\log n}} \leq e^{-n^2 \log n}, \quad e^{-2C_3 e^{-6k\gamma^2} n^2} + e^{-n^2\log n} \leq e^{-C_3 e^{-6k\gamma^2} n^2} \tag{3.6}$$

as $\varepsilon_1 \le \varepsilon_1(\gamma, k)$. Note that $\tilde{\varepsilon}_1$ and ε_2 satisfy the assumptions in Lemma 3.3(ii) for ε_1 and ε_2.

Let $\tilde{s} := s/n$, and $\tilde{r} := r + r_0$ such that $\tilde{s} = 2^{-k\tilde{r}}$. We say that

$$\text{a point } z \text{ is } \widetilde{\text{slow}} \text{ if } P^z(F_{\tilde{r}}(\sigma_{z,\tilde{s}}) \ge \tilde{\varepsilon}_1 \tilde{s}^2) \ge \varepsilon_2.$$

Denote by $\tilde{\mathcal{S}}$ the set of $\widetilde{\text{slow}}$ points.

Step 2. Suppose $\tilde{B}$ is an $\tilde{s}$-box. Applying Lemma 3.3(ii) to the $\widetilde{\text{slow}}$ points, we obtain $\mathbb{E}|\tilde{B} \cap \tilde{\mathcal{S}}| \ge 120 C_3 e^{-6k\gamma^2} \tilde{s}^2 = 2a\tilde{s}^2$, where we denote

$$a = 60 C_3 e^{-6k\gamma^2}. \tag{3.7}$$

Note that $|\tilde{B} \cap \tilde{\mathcal{S}}| \le \tilde{s}^2$, which implies that $\mathbb{E}|\tilde{B} \cap \tilde{\mathcal{S}}| = \mathbb{E}|\tilde{B} \cap \tilde{\mathcal{S}}| \mathbf{1}_{\{|\tilde{B} \cap \tilde{\mathcal{S}}| \ge a\tilde{s}^2\}} + \mathbb{E}|\tilde{B} \cap \tilde{\mathcal{S}}| \mathbf{1}_{\{|\tilde{B} \cap \tilde{\mathcal{S}}| < a\tilde{s}^2\}} \le \tilde{s}^2 \mathbb{P}(|\tilde{B} \cap \tilde{\mathcal{S}}| \ge a\tilde{s}^2) + a\tilde{s}^2$. It follows that

$$\mathbb{P}\left(|\tilde{B} \cap \tilde{\mathcal{S}}| \ge a\tilde{s}^2\right) \ge \frac{1}{\tilde{s}^2}\left(\mathbb{E}|\tilde{B} \cap \tilde{\mathcal{S}}| - a\tilde{s}^2\right) \ge a. \tag{3.8}$$

Without loss of generality, we suppose $B = [0, s)^2$. We next partition B into n^2 identical $\tilde{s}$-boxes, from which we pick those of the form $[4i\tilde{s}, (4i+1)\tilde{s}) \times [4j\tilde{s}, (4j+1)\tilde{s})$, $i, j \in \mathbb{Z} \cap [0, n/4)$, and enumerate them arbitrarily as $\tilde{B}_i$, $i = 1, \cdots, (n/4)^2$. Note that $\tilde{B}_i \cap \tilde{\mathcal{S}}$ depends on the restriction of the fine field $\psi_{\tilde{r}}$ to the $(2\tilde{s})$-box centered at $c_{\tilde{B}_i}$, and $\psi_{\tilde{r}}(w)$ is independent of $\psi_{\tilde{r}}(w')$ if $|w - w'| \ge 2\tilde{s}$. It follows that the random variables $|\tilde{B}_i \cap \tilde{\mathcal{S}}|$'s are mutually independent. Let

$$\chi_i = 1 \text{ if } |\tilde{B}_i \cap \tilde{\mathcal{S}}| \ge a\tilde{s}^2, \quad \chi_i = 0 \text{ otherwise.}$$

Then $\sum_{i=1}^{n^2/16} \chi_i \ge \varepsilon_3 s^2/(a\tilde{s}^2)$ implies $|B \cap \tilde{\mathcal{S}}| \ge a\tilde{s}^2 \times \varepsilon_3 s^2/(a\tilde{s}^2) = \varepsilon_3 s^2$. It follows that

$$\mathbb{P}(|B \cap \tilde{\mathcal{S}}| \ge \varepsilon_3 s^2) \ge \mathbb{P}\left(\sum_{i=1}^{n^2/16} \chi_i \ge \frac{\varepsilon_3 s^2}{a\tilde{s}^2}\right). \tag{3.9}$$

Now we estimate the right hand side of (3.9) via large deviations. Note that the χ_i's are Bernoulli random variables, with $P(\chi_i = 1) \ge a$, see (3.8), and therefore

$$\mathbb{E}e^{-\chi_i} = 1 - (1 - e^{-1})\mathbb{P}(\chi_i = 1) \le 1 - (1 - e^{-1})a \le \exp(-(1 - e^{-1})a).$$

Using independence and Chebyshev's inequality we get

$$\mathbb{P}\left(\sum_{i=1}^{n^2/16} \chi_i < \frac{\varepsilon_3 s^2}{a\tilde{s}^2}\right) \le \exp\left(\frac{\varepsilon_3 s^2}{a\tilde{s}^2}\right)\left(\mathbb{E}e^{-\chi_1}\right)^{n^2/16} \le \exp\left(\frac{\varepsilon_3 s^2}{a\tilde{s}^2} - \frac{n^2}{16}(1 - e^{-1})a\right). \tag{3.10}$$

Recall that $\tilde{s} = s/n$, $a = 60 C_3 e^{-6k\gamma^2}$, see (3.7), and $\varepsilon_3 \le C_3^2 e^{-12k\gamma^2} = (\frac{a}{60})^2$ by assumption. Thus,

$$\frac{\varepsilon_3 s^2}{a\tilde{s}^2} - \frac{n^2}{16}(1 - e^{-1})a \le \left(\frac{1}{60^2} - \frac{1 - e^{-1}}{16}\right) a n^2 \le -2C_3 e^{-6k\gamma^2} n^2.$$

Combining (3.10) and (3.9), we conclude that

$$\mathbb{P}(|B \cap \tilde{\mathcal{S}}| < \varepsilon_3 s^2) \le \mathbb{P}\left(\sum_{i=1}^{n^2/16} \chi_i < \frac{\varepsilon_3 s^2}{a\tilde{s}^2}\right) \le e^{-2C_3 e^{-6k\gamma^2} n^2}. \tag{3.11}$$

Step 3. Abbreviate $\sigma = \sigma_{z,s}$ and $\tilde\sigma = \sigma_{z,\tilde s}$. Recall that $z \in \mathcal{S}$ if $P^z(F_r(\sigma) \geq \varepsilon_1 s^2) \geq \varepsilon_2$ while $z \in \tilde{\mathcal{S}}$ if $P^z(F_{\tilde r}(\tilde\sigma) \geq \tilde\varepsilon_1 \tilde s^2) \geq \varepsilon_2$. Since $\tilde s < s$, it holds that $\tilde\sigma < \sigma$. Consequently, $F^z_r(\tilde\sigma) \leq F^z_r(\sigma)$. Therefore,

$$P^z(F_r(\sigma) \geq \varepsilon_1 s^2) \geq P^z(F_r(\tilde\sigma) \geq \varepsilon_1 s^2), \quad \text{for all } z. \tag{3.12}$$

We are going to compare $F^z_r(\tilde\sigma)$ with $F^z_{\tilde r}(\tilde\sigma)$, and show below that

$$\mathbb{P}\left(\mathcal{E}\right) \geq 1 - e^{-n^2 \log n}, \text{ where } \mathcal{E} = \{P^z(F_r(\tilde\sigma) \geq \varepsilon_1 s^2) \geq P^z(F_{\tilde r}(\tilde\sigma) \geq \tilde\varepsilon_1 \tilde s^2) \text{ for all } z \in B\}. \tag{3.13}$$

Combined (3.12), it follows that if $\mathcal{E}$ occurs then $z \in \tilde{\mathcal{S}} \Rightarrow z \in \mathcal{S}$, for all $z \in B$, and in particular $\mathcal{E} \subset \{B \cap \tilde{\mathcal{S}} \subseteq B \cap \mathcal{S}\}$. It follows then from (3.13) that

$$\mathbb{P}(B \cap \tilde{\mathcal{S}} \nsubseteq B \cap \mathcal{S}) \leq \mathbb{P}(\mathcal{E}^c) \leq e^{-n^2 \log n}, \tag{3.14}$$

which we will use in the next step. Before doing that, we first complete the proof of (3.13).

Let $\phi = \psi_r - \psi_{\tilde r}$, which has covariance

$$G_{r,\tilde r}(w_1, w_2) = k \log 2 \sum_{j=r}^{\tilde r - 1} A(w_1, w_2; 2^{-kj}).$$

Set

$$M = \max_{w \in \breve B}(-\phi(w)), \quad \text{where } \breve B = [-\frac{1}{2}s, \frac{3}{2}s)^2 \text{ is the } 2s\text{-box centered at } c_B.$$

Set $\hat B = 2^{kr}\breve B$, which has side length 2. Note that $A(w_1, w_2, 2^{-kj}) = A(\hat w_1, \hat w_2, 2^{-k(j-r)})$, where $\hat w_i = 2^{kr} w_i$. Therefore, $\{\phi(w), w \in \breve B\}$ is a copy of the coarse field $\{\varphi_{r_0}(\hat w), w \in \hat B\}$, with w being identified as $\hat w = 2^{kr} w$, where we recall that $r_0 = \tilde r - r$ and is defined in (3.4). By Corollary 2.4, $\mathbb{E}M \leq \sqrt{2} C_0 \sqrt{2^{kr_0} \times 2} = 2C_0\sqrt{n}$. Since $\mathbb{E}\phi(w)^2 = kr_0 \log 2 = \log n$ for all w, we have

$$\mathbb{P}(M \geq 2n \log n) \leq 2e^{-\frac{(2n\log n - 2C_0\sqrt{n})^2}{2\log n}} \leq e^{-n^2 \log n}, \tag{3.15}$$

where we use Lemma 2.2, and the last inequality holds by (3.6). Noting for all $z \in B$, the $\tilde s$-box centered at z is contained in $\breve B$, we have $X_u \in \breve B$ for $u \leq \tilde\sigma$, where we drop the superscript z in X_u. Therefore, on the event $\{M < 2n \log n\}$, it holds that for all $z \in B$,

$$\begin{aligned} F^z_r(\tilde\sigma) &= \int_0^{\tilde\sigma} e^{\gamma\psi_{\tilde r}(X_v) - \frac{\gamma^2}{2}\mathbb{E}\psi_{\tilde r}(X_v)^2} \times e^{\gamma\phi(X_v) - \frac{\gamma^2}{2}\mathbb{E}\phi(X_v)^2} dv \\ &\geq e^{-\gamma M - \frac{\gamma^2}{2} kr_0 \log 2} F_{\tilde r}(\tilde\sigma) \geq e^{-\gamma 2n \log n - \frac{\gamma^2}{2}\log n} F_{\tilde r}(\tilde\sigma), \end{aligned}$$

where in the first equality we use the independence of $\psi_{\tilde r}$ and ϕ. By the definition of $\tilde\varepsilon_1$ in (3.5),

$$P^z(F_r(\tilde\sigma) \geq \varepsilon_1 s^2) \geq P^z(F_{\tilde r}(\tilde\sigma) \geq e^{\gamma 2n\log n + \frac{\gamma^2}{2}\log n}\varepsilon_1 s^2) = P^z(F_{\tilde r}(\tilde\sigma) \geq \tilde\varepsilon_1 \tilde s^2).$$

Therefore, we conclude that $\{M < 2n \log n\} \subseteq \mathcal{E}$. This, together with (3.15), implies (3.13) and completes the proof of (3.14).

Step 4. If $|B \cap \tilde{\mathcal{S}}| \geq \varepsilon_3 s^2$ and $B \cap \tilde{\mathcal{S}} \subseteq B \cap \mathcal{S}$, we have $|B \cap \mathcal{S}| \geq \varepsilon_3 s^2$, i.e. B is slow. Hence,

$$1 - \mathbb{P}(B \text{ is slow}) \leq \mathbb{P}(|B \cap \tilde{\mathcal{S}}| < \varepsilon_3 s^2) + \mathbb{P}(B \cap \tilde{\mathcal{S}} \nsubseteq B \cap \mathcal{S}).$$

By (3.11) and (3.14), it follows that

$$\begin{aligned} 1-\mathbb{P}(B \text{ is slow}) &\leq \exp\{-2C_3 e^{-6k\gamma^2} n^2\} + \exp\{-n^2\log n\} \\ &\leq \exp\{-C_3 e^{-6k\gamma^2} n^2\} \leq \exp\{-C_3 e^{-6k\gamma^2} 2^{-2k}\kappa^2\} = \varepsilon_1^{C_3 e^{-6k\gamma^2}2^{-2k}}, \end{aligned}$$

where in the second inequality we use (3.6) and in the last two inequalities we use (3.4). This implies (ii) and completes the proof of the lemma. □

The next lemma bounds below $F_r^z(\sigma_{z,3s})$ uniformly in z in slow boxes.

Lemma 3.5. *There exists a universal positive constant C_4 such that the following holds. Suppose B is slow. Then, $P^z(F_r(\sigma_{z,3s}) \geq \varepsilon_1 s^2) \geq C_4\varepsilon_2\varepsilon_3$ for all z in the closure of B.*

Proof. Abbreviate $\sigma' = \sigma_{z,3s}$. Let $\rho_1(w,w')$ denote the heat kernel of the SBM, killed upon exiting $[0,3]^2$, at time 1. Let $C_4 := \min_{w,w'\in[0.5,2.5]^2}\rho_1(w,w')$, which is positive. Suppose that the SBM started from z hits $B\cap\mathcal{S}$ at time σ_* and point w. Since $|B\cap\mathcal{S}| \geq \varepsilon_3 s^2$, we have that $P^z(\sigma_* < \sigma') \geq C_4\varepsilon_3$. On $\sigma_* < \sigma'$, $F_r^z(\sigma') \geq \sigma$, where σ is the time that the ψ_r-LBM started from w exits $B_s(w)$. Since $w \in \mathcal{S}$, $P^w(\sigma \geq \varepsilon_1 s^2) \geq \varepsilon_2$. By the strong Markov property, $P^z(F_r(\sigma') \geq \varepsilon_1 s^2) \geq P^z(\sigma_* < \sigma', \sigma \geq \varepsilon_1 s^2) \geq C_4\varepsilon_3 \times \varepsilon_2$, which completes the proof. □

4 Lower bound

We continue to take $s := 2^{-kr} = t^{\frac{1}{1+\frac{1}{2}\gamma^2}+o(1)}$. To obtain the lower bound on the LHK, we will force the LBM $\{Y_u^x\}$, started at $x \in \mathbb{T}$, to hit $y \in \mathbb{T}$ according to the following three steps. First, we will force the LBM to hit inside $BD_r(y)$ a point which is *very fast* (a notion to be defined below), then hit inside $B(y, s^{1+\beta'})$ (where $\beta' > 0$ is a parameter to be chosen), and finally we force the LBM to hit y. We will allow time about $t/3$ for each step, and show that these steps respectively bring factors $e^{-s^{-(1+o(1))}}$, $s^{2+2\beta'+o(1)}$ and $O(1)$ for the lower bound of the heat kernel. This will give the lower bound $e^{-s^{-(1+o(1))}}s^{2+2\beta'+o(1)}$, which is $\geq \exp(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}-\varepsilon})$ as required.

The argument is naturally split according to these steps. In Section 4.1, we compute the probabilities of the first step in Lemma 4.1 and of the second one in Lemma 4.3, after introducing the notion of very fast points; in that section, r will be arbitrary, *i.e.* not tied to the value of t. We pick the value of r according to t in Section 4.2, where we will deal with the third step and show the lower bound.

4.1 Lower bound for hitting probability

Suppose $\delta > 0$, $r \geq 1$ integer, and set $s = 2^{-kr}$. Take $\delta_1 = s^{3\delta}$, $\delta_2 = s^{2\delta}$, $\delta_3 = s^{\delta}$, and define fast points/boxes with respect to the parameters δ_1, δ_2 and δ_3.

Lemma 4.1. *There exist positive constants c, $k_0 = k_0(\delta)$, $c_0 = c_0(k,\delta)$ and $r_0 = r_0(x,y,\gamma,\delta,k)$, not depending on r but possibly depending on k,γ, such that the following holds for $k \geq k_0$ and $r \geq r_0$. Suppose D is a random set (i.e. depending on h) and $D \subseteq BD_r(y)$. Let ς_1 be the hitting time of D by the LBM started from x. Then, with $\mathbb{P}$-probability at least $1 - e^{-c_0 r} - \mathbb{P}(|D| < \delta_3 s^2)$,*

$$P^x(\varsigma_1 \leq s^{1+\frac{1}{2}\gamma^2-4\delta-c\gamma\delta}) \geq e^{-s^{-(1+2\delta)}}. \tag{4.1}$$

Proof. We construct a sequence of neighboring s-boxes connecting x and y, as follows. Discretize $\mathbb{T}$ by regarding each $B \in \mathcal{BD}_r$ (equivalently, its center c_B) as a point in $\mathbb{Z}^2$. We investigate the discrete Gaussian field $\Phi := \{\varphi_r(c_B), B \in \mathcal{BD}_r\}$, together with the Bernoulli process $\Xi := \{\xi_B, B \in \mathcal{BD}_r\}$ defined by $\xi_B := 1$ iff B is fast. Next we will apply

[6, Theorem 1.7] to (Φ, Ξ). Set $N = 2^{kr}$, and correspond B, $\varphi_r(c_B)$, ξ_B respectively to $w \in \mathbb{Z}^2$, $\varphi_{N,w}$, $\xi_{N,w}$ in [6]. Then,

- Ξ is independent of Φ, since Ξ depends on the fine field while Φ depends on the coarse field.
- The collection of random variables $\{\xi_B\}_{B\in\mathcal{BD}_r}$ has finite range dependence, in particular ξ_B is independent of $\xi_{B'}$ if $|c_B - c_{B'}|_\infty > 9s$. (In the language of [6], Ξ is q-dependent for $q = 9$.)
- $\mathbb{P}(\xi_B = 1)$ is equal to a same value p for all B.

For constants $c(\geq 2), \delta, r$, we introduce the event $\mathcal{E}_1 = \mathcal{E}_1(c, \delta, r, k)$ defined as the existence of a sequence B_i, $i = 1, \cdots, I$ of s-boxes in $\mathcal{BD}_r$ satisfying the following properties:

(a) $\varphi_r(c_{B_i}) \leq (c-1)\delta kr \log 2$, $i = 1, \ldots, I$.

(b) B_i is fast (*i.e.*, $\xi_{B_i} = 1$), $i = 1, \ldots, I$.

(c) $I \leq s^{-(1+\delta)}$.

(d) $B_1 = BD_r(x)$, $B_I = BD_r(y)$, and B_{i+1} is a neighbor of B_i, i.e. $|c_{B_{i+1}} - c_{B_i}| = s$, $i = 1, \ldots, I-1$.

By Lemma 3.4, $p \geq 1 - 2s^\delta \to 1$ as $r \to \infty$. In particular, p is larger than p_1 defined in [6, Theorem 1.7], when $r \geq r_1(\delta)$. As in [6, Theorem 1.7], there exist positive constants $c(\geq 2)$, k_0, $\tilde{c}_0 = \tilde{c}_0(\delta)$ and $r_2 = r_2(x, y, \gamma, \delta, k) \geq r_1$ so that, for $k \geq k_0$ and $r \geq r_2$,

$$\mathbb{P}(\mathcal{E}_1) \geq 1 - (1-p)^{1/400} - e^{-\tilde{c}_0 r}, \tag{4.2}$$

where we use $q = 9$ and $p \to 1$ as $r \to \infty$.

Remark 4.2. (i) The space is the torus $\mathbb{T}$ here, while it is a box in [6]. One can identify the torus as $[0, 4)^2$, and consider the box $[1, 3]^2$ where we locate x and y, noting that $h(z)$ is independent of $h(w)$ if $|z - w| \geq 2$. (ii) To achieve (4.2), it is not crucial whether one uses balls $B(x, R)$ (as in our situation) or boxes $B_{2R}(x)$ (as in [6]) to define $A(x, y; R)$. That is, the proof of (4.2) is similar to that of [6, Theorem 1.7].

Let $\mathcal{E}_2$ be the event that the following properties hold.

(a′) $|\varphi_r(z) - \varphi_r(c_B)| \leq \delta kr \log 2$ for all $z \in B^*$ and $B \in \mathcal{BD}_r$.

(b′) x is fast.

By Corollary 2.5, $\mathbb{P}(\text{a}') \geq 1 - e^{-r}$. By Lemma 3.3, $\mathbb{P}(\text{b}') \geq 1 - \delta_1/\delta_2 = 1 - 2^{-k\delta r}$. Take c_0 such that $2^{\frac{1}{400}} 2^{-\frac{k\delta}{400} r} + e^{-\tilde{c}_0 r} + e^{-r} + 2^{-k\delta r} \leq e^{-c_0 r}$. Then, we have

$$\mathbb{P}(\mathcal{E}) \geq 1 - e^{-c_0 r} - \mathbb{P}(|D| < \delta_3 s^2), \text{ where } \mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \{|D| \geq \delta_3 s^2\}.$$

Next, we are going to show that (4.1) holds on $\mathcal{E}$, completing the proof. Suppose $\mathcal{E}$ holds. We will force the SBM to follow this sequence of boxes; to control the LBM time, we will force also passage through fast points, and some additional properties, as follows. Recall that $\{X_u^x\}$ is the SBM starting from x. Construct a sequence of hitting times σ_i as follows. Let $\sigma_1 = 0$. Then $X_{\sigma_1}^x = x \in B_1 \cap \mathcal{F}$ by (b′). Suppose that σ_i has been defined, such that $x_i := X_{\sigma_i}^x \in B_i \cap \mathcal{F}$. Define

$$\sigma_{i+1} := \inf\{u \geq \sigma_i : X_u^x \in A\}, \quad \text{and } \tau_i = \sigma_{i+1} - \sigma_i, \quad \text{where } A = \begin{cases} B_{i+1} \cap \mathcal{F}, & \text{if } i \leq I-2, \\ D, & \text{if } i = I-1. \end{cases}$$

Informaly, τ_i is the time it takes for the SBM to cross B_i into the next box B_{i+1} and hit a fast point.

Note that (a) together with (a′) implies that

(a″) For all $z \in \cup_i B_i^*$, $\varphi_r(z) \leq c\delta kr \log 2$.

In order to take advantage of (a$''$), we need to also control the path of the SBM when traveling from x_i to $B_{i+1} \cap \mathcal{F}$. Toward this end, define

$$\tilde{\sigma}_i = \inf\{u \geq \sigma_i : X_u^x \in \partial B_i^*\} \quad \text{and } \tilde{\tau}_i = \tilde{\sigma}_i - \sigma_i.$$

Thus, $\tilde{\tau}_i$ is the time it takes the SBM to exit B_i^* when starting at x_i. We will force the events $\tau_i \leq s^2$ and $\tau_i \leq \tilde{\tau}_i$ to ensure that the LBM stays inside B_i^* and spends a short enough time to hit $B_{i+1} \cap \mathcal{F}$.

Let $\rho_1(w, w')$ denote the heat kernel of the SBM, killed at exiting $[0,5]^2$, at time 1. Let

$$C_5 := \frac{1}{2} \min_{w,w' \in [1,4]^2} \rho_1(w, w'), \tag{4.3}$$

which is positive. Then, for any $i \geq 1$,

$$P^x(\tau_i \leq s^2 \leq \tilde{\tau}_i) \geq 2C_5\delta_3$$

since on $\mathcal{E}$, $|B_{i+1} \cap \mathcal{F}| \geq \delta_3 s^2$ by (b), and $|D| \geq \delta_3 s^2$. Let

$$\hat{\tau}_i := \inf\{u \geq 0 : X_{\sigma_i + u} \in \partial B_{6s}(x_i)\}.$$

Recall that x_i is a fast point, $\forall i \leq I - 1$. By the strong Markov property of the ψ_r-LBM,

$$P^x(F_r(\sigma_i + s^2 \wedge \hat{\tau}_i) - F_r(\sigma_i) \leq s^2/\delta_1) = P^{x_i}(F_r(s^2 \wedge \sigma_{x_i,6s}) \leq s^2/\delta_1) \geq 1 - \delta_2.$$

Therefore,

$$P^x(\tau_i \leq s^2 \leq \tilde{\tau}_i,\ F_r(\sigma_i + s^2 \wedge \hat{\tau}_i) - F_r(\sigma_i) \leq s^2/\delta_1) \geq 2C_5\delta_3 - \delta_2 \geq C_5\delta_3$$

for r larger than $r_3 := r_3(x, y, \gamma, \delta, k) \geq r_2$, where we used that $\delta_2 = o(\delta_3)$ as $r \to \infty$. By definition, $\tilde{\tau}_i \leq \hat{\tau}_i$. Hence, if $\tau_i \leq s^2 \leq \tilde{\tau}_i$, we have $\tau_i \leq s^2 \wedge \hat{\tau}_i$ thus $F_r(\sigma_{i+1}) \leq F_r(\sigma_i + s^2 \wedge \hat{\tau}_i)$, and by (a$''$),

$$F^x(\sigma_{i+1}) - F^x(\sigma_i) \leq e^{\gamma c\delta kr \log 2 - \frac{1}{2}\gamma^2 kr \log 2}\big(F_r^x(\sigma_{i+1}) - F_r^x(\sigma_i)\big).$$

Collecting the above inequalities, we have that for $i = 1, \ldots, I - 1$,

$$P^x(F(\sigma_{i+1}) - F(\sigma_i) \leq e^{\gamma c\delta kr \log 2 - \frac{1}{2}\gamma^2 kr \log 2} s^2/\delta_1) \geq C_5\delta_3. \tag{4.4}$$

Finally, note that $\varsigma_1 \leq \sum_{i=1}^{I-1}(F^x(\sigma_{i+1}) - F^x(\sigma_i))$. By (c), (4.4) and the strong Markov property of the LBM,

$$P^x(\varsigma_1 \leq I e^{\gamma c\delta kr \log 2 - \frac{1}{2}\gamma^2 kr \log 2} s^2/\delta_1)) \geq (C_5\delta_3)^I \geq e^{-s^{-(1+2\delta)}} \tag{4.5}$$

for $r \geq r_0 \geq r_3$. Note however that $I e^{\gamma c\delta kr \log 2 - \frac{1}{2}\gamma^2 kr \log 2} s^2/\delta_1 \leq s^{1+\frac{1}{2}\gamma^2 - 4\delta - c\gamma\delta}$. Together with (4.5), this completes the proof of the lemma. □

Let $\beta' > 0$ be fixed. Abbreviate $B = BD_r(y)$, and set $A = B \cap B(y, s^{1+\beta'})$. Denote by τ_A (respectively, τ^*) the times that the SBM hits A (respectively, ∂B^*). A point $z \in B$ is called *very fast* if $P^z(F_r(s^2) \leq s^{2-\delta} | \tau_A \leq s^2 \leq \tau^*) \geq 1/2$. Let $\mathcal{VF}$ denote the set of very fast points. We would like to mention that the very fast property does not imply the fast property.

Lemma 4.3. *(i)* $\mathbb{P}(|\mathcal{VF}| \geq \delta_3 s^2) \geq 1 - 3s^{\delta}$.
(ii) Let ς_2 denote the time that the LBM hits A. Then, there exists $r_1 = r_1(\delta, \gamma, k)$ such that the following holds for $r \geq r_1$. With $\mathbb{P}$-probability at least $1 - 2e^{-\frac{1}{8}\delta^2 kr \log 2}$,

$$P^z(\varsigma_2 \leq s^{2+\frac{1}{2}\gamma^2 - \delta - \gamma\delta}) \geq s^{2+2\beta'+\delta}, \quad \forall z \in \mathcal{VF}. \tag{4.6}$$

Proof. The proof of (i) is parallel to Lemma 3.4(i) combined with Lemma 3.3(i), while that of (ii) is parallel to (4.4).

(i) Set $\xi = F_r^z(s^2)$ and $\eta = P^z(\xi > s^{2-\delta}|\tau_A \le s^2 \le \tau^*)$. By a proof similar to that of Lemma 3.3(i), $\mathbb{P}(z \notin \mathcal{VF}) = \mathbb{P}(\eta > 1/2) \le 2\mathbb{E}\eta = 2E^z\big(\mathbb{P}(\xi > s^{2-\delta})|\tau_A \le s^2 \le \tau^*\big) \le 2s^\delta$ since $\mathbb{P}(\xi > s^{2-\delta}) \le s^{\delta-2}\mathbb{E}\xi = s^\delta$, for all $z \in B$. Then, $(1-2s^\delta)s^2 \le \mathbb{E}|\mathcal{VF}| \le s^2\mathbb{P}(|\mathcal{VF}| \ge \delta_3 s^2) + \delta_3 s^2$, *i.e.* $\mathbb{P}(|\mathcal{VF}| \ge \delta_3 s^2) \ge 1 - 2s^\delta - \delta_3 = 1 - 3s^\delta$, where we recall that $\delta_3 = s^\delta$.

(ii) For any $z \in \mathcal{VF}$,

$$P^z(F_r(s^2) \le s^{2-\delta}, \tau_A \le s^2 \le \tau^*) \ge \frac{1}{2}P^z(\tau_A \le s^2 \le \tau^*).$$

With C_5 defined in (4.3), we have $P^z(\tau_A \le s^2 \le \tau^*) \ge 2C_5|A| \ge 2C_5 \times \frac{1}{4}\pi s^{2(1+\beta')}$, noting that A contains at least a quarter of $B(y, s^{1+\beta'})$. It follows that, for r large enough,

$$P^z(F_r(s^2) \le s^{2-\delta}, \tau_A \le s^2 \le \tau^*) \ge \frac{C_5\pi}{4}s^{2+2\beta'} \ge s^{2+2\beta'+\delta}.$$

By Corollary 2.5, with probability $\ge 1 - 2e^{-\frac{1}{8}\delta^2 kr\log 2}$, we have $\varphi_r(w) \le \delta kr\log 2$ for all $w \in B^*$. On this event,

$$\{F_r(s^2) \le s^{2-\delta}, \tau_A \le s^2 \le \tau^*\} \Rightarrow \{\varsigma_2^z \le e^{\gamma\delta kr\log 2 - \frac{1}{2}\gamma^2 kr\log 2}s^{2-\delta}\}$$

for all $z \in B$. Noting that $e^{\gamma\delta kr\log 2-\frac{1}{2}\gamma^2 kr\log 2}s^{2-\delta} = s^{2+\frac{1}{2}\gamma^2-\delta-\gamma\delta}$ completes the proof. □

4.2 Proof of the lower bound in (1.2)

We take

$$r_t = \lceil -\frac{\log t - \log 3}{(1+\frac{1}{2}\gamma^2 - 4\delta - c\gamma\delta)k\log 2}\rceil,$$

and set $s = 2^{-kr_t}$ so that

$$2^{-k}(t/3)^{\frac{1}{1+\frac{1}{2}\gamma^2-4\delta-c\gamma\delta}} < s \le (t/3)^{\frac{1}{1+\frac{1}{2}\gamma^2-4\delta-c\gamma\delta}}. \tag{4.7}$$

The following lemma is a straight forward adaptation of [17, Corollary 5.20]. We omit the details.

Lemma 4.4. *There exists a constant $\beta = \beta(\gamma, k)$ and a positive random variable $U_0 = U_0(\gamma, k; h)$ such that for all $u \le U_0$,*

$$\inf_{z\in\mathbb{T}} \inf_{w\in\mathbb{T},|w-z|\le u^\beta} p_u^\gamma(z,w) \ge 1.$$

Set $\beta' = (1+\frac{1}{2}\gamma^2 - 4\delta - c\gamma\delta)\beta$. By (4.7), $\ell := s^{1+\beta'} \le s^{\beta'} \le s^{(1+\frac{1}{2}\gamma^2-4\delta-c\gamma\delta)\beta} \le (t/3)^\beta$. Let ς be the time the LBM hits the small ball $B(y,\ell)$. It follows that $\ell \le u^\beta$. Consequently, by strong Markov property and Lemma 4.4, it follows

$$p_t^\gamma(x,y) \ge P^x(\varsigma \le 2t/3), \quad \forall t \le U_0. \tag{4.8}$$

Next, we estimate $P^x(\varsigma \le 2t/3)$. We follow the notations in Lemma 4.1 and Lemma 4.3. Define very fast points with respect to the parameter β', and take D as $\mathcal{VF}$. Then, for any $r \ge r_0 \vee r_1$, (4.1) and (4.6) hold simultaneously, with probability $1 - e^{-c_0 r} - 3s^\delta - 2e^{-\frac{1}{8}\delta^2 kr\log 2}$. Note that $t \to 0$ is equivalent to $r_t \to \infty$. By the Borel-Cantelli Lemma, we can find $T_0 = T_0(x,y,\gamma,\varepsilon,k;h) < U_0$ such that for all $t \le T_0$, both (4.1) and (4.6) hold for $r = r_t$, and furthermore

$$e^{-s^{-(1+2\delta)}}s^{2+2\beta'+\delta} \ge \exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}-\varepsilon}\right) \tag{4.9}$$

where we take δ (according to ε) such that $\frac{1+2\delta}{1+\frac{1}{2}\gamma^2-4\delta-c\gamma\delta} < \frac{1}{1+\frac{1}{2}\gamma^2} + \varepsilon$. By the strong Markov property, $P^x(\varsigma \le 2t/3) \ge P^x(\varsigma_1 \le t/3)\min_{z\in\mathcal{VF}} P^z(\varsigma_2 \le t/3) \ge e^{-s^{-(1+3\delta)}}s^{2+2\beta'+\delta}$. This, together with (4.8) and (4.9), gives the lower bound in (1.2). □

5 Proof of the upper bound in (1.2)

We begin with the following lemma, whose proof is a slight adaptation of that of [17, Theorem 4.2]. We omit further details of the proof.

Lemma 5.1. *For any $\varepsilon > 0$ there exist $\beta = \beta(\varepsilon, \gamma, k) > 0$ and positive random constants $c_1 = c_1(h)$ and $c_2 = c_2(h)$ such that, for all $z, w \in \mathbb{T}$ and $u > 0$,*

$$p_u^\gamma(z,w) \le \frac{c_1}{u^{1+\varepsilon}} \exp\left(-c_2 \left(\frac{|z-w|}{u^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right).$$

We turn to the proof of the upper bound in (1.2). Fix α such that

$$\alpha > 1 \ \text{ and } \ (\frac{\alpha}{\beta} - 2)\frac{\beta}{\beta-1} \ge \frac{1}{1+\frac{1}{2}\gamma^2} \ ,$$

and set $u = t^\alpha$ in Lemma 5.1. Then, for $z \notin B(y, t^2)$,

$$\begin{aligned} p_{t^\alpha}^\gamma(z,y) &\le \frac{c_1}{t^{\alpha(1+\varepsilon)}} \exp\left(-c_2 \left(\frac{t^2}{t^{\alpha/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \\ &\le \frac{c_1}{t^{\alpha(1+\varepsilon)}} \exp\left(-c_2 t^{-\frac{1}{1+\frac{1}{2}\gamma^2}}\right) \le \exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon}\right), \end{aligned}$$

where the last two inequalities hold for t smaller than some $T_1(\gamma, \varepsilon, k, h)$. It follows that

$$\int_{|z-y|\ge t^2} p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) \le \exp(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon}). \tag{5.1}$$

On the other hand, again from Lemma 5.1, $p_{t^\alpha}^\gamma(z,y) \le \frac{c_1}{t^{\alpha(1+\varepsilon)}}$ for all z. Thus,

$$\int_{|z-y|<t^2} p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) \le \frac{c_1}{t^{\alpha(1+\varepsilon)}} P^x\left(|Y_{t-t^\alpha} - y| < t^2\right).$$

Assume $t^2 \le |x-y|/2$ and set

$$\varsigma := \inf\{u \ge 0 : Y_u^x \notin B(x, |x-y|/2)\}.$$

Note that $\{|Y_{t-t^\alpha} - y| < t^2\} \Rightarrow \{\varsigma \le t\}$. In Lemma 5.2 below, we will show

$$P^x(\varsigma \le t) \le \exp\left(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon}\right) \tag{5.2}$$

for t smaller than some $T_2(\gamma, k, \varepsilon; h)$. It then follows that

$$\int_{|z-y|<t^2} p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) \le \frac{c_1}{t^{\alpha(1+\varepsilon)}} \exp(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon}).$$

Combining the above inequality with (5.1), we conclude that

$$\begin{aligned} p_t^\gamma(x,y) &= \int p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) \\ &= \int_{|z-y|<t^2} p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) + \int_{|z-y|\ge t^2} p_{t-t^\alpha}^\gamma(x,z) p_{t^\alpha}^\gamma(z,y) \mu^\gamma(dz) \\ &\le (1 + \frac{c_1}{t^{\alpha(1+\varepsilon)}}) \exp(t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon}) \le \exp(t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\varepsilon}) \end{aligned}$$

for t less than some T_0. This completes the proof of the upper bound in (1.2), modulu the proof of Lemma 5.2. □

Lemma 5.2. *There exists $k_0 = k_0(\varepsilon)$ and a random variable $T_2 = T_2(\gamma, k, \varepsilon; h)$ such that, for all $k \geq k_0$ and $t < T_2$, (5.2) holds, $\mathbb{P}$-a.s.*

Proof. The proof is similar to that of Lemma 4.1. We will discretize $\mathbb{T}$ using $\mathcal{BD}_r$, and show that for $\delta > 0$ and k large enough,

$$P^x(\varsigma \leq 2^{-kr(1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta)}) \leq e^{-2^{kr(1-2\delta)}} \tag{5.3}$$

for all $r \geq r_0(\gamma, k, \delta; h)$, $\mathbb{P}$-a.s., where $c > 0$ is a constant. Then, we will pick a proper δ (according to ε) and a proper r (according to t), to obtain the lemma.

We begin by discretizing $\mathbb{T}$, fixing $r \geq 1$ and $s = 2^{-kr}$. We identify each $B \in \mathcal{BD}_r$ (equivalently, its center c_B) as a point in $\mathbb{Z}^2$ in the natural way. We next define inductively the discrete path associated with the path $\{X_u : u \leq \tilde{\varsigma}\}$, where $\{X_u\}$ is the SBM starting from x and $\tilde{\varsigma}$ is the time $\{X_u\}$ hits $\partial B(x, \frac{1}{4}|x-y|)$. We use the radius $\frac{1}{4}|x-y|$ rather than $\frac{1}{2}|x-y|$ for the convenience that we do not involve the last point in the discrete path (defined below) to $\partial B(x, \frac{1}{2}|x-y|)$.

Let $\tau_1 = 0$. Suppose τ_i has been defined. Set $B_i := BD_r(X_{\tau_i})$. Then, define

$$\tau_{i+1} := \inf\{u \geq \tau_i : X_u \in \partial B_i^*\}.$$

This procedure stops naturally when τ_{i+1} cannot be defined. We call this sequence of B_i's a discrete path from x to $\partial B(x, \frac{1}{4}|x-y|)$.

Next, set $\varepsilon_1 := s^\delta$, $\varepsilon_2 := C_3 e^{-6k\gamma^2}$, $\varepsilon_3 := C_3^2 e^{-12k\gamma^2}$, and define slow points/boxes with respect to ε_1, ε_2 and ε_3. Set $\xi_B := \mathbf{1}_{\{B \text{ is slow}\}}$. We study the discrete Gaussian field $\Phi = \{\varphi_r(c_B), B \in \mathcal{BD}_r\}$ and the Bernoulli process $\Xi = \{\xi_B, B \in \mathcal{BD}_r\}$. Note that Ξ is of finite range dependence (4-dependent in the language of [6]), and by Lemma 3.4, $P(\xi_B = 1) = p \geq 1 - 2^{-rk\delta C_3 e^{-6k\gamma^2} 2^{-2k}}$, which converges to 1 as $r \to \infty$. For (Φ, Ξ), similarly to [6, Theorem 1.5], we can find positive constants c, k_0, $\tilde{c}_0 = \tilde{c}_0(\delta)$ and $r_1 = r_1(x, y, \gamma, \delta, k)$ such that the following holds for $k \geq k_0$ and $r \geq r_1$. With probability $\geq 1 - e^{-\tilde{c}_0 r}$, we can find boxes B_{i_j}, $j = 1, \cdots, I$ in any discrete path from x to $\partial B(x, \frac{1}{4}|x-y|)$ such that $\varphi_r(c_{B_{i_j}}) \geq -(c-1)\delta kr \log 2$, $\forall j$, and the following properties hold.

(a) B_{i_j} is slow (*i.e.* $\xi_{B_{i_j}} = 1$), $\forall j$.

(b) $I \geq s^{-(1-\delta)}$.

Furthermore, by Corollary 2.5, with probability at least $1 - e^{-\tilde{c}_0 r} - e^{-r}$, we have (a), (b) and the following property (c) all hold.

(c) $\varphi_r(z) \geq -c\delta kr \log 2$, $\forall z \in B_{i_j}^*$, $\forall j$.

Remark 5.3. When a discrete path is identified as a sequence of points $v_0, v_1, \cdots$ on $\mathbb{Z}^2$, v_{i+1} may not be a neighbour of v_i. However, we have $|v_{i+1} - v_i|_\infty \leq 2$ for all i. Then, the proof in [6, Theorem 1.5] automatically extends to the current setup.

Set $\sigma_j = F_r^x(\tau_{i_j+1}) - F_r^x(\tau_{i_j})$ and $\chi_j := \mathbf{1}_{\{\sigma_j \geq \varepsilon_1 s^2\}}$. By (a) and Lemma 3.5, $P^x(\chi_j = 1) \geq C_4\varepsilon_2\varepsilon_3$ for all j, which implies that $\mathbb{E}e^{-\chi_j} \leq 1 - C_4\varepsilon_2\varepsilon_3(1-e^{-1}) \leq e^{-C_4\varepsilon_2\varepsilon_3(1-e^{-1})}$. Note that the σ_j's are mutually independent by the strong Markov property of the ψ_r-LBM, and so are the χ_j's. Therefore,

$$P^x\left(\sum_{j=1}^{I} \chi_\ell \leq \varepsilon_1 I\right) \leq (e^{\varepsilon_1}\mathbb{E}e^{-\chi_j})^I \leq e^{-(C_4\varepsilon_2\varepsilon_3(1-e^{-1})-\varepsilon_1)I} \leq e^{-\frac{1}{2}C_4\varepsilon_2\varepsilon_3 I}, \tag{5.4}$$

where we use that $\varepsilon_1 = 2^{-kr\delta} < C_4\varepsilon_2\varepsilon_3(1 - e^{-1} - \frac{1}{2})$ for all r larger than some $r_2 := r_2(\gamma, \delta) > r_1$. By (c), $\chi_j = 1$ implies that

$$F^x(\tau_{i_j+1}) - F^x(\tau_{i_j}) \geq e^{-\gamma c\delta kr \log 2 - \frac{1}{2}\gamma^2 kr \log 2}\sigma_j \geq 2^{-\gamma c\delta kr - \frac{1}{2}\gamma^2 kr}\varepsilon_1 s^2.$$

Thus, $\sum_{j=1}^{I}\chi_j > \varepsilon_1 I$ implies that

$$\varsigma > 2^{-\gamma c\delta kr-\frac{1}{2}\gamma^2 kr}\varepsilon_1 s^2 \times \varepsilon_1 I \geq 2^{-kr(1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta)}.$$

This, together with (5.4) implies that

$$P^x(\varsigma \leq 2^{-kr(1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta)}) \leq \mathbb{P}\left(\sum_{j=1}^{I}\chi_j \leq \varepsilon_1 I\right) \leq e^{-\frac{1}{2}C_4\varepsilon_2\varepsilon_3 2^{kr(1-\delta)}} \leq e^{-2^{kr(1-2\delta)}},$$

for all r larger than some $r_3 := r_3(\gamma,\delta,k) \geq r_2$. By the Borel-Cantelli Lemma, there exists a random number $r_0 = r_0(\gamma,k,\delta;h)$ such that (5.3) holds for all $r \geq r_0$, $\mathbb{P}$-a.s..

For any t, define

$$r_t := \lfloor -\frac{\log t}{\left(1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta\right)k\log 2}\rfloor.$$

Equivalently,

$$2^{kr_t} \leq t^{-\frac{1}{1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta}} < 2^{k(r_t+1)}. \tag{5.5}$$

Note that $t \to 0$ is equivalent to $r_t \to \infty$. Therefore, there exists a random constant $\tilde{T}_0 = \tilde{T}_0(\gamma,k,\delta;h)$ such that for any $t \leq \tilde{T}_0$ (equivalently, $r_t \geq r_0$), (5.3) holds for $r = r_t$. This together with (5.5) yields that

$$P^x(\varsigma \leq t) \leq \exp\left(-(2^{-k}t^{-\frac{1}{1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta}})^{1-2\delta}\right).$$

Finally, we pick δ such that $\frac{1-2\delta}{1+\frac{1}{2}\gamma^2+3\delta+c\gamma\delta} > \frac{1}{1+\frac{1}{2}\gamma^2} - \frac{1}{2}\varepsilon$, and then pick $T_0(\gamma,k,\varepsilon;h) \leq \tilde{T}_0$ such that the right hand side above is less than $\exp(-t^{-\frac{1}{1+\frac{1}{2}\gamma^2}+\frac{1}{2}\varepsilon})$, completing the proof. □

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