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Intrinsic isoperimetry of the giant component of supercritical bond percolation in dimension two*

Julian Gold[†]

Abstract

We study the isoperimetric subgraphs of the giant component $\mathbf{C}_n$ of supercritical bond percolation on the square lattice. These are subgraphs of $\mathbf{C}_n$ with minimal edge boundary to volume ratio. In contrast to the work of [8], the edge boundary is taken only within $\mathbf{C}_n$ instead of the full infinite cluster. The isoperimetric subgraphs are shown to converge almost surely, after rescaling, to the collection of optimizers of a continuum isoperimetric problem emerging naturally from the model. We also show that the Cheeger constant of $\mathbf{C}_n$ scales to a deterministic constant, which is itself an isoperimetric ratio, settling a conjecture of Benjamini in dimension two.

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1 Introduction and results

Isoperimetric problems, while among the oldest in mathematics, are fundamental to modern probability and PDE theory. The goal of an isoperimetric problem is to characterize sets of minimal boundary measure subject to an upper bound on the volume measure of the set. The Cheeger constant, introduced by Alon-Milman [3] and Tanner [36], is a way of encoding such problems. It takes its name from Cheeger's work [18] in the continuum. For (finite) graphs G, it is defined as the following minimum over subgraphs of G:

$$\Phi_G := \min \left\{ \frac{|\partial H|}{|H|} : H \subset G,\, 0 < |H| \leq |G|/2 \right\} , \tag{1.1}$$

Here ∂H is the edge boundary of H in G (the edges of G having exactly one endpoint vertex in H), $|\partial H|$ denotes the cardinality of this set, and $|H|$ denotes the cardinality of

*This research was supported in part by the National Science Foundation and by UCLA.
[†]Northwestern University, United States of America.
E-mail: gold@math.northwestern.edu

the vertex set of H. The Cheeger constant of a graph measures its robustness; it provides information about the behavior of random walks and is a useful object in spectral graph theory (see Chapter 2 of [19]). This paper is concerned with isoperimetric properties of random graphs arising from bond percolation in $\mathbb{Z}^2$.

Bond percolation is defined as follows: view $\mathbb{Z}^2$ as a graph with standard nearest-neighbor graph structure and form the probability space $(\{0,1\}^{\mathrm{E}(\mathbb{Z}^2)}, \mathcal{F}, \mathbb{P}_p)$ for the *percolation parameter* $p \in [0,1]$. Here $\mathcal{F}$ denotes the product σ-algebra on $\{0,1\}^{\mathrm{E}(\mathbb{Z}^2)}$ and $\mathbb{P}_p$ is the product Bernoulli measure associated to p. Elements of this probability space are written as $\omega = (\omega_e)_{e\in\mathrm{E}(\mathbb{Z}^2)}$ and are called *percolation configurations*. An edge e is *open* in the configuration ω if $\omega_e = 1$ and is *closed* otherwise. For each ω, the edges open in ω determine a subgraph of $\mathbb{Z}^2$, denoted $[\mathbb{Z}^2]^\omega$. Under the probability measure $\mathbb{P}_p$, $[\mathbb{Z}^2]^\omega$ is a random subgraph of $\mathbb{Z}^2$.

Connected components of $[\mathbb{Z}^2]^\omega$ are *open clusters*, or simply *clusters*. Bond percolation on $\mathbb{Z}^2$ exhibits a well known (Grimmett [25] is a standard reference) phase transition: there is $p_c(2) \in (0,1)$ so that $p > p_c(2)$ implies there is a unique infinite open cluster $\mathbb{P}_p$-almost surely, while $p < p_c(2)$ implies there is no infinite open cluster $\mathbb{P}_p$-almost surely. It is well known [26] that $p_c(2) = 1/2$. We focus on the supercritical ($p > p_c(2)$) regime, writing $\mathbf{C}_\infty = \mathbf{C}_\infty(\omega)$ for the almost surely unique infinite cluster. For $p > p_c(2)$, the quantity $\theta_p := \mathbb{P}_p(0 \in \mathbf{C}_\infty)$ is positive, and is the *density* of $\mathbf{C}_\infty$ in $\mathbb{Z}^2$.

1.1 A conjecture

It is possible to study the geometry of $\mathbf{C}_\infty$ using the Cheeger constant: define $\widetilde{\mathbf{C}}_n := \mathbf{C}_\infty \cap [-n,n]^2$, and define the *giant component* $\mathbf{C}_n$ to be the largest connected component of $\widetilde{\mathbf{C}}_n$. The random variable $\Phi_n := \Phi_{\mathbf{C}_n}$ is central to this paper. It is known (Benjamini and Mossel [6], Mathieu and Remy [30], Rau [34], Berger, Biskup, Hoffman and Kozma [7] and Pete [31]) that $\Phi_n \asymp n^{-1}$ as $n \to \infty$, prompting the following conjecture of Benjamini, which we state in all dimensions $d \geq 2$.

Conjecture 1.1. (Benjamini) Let $d \geq 2$ and $p > p_c(d)$. The limit

$$\lim_{n\to\infty} n\Phi_{\mathbf{C}_n} \tag{1.2}$$

exists $\mathbb{P}_p$-almost surely as a deterministic constant in $(0,\infty)$.

Procaccia and Rosenthal [33] showed for $d \geq 2$ that $\mathbb{V}\mathrm{ar}(n\Phi_n) \leq cn^{2-d}$, with $c(p,d) > 0$. Biskup, Louidor, Procaccia and Rosenthal [8] settled Conjecture 1.1 in $d = 2$ for a natural modification $\widetilde{\Phi}_n$ of Φ_n. The results of [8] go beyond resolving Conjecture 1.1 for $\widetilde{\Phi}_n$: the random variables $\widetilde{\Phi}_n$ encode a sequence of discrete, random isoperimetric problems, whose optimizers are the subgraphs of $\widetilde{\mathbf{C}}_n$ realizing the minimum defining $\widetilde{\Phi}_n$. The main result of [8] is that these optimizers, upon rescaling, tend almost surely (with respect to Hausdorff distance) to a translate of a deterministic shape, a convex subset of $[-1,1]^2$ whose two-dimensional Lebesgue measure is half that of $[-1,1]^2$. This limit shape, called the *Wulff shape* and denoted W_p, is the solution to a deterministic isoperimetric problem in the continuum, posed for rectifiable subsets of $[-1,1]^2$.

We settle Conjecture 1.1 for the original Cheeger constant Φ_n using the strategy of [8]. The distinction between Φ_n and the modified Cheeger constant $\widetilde{\Phi}_n$ is that, in the latter object, the edge boundary of a subgraph $H \subset \mathbf{C}_n$ is taken in the full infinite cluster $\mathbf{C}_\infty$ instead of just $\mathbf{C}_n$. This modification simplifies the nature of the limiting isoperimetric problem, which is the analogue of the standard Euclidean isoperimetric problem for an anisotropic perimeter functional. In our case, a *restricted perimeter* functional replaces the perimeter functional, reflecting the fact that Φ_n does not “see" edges outside the box $[-n,n]^2$.

1.2 The general form of the limiting variational problem

A *curve* λ in the unit square $[-1,1]^2$ is the image of a continuous function $\lambda : [0,1] \to [-1,1]^2$. A curve λ is *closed* if $\lambda(0) = \lambda(1)$ in any parametrization, *Jordan* if it is closed and one-to-one on $[0,1)$ and *rectifiable* if there is a parametrization of λ such that

$$\text{length}(\lambda) := \sup_{n\in\mathbb{N}} \sup_{t_1<\cdots<t_n\in[0,1]} \sum_{j=1}^{n} |\lambda(t_j) - \lambda(t_{j-1})|_2 < \infty \,. \tag{1.3}$$

Many curves considered in this paper will be Jordan, and we often conflate a curve λ with its image, denoted $\mathsf{image}(\lambda)$. We will use greater care in Section 3, where the variational problem (1.6) is studied. The setting of this variational problem is the following class $\mathcal{R}$ of sets:

$$\mathcal{R} := \left\{ R \subset [-1,1]^2 \; : \begin{array}{l} R \text{ is compact, } R^\circ \neq \emptyset,\ \partial R \text{ is a finite union of rectifiable Jordan} \\ \text{curves, and the intersection of any two such curves is } \mathcal{H}^1\text{-null} \end{array} \right\}, \tag{1.4}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, and where R° denotes the interior of R. Given a norm τ on $\mathbb{R}^2$, define the restricted perimeter functional $\mathcal{I}_\tau$ on $R \in \mathcal{R}$ via

$$\mathcal{I}_\tau(\partial R) := \int_{\partial R \cap (-1,1)^2} \tau(n_x) \mathcal{H}^1(dx) \,, \tag{1.5}$$

where n_x is the normal vector to $\partial R \cap (-1,1)^2$ which exists at $\mathcal{H}^1$-almost every point on the curves $\partial R \cap (-1,1)^2$. Using $\mathcal{I}_\tau$, form the following variational problem of central interest:

$$\text{minimize: } \frac{\mathcal{I}_\tau(\partial R)}{\text{Leb}(R)}\,, \qquad \text{subject to: } \text{Leb}(R) \leq 2 \,. \tag{1.6}$$

Here $R \in \mathcal{R}$, and Leb is the two-dimensional Lebesgue measure.

1.3 Results

Let $\mathcal{G}_n$ be the set of *Cheeger optimizers*, the subgraphs of $\mathbf{C}_n$ realizing the minimum defining Φ_n. Recall that the Hausdorff metric on (non-empty) compact subsets of $[-1,1]^2$ is defined as follows: given $A, B \subset [-1,1]^2$ compact,

$$\text{d}_H(A,B) := \max\left(\sup_{x\in A} \inf_{y\in B} \left|x-y\right|_\infty, \sup_{y\in B} \inf_{x\in A} \left|x-y\right|_\infty \right) \,, \tag{1.7}$$

where for $x, y \in \mathbb{R}^2$ and $p \in [1,\infty]$, $|x-y|_p$ denotes the ℓ^p-distance between x and y. The following shape theorem is the first of our main results.

Theorem 1.2. Let $d = 2$ and let $p > p_c(2)$. There is a norm β_p on $\mathbb{R}^2$ with non-empty collection of optimizers $\mathcal{R}_p$ to the associated variational problem (1.6) so that

$$\max_{G_n\in\mathcal{G}_n} \inf_{E\in\mathcal{R}_p} \text{d}_H\left(n^{-1}G_n \,,\, E\right) \xrightarrow[n\to\infty]{} 0 \tag{1.8}$$

holds $\mathbb{P}_p$-almost surely.

The collection $\mathcal{R}_p$ inherits symmetries from the lattice and the square domain, and in particular $\mathcal{R}_p$ is invariant under rotations by $\pi/2$. This is discussed further in Section 3, while the relation between $\mathcal{R}_p$ and the limit shape appearing in [8] is the first open problem discussed in Section 1.6. The following definitions link Theorem 1.2 with the limit in Conjecture 1.1.

Definition 1.3. Let β_p be the norm in Theorem 1.2, which is the norm defined in [8]. Given $R \in \mathcal{R}$, define the ratio

$$\frac{\mathcal{I}_{\beta_p}(\partial R)}{\mathrm{Leb}(R)} \tag{1.9}$$

to be the *conductance* of R. Define the constant φ_p as

$$\varphi_p := \inf\left\{ \frac{\mathcal{I}_{\beta_p}(\partial R)}{\mathrm{Leb}(R)} \,:\, R \in \mathcal{R},\, \mathrm{Leb}(R) \leq 2 \right\}. \tag{1.10}$$

The two appearing in (1.10) and (1.6) is half the area of $[-1,1]^2$ coming from the 2 in the denominator of (1.1). Theorem 1.4 settles Conjecture 1.1 in dimension two and is the second of our main results.

Theorem 1.4. Let $d = 2$ and let $p > p_c(2)$. Then $\mathbb{P}_p$-almost surely,

$$\lim_{n\to\infty} n\Phi_n = \frac{\varphi_p}{\theta_p}, \tag{1.11}$$

where $\theta_p = \mathbb{P}_p(0 \in \mathbf{C}_\infty)$, and where $\varphi_p \in (0,\infty)$ is defined in (1.10).

Definition 1.5. For U a subgraph of $\mathbf{C}_n$, write $\partial^n U$ for the edge boundary of U in $\mathbf{C}_n$. This is the *open edge boundary of U in $\mathbf{C}_n$*. Let $\partial^\infty U$ be the edge boundary of U in all of $\mathbf{C}_\infty$, which we call the *open edge boundary of U*. The *n-conductance* of U is $|\partial^n U|/|U|$, and the *conductance* of U is $|\partial^\infty U|/|U|$.

Remark 1.6. Theorem 1.2 says the optimizers to the variational problems encoded by the Φ_n scale to the optimizers of (1.6) for $\tau = \beta_p$. The random variable Φ_n is the n-conductance of any $G_n \in \mathcal{G}_n$. Theorem 1.4 says that these n-conductances scale to the optimal conductance (1.10) of the continuum problem (1.6) for the norm β_p.

1.4 Outline

In Section 2, we recall the definition of β_p from [8], and we reintroduce the notion of *right-most paths* used to define β_p. We collect properties of the norm and of right-most paths. In Section 3, we study the variational problem (1.6) for $\tau = \beta_p$. The main outputs are existence and stability results.

In Section 4, we show the conductance of any $R \in \mathcal{R}$ with $\mathrm{Leb}(R) \leq 2$ yields upper bounds on Φ_n with high probability. This uses tools from Section 2 to pass from a nice object in the continuum to a subgraph of $\mathbf{C}_n$. We relate the conductances of these two objects, ultimately showing for any $\epsilon > 0$ that $n\Phi_n \leq (1+\epsilon)\varphi_p$ with high probability.

In Section 5, we move in the other direction, using each Cheeger optimizer $G_n \in \mathcal{G}_n$ to build $R \in \mathcal{R}$ with $\mathrm{d}_H(G_n, nR)$ small and with comparable conductance. We show the conductance of R is at least $(1-\epsilon)\varphi_p$, yielding a high probability lower bound on Φ_n. This settles Theorem 1.4. We then use the stability result of Section 3 with the main result of Section 4 to see that it is rare for G_n to be far from $\mathcal{R}_p$, settling Theorem 1.2.

1.5 Discussion and context

We use many of the tools developed in [8], and as such, our work falls under the umbrella of the Wulff construction program. This was initiated in the early 1990s independently by Dobrushin, Kotecký and Shlosman [20] in the Ising model and by Alexander, Chayes and Chayes [1] in percolation, both on the square lattice.

These works characterized the asymptotic shape of a large droplet of one phase of the model (for instance, a large finite open cluster in supercritical bond percolation). The probability of such an event decays rapidly in the size of the droplet, thus large deviation theory plays a role in the analysis and is key to defining a model-dependent norm τ. Though the large droplets are not the minimizers of any isoperimetric problem, their limit shape is the minimizer of

$$\text{minimize: } \frac{\text{length}_\tau(\partial R)}{\text{Leb}(R)}, \quad \text{subject to: } \text{Leb}(R) \leq c \tag{1.12}$$

for some constant $c > 0$, where $\text{length}_\tau(\partial R)$ is defined as in (1.5) but with the integral taken over all of ∂R. The solution to (1.12) is easily constructed and was postulated by Wulff [41] in 1901; it is a convex subset of $\mathbb{R}^2$ depending on τ. This solution is known to be unique up to translations and modifications on a null set thanks to the substantial work of Taylor [37, 38, 39], whose results hold in all dimensions at least two.

In contrast, the problem (1.6) has attracted far less attention. The shapes of droplets in the presence of a boundary, a single infinite wall, have been studied in the context of the Ising model [32, 12] using the Winterbottom construction [40]. This construction has been generalized further in a paper of Kotecký and Pfister [28], and related problems have been studied by Schlosman [35]. However, with an infinite and flat boundary, one can exploit dilation and reflection arguments (when the norm in question has the right symmetries), and this allows one to compare such problems to the unrestricted version (1.12). While we can and do use some dilation and reflection arguments in the analysis of (1.6), the finiteness of the domain complicates and limits these: for instance, we can only enlarge a shape attached to $\partial[-1,1]^2$ if it does not break through the box in the process, and we must be careful that the correct portions of the boundary of a shape remain attached $\partial[-1,1]^2$. This culminates in a lack of homothety in the solutions of (1.6) as we allow the upper bound on the area to vary, leading to a slight shift in strategy for the probabilistic arguments given later. For more details, see Remark 5.9.

The Wulff construction has been successfully employed in dimensions strictly larger than two [13, 9, 10, 15, 16], though with significant technical overhead due to geometric complications arising in higher dimensions. More details can be found in Section 5.5 of [14] and in [11]. The present work, as well as that of [8], differs from the above in that we work in an event of full probability, and that we are faced with a collection of isoperimetric problems at the discrete level. The variational problem in the continuum considered here is a limit of these discrete problems.

1.6 Open problems

We remark on several future directions:

(1) We find it desirable to classify elements of $\mathcal{R}_p$ in terms of the Wulff shape W_p, the limit shape obtained in [8] and the solution to the unrestricted isoperimetric problem (1.12) for the norm β_p. Based on work of Kotecký and Pfister [28] and Schlosman [35], we conjecture that the collection $\mathcal{R}_p$ consists of quarter-Wulff shapes or their complements in the square. Answering such questions may require a better understanding of the regularity of the norm. Questions regarding the regularity and strict convexity of β_p are interesting in their own right and touch on open problems in first-passage percolation (see for instance Chapter 2 of [5]). We remark that the shapes W_p were shown [22] to depend continuously on the parameter p, and we expect this continuity to hold in our setting as well.

(2) Instead of studying the largest connected component of $\mathbf{C}_\infty \cap [-n,n]^2$, we can fix a Jordan domain $\Omega \subset \mathbb{R}^2$ and consider the Cheeger constant of the largest connected component of $\mathbf{C}_\infty \cap n\Omega$. The argument in this paper is likely robust enough that both Cheeger asymptotics and a shape theorem can be deduced in this case (perhaps

depending on the convexity of Ω). This problem is similar in flavor to work of Cerf and Théret [17], in which the shapes of minimal cutsets in first passage percolation are studied for more general domains.

(3) A sharp limit and related shape theorem were recently obtained [24] for the modified Cheeger constant in dimensions three and higher. It is likely that by combining the techniques of [24] and the present paper, one can prove analogues of Theorem 1.2 and Theorem 1.4 for the giant component in dimensions larger than two.

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2 The boundary norm

The motivation for the construction of β_p goes back to a postulate of Gibbs [23]: that one phase of matter immersed in another will arrange itself so that the surface energy between the two phases is minimized. By regarding each $G_n \in \mathcal{G}_n$ as a droplet immersed in $\mathbf{C}_n \setminus G_n$, we can study the interface between these two “phases" and attempt to extract a surface energy.

Our tool for studying these interfaces are right-most paths, introduced in [8]. Each Cheeger optimizer G_n may be expressed using finitely many right-most circuits, which together represent the boundary of G_n and hence the total interface between G_n and $\mathbf{C}_n \setminus G_n$. We assign a configuration dependent weight to each right-most path, so that the combined weight of all right-most circuits making up the boundary of G_n is exactly $|\partial^\infty G_n|$.

Given $v \in \mathbb{S}^1$, the value $\beta_p(v)$ encodes the asymptotic minimal weight of a right-most path joining two vertices $x, y \in \mathbb{Z}^d$ with $y - x$ a large multiple of v. Thus, the norm β_p encodes the surface energy minimization taking place locally at the boundary of each G_n.

2.1 Right-most paths

Consider the graph $\mathbb{Z}^2 = (\mathrm{V}(\mathbb{Z}^2), \mathrm{E}(\mathbb{Z}^2))$. Given $x, y \in \mathrm{V}(\mathbb{Z}^2)$, a *path from x to y* is an alternating sequence of vertices and edges $\gamma = (x_0, e_1, x_1, \dots, e_m, x_m)$ such that e_i joins x_{i-1} with x_i for $i \in \{1, \dots, m\}$, and such that $x_0 = x$ and $x_m = y$. The *length* of γ, denoted $|\gamma|$, is m. If $x_0 = x_m$, the path is said to be a *circuit*.

It is useful to regard edges in a given path γ as oriented, so that the edge e_i starting at x_{i-1} and ending at x_i, denoted $\langle x_{i-1}, x_i \rangle$, is considered distinct from the edge starting at x_i and ending at x_{i-1}, denoted $\langle x_i, x_{i-1} \rangle$. A path γ in $\mathbb{Z}^2$ is *simple* if no oriented edge is used twice. Given paths $\gamma_1 = (x_0, e_1, \dots, e_m, x_m)$ and $\gamma_2 = (y_0, f_1, \dots, f_k, y_k)$ with $x_m = y_0$, define the *concatenation* of γ_1 and γ_2, denoted $\gamma_1 * \gamma_2$, to be the path $(x_0, e_1, \dots, e_m, x_m, f_1 \dots, f_k, y_k)$.

Definition 2.1. Let γ be a path in $\mathbb{Z}^d$ and let x_i be a vertex in γ with x_{i-1} and x_{i+1} well-defined. The *right-boundary edges at x_i* are obtained by enumerating all oriented edges which start at x_i, beginning with but not including $\langle x_i, x_{i-1} \rangle$, proceeding in a counter-clockwise manner and ending with but not including $\langle x_i, x_{i+1} \rangle$. If either x_{i-1} or x_{i+1} is not well-defined, the right-most boundary edges at x_i are defined to be the empty set. The *right-boundary of* γ, denoted $\partial^+\gamma$, is the union of all right-boundary edges at each vertex of γ.

Definition 2.2. A path $\gamma = (x_0, e_1, x_1, \dots, e_m, x_m)$ is said to be *right-most* if it is simple, and if no e_i is an element of $\partial^+\gamma$.

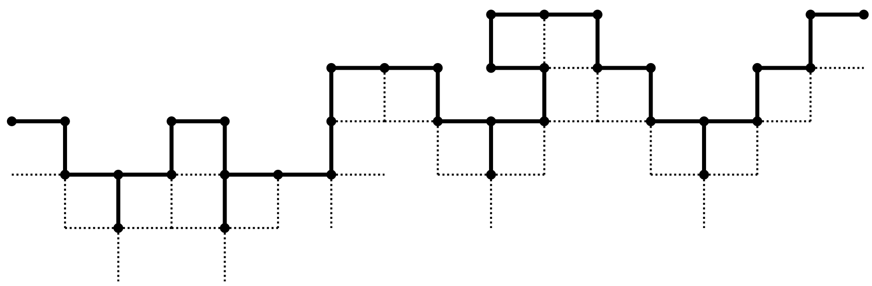

Figure 1: In black, a right-most path which begins on the left and ends on the right. The dotted edges are the right-most boundary of this path.

Definition 2.3. We assign configuration-dependent weights to right-most paths. Define the edge-sets

$$\mathfrak{b}(\gamma) := \left\{ e \in \partial^+\gamma \ : \omega(e) \text{ is open} \right\}, \tag{2.1}$$

$$\mathfrak{b}^n(\gamma) := \left\{ e \in \mathfrak{b}(\gamma) \ : e \subset [-n, n]^2 \right\}, \tag{2.2}$$

and refer to $|\mathfrak{b}(\gamma)|$ and $|\mathfrak{b}^n(\gamma)|$ respectively as the $\mathbf{C}_\infty$*-length* of γ and the $\mathbf{C}_n$*-length* of γ.

Remark 2.4. As we will see in Lemma 2.10, the boundary of a subgraph U of $\mathbf{C}_n$ may be expressed as a collection of right-most circuits. The total $\mathbf{C}_\infty$-length of these circuits will correspond to the size of $\partial^\infty U$, and the total $\mathbf{C}_n$-length of these circuits will correspond to the size of $\partial^n U$.

Following [8], let $\mathcal{R}(x, y)$ denote the collection of all right-most paths joining x to y. If vertices x and y are joined by an open path (and hence joined by an open right-most path) in the configuration ω, define the *right-boundary distance* from x to y as

$$b(x, y) := \inf \left\{ \mathfrak{b}(\gamma) \ : \gamma \in \mathcal{R}(x, y),\ \gamma \text{ uses only open edges} \right\}. \tag{2.3}$$

Remark 2.5. It is convenient to allow b to act on points in $\mathbb{R}^2$ by assigning to each $x \in \mathbb{R}^2$ a "nearest" point $[x]$ in $\mathbf{C}_\infty$. To do this, we augment our probability space to support a collection $\{\eta_x : x \in \mathbb{Z}^2\}$ of i.i.d. random variables uniform on $[0, 1]$ and independent of the Bernoulli random variables used to define the bond percolation. Given $x \in \mathbb{R}^2$, let $[x]$ be the nearest (in ℓ^∞-sense) vertex in $\mathbf{C}_\infty$ to x, breaking ties using the η_x if necessary.

One can establish high-probability closeness of any $x \in \mathbb{R}^2$ with $[x]$ using a duality argument; the following is Lemma 2.7 of [8].

Lemma 2.6. Suppose $p > p_c(2)$. There are positive constants $c_1(p), c_2(p)$ so that for all $x \in \mathbb{Z}^2$ and all $r > 0$,

$$\mathbb{P}_p\Big(|[x] - x|_2 > r \Big) \le c_1 \exp\Big(-c_2 r \Big). \tag{2.4}$$

2.2 Properties of right-most paths

Before defining β_p, we mention some useful properties of right-most paths, recalling in particular the correspondence between right-most paths and simple paths in the medial graph of $\mathbb{Z}^2$. Given a planar graph $G = (\mathrm{V}, \mathrm{E})$, the *medial graph* $G_\sharp = (\mathrm{V}_\sharp, \mathrm{E}_\sharp)$ is the graph with vertices $\mathrm{V}_\sharp = \mathrm{E}$, and with any two vertices in $\mathrm{V}_\sharp$ adjacent in $G_\sharp$ if the corresponding edges of G are adjacent in a face of G.

An *interface* is an edge self-avoiding oriented path in $\mathbb{Z}^2_\sharp$, which does not use its initial or terminal vertex more than once, except to close a circuit. There is a correspondence between interfaces and right-most paths: an interface $\partial = (e_1, \dots, e_m)$, written as a sequence of vertices in $\mathbb{Z}^2_\sharp$, either reflects on a given edge e_i or cuts through a given edge.

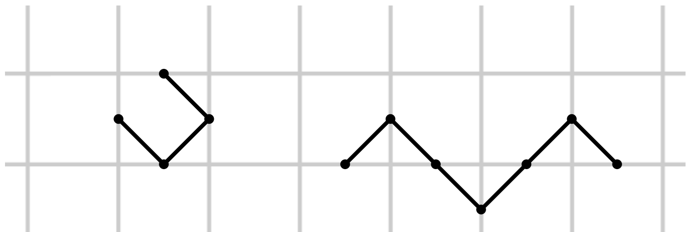

Figure 2: The medial path of length three on the left reflects on each edge. On the right, the medial path of length six cuts through each edge.

More rigorously, an interface $\partial = (e_1, \dots, e_m)$ *reflects* on e_i (for $i \in \{2, \dots, m-1\}$) if e_{i-1} and e_{i+1} are on the boundary of the same face of $\mathbb{Z}^2$, and ∂ *cuts through* e_i otherwise. The following proposition (Proposition 2.3 of [8]) provides a fundamental correspondence between interfaces and right-most paths.

Proposition 2.7. For each interface $\partial = (e_1, \dots, e_m)$, the subsequence $(e_{k_1}, \dots, e_{k_n})$ of edges not cut through by ∂ forms a right-most path γ. This mapping is one-to-one and onto the set of all right-most paths. In particular, γ is a right-most circuit if and only if ∂ is a circuit in the medial graph. Finally, the edges of $\partial \setminus (e_{k_1}, \dots, e_{k_n})$ (oriented properly) form $\partial^+\gamma$.

Remark 2.8. Interfaces may be perturbed via "corner-rounding" to simple curves in $\mathbb{R}^2$, as illustrated at the bottom of Figure 3. In particular, if γ is a right-most circuit, it may be identified with a rectifiable Jordan curve λ_∂ built from the interface ∂ corresponding to γ via Proposition 2.7.

Definition 2.9. Let λ be a rectifiable curve and for $x \notin \lambda$, let $w_\lambda(x)$ denote the winding number of λ around x. Define

$$\mathsf{hull}(\lambda) := \lambda \cup \left\{ x \notin \lambda \, : w_\lambda(x) \text{ is odd} \right\}, \tag{2.5}$$

A fundamental property of right-most circuits is that they may be used to "carve out" subgraphs of $\mathbf{C}_n$. This is done in a way which conveniently links the total length of the circuits with the edge boundary of the subgraph, see Remark 2.4. Let $\mathcal{U}_n$ denote the collection of *connected* subgraphs of $\mathbf{C}_\infty \cap [-n, n]^2$ determined by their vertex set. Given an interface ∂ corresponding to a right-most circuit, let λ_∂ be the Jordan curve obtained from ∂ by rounding the corners, and write $\mathsf{hull}(\partial)$ for $\mathsf{hull}(\lambda_\partial)$. The following decomposition is crucial, though we leave the proof of this lemma to the very end of the appendix.

Lemma 2.10. Let $U \in \mathcal{U}_n$. The graph $\mathbf{C}_\infty \setminus U$ consists of a unique infinite connected component and finitely many finite connected components $\Lambda_1, \dots, \Lambda_m$. There are open,

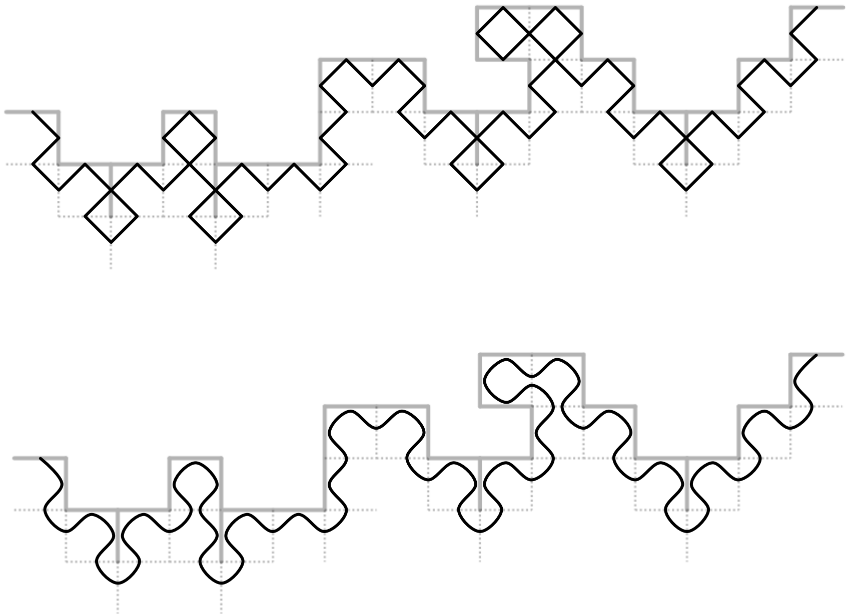

Figure 3: Above: the correspondence of Proposition 2.7, built from the right-most path in Figure 1. Below: the perturbed interface is a simple curve.

counter-clockwise oriented right-most circuits $\gamma \subset U$ and $\gamma_j \subset \Lambda_j$ for each $j \in \{1, \dots, m\}$ satisfying (1) – (4) below:

(1) $\partial, \partial_1, \dots, \partial_m$ are disjoint,
(2) $\mathfrak{b}(\gamma) \cup \left(\bigsqcup_{j=1}^m \mathfrak{b}(\gamma_j) \right) = \partial^\infty U$,
(3) $U = \left[\mathsf{hull}(\partial) \setminus \left(\bigsqcup_{j=1}^m \mathsf{hull}(\partial_j) \right) \right] \cap \mathbf{C}_\infty$,
(4) For each $j \in \{1, \dots, m\}$, we have $\Lambda_j = \mathsf{hull}(\partial_j) \cap \mathbf{C}_\infty$,

where ∂ is the counter-clockwise interface corresponding to γ, and where each ∂_j is the counter-clockwise interface corresponding to γ_j.

The final input on right-most paths we include is Proposition 2.9 of [8], which tells us $|\gamma|$ and $|\mathfrak{b}(\gamma)|$ are comparable when $|\gamma|$ is sufficiently large. This enables us to pass from discrete sets with reasonably sized open edge boundaries to rectifiable sets in the continuum.

Proposition 2.11. Let $p > p_c(2)$. There are positive constants α, c_1, c_2 depending only on p such that for all $n \geq 0$, we have

$$\mathbb{P}_p\Big(\exists \gamma \in \bigcup_{x \in \mathbb{Z}^2} \mathcal{R}(0, x) \ : |\gamma| \geq n, |\mathfrak{b}(\gamma)| \leq \alpha n \Big) \leq c_1 \exp(-c_2 n) \,. \tag{2.6}$$

2.3 The norm

We now use right-most paths to define the norm β_p on $\mathbb{R}^2$, and we aggregate several useful results from [8]. The following is the main result (Theorem 2.1 and Proposition 2.2) of Section 2 in [8], which we state verbatim.

Theorem 2.12. Let $p > p_c(2)$, and let $x \in \mathbb{R}^2$. The limit

$$\beta_p(x) := \lim_{n\to\infty} \frac{b([0],[nx])}{n} \tag{2.7}$$

exists $\mathbb{P}_p$-almost surely and is non-random, non-zero (when $x \neq 0$) and finite. The limit also exists in L^1 and the convergence is uniform on $\{x \in \mathbb{R}^2 \,:\, |x|_2 = 1\}$. Moreover,

(1) β_p is homogeneous, i.e. $\beta_p(cx) = |c|\beta_p(x)$ for all $x \in \mathbb{R}^2$ and all $c \in \mathbb{R}$,
(2) β_p obeys the triangle inequality

$$\beta_p(x+y) \leq \beta_p(x) + \beta_p(y)\,, \tag{2.8}$$

(3) β_p inherits the symmetries of $\mathbb{Z}^2$; for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\beta_p\big((x_1,x_2)\big) = \beta_p\big((x_2,x_1)\big) = \beta_p\big((\pm x_1, \pm x_2)\big) \tag{2.9}$$

for any choice of the signs $\pm$.

Remark 2.13. Theorem 2.12 tells us β_p defines a norm on $\mathbb{R}^2$, and that this norm inherits the symmetries of $\mathbb{Z}^2$. It is first proved by appealing to the subadditive ergodic theorem, but can also be deduced from concentration estimates developed in Section 3 of [8], recalled below.

The first concentration estimate we record is measure theoretic, it is Theorem 3.1 of [8].

Theorem 2.14. Let $p > p_c(2)$. For each $\epsilon > 0$, there are positive constants $c_1(p,\epsilon), c_2(p,\epsilon)$ so that for all $x, y \in \mathbb{Z}^2$,

$$\mathbb{P}_p\left(\left|\frac{b([x],[y])}{\beta_p(y-x)} - 1\right| > \epsilon\right) \leq c_1 \exp\Big(-c_2 \log^2 |y-x|_2\Big)\,. \tag{2.10}$$

We also require a result on the geometric concentration of right-most paths; namely that right-most paths which are almost optimal are geometrically close to the straight line joining their endpoints. Given $x, y \in \mathbf{C}_\infty$, say that $\gamma \in \mathcal{R}(x,y)$ is ϵ-*optimal* if

$$\mathfrak{b}(\gamma) - b(x,y) \leq \epsilon |y-x|_2\,, \tag{2.11}$$

and write $\Gamma_\epsilon(x,y)$ for the set of ϵ-optimal paths in $\mathcal{R}(x,y)$. The following is Proposition 3.2 of [8].

Proposition 2.15. Let $p > p_c(2)$. There are positive constants α, c_1, c_2 so that for all $x, y \in \mathbb{Z}^2$,

(1) For any $t > \alpha|x-y|_2$,

$$\mathbb{P}_p\Big(\exists \gamma \in \Gamma_0([x],[y]) \,:\, |\gamma| > t\Big) \leq c_1 \exp\Big(-c_2 t\Big)\,. \tag{2.12}$$

(2) For all $\epsilon > 0$, once $|y-x|$ is sufficiently large depending on ϵ,

$$\mathbb{P}_p\Big(\forall \gamma \in \Gamma_\epsilon([x],[y]) \,:\, \mathrm{d}_H\big(\gamma, \mathsf{poly}(x,y)\big) > \epsilon|y-x|_2\Big) \leq c_1 \exp\Big(-c_2 \log^2(|y-x|_2)\Big)\,, \tag{2.13}$$

where $\mathsf{poly}(x,y)$ is the linear segment connecting x and y.

3 The variational problem

Having reintroduced β_p in Section 2, we now discuss the variational problem (1.6) specialized to $\tau = \beta_p$, though we stress that throughout most of this section, nothing about β_p is used other than that it is a norm. In a few instances, we appeal to the symmetries of β_p given by the third statement of Theorem 2.12. We need two results in order to prove Theorem 1.2 and Theorem 1.4: an existence result and a stability result. We write the functional defined in (1.5) for $\tau = \beta_p$ as $\mathcal{I}_p$, and for $R \in \mathcal{R}$, and refer to $\mathcal{I}_p(\partial R)$ as the *surface energy of* R. Define the β_p*-length* of a rectifiable curve $\lambda : [0,1] \to \mathbb{R}^2$:

$$\mathrm{length}_{\beta_p}(\lambda) := \sup_{n\in\mathbb{N}} \sup_{t_1<\cdots<t_n\in[0,1]} \sum_{j=1}^{n} \beta_p(\lambda(t_j) - \lambda(t_{j-1}))\,. \tag{3.1}$$

It is necessary to consider a family of variational problems related to (1.6). For $\alpha \in [-1,1]$, define the following isoperimetric problem for sets $R \in \mathcal{R}$:

$$\text{minimize: } \frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)}\,, \qquad \text{subject to: } \mathrm{Leb}(R) \leq 2+\alpha \tag{3.2}$$

The minimal value for (3.2) is

$$\varphi_p^{(2+\alpha)} := \inf\left\{\frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} \;: \mathrm{Leb}(R) \leq 2+\alpha\,, R\in\mathcal{R}\right\}\,, \tag{3.3}$$

and the set of optimizers for (3.2) is defined below as

$$\mathcal{R}_p^{(2+\alpha)} := \left\{R\in\mathcal{R} \;: \mathrm{Leb}(R)\leq 2+\alpha\,, \frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} = \varphi_p^{(2+\alpha)}\right\}\,. \tag{3.4}$$

In our new notation, the constant φ_p introduced in (1.10) is written $\varphi_p^{(2)}$ in this section, and the collection of optimizers $\mathcal{R}_p$ introduced in Theorem 1.2 is denoted $\mathcal{R}_p^{(2)}$.

3.1 Sets of finite perimeter

We extend the problem (3.2) to a larger class of sets, proving existence within this class and then recovering a representative in $\mathcal{R}$. Let $E \subset [-1,1]^2$ be Borel and define the *perimeter* of E, denoted $\mathrm{per}(\partial E)$, as

$$\mathrm{per}(\partial E) := \sup\left(\int_E \mathrm{div}(f)dx \;: f\in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)\,, |f|_2\leq 1\right)\,, \tag{3.5}$$

and say that E is a *set of finite perimeter* if $\mathrm{per}(\partial E) < \infty$. Let $\mathcal{C}$ denote the collection of all sets of finite perimeter (after Caccioppoli) contained in $[-1,1]^2$. Given $E \in \mathcal{C}$, define the β_p*-perimeter* of E similarly:

$$\mathrm{per}_{\beta_p}(\partial E) := \sup\left(\int_E \mathrm{div}(f)dx \;: f\in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)\,, \beta_p^*(f)\leq 1\right)\,, \tag{3.6}$$

where β_p^* is the dual norm to β_p. Finally, define the surface energy of $E \in \mathcal{C}$ as:

$$\mathcal{I}_p(\partial E) := \sup\left(\int_E \mathrm{div}(f)dx \;: f\in C_c^\infty((-1,1)^2,\mathbb{R}^2)\,, \beta_p^*(f)\leq 1\right)\,. \tag{3.7}$$

Remark 3.1. Each $R \in \mathcal{R}$ is an element of $\mathcal{C}$, and the surface energy of R defined in (1.5) agrees with the surface energy of E, defined in (3.7). This enables us to extend the variational problem (3.2) to sets of finite perimeter, and given $E \in \mathcal{C}$, we call $\mathcal{I}_p(\partial E)/\mathrm{Leb}(E)$ the *conductance* of E, which is consistent with the terminology in the introduction. Note that, though the distributional nature of the definitions (3.5), (3.6) and (3.7) may appear unintuitive, they are linked to the more natural definition of the surface energy (1.5) through the divergence theorem. One can use this to show that (1.5) and (3.7) agree on sets with, for instance, smooth boundaries.

We introduce the optimal value and set of optimizers corresponding to the variational problem over this wider class of sets. Define

$$\psi_p^{(2+\alpha)} := \inf\left\{\frac{\mathcal{I}_p(\partial E)}{\mathrm{Leb}(E)} \;: \mathrm{Leb}(E) \le 2+\alpha\,, E \in \mathcal{C}\right\}, \tag{3.8}$$

with the convention that zero divided by zero is infinity. Also define

$$\mathcal{C}_p^{(2+\alpha)} := \left\{E \in \mathcal{C} \;: \mathrm{Leb}(E) \le 2+\alpha\,, \frac{\mathcal{I}_p(\partial E)}{\mathrm{Leb}(E)} = \psi_p^{(2+\alpha)}\right\}. \tag{3.9}$$

Lower semicontinuity is a fundamental feature of the perimeter and surface energy functionals (see for instance Section 14.2 of [14]).

Lemma 3.2. Let $E_k \in \mathcal{C}$ be a sequence converging in L^1-sense to E. Then

(1) $\mathrm{per}(\partial E) \le \liminf_{k\to\infty} \mathrm{per}(\partial E_k)$,
(2) $\mathrm{per}_{\beta_p}(\partial E) \le \liminf_{k\to\infty} \mathrm{per}_{\beta_p}(\partial E_k)$,
(3) $\mathcal{I}_p(\partial E) \le \liminf_{k\to\infty} \mathcal{I}_p(\partial E_k)$,

so that if $\mathrm{per}(\partial E_k)$ is uniformly bounded in k, we have $E \in \mathcal{C}$.

We now introduce some terminology in order to state a result linking the classes $\mathcal{R}$ and $\mathcal{C}$.

Definition 3.3. Given $E \subset [-1,1]^2$ Borel, define the *upper density* of E at $x \in \mathbb{R}^2$ as

$$D^+(E,x) := \limsup_{r\to 0} \frac{\mathrm{Leb}(E \cap B(x,r))}{\mathrm{Leb}(B(x,r))}, \tag{3.10}$$

and define the *essential boundary* of E as

$$\partial^* E := \left\{x \in \mathbb{R}^2 \;: D^+(E,x) > 0,\; D^+(\mathbb{R}^2 \setminus E, x) > 0\right\} \tag{3.11}$$

Definition 3.4. Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. Say E is *decomposable* if there is a partition of E into $A, B \subset \mathbb{R}^2$ so that $\mathrm{Leb}(A)$ and $\mathrm{Leb}(B)$ are strictly positive and so that $\mathrm{per}(\partial E) = \mathrm{per}(\partial A) + \mathrm{per}(\partial B)$. Say that E is *indecomposable* if it is not decomposable.

Recall that given a Jordan curve λ, we defined the compact set $\mathsf{hull}(\lambda)$ in (2.5). We write $\mathsf{hull}(\lambda)^\circ$ for the interior of this compact set. The following result, originally due to Fleming and Federer, allows us to think of $\partial^* E$ for $E \in \mathcal{C}$ as a countable collection of rectifiable Jordan curves. The version we state is taken from Corollary 1 of [4], and is illustrated by Figure 4.

Proposition 3.5. Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. There is a unique decomposition of $\partial^* E$ into rectifiable Jordan curves $\{\lambda_i^+, \lambda_j^- \;: i, j \in \mathbb{N}\}$ (modulo $\mathcal{H}^1$-null sets) so that

(1) For $i \ne k \in \mathbb{N}$, $\mathsf{hull}(\lambda_i^+)^\circ$ and $\mathsf{hull}(\lambda_k^+)^\circ$ are either disjoint, or one is contained in the other. Likewise, for $i \ne k \in \mathbb{N}$, $\mathsf{hull}(\lambda_i^-)^\circ$ and $\mathsf{hull}(\lambda_k^-)^\circ$ are either disjoint, or one is contained in the other. Each $\mathsf{hull}(\lambda_j^-)^\circ$ is contained in one of the $\mathsf{hull}(\lambda_i^+)^\circ$.

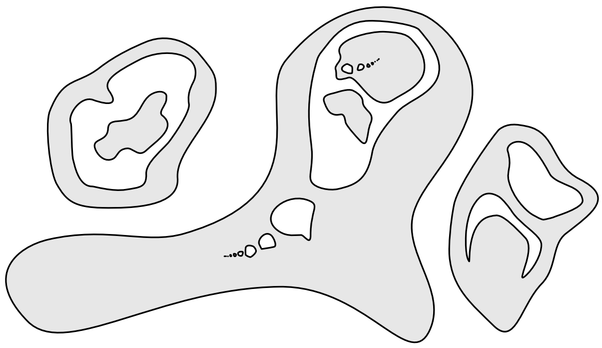

Figure 4: In grey is a set of finite perimeter, its boundary contours exhibit a tree structure.

(2) $\mathrm{per}(\partial E) = \sum_{i=1}^{\infty} \mathcal{H}^1(\lambda_i^+) + \sum_{j=1}^{\infty} \mathcal{H}^1(\lambda_j^-)$.

(3) If $\mathsf{hull}(\lambda_i^+)^\circ \subset \mathsf{hull}(\lambda_j^+)^\circ$ for $i \neq j$, then for some λ_k^-, we have $\mathsf{hull}(\lambda_i^+)^\circ \subset \mathsf{hull}(\lambda_k^-)^\circ \subset \mathsf{hull}(\lambda_j^+)^\circ$. Likewise, if $\mathsf{hull}(\lambda_i^-)^\circ \subset \mathsf{hull}(\lambda_j^-)^\circ$ for $i \neq j$, there is some λ_k^+ with $\mathsf{hull}(\lambda_i^-)^\circ \subset \mathsf{hull}(\lambda_k^+)^\circ \subset \mathsf{hull}(\lambda_j^-)^\circ$.

(4) For $i \in \mathbb{N}$, let $L_i = \{j\ : \mathsf{hull}(\lambda_j^-)^\circ \subset \mathsf{hull}(\lambda_i^+)^\circ\}$, and set

$$Y_i = \mathsf{hull}(\lambda_i^+) \setminus \left(\bigcup_{j \in L_i} \mathsf{hull}(\lambda_j^-)^\circ \right) . \tag{3.12}$$

The sets Y_i are indecomposable with $\mathcal{H}^1$-null intersection, and moreover $\bigcup_{j=1}^{\infty} Y_j$ is equivalent to E modulo Lebesgue null sets.

Proposition 3.5 says sets of finite perimeter are in a sense extensions of the class $\mathcal{R}$ to sets whose boundary consists of countably many Jordan arcs instead of finitely many. Thus, it is reasonable that the theory of such sets comes into play when discussing limits of sets in $\mathcal{R}$.

3.2 Existence

We now show that $\mathcal{R}_p^{(2+\alpha)}$ is non-empty for all $\alpha \in [-1, 1]$: we use standard arguments to show $\mathcal{C}_p^{(2+\alpha)}$ is non-empty, and then we recover elements of $\mathcal{R}$ from sets in $\mathcal{C}_p^{(2+\alpha)}$. We begin with the observation that optimal Jordan domains must have full area.

Lemma 3.6. Let $\alpha \in [-1, 1]$. Let $R \in \mathcal{R}$ be such that $\mathrm{Leb}(R) < 2 + \alpha$ and such that $R = \mathsf{hull}(\lambda)$ for a rectifiable Jordan curve $\lambda \subset [-1, 1]^2$. Then there is $R' \in \mathcal{R}$ with $\mathrm{Leb}(R') = 2 + \alpha$ and $R' = \mathsf{hull}(\lambda')$ for a rectifiable Jordan curve $\lambda' \subset [-1, 1]^2$ with

$$\frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} > \frac{\mathcal{I}_p(\partial R')}{\mathrm{Leb}(R')} \,. \tag{3.13}$$

Proof. Let $R \in \mathcal{R}$ be as above, and consider the open set $A = (-1, 1)^2 \setminus R$. We consider three cases. Throughout, we dilate and translate subsets of $[-1, 1]^2$, and find the following remark useful to mention. Let $\widetilde{R}$ be a dilation of R with strictly larger area, so that $\widetilde{R}$ is a translate of λR for $\lambda > 1$. Suppose that $\widetilde{R}$ is contained in $[-1, 1]^2$ (making it an element of $\mathcal{R}$). Then, because (1.5) scales at most linearly in λ, it follows that the conductance of $\widetilde{R}$ is strictly less than the conductance of R.

Case I: In the first case, each connected component A' of A is such that $\partial A'$ intersects the interior of at most two adjacent sides of $\partial[-1,1]^2$ non-trivially. To be clear, $\partial[-1,1]^2$ is the union of four line-segments, $\ell_1,\dots,\ell_4$, each closed in the subspace topology inherited from $\partial[-1,1]^2$. Writing $\ell_1^\circ,\dots,\ell_4^\circ$ for the interiors of these in the subspace topology, the first case requires that $\partial A'$ has non-empty intersection with at most two of the ℓ_i° for $i=1,\dots,4$.

In this case, shrink the connected components of A to form a new open set of arbitrarily small volume, and whose surface energy is at most that of A. By complementation, we recover R' with the desired properties.

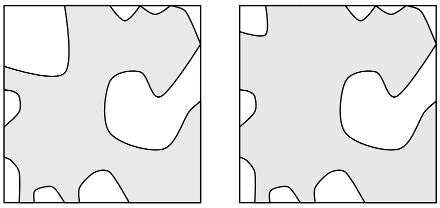

Figure 5: On the left, the original set $R\in\mathcal{R}$ in grey. On the right, the set $R'\in\mathcal{R}$ obtained through the procedure described in **Case I**.

Case II: In the second case, there is a connected component A' of A such that $\partial A'$ intersects the interior of exactly three sides of $[-1,1]^2$ non-trivially. As R is connected, $\partial A'\cap(-1,1)^2$ is a single arc joining two opposing faces of the square. This arc may be translated until it touches one of the other faces of the square, yielding sets of the desired form with larger area. If the measure of these sets surpasses $2+\alpha$ before the arc reaches the boundary, we are content. Otherwise, we have built a set handled by the previous case (after performing the same procedure on at most one other arc, perhaps).

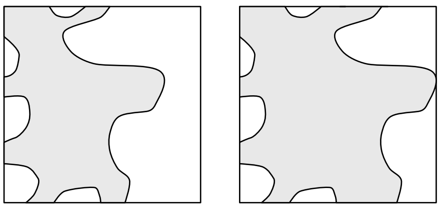

Figure 6: On the left, the original $R\in\mathcal{R}$ in grey. On the right, R' is obtained by “sliding" one of the contours along the boundary of the box.

Case III: As R is connected, no connected component A' of A has the property that $\partial A'$ intersects only the interiors of two opposite sides of $[-1,1]^2$ non-trivially. Thus the last case to consider is that there is a connected component A' of A where $\partial A'$ intersects the interior of all four sides of $[-1,1]^2$ non-trivially. In this case, ∂R intersects the interiors of at most two adjacent sides of $[-1,1]^2$ non-trivially. Dilate R about the corner it contains or the side it rests against until we either have a set of the desired measure or we have a set falling into one of the preceding cases.

This completes the proof. □

Lemma 3.6 implies that optimal sets of finite perimeter also have full area.

Lemma 3.7. Let $\alpha\in[-1,1]$, and let $E\in\mathcal{C}$ with either $\mathrm{Leb}(E)<2+\alpha$, or $\mathrm{Leb}(E)\le 2+\alpha$

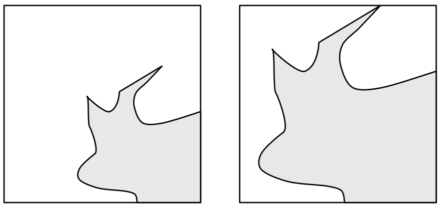

Figure 7: On the left, $R \in \mathcal{R}$ is in grey. On the right, $R' \in \mathcal{R}$ is obtained by dilating R.

and E decomposable. There is $E' \in \mathcal{C}$ with $\mathrm{Leb}(E') = 2 + \alpha$ so that

$$\frac{\mathcal{I}_p(\partial E)}{\mathrm{Leb}(E)} > \frac{\mathcal{I}_p(\partial E')}{\mathrm{Leb}(E')} \,. \tag{3.14}$$

Proof. The case that $\mathrm{Leb}(E) \le 2 + \alpha$ and E is decomposable is an immediate corollary of the case $\mathrm{Leb}(E) < 2 + \alpha$, so we assume $\mathrm{Leb}(E) < 2 + \alpha$. Recall from Proposition 3.5 that E is equivalent up to a Lebesgue-null set to $\bigcup_{i=1}^{\infty} Y_i$, where each Y_i is defined in (3.12). Because the Y_i are disjoint and lie within $[-1,1]^2$, planarity implies that all but finitely many Y_i touch zero, one or two adjacent sides of $\partial[-1,1]^2$, in the sense used in the proof of Lemma 3.6. We use this to show the following *claim*: that the conductance of the Y_i tend to ∞ with i.

Given Y_i not touching any side of $\partial[-1,1]^2$, use that β_p is a norm on $\mathbb{R}^2$ along with the Euclidean isoperimetric inequality to deduce $\mathcal{I}_p(Y_i)/\mathrm{Leb}(Y_i)$ is bounded from below by $c(\mathrm{Leb}(Y_i))^{-1/2}$ for some $c(p) > 0$. When Y_i touches only one side of $\partial[-1,1]^2$, we can *reflect* Y_i over this side to produce a set to which the previous argument can be applied. When Y_i touches two adjacent sides of $\partial[-1,1]^2$, reflecting twice puts us in the original case, and we have the same lower bound on the conductance of Y_i. Note however that because $\mathrm{Leb}(E) < \infty$, we have $\lim_{i\to\infty} \mathrm{Leb}(Y_i) = 0$, and using this with the lower bound on the conductance, the above claim follows. We remark that we have used the symmetries of β_p given by (3) of Theorem 2.12, and that this argument is later used to prove Lemma 3.10.

For $N \ge 1$, let us write X_N for $\bigcup_{i=N}^{\infty} Y_i$. Then, by Proposition 3.5, we have

$$\frac{\mathcal{I}_p(X_N)}{\mathrm{Leb}(X_N)} = \frac{\sum_{i=N}^{\infty} \mathcal{I}_p(Y_i)}{\sum_{i=N}^{\infty} \mathrm{Leb}(Y_i)} \tag{3.15}$$

For N sufficiently large, the above argument also implies that the conductance of X_N tends to ∞ with N. We now appeal to the elementary inequality $\frac{a+b}{c+d} \ge \min(\frac{a}{c}, \frac{b}{d})$ holding for positive a, b, c and d: given E with $\mathrm{Leb}(E) < 2 + \alpha$ decomposable, we find for any $N \ge 1$ that

$$\frac{\mathcal{I}_p(E)}{\mathrm{Leb}(E)} \ge \min\left(\frac{\mathcal{I}_p(Y_1)}{\mathrm{Leb}(Y_1)}, \dots, \frac{\mathcal{I}_p(Y_N)}{\mathrm{Leb}(Y_N)}, \frac{\mathcal{I}_p(X_{N+1})}{\mathrm{Leb}(X_{N+1})}\right), \tag{3.16}$$

and, thanks to the diverging conductances of the Y_i and the X_N, it follows there is some Y_m whose conductance is at most that of E.

Using Proposition 3.5 once more, Y_m may be represented by rectifiable Jordan arcs λ and $\{\lambda_j\}_{j\ge1}$ so that up to a Lebesgue-null set, $Y_m = \mathsf{hull}(\lambda) \setminus \bigcup_{j\ge1} \mathsf{hull}(\lambda_j)^\circ$. As the curves λ, λ_j have $\mathcal{H}^1$-null intersection, the sets $\mathsf{hull}(\lambda_j)^\circ$ are pairwise disjoint. Under the hypothesis that $\mathrm{Leb}(E) < 2 + \alpha$, we may then shrink the curves λ_j one by one to produce a set E' from Y_m having *strictly* smaller conductance. After shrinking all such interior curves, it may be that the area of E' is still not $2+\alpha$. But this is precisely the setting of of Lemma 3.6, which handles sets of finite perimeter represented by a single Jordan curve. □

We may now deduce that the collection of optimizers for (3.4) is non-empty within the class of sets of finite perimeter.

Lemma 3.8. The set of optimizers $\mathcal{C}_p^{(2+\alpha)}$ for the variational problem (3.4) is non-empty.

Proof. Let $E_k \in \mathcal{C}$ be a sequence of sets of finite perimeter such that

$$\frac{\mathcal{I}_p(\partial E_k)}{\mathrm{Leb}(E_k)} \to \psi_p^{(2+\alpha)} . \tag{3.17}$$

By Lemma 3.7, we lose no generality supposing $\mathrm{Leb}(E_k) = 2+\alpha$ for each k. As $\psi_p^{(2+\alpha)}$ is clearly finite, the perimeters of the E_k are uniformly bounded. Appealing to Theorem 12.26 of Maggi's book [29], we pass to a subsequence of the E_k converging to some $E \subset [-1,1]^2$ in L^1-sense. By Lemma 3.2, it follows that E is a set of finite perimeter with $\mathrm{Leb}(E) = 2+\alpha$ and $\mathcal{I}_p(\partial E) \le \liminf_{k\to\infty} \mathcal{I}_p(\partial E_k)$. Thus the conductance of E is at most $\psi_p^{(2+\alpha)}$, which implies $E \in \mathcal{C}_p^{(2+\alpha)}$. □

We may now deduce that $\mathcal{R}_p^{(2+\alpha)}$ is non-empty for $\alpha \in [-1,1]$, among other things. The following is the main result of this subsection.

Corollary 3.9. Let $\alpha \in [-1,1]$.

(1) If $E \in \mathcal{C}_p^{(2+\alpha)}$, then E is indecomposable and $\mathrm{Leb}(E) = 2+\alpha$.
(2) $E \in \mathcal{C}_p^{(2+\alpha)}$ if and only if $E^c \in \mathcal{C}_p^{(2-\alpha)}$.
(3) $\frac{2+\alpha}{2-\alpha}\psi_p^{(2+\alpha)} = \psi_p^{(2-\alpha)}$.
(4) Each $E \in \mathcal{C}_p^{(2+\alpha)}$ is equivalent up to a Lebesgue-null set to some $R \in \mathcal{R}$. Thus, $\mathcal{R}_p^{(2+\alpha)}$ is non-empty and $\varphi_p^{(2+\alpha)} = \psi_p^{(2+\alpha)}$.
(5) If $E \in \mathcal{C}_p^{(2+\alpha)}$, there are rectifiable Jordan curves $\lambda, \lambda' \subset [-1,1]^2$ so that up to Lebesgue-null sets, $E = \mathsf{hull}(\lambda)$ and $E^c = \mathsf{hull}(\lambda')$. Moreover, $\lambda \cap \lambda'$ is a simple rectifiable curve joining distinct points on $\partial[-1,1]^2$.

Proof. The first assertion is an immediate consequence the inequality $\frac{a+b}{c+d} \ge \min(\frac{a}{c}, \frac{b}{d})$ (valid for a,b,c and d positive) and of Lemma 3.7. Because each $E \in \mathcal{C}_p^{(2+\alpha)}$ satisfies $\mathrm{Leb}(E) = 2+\alpha$, and because $\mathcal{I}_p(\partial E) = \mathcal{I}_p(\partial E^c)$, the second and third assertions follow. Thus, whenever $E \in \mathcal{C}_p^{(2+\alpha)}$, both E and E^c are indecomposable. By Proposition 3.5, either E or E^c is equivalent to $\mathsf{hull}(\lambda)$ for some rectifiable Jordan curve $\lambda \subset [-1,1]^2$, and the fourth assertion follows.

Turning our attention to the fifth assertion, suppose $E \in \mathcal{C}_p^{2+\alpha}$. By assertion (2), $E^c \in \mathcal{C}_p^{2-\alpha}$, and assertion (1) implies there are rectifiable Jordan curves $\lambda, \lambda' \subset [-1,1]^2$ with $E = \mathsf{hull}(\lambda)$ and $E^c = \mathsf{hull}(\lambda')$ up to Lebesgue-null sets. Otherwise, appealing to the decomposition of Proposition 3.5, either E or E^c would be decomposable.

Without loss of generality, $\mathrm{Leb}(E) \le 2$ (otherwise take E^c). Represent E as $\mathsf{hull}(\lambda)$ for a rectifiable Jordan curve $\lambda \subset [-1,1]^2$; we claim that $\mathcal{H}^1(\lambda \cap \partial[-1,1]^2) > 0$: this follows from the fact that if $\mathcal{H}^1(\lambda \cap \partial[-1,1]^2) = 0$, the curve λ at best can be the boundary of (a dilate of) the area two Wulff shape W_p (this is the limit shape of [8] which is the unique solution, up to translation, of the unrestricted isoperimetric problem associated to the norm β_p). However, this shape is not optimal. For instance, a suitably dilated quarter-Wulff shape has strictly better conductance. Consequently, if λ' represents E^c, it follows that $\lambda \cap \lambda'$ is simple, rectifiable and joins distinct points on $\partial[-1,1]^2$. □

Let us include one last result to be used in the proof of Theorem 1.2, and which guarantees the non-degeneracy of the limit in Theorem 1.4.

Lemma 3.10. For each $\alpha \in [-1,1]$, we have $\varphi_p^{(2+\alpha)} > 0$. Moreover, for each $\alpha, \alpha' \in [-1,1]$ with $\alpha > \alpha'$, we have the strict monotonicity $\varphi_p^{(2+\alpha')} > \varphi_p^{(2+\alpha)}$.

Proof. Strict monotonicity follows from Lemma 3.7. It suffices to show $\varphi_p^{(3)}$ is positive; this follows from the fifth assertion of Corollary 3.9. Given $R \in \mathcal{R}_p^{(2+\alpha)}$, point (3.9) implies $\partial R \cap (-1,1)^2$ is a simple rectifiable curve η joining distinct points on the boundary of $\partial[-1,1]^2$. There are three short cases, with the first two implicitly using the symmetries of β_p given in (3) of Theorem 2.12.

Case I: Suppose the endpoints of η lie on the same side of $\partial[-1,1]^2$. Thus, either R or R^c intersects at most one side of $[-1,1]^2$, and we let A denote the set among R and R^c with this property. Reflect A about the side it borders yielding a set A' with twice the area, and with $\mathcal{I}_p(\partial A') = 2\mathcal{I}_p(\eta) \equiv 2\text{length}_{\beta_p}(\eta)$. As $\mathrm{Leb}(A) \geq 1$, the the standard Euclidean isoperimetric inequality implies

$$\mathcal{I}_p(\eta) \equiv \text{length}_{\beta_p}(\eta) \geq \frac{c}{\sqrt{2}}\beta_p^{\min}\,, \tag{3.18}$$

where $c > 0$ is some absolute constant, and where $\beta_p^{\min}$ is the minimum of β_p over the unit circle.

Case II: In the second case, we suppose the endpoints of η lie on two adjacent sides of $\partial[-1,1]^2$. Either R or R^c intersects only these two sides of the square, and as before we let A denote the set among R and R^c with this property. We proceed as before, except we now reflect twice, obtaining A' with four times the volume of A, and with $\mathcal{I}_p(\partial A') = 4\mathcal{I}_p(\eta) \equiv 4\text{length}_{\beta_p}(\eta)$. Thus,

$$\mathcal{I}_p(\eta) \equiv \text{length}_{\beta_p}(\eta) \geq \frac{c}{2}\beta_p^{\min}\,, \tag{3.19}$$

with c and $\beta_p^{\min}$ as above.

Case III: In the final case, η joins points on two opposing sides of $\partial[-1,1]^2$. Clearly, $\mathcal{I}_p(\eta) \equiv \text{length}_{\beta_p}(\eta) \geq 2\beta_p^{\min}$, where the two arises as the Euclidean distance between opposite sides of the square.

In each case, we conclude that $\mathcal{I}_p(\partial R) = \mathcal{I}_p(\eta) > 0$, completing the proof. □

3.3 Stability for connected sets

Now that we have shown the set $\mathcal{R}_p^{(2+\alpha)}$ is non-empty, we show a stability result with respect to the d_H-metric. First, some preliminary results.

Lemma 3.11. Let $\alpha \in (-1,1)$. Suppose that $E_k \in \mathcal{C}$ are such that $\mathrm{Leb}(E_k) \leq 2+\alpha$ and the conductances of the E_k tend to $\varphi_p^{(2+\alpha)}$. Then $\liminf_{k\to\infty} \mathrm{Leb}(E_k) > 0$, and if $E_k \to E$ in L^1-sense, we have $E \in \mathcal{C}_p^{(2+\alpha)}$.

Proof. Let $\alpha' \in (-1,1)$ be strictly less than α. If $\mathrm{Leb}(E_k) \to 0$, we would have $\varphi_p^{(2+\alpha)} \geq \varphi_p^{(2+\alpha')}$, which contradicts Lemma 3.10. Thus if the E_k tend to $E \subset [-1,1]^2$ in L^1-sense, it follows that $\mathrm{Leb}(E) > 0$. By Lemma 3.2, we have

$$\varphi_p^{(2+\alpha)} = \liminf_{k\to\infty} \frac{\mathcal{I}_p(E_k)}{\mathrm{Leb}(E_k)} \geq \frac{\mathcal{I}_p(E)}{\mathrm{Leb}(E)}\,, \tag{3.20}$$

and thus $E \in \mathcal{C}_p^{(2+\alpha)}$. □

For $E \in \mathcal{C}$ indecomposable, Proposition 3.5 tells us that E is equivalent (up to a Lebesgue-null set) to $\mathsf{hull}(\lambda) \setminus \left(\bigcup_{j\geq 1} \mathsf{hull}(\lambda_j)^\circ\right)$ for $\lambda, \lambda_j \subset [-1,1]^2$ rectifiable Jordan curves. Given $E \in \mathcal{C}$ indecomposable, define $\widehat{E} := \mathsf{hull}(\lambda)$, where λ corresponds to E as above. The next result tells us that if a sequence E_k of indecomposable sets of finite perimeter tend to an optimal set, the size of the "holes" in these sets must tend to zero.

Lemma 3.12. Let $\alpha \in (-1,1)$. Let $E_k \in \mathcal{C}$ be indecomposable with $\mathrm{Leb}(E_k) \leq 2+\alpha$ for all $k \geq 1$. Suppose that the E_k tend to $E \in \mathcal{C}_p^{(2+\alpha)}$ in L^1-sense. Then as $k \to \infty$, we have $\mathrm{Leb}(\widehat{E}_k \setminus E_k) \to 0$.

Proof. Suppose not: we lose no generality supposing that $\mathrm{Leb}(\widehat{E}_k \setminus E_k) \geq \epsilon$ for all k and some $\epsilon > 0$. Moreover, by Lemma 3.11, we also lose no generality supposing $\mathrm{Leb}(E_k) \geq 2+\alpha-\epsilon/2$ for all k (using that each $E \in \mathcal{C}_p^{(2+\alpha)}$ satisfies $\mathrm{Leb}(E) = 2+\alpha$). We may take ϵ small enough so that $\alpha' := \alpha + \epsilon/2 \in (-1,1)$.

Note that the E_k^c also converge in L^1-sense to $E^c \in \mathcal{C}_p^{(2-\alpha)}$. The sets E_k^c however are not indecomposable by hypothesis: let A_k be the component of E_k^c of smallest conductance, so that the conductance of E_k^c serves as an upper bound for the conductance of A_k. But our hypotheses on the volumes of $\widehat{E}_k$ and E_k ensure that $\mathrm{Leb}(A_k) \leq 2-\alpha-\epsilon/2$, which implies that $\varphi_p^{(2-\alpha')} \leq \varphi_p^{(2-\alpha)}$, contradicting Lemma 3.10. □

Heuristically, the above lemma allows us to replace a sequence of sets in $\mathcal{R}$ by Jordan domains. The next result tells us that a sequence of Jordan domains converging in the correct sense to an element of $\mathcal{C}_p^{(2+\alpha)}$ has a limit in $\mathcal{R}$.

Lemma 3.13. Let $R_k \in \mathcal{R}$ be a sequence such that $R_k = \mathsf{hull}(\lambda_k)$ for rectifiable Jordan curves $\lambda_k \subset [-1,1]^2$, and suppose that the conductances of the R_k tend to $\varphi_p^{(2+\alpha)}$. Suppose also that $R_k \to K$ both in L^1-sense and in d_H-sense, where $K \subset [-1,1]^2$ is compact and $K \in \mathcal{C}_p^{(2+\alpha)}$. Then $K \in \mathcal{R}_p^{(2+\alpha)}$.

Proof. In this proof, we carefully distinguish curves (continuous functions from $[0,1]$ into $[-1,1]^2$ taking the same value at 0 and 1) from their images. Given a curve $\lambda : [0,1] \to [-1,1]^2$, let $\mathsf{image}(\lambda)$ denote the image of λ. Let λ_k' be an arc length parametrization of ∂R_k; a continuous function $\lambda_k' : [0, \mathrm{per}(\partial R_k)] \to [-1,1]^2$ with Lipschitz constant one which takes the same value at the endpoints of $[0, \mathrm{per}(\partial R_k)]$. As $K \in \mathcal{C}_p^{(2+\alpha)}$, the perimeters of the ∂R_k are uniformly bounded, thus we may linearly reparametrize each λ_k' to produce a sequence of curves $\lambda_k : [0,1] \to [-1,1]^2$, with λ_k parametrizing ∂R_k, and with a uniform bound on the Lipschitz constant across all k. Invoking Arzela-Ascoli and passing to a subsequence, we find that the λ_k tend uniformly to a rectifiable curve λ.

By appealing to the definition of the hull of a curve (using winding number), we find that $\mathsf{hull}(\lambda_k) \to \mathsf{hull}(\lambda)$ in d_H-sense, thus $K \equiv \mathsf{hull}(\lambda)$. Let $\widetilde{\lambda} : [0,1] \to (-1,1)^2$ be a reparametrization of λ of constant speed, so that $K = \mathsf{hull}(\widetilde{\lambda})$ also. Suppose that $\widetilde{\lambda}$ is not a simple curve, and moreover suppose there is $x \in (-1,1)^2$ such that $|\widetilde{\lambda}^{-1}(x)| > 1$. Let $s < t \in [0,1]$ be such that $x = \widetilde{\lambda}(s) = \widetilde{\lambda}(t)$. Let us write $\zeta_1 := \widetilde{\lambda}|_{[s,t)}$ and $\zeta_2 := \widetilde{\lambda}|_{[0,s]\cup(t,1]}$, so that both ζ_1 and ζ_2 are closed curves.

As $K \in \mathcal{C}_p^{(2+\alpha)}$, the set K must be indecomposable with indecomposable complement. It follows that $\mathsf{hull}(\widetilde{\lambda})^\circ$ is either $\mathsf{hull}(\zeta_1)^\circ$ or $\mathsf{hull}(\zeta_2)^\circ$. As $x \in (-1,1)^2$, we also have that $\mathcal{I}_p(\widetilde{\lambda}) > \mathcal{I}_p(\zeta_1)$ and $\mathcal{I}_p(\widetilde{\lambda}) > \mathcal{I}_p(\zeta_2)$. Without loss of generality then, we have

$$\frac{\mathcal{I}_p(\partial K)}{\mathrm{Leb}(K)} \leq \frac{\mathcal{I}_p(\zeta_1)}{\mathrm{Leb}(K)} < \frac{\mathcal{I}_p(\widetilde{\lambda})}{\mathrm{Leb}(K)} \leq \varphi_p^{(2+\alpha)}, \tag{3.21}$$

where the right-most inequality follows from lower semicontinuity of the surface energy Lemma 3.2 (and the hypothesis that the conductances of the R_k tend to the optimal value). This is a contradiction. Thus, if $|\widetilde{\lambda}^{-1}(x)| > 1$, it must be that $x \in \partial[-1,1]^2$, and there exists a Jordan curve $\lambda' \subset [-1,1]^2$ such that $\mathsf{hull}(\lambda') = \mathsf{hull}(\widetilde{\lambda}) = K$. We conclude that $K \in \mathcal{R}_p^{(2+\alpha)}$. □

Lemma 3.13 essentially allows us to recover some regularity of a suitable limit of Jordan domains. We now use this to show that the collections $\mathcal{R}_p^{(2)}$ and $\mathcal{R}_p^{(2+\alpha)}$ are close when α is small.

Lemma 3.14. Let $\alpha \in (0,1]$. As $\alpha \to 0$, we have $\mathrm{d}_H(\mathcal{R}_p^{(2+\alpha)}, \mathcal{R}_p^{(2)}) \to 0$.

Proof. Let $\alpha_k \in (0,1]$ be a sequence tending to zero as $k \to \infty$. Let $R_k \in \mathcal{R}_p^{(2+\alpha_k)}$. By Corollary 3.9 (3.9), there are rectifiable Jordan curves $\lambda_k \subset [-1,1]^2$ with $R_k = \mathsf{hull}(\lambda_k)$. By Corollary 3.9 (3.9), the conductances of the R_k tend to $\varphi_p^{(2)}$.

The non-empty compact subsets of $[-1,1]^2$ form a compact metric space when equipped with the d_H-metric. We pass to a subsequence (twice, using this compactness and Theorem 12.26 of [29]) so that $R_k \to K$ in d_H-sense and in L^1-sense, where $K \subset [-1,1]^2$ is compact. As $\mathrm{Leb}(R_k) \to 2$ as $k \to \infty$, the lower semicontinuity of the surface energy (Lemma 3.2) implies $K \in \mathcal{C}_p^{(2)}$. We apply Lemma 3.13 to conclude that $K \in \mathcal{R}_p^{(2)}$ to complete the proof. □

The following is the first of two stability results, and is a precursor to the main result in this subsection.

Proposition 3.15. Let $\alpha \in (-1,1)$ and let $\epsilon > 0$. There is $\delta = \delta(\alpha,\epsilon) > 0$ so that whenever $R \in \mathcal{R}$ is connected with $\mathrm{Leb}(R) \leq 2+\alpha$ and $\mathrm{d}_H(R, \mathcal{R}_p^{(2+\alpha)}) > \epsilon$, we have

$$\frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} \geq \varphi_p^{(2+\alpha)} + \delta \tag{3.22}$$

Proof. Suppose not. Then there is a sequence $R_k \in \mathcal{R}$ of connected sets with $\mathrm{Leb}(R_k) \leq 2+\alpha$, and with $\mathrm{d}_H(R_k, \mathcal{R}_p^{(2+\alpha)}) > \epsilon$ and

$$\frac{\mathcal{I}_p(\partial R_k)}{\mathrm{Leb}(R_k)} \to \varphi_p^{(2+\alpha)} \,. \tag{3.23}$$

Suppose first that for each k, $R_k = \mathsf{hull}(\lambda_k)$, where $\lambda_k \subset [-1,1]^2$ is a rectifiable Jordan curve. By Theorem 12.26 of [29], and by the compactness of the set of non-empty compact subsets of $[-1,1]^2$ in the metric d_H, we lose no generality supposing $R_k \to K \subset [-1,1]^2$ compact, where the convergence takes place both in L^1-sense and in d_H-sense. By Lemma 3.11, it follows that $K \in \mathcal{C}_p^{(2+\alpha)}$, and by Lemma 3.13, it then follows that $K \in \mathcal{R}_p^{(2+\alpha)}$, which is a contradiction.

Let us then suppose that none of the R_k are of the form $\mathsf{hull}(\lambda_k)$ for a sequence of rectifiable Jordan curves $\lambda_k \subset [-1,1]^2$, so that for each k, we have $\widehat{R}_k \neq R_k$. We appeal to the same compactness argument as above, and suppose that the R_k tend to $K \subset [-1,1]^2$ compact both in L^1-sense and in d_H-sense. As before, Lemma 3.11 tells us $K \in \mathcal{C}_p^{(2+\alpha)}$. We then use Lemma 3.12 to deduce that $\mathrm{Leb}(\widehat{R}_k \setminus R_k) \to 0$.

As the conductances of the R_k tend to $\varphi_p^{(2+\alpha)}$, and as $\varphi_p^{(2+\alpha+\epsilon)} \to \varphi_p^{(2+\alpha)}$ as $\epsilon \to 0$ (from (3) and (4) of Corollary 3.9), the diameter of any connected component of $\widehat{R}_k \setminus R_k$ must also tend to zero. Thus, as $k \to \infty$, we have that $\mathrm{d}_H(\widehat{R}_k, R_k) \to 0$, and we may then realize $K \in \mathcal{C}_p^{(2+\alpha)}$ as the L^1- and d_H-limit of the $\widehat{R}_k$. As each $\widehat{R}_k$ is the hull of a rectifiable Jordan curve, we may now use Lemma 3.13 to deduce that $K \in \mathcal{R}_p^{(2+\alpha)}$, which is again a contradiction. □

Our second stability result upgrades Proposition 3.15, telling us that δ does not tend to zero with α. It is instrumental to the proof of Theorem 1.2.

Corollary 3.16. Let $\alpha \in (0,1]$ and let $\epsilon > 0$. There is $\delta = \delta(\epsilon, \alpha) > 0$ so that whenever $R \in \mathcal{R}$ is connected with $\mathrm{Leb}(R) \leq 2 + \alpha$ and $\mathrm{d}_H(R, \mathcal{R}_p^{(2+\alpha)}) > \epsilon$, we have

$$\frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} \geq \varphi_p^{(2+\alpha)} + \delta \,, \tag{3.24}$$

and where δ stays bounded away from zero as $\alpha \to 0$.

Proof. Let $\epsilon > 0$ and let α_k be a sequence in $(0,1]$ tending to zero as $k \to \infty$. Let $\tilde{\delta}(\alpha_k, \epsilon)$ be the supremum of all $\delta > 0$ for which Proposition 3.15 is valid for the parameters α_k and ϵ. Then, for each k, there are connected sets $R_k \in \mathcal{R}$ with $\mathrm{Leb}(R_k) \leq 2 + \alpha_k$ so that $\mathrm{d}_H(R_k, \mathcal{R}_p^{(2+\alpha_k)}) \geq \epsilon$ and

$$\frac{\mathcal{I}_p(\partial R_k)}{\mathrm{Leb}(R_k)} \leq \varphi_p^{(2+\alpha_k)} + 2\tilde{\delta}(\alpha_k, \epsilon) \,. \tag{3.25}$$

Suppose for the sake of contradiction that $\tilde{\delta}(\alpha_k, \epsilon) \to 0$ as $k \to \infty$. Then the conductances of the R_k tend to $\varphi_p^{(2)}$. Passing to a subsequence, we may assume that $R_k \to K$ compact with $K \in \mathcal{C}_p^{(2+\alpha)}$, where the convergence takes place both in L^1-sense and in d_H-sense. If each R_k is the hull of a rectifiable Jordan curve, we may invoke Lemma 3.13 to deduce that $K \in \mathcal{R}_p^{(2+\alpha)}$. If not, we may proceed as in the proof of Proposition 3.15, replacing each R_k by $\widehat{R}_k$ to deduce the same result.

Thus, the R_k get arbitrarily close in d_H-sense to $\mathcal{R}_p^{(2)}$, so that for all k sufficiently large, $\mathrm{d}_H(R_k, \mathcal{R}_p^{(2)}) \leq \epsilon/4$. Thanks to Lemma 3.14, we may also find k sufficiently large so that $\mathrm{d}_H(\mathcal{R}_p^{(2+\alpha_k)}, \mathcal{R}_p^{(2)}) < \epsilon/4$. This contradicts the fact that $\mathrm{d}_H(R_k, \mathcal{R}_p^{(2+\alpha_k)}) > \epsilon$. ☐

4 Continuous to discrete: upper bounds

In this section, we show that given $R \in \mathcal{R}$ with $\mathrm{Leb}(R) \leq 2$, there are high probability upper bounds on $n\Phi_n$ in terms of the conductance of R. We first show this for polygons and then use approximation to pass to more general sets.

4.1 From simple polygons to discrete sets

A *convex polygon* in $\mathbb{R}^2$ is a compact subset of $\mathbb{R}^2$ having non-empty interior which may be written as the intersection of finitely many closed half-spaces. A *polygon* is any subset of $\mathbb{R}^2$ which may be written as a finite union of convex polygons.

Recall (from the statement of Proposition 2.15) that given $x, y \in \mathbb{R}^2$, we use $\mathsf{poly}(x,y)$ to denote the linear segment joining x and y. Given a sequence of points $x_1, \ldots, x_m$, we define

$$\mathsf{poly}(x_1, \ldots, x_m) := \mathsf{poly}(x_1, x_2) * \cdots * \mathsf{poly}(x_{m-1}, x_m) \,, \tag{4.1}$$

where “$*$” denotes concatenation of these curves. A *polygonal curve* is any curve of the form (4.1) for some $x_1, \ldots, x_m \in \mathbb{R}^2$ and some $m \in \mathbb{N}$ (we return to being vague about the parametrization). Polygons may be defined from polygonal curves in a natural way; we say a polygon is *simple* if it may be written as the hull of a simple polygonal circuit. The first proposition of this section associates a discrete set to any simple polygon in a convenient way.

Remark 4.1. In this section and the next we will be somewhat cavalier with notation. In particular, for $R \in \mathcal{R}$, the dilated set nR is not in general contained in $[-1,1]^2$. The surface energy of nR, denoted $\mathcal{I}_p(n\partial R)$ is defined to be $n\mathcal{I}_p(\partial R)$. We employ a similar convention for curves.

Proposition 4.2. Let $p > p_c(2)$ and let $\epsilon > 0$. Let $P \subset [-1,1]^2$ be a simple non-degenerate polygon. There are positive constants $c_1(p,P,\epsilon)$ and $c_2(p,P,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, there is a rectifiable circuit $\lambda \equiv \lambda(P) \subset [-1,1]^2$ so that

(1) $\mathrm{d}_H(n\partial P, n\lambda) \leq \epsilon n$,
(2) $\mathcal{I}_p(n\partial P) \geq (1-\epsilon)|\partial^n[\mathsf{hull}(n\lambda) \cap \mathbf{C}_n]|$.

Proof. **Step I: (Aggregation of high probability events)** Let $x_1, \dots, x_m$ be the corners of nP, so that we have

$$nP = \mathsf{hull}(\mathsf{poly}(x_1, \dots, x_m)), \tag{4.2}$$

where $x_m \equiv x_1$, and where the circuit $\mathsf{poly}(x_1, \dots, x_m)$ is oriented counter-clockwise. Let $\mathcal{E}_1$ be the high probability event from Lemma 2.6 that for each $i \in \{1, \dots, m\}$, we have $|[x_i] - x_i|_2 \leq \log^2 n$. Say x_i is an *interior point* if $x_i \in (-n,n)^2$, and that it is a *boundary point* otherwise. For n sufficiently large, the Euclidean ball $B_{2\log^2 n}(x_i)$ is contained in $(-n,n)^2$ for each interior point x_i. For such n and within $\mathcal{E}_1$, we have $[x_i] \in (-n,n)^2$ for each interior x_i.

For $\delta > 0$, define the high probability event $\mathcal{E}_2(\delta)$ via

$$\mathcal{E}_2(\delta) := \bigcap_{i=1}^{m-1} \left\{ \exists \gamma \in \Gamma_\delta(x_i, x_{i+1}) \ : \mathrm{d}_H\Big(\gamma, \mathsf{poly}(x_i, x_{i+1})\Big) \leq \delta |x_{i+1} - x_i|_2 \right\}, \tag{4.3}$$

so that $\mathcal{E}_2(\delta)^c$ is subject to the bounds in Proposition 2.15. Additionally, define

$$\mathcal{E}_3(\delta) := \bigcap_{i=1}^{m-1} \left\{ \left| \frac{b([x_i],[x_{i+1}])}{\beta_p(x_{i+1} - x_i)} - 1 \right| > \delta \right\}, \tag{4.4}$$

so that $\mathcal{E}_3(\delta)^c$ is subject to the bounds in Theorem 2.14. For the remainder of the proof, work within the intersection $\mathcal{E}_1 \cap \mathcal{E}_2(\delta) \cap \mathcal{E}_3(\delta)$.

Step II: (Constructing λ) Select $\gamma_i \in \Gamma_\delta(x_i, x_{i+1})$ with $\mathrm{d}_H(\gamma_i, \mathsf{poly}(x_i, x_{i+1})) < \delta |x_{i+1} - x_i|_2$ for each $i \in \{1, \dots, m-1\}$. Each γ_i may be identified with an interface ∂_i via the correspondence in Proposition 2.7.

A linear segment $\mathsf{poly}(x_i, x_{i+1})$ is an *interior segment* if at least one of x_i or x_{i+1} is an interior point, and otherwise it is a *boundary segment*. If $\mathsf{poly}(x_i, x_{i+1})$ is a boundary segment, set $\lambda_i := \mathsf{poly}(x_i, x_{i+1})$, otherwise, via "corner-rounding" (see Remark 2.8), regard ∂_i as a simple curve and set $\lambda_i := \partial_i$. If the endpoint of λ_i is not equal to the starting point of λ_{i+1}, let $\widetilde{\lambda}_i$ be the linear segment joining these two points. If λ_i ends at the starting point of λ_{i+1}, let $\widetilde{\lambda}_i$ be the degenerate linear segment at this endpoint. Define the circuit $n\lambda$ as the concatenation of these curves in the proper order:

$$n\lambda := \lambda_1 * \widetilde{\lambda}_1 * \lambda_2 * \widetilde{\lambda}_2 * \cdots * \lambda_m * \widetilde{\lambda}_m, \tag{4.5}$$

and write H_n for $\mathsf{hull}(n\lambda) \cap \mathbf{C}_n$. Let E_i be the set of all edges of $\mathbb{Z}^2$ contained in the Euclidean ball $B_{2\log^2 n}(x_i)$, so that by construction of H_n,

$$|\partial^n H_n| \leq \sum_{\substack{i \,:\, \mathsf{poly}(x_i, x_{i+1}) \\ \text{is interior}}} |\mathfrak{b}(\gamma_i)| + \sum_{i=1}^{m} |E_i|. \tag{4.6}$$

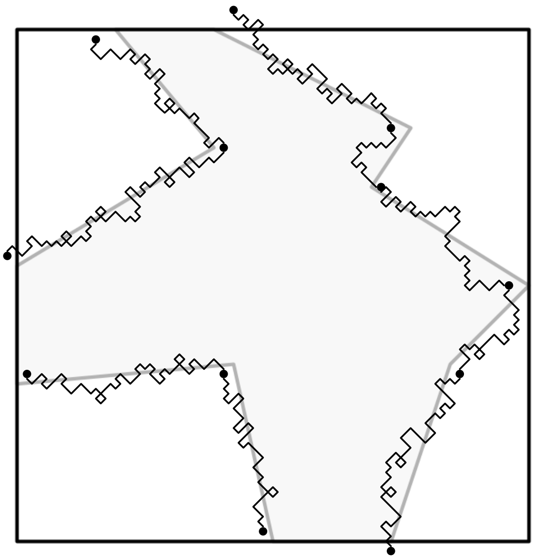

Figure 8: The polygon nP is in grey. The black dots are the $[x_i]$, and the contours joining these dots are the $\partial_i \equiv \lambda_i$ corresponding to the interior segments $\mathsf{poly}(x_i, x_{i+1})$.

Step III: (Controlling $|\partial^n H_n|$) We build off (4.6) and use that each γ_i is δ-optimal (see (2.11)),

$$|\partial^n H_n| \leq \sum_{\substack{i\,:\,\mathsf{poly}(x_i,x_{i+1})\\ \text{is interior}}} \Big(b([x_i],[x_{i+1}]) + \delta |x_{i+1}-x_i|_2 \Big) + \sum_{i=1}^{m} |E_i|\,, \tag{4.7}$$

$$\leq \left(\sum_{\substack{i\,:\,\mathsf{poly}(x_i,x_{i+1})\\ \text{is interior}}} b([x_i],[x_{i+1}]) \right) + 2mn\delta + C\log^4 n\,, \tag{4.8}$$

for some absolute positive constant C. As we are within $\mathcal{E}_2(\delta)$, for n sufficiently large we have

$$|\partial^n H_n| \leq \left(\sum_{\substack{i\,:\,\mathsf{poly}(x_i,x_{i+1})\\ \text{is interior}}} (\beta_p(x_{i+1}-x_i) + n\delta) \right) + 4mn\delta\,, \tag{4.9}$$

$$\leq \mathcal{I}_p(n\partial P) + 8mn\delta\,. \tag{4.10}$$

Step IV: (Wrapping up) Given $\epsilon > 0$, we may choose δ sufficiently small depending on P and ϵ so that from (4.10), we have

$$\mathcal{I}_p(n\partial P) \geq (1-\epsilon)|\partial^n H_n|\,. \tag{4.11}$$

Finally, the construction of λ from the γ_i ensures that

$$\mathrm{d}_H(nP, n\lambda) \leq 2\delta \max_{i=1}^{m-1} |x_{i+1}-x_i|_2\,, \tag{4.12}$$

and we take δ smaller if necessary to complete the proof. □

4.2 Upper bounds on $n\Phi_n$ using simple polygons

We now use the output of Proposition 4.2 to construct a discrete approximate to more general connected polygons. We also relate the volume of the discrete approximate to the volume of this polygon.

Proposition 4.3. Let $p > p_c(2)$ and let $\epsilon > 0$. Let $P \subset [-1,1]^2$ be a simple non-degenerate polygon. There are positive constants $c_1(p,P,\epsilon)$ and $c_2(p,P,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1\exp(-c_2\log^2 n)$, there is subgraph $H_n \equiv H_n(P) \subset \mathbf{C}_n$ with

(1) $|\theta_p \mathrm{Leb}(nP) - |H_n|| \leq \epsilon \mathrm{Leb}(nP)$,
(2) $\mathcal{I}_p(n\partial P) \geq (1-\epsilon)|\partial^n H_n|$.

Proof. **Step I: (Identifying H_n)** Let ρ be the simple polygonal circuit with $P = \mathsf{hull}(\rho)$. Work within the high probability event from Proposition 4.2 that there is a circuit $\lambda \subset [-1,1]^2$ with

(1) $\mathrm{d}_H(n\rho, n\lambda) \leq \delta n$.
(2) $\mathcal{I}_p(n\rho) \geq (1-\delta)|\partial^n[\mathsf{hull}(n\lambda) \cap \mathbf{C}_n]|$

Write R for $\mathsf{hull}(\lambda)$ and define $H_n := nR \cap \mathbf{C}_n$. By (4.2), the graph H_n has the second desired property:

$$\mathcal{I}_p(n\partial P) \geq (1-\delta)|\partial^n H_n| \,. \tag{4.13}$$

Step II: (Controlling the volume of H_n from above) We control the volume of H_n by appealing to Proposition A.2. Let $k \in \mathbb{N}$ and let S_k denote the set of half-open dyadic squares at the scale k which are contained in $[-1,1]^2$; these are translates of $[-2^{-k}, 2^{-k})^2$. For $\delta' > 0$ and $S \in \mathsf{S}_k$, define the event

$$\mathcal{E}_S(\delta') := \left\{ \frac{|\mathbf{C}_\infty \cap nS|}{\mathrm{Leb}(nS)} \in \Big((1-\delta')\theta_p, (1+\delta')\theta_p\Big) \right\} , \tag{4.14}$$

and let $\mathcal{E}_{\mathrm{vol}}(\delta')$ be the intersection of $\mathcal{E}_S(\delta')$ over all $S \in \mathsf{S}_k$. From now on, work within the event $\mathcal{E}_{\mathrm{vol}}(\delta')$. Using Proposition A.2 with a union bound, there are $c_1, c_2 > 0$ depending on p and δ' with

$$\mathbb{P}_p(\mathcal{E}_{\mathrm{vol}}(\delta')) \leq 2^{2k+2} c_1 \exp(-c_2 2^{-k} n) \tag{4.15}$$

Let $N_{2\delta}$ be the closed 2δ-neighborhood (with respect to Euclidean distance) of ∂P. Let S_k^- be the squares of S_k contained in $P \setminus N_{2\delta}$, and let S_k^+ be the squares of S_k having non-empty intersection with $P \cup N_{2\delta}$. Here we assume δ is small enough and k is large enough for S_k^- to be non-empty. Thanks to the construction of H_n, we have

$$|H_n| \leq \sum_{S \in \mathsf{S}_k^+} |nS \cap \mathbf{C}_\infty| + Cn \,, \tag{4.16}$$

where C is some absolute constant, and the term Cn directly above accounts for the vertices of $\mathbb{Z}^2$ in $\partial[-n,n]^2$, which we must be mindful of as the squares $S \in \mathsf{S}_k$ are half-open. Choose k large enough depending on δ' and P so that

$$(1-\delta')\mathrm{Leb}(P) \leq \sum_{S \in \mathsf{S}_k^-} \mathrm{Leb}(S) \leq \sum_{S \in \mathsf{S}_k^+} \mathrm{Leb}(S) \leq (1+\delta')\mathrm{Leb}(P) \,. \tag{4.17}$$

For n sufficiently large, it follows from (4.16), (4.17), and that we are within $\mathcal{E}_{\mathrm{vol}}(\delta')$ that

$$|H_n| \leq (1+2\delta')^2 \theta_p \mathrm{Leb}(nP) \,. \tag{4.18}$$

Step III: (Controlling the volume of H_n from below) Work within the following high probability event from Proposition A.3 for the remainder of the proof:

$$\Big\{\mathbf{C}_\infty \cap [-n+\log^2 n, n-\log^2 n] = \mathbf{C}_n \cap [-n+\log^2 n, n-\log^2 n]\Big\}. \tag{4.19}$$

From the construction of H_n and the disjointness of the squares in S_k, we find

$$|H_n| \geq \sum_{S\in\mathsf{S}_k^-} |\mathbf{C}_n \cap nS| \geq \left(\sum_{S\in\mathsf{S}_k^-} |\mathbf{C}_\infty \cap nS|\right) - |\mathbf{C}_\infty \cap [-n,n]^2 \setminus \mathbf{C}_n|\,. \tag{4.20}$$

Using that we are within (4.19) and taking n sufficiently large, we find

$$|H_n| \geq (1-2\delta')\sum_{S\in\mathsf{S}_k^-}\theta_p \mathrm{Leb}(nS)\,, \tag{4.21}$$

$$\geq (1-2\delta')(1-\delta')\theta_p\mathrm{Leb}(nP)\,. \tag{4.22}$$

where we have taken n sufficiently large to obtain the second line above, and where the last line follows from (4.17). We choose δ, δ' sufficiently small to complete the proof. □

We now use Proposition 4.3 to obtain upper bounds on Φ_n in terms of the conductance of simple, non-degenerate polygons.

Corollary 4.4. Let $p > p_c(2)$ and let $\epsilon > 0$. Let $P \subset [-1,1]^2$ be a simple, non-degenerate polygon with $\mathrm{Leb}(P) < 2$. There are positive constants $c_1(p,P,\epsilon)$ and $c_2(p,P,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1\exp(-c_2\log^2 n)$,

$$n\Phi_n \leq (1+\epsilon)\frac{\mathcal{I}_p(\partial P)}{\theta_p \mathrm{Leb}(P)}\,. \tag{4.23}$$

Proof. Define $\epsilon' := 2 - \mathrm{Leb}(P)$ and let $\delta > 0$. By combining Proposition A.2 with Proposition A.3, we obtain positive constants $c_1(p,\delta)$ and $c_2(p,\delta)$ so that the probability of the event

$$\left\{\frac{|\mathbf{C}_n|}{(2n)^2} \in \Big((1-\delta)\theta_p, (1+\delta)\theta_p\Big)\right\} \tag{4.24}$$

is at least $1 - c_1\exp(-c_2\log^2 n)$. Work within this high probability event, and additionally work within the high probability event from Proposition 4.3 that there is $H_n \subset \mathbf{C}_n$ satisfying

(1) $|\theta_p\mathrm{Leb}(nP) - |H_n|| \leq \delta\mathrm{Leb}(nP)$,
(2) $\mathcal{I}_p(n\partial P) \geq (1-\delta)|\partial^n H_n|$.

Thus, $|H_n| \leq (\theta_p+\delta)(2-\epsilon')n^2$. Using (4.24) and choosing δ small enough depending on ϵ' so that $2(1-\delta)\theta_p \geq (\theta_p+\delta)(2-\epsilon')$, we find $|H_n| \leq |\mathbf{C}_n|/2$, and conclude that with high probability,

$$\Phi_n \leq \frac{|\partial^n H_n|}{|H_n|} \leq \frac{\frac{1}{1-\delta}\mathcal{I}_p(nP)}{(\theta_p-\delta)\mathrm{Leb}(nP)}\,, \tag{4.25}$$

which completes the proof, taking δ smaller if necessary. □

4.3 The optimal upper bound on $n\Phi_n$

We now exhibit a high probability upper bound on $n\Phi_n$ using the optimal conductance of φ_p defined in (1.10). We introduce results allowing us to approximate rectifiable Jordan curves by simple polygonal circuits. The following consolidates Lemma 4.3 and Lemma 4.4 of [8].

Proposition 4.5. Let λ be a rectifiable curve in $\mathbb{R}^2$ starting at x and ending at y. Let $\epsilon > 0$. There is a simple polygonal curve ρ starting at x and ending at y such that (1) and (2) hold:

(1) $\mathrm{d}_H(\lambda, \rho) \leq \epsilon\,,$
(2) $\mathsf{length}_{\beta_p}(\lambda) + \epsilon \geq \mathsf{length}_{\beta_p}(\rho)\,.$

Furthermore, if λ is a closed curve (i.e. $x = y$), ρ can additionally be taken to satisfy (3):

(3) $\mathrm{Leb}(\mathsf{hull}(\lambda)\,\Delta\,\mathsf{hull}(\rho)) \leq \epsilon\,.$

Remark 4.6. We remark that, in Proposition 4.5, if the curve λ is contained in $[-1,1]^2$, one can easily arrange that the polygonal approximate ρ is also contained in $[-1,1]^2$.

The following is a nearly immediate consequence Proposition 4.5, so we omit the proof.

Corollary 4.7. Let $\lambda \subset [-1,1]^2$ be a rectifiable Jordan curve such that $\lambda = \lambda_1 * \lambda_2$, where λ_1 and λ_2 are simple curves with $\lambda_1 \subset \partial[-1,1]^2$, and such that every point on the curve λ_2 except the endpoints lies in $(-1,1)^2$. Let $\epsilon > 0$. There is a simple polygonal circuit $\rho \subset [-1,1]^2$ so that

(1) $\mathrm{d}_H(\lambda, \rho) \leq \epsilon\,,$
(2) $\mathcal{I}_p(\lambda) + \epsilon \geq \mathcal{I}_p(\rho)\,,$
(3) $\mathrm{Leb}(\mathsf{hull}(\lambda)\,\Delta\,\mathsf{hull}(\rho)) \leq \epsilon\,.$

Remark 4.8. If instead of a decomposition of λ into two curves as in Corollary 4.7, we express λ as a concatenation of finitely many curves, each having the properties of λ_1 or λ_2, the conclusion of Corollary 4.7 still holds. That is, for such λ, we may find a polygonal circuit ρ for which (4.7) – (4.7) hold.

We are now equipped to prove Theorem 4.9, the main theorem of the section.

Theorem 4.9. There are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$,

$$n\Phi_n \leq (1+\epsilon)\varphi_p\,, \tag{4.26}$$

where φ_p is defined in (1.10).

Proof. Let $R \in \mathcal{R}_p$. By Corollary 3.9, we lose no generality taking $R = \mathsf{hull}(\lambda)$, with λ as in the statement of Corollary 4.7. Because of the structure of λ, we may slightly shrink R (against the wall or walls it rests on) to a set R_ϵ with $\mathrm{Leb}(R_\epsilon) = (1-\epsilon)^2\mathrm{Leb}(R)$ and $\mathcal{I}_p(R_\epsilon) = (1-\epsilon)\mathcal{I}_p(R)$ for some $\epsilon > 0$ which can be taken arbitrarily small. Let λ_ϵ be the circuit with $R_\epsilon = \mathsf{hull}(\lambda_\epsilon)$, and for $\delta > 0$, apply Corollary 4.7 to λ_ϵ to find a simple polygonal circuit $\rho \subset [-1,1]^2$ so that

(1) $\mathrm{d}_H(\lambda_\epsilon, \rho) \leq \delta\,,$
(2) $\mathcal{I}_p(\lambda_\epsilon) + \delta \geq \mathcal{I}_p(\rho)\,,$
(3) $\mathrm{Leb}(\mathsf{hull}(\lambda_\epsilon)\,\Delta\,\mathsf{hull}(\rho)) \leq \delta\,.$

For δ small enough depending on ϵ, we may apply Corollary 4.4 to $P := \mathsf{hull}(\rho)$ using the parameter ϵ, deducing that with high probability,

$$n\Phi_n \le (1+\epsilon)\frac{\mathcal{I}_p(\partial P)}{\theta_p \mathrm{Leb}(P)}\,, \tag{4.27}$$

$$\le (1+\delta)\frac{\mathcal{I}_p(\partial R_\epsilon)+\delta}{\theta_p(\mathrm{Leb}(R_\epsilon)-\delta)}\,, \tag{4.28}$$

The proof is complete upon choosing δ to depend suitably on ϵ. □

5 Discrete to continuous objects: lower bounds

We construct tools which allow us to pass from a subgraph of $\mathbf{C}_n$ to a connected polygon of comparable conductance. By Lemma 2.10, the boundary of a subgraph of $\mathbf{C}_n$ may be thought of as a finite collection of open right-most circuits. Our first goal is then to construct an approximating polygonal curve to any open right-most path.

5.1 Extracting polygonal curves from right-most paths

Our first result enables us to pass from open right-most paths of sufficient length to polygonal curves. We omit the proof, as it follows directly from the proof of Proposition 4.1 in [8] and Proposition 4.5.

Lemma 5.1. Let $p > p_c(2)$ and let $\epsilon > 0$. There are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that for all $n \ge 1$, with probability at least $1 - c_1\exp(-c_2\log^2 n)$, whenever $\gamma \subset [-n,n]^2$ is an open right-most path with $|\gamma| \ge n^{1/32}$, there is a simple polygonal curve $\rho = \rho(\gamma) \subset [-1,1]^2$ with

(1) $\mathrm{d}_H(\gamma, n\rho) \le n^{1/64}$,
(2) $|\mathfrak{b}(\gamma)| \ge (1-\epsilon)\mathrm{length}_{\beta_p}(n\rho)$.

Our second result allows us to pass from right-most circuits of sufficient length to polygonal circuits. The boundary of $[-1,1]^2$ now comes into play: we bound the surface energy of the polygonal circuit (instead of the β_p-length) in terms of the $\mathbf{C}_n$-length of the right-most circuit (as opposed to the $\mathbf{C}_\infty$-length).

Lemma 5.2. Let $p > p_c(2)$ and let $\epsilon > 0$. There are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that with probability at least $1 - c_1\exp(-c_2\log^2 n)$, whenever $\gamma \subset [-n,n]^2$ is an open right-most circuit with $|\gamma| \ge n^{1/4}$, there is a simple polygonal circuit $\rho = \rho(\gamma) \subset [-1,1]^2$ with

(1) $\mathrm{d}_H(\gamma, n\rho) \le n^{1/16}$,
(2) $|\mathfrak{b}^n(\gamma)| \ge (1-\epsilon)\mathcal{I}_p(n\rho)$.

Moreover, if $\gamma \subset (-n,n)^2$, we may replace (2) above with

(3) $|\mathfrak{b}^n(\gamma)| \ge (1-\epsilon)\mathrm{length}_{\beta_p}(n\rho)$.

Proof. Let $\gamma \subset [-n,n]^2$ be an open right-most circuit with $|\gamma| \ge n^{1/4}$, and express γ as an alternating sequence of vertices and edges

$$\gamma = (x_0, e_1, x_1, e_2, x_2, \ldots, e_m, x_m)\,, \tag{5.1}$$

where $x_0 = x_m$.

Step I: (Decomposition of γ) Say that x_i is a *boundary vertex* if $x \in \partial[-n,n]^2$ and that x_i is an *interior vertex* otherwise. If no x_i in γ is a boundary vertex, this Lemma follows from Proposition 4.1 in [8]. As γ is a circuit, we lose no generality supposing x_0 is a boundary vertex. Let $\widetilde{x}_0, \dots, \widetilde{x}_\ell$ enumerate the boundary vertices of γ ordered in terms of increasing index in (5.1). For $j \in \{1, \dots, \ell\}$, let γ_j be the subpath of γ starting at $\widetilde{x}_{j-1}$ and ending at $\widetilde{x}_j$. Each γ_j is right-most and has the property that only the endpoints of γ_j are boundary vertices.

Say γ_j is *long* if $|\gamma_j| \geq n^{1/32}$, and that it is *short* otherwise. For each γ_j, let γ_j' denote the unique self-avoiding path of edges contained in $\partial[-n,n]^2$ whose starting and ending points are those of γ_j.

Step II: (Polygonal approximation) Work within the high probability event from Lemma 5.1 for $\epsilon > 0$. For each long γ_j, there is then a simple polygonal curve $\rho_j \subset [-1,1]^2$ satisfying

(1) $\mathrm{d}_H(\gamma_j, n\rho_j) \leq n^{1/64}$,

(2) $|\mathfrak{b}(\gamma_j)| \geq (1-\epsilon)\mathrm{length}_{\beta_p}(n\rho_j)$.

If γ_j is short, regard γ_j' as a polygonal curve $n\rho_j \subset \partial[-n,n]^2$ joining $\widetilde{x}_{j-1}$ with $\widetilde{x}_j$. Thus, each γ_j gives rise to a simple polygonal curve $\rho_j \subset [-1,1]^2$ in one of two ways, according to $|\gamma_j|$. Let ρ' be the concatenation of the ρ_j in the proper order, $\rho' := \rho_1 * \cdots * \rho_\ell$, so that ρ' is a polygonal circuit. We claim ρ' has the desired properties.

We first check d_H-closeness of $n\rho'$ and γ. If γ_j is short, any vertex in γ_j has an ℓ^∞-distance of at most $2n^{1/32}$ to $\widetilde{x}_j$, and likewise any vertex in γ_j' has an ℓ^∞-distance of at most $2n^{1/32}$ to $\widetilde{x}_j$. Thus $\mathrm{d}_H(\gamma_j, n\rho_j) \leq 4n^{1/32}$ when γ_j is short. For γ_j long, (5.1) gives even better control, and consequently,

$$\mathrm{d}_H(\gamma, n\rho') \leq 4n^{1/32} + n^{1/64}\,. \tag{5.2}$$

We now turn to $\mathcal{I}_p(n\rho')$. Using the decomposition $\gamma = \gamma_1 * \cdots * \gamma_\ell$ and the construction of ρ',

$$|\mathfrak{b}^n(\gamma)| \geq \sum_{j\,:\,\gamma_j \text{ long}} |\mathfrak{b}(\gamma_j)| \;\; \geq (1-\epsilon) \sum_{j\,:\,\gamma_j \text{ long}} \mathrm{length}_{\beta_p}(n\rho_j)\,, \tag{5.3}$$

$$\geq (1-\epsilon)\mathcal{I}_p(n\rho')\,, \tag{5.4}$$

where we have used (5.1) in the first line above.

Step III: (Perturbation) It remains to perturb ρ' to a simple polygonal circuit. Let $\delta > 0$, and apply Corollary 4.7 (and Remark 4.8) to ρ' with this δ, so that by (5.2) we have

$$\mathrm{d}_H(\gamma, n\rho) \leq 4n^{1/32} + n^{1/64} + \delta n\,, \tag{5.5}$$

and by (5.4) we have

$$|\mathfrak{b}^n(\gamma)| \geq (1-\epsilon)(\mathcal{I}_p(n\rho) - \delta n). \tag{5.6}$$

The proof is complete upon setting $\delta = \min(n^{(1/32)-1}, \epsilon\mathcal{I}_p(\rho'))$, adjusting ϵ and taking n larger if necessary. In the case that γ contains no boundary vertices, we split γ into a concatenation of two long right-most paths and proceed as above. □

5.2 Interlude: optimizers are of order n^2

In arguments to come, it will be important to know that with high probability, each Cheeger optimizer has size on the order of n^2. First, we record that Φ_n is at most a constant times n^{-1} with high probability.

Proposition 5.3. Let $p > p_c(2)$. There are positive constants $c(p), c_1(p), c_2(p) > 0$ so that with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, we have $\Phi_n \leq cn^{-1}$.

Proof. This is an immediate consequence of Theorem 4.9, though we remark that this admits an elementary proof using only Proposition A.2 and Proposition A.3. □

We now deduce that with high probability, each Cheeger optimizer is large.

Proposition 5.4. Let $p > p_c(2)$. There are positive constants $c_1(p), c_2(p), \alpha(p)$ so that with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, we have

$$\min_{G_n \in \mathcal{G}_n} |G_n| \geq \alpha n^2 \,. \tag{5.7}$$

Proof. We make two assumptions which will be justified at the end of the proof.

(1) G_n is connected.

(2) $|G_n| \leq |\mathbf{C}_n|/2 - n^{1/8}$

Use Lemma 2.10 and (5.2) to identify a right-most circuit γ as in the statement of Lemma 2.10, which we express as an alternating sequence of vertices and edges:

$$\gamma = (x_0, e_1, x_1, e_2, x_2, \dots, e_m, x_m) \,, \tag{5.8}$$

where $x_0 = x_m$. Say x_i is a *boundary vertex* if $x_i \in \partial[-n, n]^2$ and that it is an *interior vertex* otherwise. In the case that γ contains no boundary edges, the proposition follows by combining Proposition A.4 with Proposition 5.3.

Henceforth, we suppose γ contains at least one boundary vertex, and we follow ***Step I*** in the proof of Lemma 5.2. Without loss of generality, x_0 is then a boundary vertex and we let $\widetilde{x}_0, \dots, \widetilde{x}_\ell$ enumerate the boundary vertices of γ in terms of increasing order in (5.8). For $j \in \{1, \dots, \ell\}$, we let γ_j be the subpath of γ which begins at x_{j-1} and ends at x_j. As before, we note that each γ_j is right-most and that only the endpoints of γ_j are boundary vertices. We say that γ_j is *long* if $|\gamma_j| \geq n^{1/32}$ and that γ_j is *short* otherwise.

We claim that no γ_j can be short. To see this, let $\widetilde{\gamma}_j$ be the right-most path defined by the sequence of edges, each contained in $\partial[-n, n]^2$, and which begin at $\widetilde{x}_j$ and end at $\widetilde{x}_{j-1}$. Let ∂_j be the counter-clockwise interface corresponding to $\gamma_j * \widetilde{\gamma}_j$, and observe that

$$
\begin{aligned}
|\mathsf{hull}(\partial_j) \cap \mathbf{C}_n| &\leq \mathrm{Leb}(\mathsf{hull}(\partial_j)) + c|\gamma_j * \widetilde{\gamma}_j| \,, && (5.9)\\
&\leq c\mathsf{length}(\partial_j)^2 + c|\gamma_j * \widetilde{\gamma}_j| \,, && (5.10)\\
&\leq cn^{1/16} < n^{1/8} \,. && (5.11)
\end{aligned}
$$

Here, c is an absolute constant which is allowed to change from line to line, and we have used the standard Euclidean isoperimetric inequality to obtain the second line. The third line follows from the assumption that γ_j is short and by taking n large. Writing $G_n' := G_n \cup [\mathsf{hull}(\partial_j) \cap \mathbf{C}_n]$, and using (5.2), we have that $|G_n'| \leq |\mathbf{C}_n|/2$ and that the conductance of G_n' is strictly smaller than that of G_n. This is a contradiction, so our claim that no γ_j can be short holds.

By Proposition 2.11, it is a high-probability event that $|\mathfrak{b}^n(\gamma_j)| \geq \alpha|\gamma_j|$. Thus, writing ∂ for the interface corresponding to γ, it follows that

$$|\partial^n G_n| \geq |\mathfrak{b}^n(\gamma)| \geq c\mathcal{H}^1\big(\partial \cap (-n,n)^2\big)\,, \tag{5.12}$$

$$\geq c\mathrm{Leb}\big(\mathsf{hull}(\partial) \cap [-n,n]^2\big)^{1/2} \tag{5.13}$$

$$\geq c|G_n|^{1/2}\,, \tag{5.14}$$

where we've used the isoperimetric inequality to obtain the second line, and where the constant $c > 0$ changes from line to line.

It remains to address our assumptions (5.2) and (5.2). If $|G_n| \geq |\mathbf{C}_n|/2 - n^{1/8}$, Proposition A.2 and Proposition A.3 together imply $|G_n| \geq cn^2$ with high probability. Finally, any G_n is a disjoint union of connected Cheeger optimizers, so the lower bounds on the connected Cheeger optimizers suffice. □

5.3 Approximating discrete sets via polygons

Now that we have tools for converting right-most circuits to polygonal circuits, we use the decomposition given by Lemma 2.10 to pass from subgraphs of $\mathbf{C}_n$ to connected polygons. To relate the conductances of these objects, we require a mild isoperimetric assumption on the subgraph of $\mathbf{C}_n$ in question.

Recall that $\mathcal{U}_n$ denotes the collection of connected subgraphs of $\mathbf{C}_n$ inheriting their graph structure from $\mathbf{C}_n$. Given a decomposition of $U \in \mathcal{U}_n$ as in Lemma 2.10, define

$$\mathsf{d\text{-}per}(U) := |\gamma| + \sum_{j=1}^{m} |\gamma_j|, \tag{5.15}$$

which may be thought of as the "full" perimeter of U. Also define

$$\mathsf{vol}(U) := \mathsf{hull}(\partial) \setminus \left(\bigsqcup_{j=1}^{m} \mathsf{hull}(\partial_j) \right)\,, \tag{5.16}$$

where ∂ and the ∂_j are the interfaces corresponding to the right-most circuits γ, γ_j.

Definition 5.5. Say that $U \in \mathcal{U}_n$ is *well-proportioned* if

$$\mathsf{d\text{-}per}(U) \leq \mathrm{Leb}(\mathsf{vol}(U))^{2/3}\,. \tag{5.17}$$

The following coarse-graining result says that with high probability, each $U \in \mathcal{U}_n$ is d_H-close to $\mathsf{vol}(U)$. Moreover, if $U \in \mathcal{U}_n$ is well-proportioned and sufficiently large, we may deduce U has "typical" density within $\mathsf{vol}(U)$.

Lemma 5.6. Let $p > p_c(2)$ and let $\epsilon > 0$. There are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$,

$$\mathrm{d}_H(U, \mathsf{vol}(U)) \leq \log^4 n\,. \tag{5.18}$$

Moreover, whenever $U \in \mathcal{U}_n$ satisfies

(1) U is well-proportioned,
(2) $\mathrm{Leb}(\mathsf{vol}(U)) \geq \log^{20} n$,

we have

$$\left| \frac{|U|}{\mathrm{Leb}(\mathsf{vol}(U))} - \theta_p \right| < \epsilon\,. \tag{5.19}$$

Proof. The density statement (5.19) is furnished by Lemma 5.3 of [8]. In the proof of this lemma, one obtains (5.18) even without hypotheses (1) and (2); we supply these details below.

Let $\epsilon > 0$, and define $r := \lfloor \log^2 n \rfloor$. For $u \in \mathbb{Z}^2$, define the square $S_u := (2r)u + [-r, r)^2$, and use Proposition A.2 with a union bound to find positive constants $c_1(p, \epsilon)$ and $c_2(p, \epsilon)$ so that

$$\mathcal{A}_n := \left\{ u \in \mathbb{Z}^2,\, S_u \cap [-n, n]^2 \neq \emptyset \implies \left| \frac{|\mathbf{C}_\infty \cap S_u|}{\mathrm{Leb}(S_u)} - \theta_p \right| < \epsilon \right\} \tag{5.20}$$

satisfies $\mathbb{P}_p(\mathcal{A}_n) \geq 1 - c_1 \exp(-c_2 \log^2 n)$. Given $U \in \mathcal{U}_n$, let (γ, ∂) and $\{(\gamma_j, \partial_j)\}_{j=1}^m$ be pairs of corresponding right-most and interface circuits for U, as in Lemma 2.10. Together, these circuits determine $\mathsf{vol}(U)$, defined in (5.16). Define two collections of squares:

$$\mathsf{S}_1 := \left\{ S_u : u \in \mathbb{Z}^2,\, S_u \cap \partial \mathsf{vol}(U) \neq \emptyset \right\}, \tag{5.21}$$

$$\mathsf{S}_2 := \left\{ S_u : u \in \mathbb{Z}^2,\, S_u \subset \big(\mathsf{vol}(U) \setminus \partial \mathsf{vol}(U)\big) \right\}, \tag{5.22}$$

and let $y \in \mathsf{vol}(U)$. As the S_u partition $\mathbb{R}^2$, it follows that y lives in exactly one S_u, which is then either in S_1 or S_2. If $S_u \in \mathsf{S}_1$, there is $u' \in \mathbb{Z}^2$ with $|u - u'|_\infty \leq 1$ so that S_u contains a vertex in γ or some γ_j. In this case, $\mathrm{dist}_\infty(y, U) \leq 4 \log^2 n$. On the other hand, if $B_u \in \mathsf{S}_2$, working within $\mathcal{A}_n$, we find $S_u \cap \mathbf{C}_\infty \subset U$ is non-empty and hence that $\mathrm{dist}_\infty(y, U) \leq 4 \log^2 n$. As $U \subset \mathsf{vol}(U)$, it follows from the above observations that $\mathrm{d}_H(U, \mathsf{vol}(U)) \leq \log^4 n$, for n sufficiently large. □

Given $U \in \mathcal{U}_n$, we will build a polygonal approximate from a collection of simple polygonal circuits. It is convenient to introduce the following construction, used in Lemma 5.8 which is in turn used in the proof of Proposition 5.10 below.

Definition 5.7. Given polygonal curves $\rho, \rho_1, \dots, \rho_m \subset \mathbb{R}^2$, we define the set $\mathsf{hull}(\rho, \rho_1, \dots, \rho_m)$ to be the union of $\rho \cup \rho_1 \cup \dots \cup \rho_m$ with

$$\left\{ x \in \mathbb{R}^2 \setminus \left(\rho \cup \bigcup_{j=1}^m \rho_j \right) : w_\rho(x) - \left(\sum_{j=1}^m w_{\rho_j}(x) \right) \text{ is odd} \right\}, \tag{5.23}$$

where we recall $w_\rho(x), w_{\rho_j}(x)$ are the winding numbers of these curves about x.

Note that, in general, $\mathsf{hull}(\rho, \rho_1, \dots, \rho_m)$ is not a polygon (for instance with ρ a square, and ρ_1 a smaller square contained in ρ with $\mathcal{H}^1(\rho \cap \rho_1) > 0$), though it is when the curves ρ, ρ_j are in general position.

Lemma 5.8. Let $R \in \mathcal{R}$ be connected, with ∂R consisting of the Jordan curves $\lambda, \lambda_1, \dots, \lambda_m$. Let $\delta > 0$ and let $\rho, \rho_1, \dots, \rho_m \subset [-1, 1]^2$ be simple polygonal circuits with $\mathrm{d}_H(\lambda, \rho) \leq \delta$ and with $\mathrm{d}_H(\lambda_j, \rho_j) \leq \delta$ for each j. We suppose that δ is small enough so that $\mathsf{hull}(\rho_j)^\circ \cap \mathsf{hull}(\rho)^\circ$ is non-empty for each j. There are simple polygonal circuits $\rho', \rho'_1, \dots, \rho'_m \subset [-1, 1]^2$ so that

(1) $\mathrm{d}_H(\rho, \rho') \leq \delta$ and $\mathrm{d}_H(\rho_j, \rho'_j) \leq \delta$ for each j,
(2) $P := \mathsf{hull}(\rho', \rho'_1, \dots, \rho'_m)$ is a connected polygon,
(3) $\mathrm{d}_H(R, P) \leq 2\delta$
(4) $\mathcal{I}_p(\rho) + \mathcal{I}_p(\rho_1) + \dots + \mathcal{I}_p(\rho_m) + \delta \geq \mathcal{I}_p(\partial P)$.

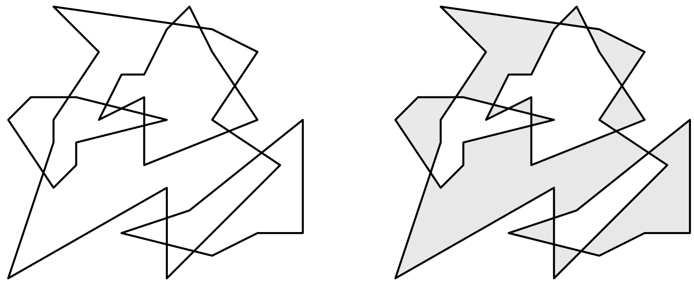

Figure 9: On the left, the curves $\rho, \rho_1, \rho_2, \rho_3$. On the right, $\mathsf{hull}(\rho, \rho_1, \dots, \rho_3)$. As these curves are in general position, $\mathsf{hull}(\rho, \rho_1, \dots, \rho_3)$ is a polygon.

Proof. Appealing to the continuity of β_p, perturb each $\rho, \rho_1, \dots, \rho_m$ to a collection $\rho', \rho_1', \dots, \rho_m'$ of simple polygonal curves in general position with respect to each other satisfying (5.8) and (5.8). Take δ smaller if necessary, and use the hypotheses of the lemma to execute this perturbation in such a way that $\mathsf{hull}(\rho_j')^\circ \cap \mathsf{hull}(\rho')^\circ$ is non-empty for each j. Together with the transversality of the ρ', ρ_j', it follows that $\mathsf{hull}(\rho', \rho_1', \dots, \rho_m')$ is a connected polygon, settling (5.8) (connectedness can be established by inducting on the number m of polygonal curves $\rho_1', \dots, \rho_m'$).

We turn our attention to the Hausdorff distance between R and $P := \mathsf{hull}(\rho', \rho_1', \dots, \rho_m')$. Let $x \in R$. If $x \in \partial R$, there is $y \in \partial P$ a distance of at most 2δ from x. If $x \in R$ and $x \notin P$, we appeal to the definition of hull (using winding number) to deduce that x is at most 2δ from ∂P. A symmetric argument starting with $x \in P$ settles (5.8). □

Remark 5.9. Proposition 5.10 below is our first tool for passing from elements of $\mathcal{U}_n$ to connected polygons. We remark on how this result differs from its counterparts in [8]: Proposition 4.1 and Proposition 5.4. The latter results only deal with outer boundary circuits; this is a viable strategy in [8] because one can leverage knowledge of the unrestricted isoperimetric problem (1.12). In particular, as the constant c in (1.12) varies, solutions remain dilations of the Wulff shape, which in turn gives a linear relationship between optimal conductances. This homothety is lost in the restricted problem (1.6), preventing straightforward estimates like (5.15) in [8] from going through. Such an estimate allows one, in the setting of [8], to take a Cheeger optimizer with many large holes and to fill these holes, producing a continuum set whose conductance can ultimately be compared to the optimal conductance for the area bound $c = 2$. In our case, we need a result allowing large holes to pass to the continuum, where they can be ruled out using the work of Section 3 and Section 4.

Proposition 5.10. Let $p > p_c(2)$ and let $\epsilon > 0$. There are positive constants $c_1(p, \epsilon)$ and $c_2(p, \epsilon)$ so that with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, whenever $U \in \mathcal{U}_n$ satisfies

(1) U is well-proportioned,
(2) $\mathrm{Leb}(\mathsf{vol}(U)) \geq n^{7/4}$,
(3) $|\partial^\infty U| \leq Cn$.

there is a connected polygon $P = P(U) \in \mathcal{R}$ so that

(1) $\mathrm{d}_H(U, nP) \leq n^{1/2}$,
(2) $\big||U| - \theta_p \mathrm{Leb}(nP)\big| \leq \epsilon |U|$,
(3) $|\partial^n U| \geq (1 - \epsilon)\mathcal{I}_p(n\partial P)$.

Proof. Let $U \in \mathcal{U}_n$. Using Lemma 2.10, form the pairs of right-most and interface circuits (γ, ∂) and $\{(\gamma_j, \partial_j)\}_{j=1}^m$ associated to U. View the interfaces ∂, ∂_j as Jordan curves (via "corner-rounding," see Remark 2.8). Recall that we denoted $\mathsf{hull}(\partial_j) \cap \mathbf{C}_\infty$ as Λ_j, and that the Λ_j are the finite connnected components of $\mathbf{C}_\infty \setminus U$. We say Λ_j is *large* if $|\Lambda_j| \geq n^{1/2}$ and that it is *small* otherwise.

Step I: (Filling of small components) Let $(\widetilde{\gamma}_1, \widetilde{\partial}_1) \dots, (\widetilde{\gamma}_\ell, \widetilde{\partial}_\ell)$ enumerate the pairs of right-most circuits and corresponding interfaces associated to the large components Λ_j. Define

$$R := \mathsf{hull}(\partial) \setminus \left(\bigsqcup_{i=1}^{\ell} \mathsf{hull}(\widetilde{\partial}_i)^\circ \right), \tag{5.24}$$

and let $\widetilde{U} := R \cap \mathbf{C}_\infty$ (hence, $R = \mathsf{vol}(\widetilde{U})$). Observe that $\widetilde{U}$ is well-proportioned because U is. By construction, $\widetilde{U}$ is close to U both in d_H-sense and in volume. To see this, observe that the open edge boundaries of each Λ_j are disjoint and are each subsets of $\partial^\infty U$. The hypothesis $|\partial^\infty U| \leq Cn$ implies

$$|\widetilde{U} \setminus U| \leq Cn^{3/2}, \tag{5.25}$$

and it is immediate that

$$\mathrm{d}_H(U, \widetilde{U}) \leq n^{1/2}. \tag{5.26}$$

Step II: (Constructing a polygon P) We use Lemma 5.2 and Lemma 5.8 to build a suitable polygon from $\widetilde{U}$. By Lemma A.1, for each large $\widetilde{\gamma}_i$, we have $|\widetilde{\gamma}_i| \geq n^{1/8}$ for n sufficiently large, and likewise that $|\gamma| \geq n^{1/8}$. Work within the high probability event from Lemma 5.2, taking simple polygonal circuits $\rho_i \subset [-1,1]^2$ for each large $\widetilde{\gamma}_i$ so that

(1) $\mathrm{d}_H(\widetilde{\partial}_i, n\rho_i) \leq 2n^{1/16}$,
(2) $|\mathfrak{b}^n(\widetilde{\gamma}_i)| \geq (1-\epsilon)\mathcal{I}_p(n\rho_i)$,

as well as a polygonal circuit $\rho \subset [-1,1]^2$ corresponding to γ with

(3) $\mathrm{d}_H(\partial, n\rho) \leq n^{1/16}$,
(4) $|\mathfrak{b}^n(\gamma)| \geq (1-\epsilon)\mathcal{I}_p(n\rho)$.

If there are no large components, simply define $P := \mathsf{hull}(\rho)$. Otherwise, define P differently below by using Lemma 5.8 to find polygonal circuits $\rho', \rho'_1, \dots, \rho'_\ell \subset [-1,1]^2$ with

(5) $\mathrm{d}_H(n\rho, n\rho') \leq n^{1/16}$ and $\mathrm{d}_H(n\rho_i, n\rho'_i) \leq n^{1/16}$ for each i,
(6) $P := \mathsf{hull}(\rho', \rho'_1, \dots, \rho'_\ell)$ is a connected polygon,
(7) $\mathrm{d}_H(R, P) \leq 2n^{1/16}$
(8) $\mathcal{I}_p(\rho) + \mathcal{I}_p(\rho_1) + \cdots + \mathcal{I}_p(\rho_\ell) + n^{-15/16} \geq \mathcal{I}_p(\partial P)$.

In either case, we will show the polygon $P \subset [-1,1]^2$ has the desired properties.

Step III: (Controlling $\mathcal{I}_p(\partial P)$) In the first case that $P = \mathsf{hull}(\rho)$, we find

$$|\partial^n U| \geq |\partial^n \widetilde{U}| = |\mathfrak{b}^n(\gamma)| \geq (1-\epsilon)\mathcal{I}_p(n\partial P), \tag{5.27}$$

which is satisfactory. Thus we may suppose the set of large components is non-empty. Let $\alpha > 0$ be as in the statement of Proposition 2.11 and let

$$\mathcal{E}_n := \left\{ \exists \gamma \in \bigcup_{\substack{x_0 \in [-n,n]^2 \cap \mathbb{Z}^2 \\ x \in \mathbb{Z}^2}} \mathcal{R}(x_0, x) \ : n^{1/8} \leq |\gamma| \leq 100n^2, \ |\mathfrak{b}(\gamma)| \leq \alpha|\gamma| \right\}, \tag{5.28}$$

so that Proposition 2.11 with a union bound gives positive constants $c_1(p)$ and $c_2(p)$ so that $\mathbb{P}_p(\mathcal{E}_n) \leq c_1 \exp(-c_2 n)$. Work in $\mathcal{E}_n^c$ for the remainder of the proof, and use that $\mathfrak{b}(\widetilde{\gamma}_i) = \mathfrak{b}^n(\widetilde{\gamma}_i)$ along with the bound $|\widetilde{\gamma}_i| \geq n^{1/8}$:

$$(1+\epsilon)|\partial^n \widetilde{U}| = (1+\epsilon)\left(|\mathfrak{b}^n(\gamma)| + \sum_{i=1}^{\ell} |\mathfrak{b}^n(\widetilde{\gamma}_i)|\right), \tag{5.29}$$

$$\geq |\mathfrak{b}^n(\gamma)| + \sum_{i=1}^{\ell} |\mathfrak{b}^n(\widetilde{\gamma}_i)| + n^{1/16}, \tag{5.30}$$

for n sufficiently large. Continuing from (5.30), let us use (5.3), (5.3) and (5.3):

$$|\partial^n U| \geq |\partial^n \widetilde{U}| \geq \frac{1}{1+\epsilon}\left(|\mathfrak{b}^n(\gamma)| + \sum_{i=1}^{\ell} |\mathfrak{b}^n(\widetilde{\gamma}_i)| + n^{1/16}\right), \tag{5.31}$$

$$\geq \frac{1-\epsilon}{1+\epsilon}\left(\mathcal{I}_p(n\rho) + \sum_{i=1}^{\ell} \mathcal{I}_p(n\rho_i) + n^{1/16}\right), \tag{5.32}$$

$$\geq \frac{1-\epsilon}{1+\epsilon}\mathcal{I}_p(n\partial P), \tag{5.33}$$

so P has the desired properties as far as the surface energy in this case as well.

Step IV: (d_H-closeness of nP and $\widetilde{U}$) Let $\mathcal{A}_n$ be the high probability event from Lemma 5.6, and work within this event for the remainder of the proof. If the collection of large components is empty, $P = \mathsf{hull}(\rho)$ implies $\mathrm{d}_H(R, nP) \leq n^{1/16}$. As $R = \mathsf{vol}(\widetilde{U})$, it follows from working in $\mathcal{A}_n$ that

$$\mathrm{d}_H(\widetilde{U}, nP) \leq n^{1/16} + \log^4 n. \tag{5.34}$$

On the other hand, if the collection of large components is non-empty, (5.3) implies

$$\mathrm{d}_H(\widetilde{U}, nP) \leq 2n^{1/16} + \log^4 n, \tag{5.35}$$

as desired.

Step V: (Controlling the volume of P) Let $r = \lceil n^{1/16} \rceil$, and for $x \in \mathbb{Z}^d$ let $B_x = x + [-2r, 2r]^2$. Let $V(\widetilde{U})$ denote the vertices of $\mathbb{Z}^2$ contained in the union of paths $\gamma \cup \bigcup_{i=1}^{\ell} \widetilde{\gamma}_i$. In either construction of P, we have

$$nP \,\Delta\, R \subset \bigcup_{x \in V(\widetilde{U})} B_x, \tag{5.36}$$

so that

$$\mathrm{Leb}(nP \,\Delta\, R) \leq 100 n^{1/16} \left[\mathsf{d\text{-}per}(\widetilde{U})\right], \tag{5.37}$$

$$\leq 100 n^{1/16} \left[\mathrm{Leb}(\mathsf{vol}(\widetilde{U}))\right]^{2/3}, \tag{5.38}$$

as $\widetilde{U}$ is well-proportioned. As $\widetilde{U}$ is also sufficiently large and we are within $\mathcal{A}_n$, we also have $\big||\widetilde{U}| - \theta_p \mathrm{Leb}(R)\big| \leq \epsilon \mathrm{Leb}(R)$, thus

$$\mathrm{Leb}(nP \,\Delta\, R) \leq 100 n^{1/16} \left[\frac{|\widetilde{U}|}{\theta_p - \epsilon}\right]^{2/3} \leq \epsilon |\widetilde{U}|, \tag{5.39}$$

for n sufficinently large. It follows that

$$\big||\widetilde{U}| - \theta_p \mathrm{Leb}(nP)\big| \leq \big||\widetilde{U}| - \theta_p \mathrm{Leb}(R)\big| + \epsilon|\widetilde{U}|\,, \tag{5.40}$$

$$\leq \left(\frac{\epsilon}{\theta_p - \epsilon} + \epsilon\right)|\widetilde{U}|\,. \tag{5.41}$$

Step VI: (Wrapping up) Using (5.25), we have

$$\big||U| - \theta_p \mathrm{Leb}(nP)\big| \leq \left(\frac{\epsilon}{\theta_p - \epsilon} + \epsilon\right)(|U| + Cn^{3/2}) + Cn^{3/2}\,, \tag{5.42}$$

$$\leq C'\epsilon|U|\,, \tag{5.43}$$

for some $C' > 0$ and when n is taken sufficiently large. By (5.26) and either (5.34) or (5.35), we also have $\mathrm{d}_H(U, nP) \leq n^{1/2}$ for n sufficiently large. Finally, recall that from either (5.27) or (5.33) we have $|\partial^n U| \geq \frac{1-\epsilon}{1+\epsilon}\mathcal{I}_p(\partial nP)$. The proof is complete upon adjusting ϵ. □

We now apply Proposition 5.10 to connected Cheeger optimizers. Let us define

$$\mathcal{G}_n^* := \big\{G_n \in \mathcal{G}_n\ : G_n \text{ is connected}\big\}\,. \tag{5.44}$$

Proposition 5.11. Let $p > p_c(2)$. There are positive constants $c_1(p,\epsilon), c_2(p,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1\exp(-c_2\log^2 n)$, for each $G_n \in \mathcal{G}_n^*$, there is a connected polygon $P_n \equiv P(G_n,\epsilon) \subset [-1,1]^2$ satisfying

(1) $\mathrm{d}_H(G_n, nP_n) \leq 2n^{1/2}$,
(2) $\big||G_n| - \theta_p \mathrm{Leb}(nP_n)\big| \leq \epsilon|G_n|$,
(3) $|\partial^n G_n| \geq (1-\epsilon)\mathcal{I}_p(n\partial P_n)$.

Proof. Work in the high probability event from Proposition 5.4 that for some $\alpha_1 > 0$, we have $\min_{G_n\in\mathcal{G}_n}|G_n| \geq \alpha_1 n^2$. With Proposition 5.3, we find that $\max_{G_n\in\mathcal{G}_n}|\partial^n G_n| \leq \alpha' n$ for some $\alpha' > 0$. As $|\partial^\infty G_n \backslash \partial^n G_n| \leq 8n$ for all $G_n \in \mathcal{G}_n$, it follows that $\max_{G_n\in\mathcal{G}_n}|\partial^\infty G_n| \leq \alpha_2 n$ for some $\alpha_2 > 0$. Fix $G \equiv G_n \in \mathcal{G}_n^*$, and observe that $G \in \mathcal{U}_n$.

Consider the pairs of right-most and interface circuits (γ,∂) and $\{(\gamma_j,\partial_j)\}_{j=1}^m$ giving rise to $\mathsf{vol}(G)$, and let Λ_j denote $\mathsf{hull}(\partial_j)\cap\mathbf{C}_\infty$. Say that Λ_j is *large* if $|\Lambda_j| \geq n^{1/2}$ and that Λ_j is *small* otherwise. Define

$$\widetilde{G} := \left[\mathsf{hull}(\partial) \setminus \left(\bigsqcup_{j\,:\,\Lambda_j \text{ large}} \mathsf{hull}(\partial_j)\right)\right] \cap \mathbf{C}_\infty\,, \tag{5.45}$$

As in the proof of Propostion 5.10, we observe $\widetilde{G}$ is close to G both in d_H-sense and in volume; as $|\partial^\infty G_n| \leq \alpha_2 n$, we find

$$|\widetilde{G}\setminus G| \leq \alpha_2 n^{3/2} \quad \text{and} \quad \mathrm{d}_H(\widetilde{G}, G) \leq n^{1/2}. \tag{5.46}$$

Step I: (Controlling $\mathsf{d\text{-}per}(\widetilde{G})$) The isoperimetric inequality (Lemma A.1) implies $|\gamma_j| \geq n^{1/8}$ for any Λ_j which is large. Likewise, because $|G| \geq \alpha_1 n^2$, we also have $|\gamma| \geq n^{1/8}$. Let $\alpha > 0$ be as in the statement of Proposition 2.11 and let $\mathcal{E}_n$ be the event in (5.28). Work in the high probability event $\mathcal{E}_n^c$ for the remainder of the proof, so that $|\mathfrak{b}(\gamma)| \geq \alpha|\gamma|$ and $|\mathfrak{b}(\gamma_j)| \geq \alpha|\gamma_j|$ for each large $|\Lambda_j|$. It follows that

$$\mathsf{d\text{-}per}(\widetilde{G}) \leq \frac{\alpha_2}{\alpha} n\,. \tag{5.47}$$

Step II: (Showing $\mathrm{Leb}(\mathsf{vol}(\widetilde{G}))$ is on the order of n^2) By construction, for some $C > 0$,

$$\mathrm{Leb}(\mathsf{vol}(\widetilde{G})) \geq |\mathsf{vol}(\widetilde{G}) \cap \mathbb{Z}^2| - C\mathsf{d\text{-}per}(\widetilde{G})\,, \tag{5.48}$$

$$\geq |G| - C\mathsf{d\text{-}per}(\widetilde{G}) \geq \frac{\alpha_1}{2} n^2\,, \tag{5.49}$$

for n sufficiently large. We conclude $\widetilde{G}$ is well-proportioned and satisfies $\mathrm{Leb}(\mathsf{vol}(\widetilde{G})) \geq n^{7/4}$ when n is large enough. Moreover, $\partial^\infty \widetilde{G} \subset \partial^\infty G$, so that $|\partial^\infty \widetilde{G}| \leq \alpha_2 n$, and $\widetilde{G}$ satisfies all necessary prerequisites of Proposition 5.10. We apply this proposition to complete the proof, using that $\partial^n \widetilde{G} \subset \partial^n G$. □

5.4 Proofs of main theorems

We begin by proving a precursor to Theorem 1.2 for connected Cheeger optimizers.

Proposition 5.12. Let $p > p_c(2)$ and let $\epsilon > 0$. There are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that for all $n \geq 1$, with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, we have

$$\max_{G_n \in \mathcal{G}_n^*} \mathrm{d}_H(n^{-1}G_n, \mathcal{R}_p) \leq \epsilon\,. \tag{5.50}$$

We emphasize that the maximum directly above runs over $\mathcal{G}_n^*$.

Proof. Let $\epsilon > 0$, and define the event

$$\mathcal{E}^{(n)}(\epsilon) := \Big\{ \exists G_n \in \mathcal{G}_n^* \,:\, \mathrm{d}_H(n^{-1}G_n, \mathcal{R}_p) > \epsilon \Big\} \tag{5.51}$$

Let $\epsilon' > 0$ to be determined later, and let $\mathcal{A}_1^{(n)}(\epsilon')$ be the event from Proposition 5.11 that for each $G_n \in \mathcal{G}_n^*$, there is a connected polygon $P_n \subset [-1,1]^2$ so that

(1) $\mathrm{d}_H(G_n, nP_n) \leq 2n^{1/2}$,
(2) $\big| |G_n| - \theta_p \mathrm{Leb}(nP_n) \big| \leq \epsilon' |G_n|$,
(3) $|\partial^n G_n| \geq (1-\epsilon')\mathcal{I}_p(n\partial P_n)$,

We first bound $\mathrm{Leb}(P_n)$ from above within the intersection of $\mathcal{A}_1^{(n)}(\epsilon')$ and another high probability event. Let

$$\mathcal{A}_2^{(n)}(\epsilon') := \left\{ \frac{|\mathbf{C}_n|}{(2n)^2} \in \Big((1-\epsilon')\theta_p, (1+\epsilon')\theta_p \Big) \right\}\,, \tag{5.52}$$

so that by Proposition A.2 and Proposition A.3, there are positive constants $c_1(p,\epsilon'), c_2(p,\epsilon')$ with $\mathbb{P}(\mathcal{A}_2^{(n)}(\epsilon')^c) \leq c_1 \exp(-c_2 \log^2 n)$. In the intersection $\mathcal{A}_1^{(n)}(\epsilon') \cap \mathcal{A}_2^{(n)}(\epsilon')$ and using (5.4), we have

$$\max_{G_n \in \mathcal{G}_n^*} \mathrm{Leb}(P_n) \leq 2(1+\epsilon')^2\,, \tag{5.53}$$

and choose $\alpha = \alpha(\epsilon') > 0$ so that $2 + \alpha = 2(1+\epsilon')^2$. Corollary 3.16 gives $\delta = \delta(\epsilon, \alpha) > 0$ so that when $R \in \mathcal{R}$ is connected with $\mathrm{Leb}(R) \leq 2 + \alpha$ and $\mathrm{d}_H(R, \mathcal{R}_p^{(2+\alpha)}) > \epsilon/100$,

$$\frac{\mathcal{I}_p(\partial R)}{\mathrm{Leb}(R)} > \varphi_p^{(2+\alpha)} + \delta\,. \tag{5.54}$$

Now take ϵ' small enough so that $\mathrm{d}_H(\mathcal{R}_p^{(2+\alpha)}, \mathcal{R}_p) \leq \epsilon/4$ (using Lemma 3.14). For this ϵ', within $\mathcal{E}_n(\epsilon) \cap \mathcal{A}_1^{(n)}(\epsilon') \cap \mathcal{A}_2^{(n)}(\epsilon')$ and for n sufficiently large (using (5.4)), the following event occurs

$$\Big\{ \mathrm{d}_H(P_n, \mathcal{R}_p^{(2+\alpha)}) > \epsilon/4 \Big\}\,, \tag{5.55}$$

so that by (5.54), (5.4) and (5.4), we have

$$n\Phi_n \geq (1-\epsilon')^2\theta_p^{-1}\frac{\mathcal{I}_p(\partial P_n)}{\mathrm{Leb}(P_n)}\,, \tag{5.56}$$

$$\geq (1-\epsilon')^2\theta_p^{-1}\left[\varphi_p^{(2+\alpha)}+\delta\right], \tag{5.57}$$

within $\mathcal{E}_n(\epsilon)\cap\mathcal{A}_1^{(n)}(\epsilon')\cap\mathcal{A}_2^{(n)}(\epsilon')$. Working within this intersection, use Corollary 3.9 to deduce

$$n\Phi_n \geq (1-\epsilon')^2\theta_p^{-1}\left(\frac{2-\alpha}{2+\alpha}\varphi_p^{(2-\alpha)}+\delta\right), \tag{5.58}$$

$$\geq (1-\epsilon')^2\theta_p^{-1}\left(\frac{2-\alpha}{2+\alpha}\varphi_p+\delta\right), \tag{5.59}$$

$$\geq \theta_p^{-1}\left(\varphi_p+\delta/2\right), \tag{5.60}$$

where we have taken ϵ' sufficiently small (depending on δ and hence ϵ) to obtain the last line, and where we emphasize that by Corollary 3.16, δ crucially does not go to zero with ϵ'.

$$\mathbb{P}_p(\mathcal{E}_n(\epsilon)) \leq \mathbb{P}_p(\mathcal{A}_1^{(n)}(\epsilon')^c)+\mathbb{P}_p(\mathcal{A}_2^{(n)}(\epsilon')^c)+\mathbb{P}_p\left(n\Phi_n\geq\theta_p^{-1}\left(\varphi_p+\delta/2\right)\right) \tag{5.61}$$

We have established that $\mathcal{A}_1^{(n)}(\epsilon')^c$ and $\mathcal{A}_2^{(n)}(\epsilon')^c$ are low-probability events; we bound the last term in (5.61) using Theorem 4.9 to complete the proof. □

Proof of Theorem 1.4. Let $\delta>0$, and let $\mathcal{A}_1^{(n)}(\delta)$ and $\mathcal{A}_2^{(n)}(\delta)$ be the high-probability events from the proof of Proposition 5.12 for the parameter δ in place of ϵ'. Within $\mathcal{A}_1^{(n)}(\delta)\cap\mathcal{A}_2^{(n)}(\delta)$, for each $G_n\in\mathcal{G}_n^*$ there is a connected polygon $P_n\subset[-1,1]^2$ satisfying

(1) $\mathrm{Leb}(P_n)\leq 2(1+\delta)^2$,
(2) $\big||G_n|-\theta_p\mathrm{Leb}(nP_n)\big|\leq\delta|G_n|$,
(3) $|\partial^n G_n|\geq(1-\delta)\mathcal{I}_p(n\partial P_n)$,

and as before we define $\alpha=\alpha(\delta)>0$ so that $2(1+\delta)^2=2+\alpha$. In $\mathcal{A}_1^{(n)}(\delta)\cap\mathcal{A}_2^{(n)}(\delta)$ we have

$$n\Phi_n \geq (1-\delta)^2\frac{\mathcal{I}_p(\partial P_n)}{\theta_p\mathrm{Leb}(P_n)}\,, \tag{5.62}$$

$$\geq (1-\delta)^2\frac{\varphi_p^{(2+\alpha)}}{\theta_p}\,, \tag{5.63}$$

$$\geq \frac{(1-\delta)^2(2-\alpha)}{2+\alpha}\frac{\varphi_p}{\theta_p}\,, \tag{5.64}$$

where we have used Corollary 3.9 and the fact that $\varphi_p^{(2-\alpha)}\geq\varphi_p$ to obtain the last line. Thus, for $\epsilon>0$, we may take δ and hence α sufficiently small so that $n\Phi_n\geq(1-\epsilon)(\varphi_p/\theta_p)$ (within $\mathcal{A}_1^{(n)}(\delta)\cap\mathcal{A}_2^{(n)}(\delta)$). Use Theorem 4.9 to conclude that for all $n\geq1$, there are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that

$$(1+\epsilon)\varphi_p\geq n\Phi_n\geq(1-\epsilon)\varphi_p \tag{5.65}$$

with probability at least $1-c_1\exp(-c_2\log^2 n)$. We apply Borel-Cantelli to complete the proof. □

Proof of Theorem 1.2. Our strategy is to show that each $G_n \in \mathcal{G}_n^*$ is large. By Lemma 3.10, we have $\varphi_p^{(7/4)} > \varphi_p$. Let $\epsilon > 0$ be small enough so that $\varphi^{(7/4)} > (1+\epsilon)\varphi_p$, and choose δ depending on this ϵ so that

$$(1-\delta)^2 \varphi_p^{(7/4)} \geq (1+\epsilon)\varphi_p \,. \tag{5.66}$$

For this δ, work in the intersection $\mathcal{A}_1^{(n)}(\delta) \cap \mathcal{A}_2^{(n)}(\delta)$, the events introduced in the proof of Proposition 5.12, so that for each $G_n \in \mathcal{G}_n^*$, there is a connected polygon $P_n \subset [-1,1]^2$ with

(1) $\mathrm{d}_H(G_n, nP_n) \leq 2n^{1/2}$,
(2) $\big||G_n| - \theta_p \mathrm{Leb}(nP_n)\big| \leq \delta |G_n|$,
(3) $|\partial^n G_n| \geq (1-\delta)\mathcal{I}_p(n\partial P_n)$,

Thus by (5.4), (5.4) and (5.66)

$$\mathcal{A}_1^{(n)}(\delta) \cap \mathcal{A}_2^{(n)}(\delta) \cap \{\exists G_n \in \mathcal{G}_n^* \, : \mathrm{Leb}(P_n) \leq 7/4\} \subset \left\{ n\Phi_n \geq (1-\delta)^2 \varphi_p^{(7/4)} \right\} , \tag{5.67}$$

$$\subset \left\{ n\Phi_n \geq (1+\epsilon)\varphi_p \right\} . \tag{5.68}$$

Let us write $\mathcal{F}_n(\epsilon)$ for the complement of the event in (5.68). Theorem 4.9 tells us $\mathcal{F}_n(\epsilon)$ occurs with high probability, so that on the intersection $\mathcal{A}_1^{(n)}(\delta) \cap \mathcal{A}_2^{(n)}(\delta) \cap \mathcal{F}_n(\epsilon)$, we have

$$\min_{G_n \in \mathcal{G}_n^*} \mathrm{Leb}(P_n) > 7/4 \,, \tag{5.69}$$

and hence by (5.4),

$$\min_{G_n \in \mathcal{G}_n^*} |G_n| \geq \frac{1}{1+\delta} \theta_p \left(\frac{7}{4}\right) n^2 \,. \tag{5.70}$$

As we are working within $\mathcal{A}_2^{(n)}(\delta)$, we also have $|\mathbf{C}_n| \leq 4n^2\theta_p(1+\delta)$, so that from (5.70) and by taking δ smaller if necessary, we find

$$\min_{G_n \in \mathcal{G}_n^*} |G_n| \geq \left(\frac{5}{16}\right) |\mathbf{C}_n| \,. \tag{5.71}$$

The inequality $\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)$ tells us that each $G_n \in \mathcal{G}_n$ is a disjoint union of elements of $\mathcal{G}_n^*$. The constraint $|G_n| \leq |\mathbf{C}_n|/2$ and (5.71) tell us that:

$$\mathcal{A}_1^{(n)}(\delta) \cap \mathcal{A}_2^{(n)}(\delta) \cap \mathcal{F}_n(\epsilon) \subset \left\{ \mathcal{G}_n^* \equiv \mathcal{G}_n \right\} , \tag{5.72}$$

as any disconnected $G_n \in \mathcal{G}_n$ would consist of at least two disjoint connected Cheeger optimizers, but (5.71) implies that the volume of G_n would then strictly exceed half of $|\mathbf{C}_n|$.

Thus, on the intersection of $\mathcal{A}_1^{(n)}(\delta) \cap \mathcal{A}_2^{(n)}(\delta) \cap \mathcal{F}_n(\epsilon)$ and the high-probability event from Proposition 5.12, we find that there are positive constants $c_1(p,\epsilon)$ and $c_2(p,\epsilon)$ so that for each $n \geq 1$, with probability at least $1 - c_1 \exp(-c_2 \log^2 n)$, we have

$$\max_{G_n \in \mathcal{G}_n} \mathrm{d}_H(n^{-1}G_n, \mathcal{R}_p) \leq \epsilon \,, \tag{5.73}$$

where we emphasize the above maximum now runs over all of $\mathcal{G}_n$. The proof is complete upon applying Borel-Cantelli. □

A Percolation inputs and miscellany

Recall that $\mathcal{U}_n$ denotes the connected subgraphs of $\mathbf{C}_\infty \cap [-1,1]^2$ which are defined by their vertex set. For $U \in \mathcal{U}_n$, Lemma 2.10 furnishes pairs of right-most circuits and corresponding interfaces $(\gamma, \partial), (\gamma_1, \partial_1), \ldots, (\gamma_m, \partial_m)$ which "carve" U out of $\mathbf{C}_\infty$. Recall that we used these pairs to define the value $\mathsf{d\text{-}per}(U)$ in (5.15) and the set $\mathsf{vol}(U)$ in (5.16). Recall that we identify the interfaces $\partial, \partial_1, \ldots, \partial_m$ with simple closed curves, see Remark 2.8.

Lemma A.1. There is $c > 0$ so that for all $n \geq 1$ and for all $U \in \mathcal{U}_n$,

$$\mathsf{d\text{-}per}(U) \geq c\mathrm{Leb}(\mathsf{vol}(U))^{1/2} . \tag{A.1}$$

Proof. Using the correspondence of Proposition 2.7, we find constants $c_1, c_2 > 0$ so that whenever γ' is a right-most circuit with corresponding interface ∂', we have

$$c_1|\gamma'| \leq \mathrm{length}(\partial') \leq c_2|\gamma'| , \tag{A.2}$$

where we view ∂' as a simple circuit in $\mathbb{R}^2$. As the circuits $\partial, \partial_1, \ldots, \partial_m$ make up the boundary of the set $\mathsf{vol}(U)$, the standard Euclidean isoperimetric inequality gives $c > 0$ so that

$$\mathrm{length}(\partial) + \sum_{i=1}^{m} \mathrm{length}(\partial_i) \geq c\mathrm{Leb}(\mathsf{vol}(U))^{1/2} . \tag{A.3}$$

The proof is complete upon combining (A.2) with (A.3). □

The next three results are more general percolation inputs. The following result of Durrett and Schonmann ([21] Theorems 2 and 3) controls the density of $\mathbf{C}_\infty$ within large boxes.

Proposition A.2. Let $p > p_c(2)$, let $\epsilon > 0$ and let $r > 0$, and let $B_r \subset \mathbb{R}^2$ be a translate of $[-r, r)^2$. There are positive constants c_1, c_2 depending on p and ϵ so that

$$\mathbb{P}_p\left(\frac{|\mathbf{C}_\infty \cap B_r|}{(2r)^2} \notin (\theta_p - \epsilon, \theta_p + \epsilon)\right) \leq c_1 \exp\Big(-c_2 r\Big) . \tag{A.4}$$

The next result, due to Benjamini and Mossel, allows us to pass from $\widetilde{\mathbf{C}}_n = \mathbf{C}_\infty \cap [-n,n]^2$ to $\mathbf{C}_n$ (see Proposition 1.2 of [6] and Lemma 5.2 of [8]).

Proposition A.3. Let $p > p_c(2)$. There is a positive constant $c(p)$ such that for all $n \geq 1$, with probability at least $1 - \exp(-C \log^2 n)$, and for any $n' \leq n - \log^2 n$, we have

$$\mathbf{C}_\infty \cap [-n', n']^2 = \mathbf{C}_n \cap [-n', n']^2 . \tag{A.5}$$

Finally we need Proposition A.2 of [7], which we state in dimension two only.

Proposition A.4. Let $p > p_c(2)$. There are positive constants $c_1(p), c_2(p)$ and $\widetilde{\alpha}(p)$ so that for all $t > 0$,

$$\mathbb{P}_p(\exists \Lambda \subset \mathbf{C}_\infty, \omega\text{-connected}, 0 \in \Lambda, |\Lambda| \geq t^2, |\partial^\infty \Lambda| < \widetilde{\alpha}|\Lambda|^{1/2}) \leq c_1 \exp(-c_2 t) . \tag{A.6}$$

The last task of the appendix is to provide a justification for the circuit decomposition used throughout the paper.

Proof of Lemma 2.10. Let us define the *vertex boundary* of a finite subgraph $U \subset \mathbb{Z}^2$, denoted $\Delta_v U$, as the collection of vertices x in U for which there is a path in $\mathbb{Z}^2$ from x to ∞ using no other vertices of U.

Now, fix a percolation configuration ω, and consider a finite, connected subgraph $U \subset \mathbf{C}_\infty$ such that $\mathbf{C}_\infty \setminus U$ consists of a single, infinite connected component. Moreover, we stipulate that the edge set of U is determined by the vertex set, so that if two vertices in U are joined by an open edge, this edge lies in the edge set of U. We *claim* there is an open right-most circuit $\gamma \subset U$ whose corresponding counter-clockwise interface ∂ (in the sense of Proposition 2.7) satisfies

(1) $\mathfrak{b}(\gamma) = \partial^\infty U$.
(2) $U = \mathsf{hull}(\partial)$.

We verify this claim by induction on the cardinality of the vertex set of U, beginning with a base case of $|U| = 2$. For any vertex x, define the diamond $D(x) := x + D$, where D is the ℓ^1-unit ball. Denote the vertices of U as x_1 and x_2, and consider the boundary of the set $D(x_1) \cup D(x_2)$. This boundary can naturally be identified with a counter-clockwise oriented circuit of length eight in the medial graph $\mathbb{Z}^2_\#$, denoted ∂. The correspondence Proposition 2.7 yields a right-most circuit γ, which in this case has length two, and traverses both orientations of the single edge in U. One easily checks that $\mathfrak{b}(\gamma) = \partial^\infty U$ in any percolation configuration, and that $\mathsf{hull}(\partial) = U$, so that (1) and (2) above hold.

For the inductive step, consider U with the aforementioned properties, and with $|U| = n+1$. Let $x \in \Delta_v U$, and note that x has either one or two neighbors in $\Delta_v U$. Let us form two cases.

Case I: Suppose there is $x \in \Delta_v U$ with only one other neighbor in $\Delta_v U$. Let $U' = U \setminus x$, and observe that U' has the same properties as U: it is connected, its edge set is determined by its vertex set, and moreover its complement in $\mathbf{C}_\infty$ is a single infinite graph. We apply our inductive hypothesis to obtain a pair (γ', ∂') satisfying (1) and (2) above. The curve $\partial D(x)$ is simple, and may be viewed as a circuit in the medial graph. We perturb ∂' locally to a new interface ∂ with the desired properties by *inserting* this circuit into ∂' at the natural place.

Case II: If no $x \in \Delta_v U$ has only one other neighbor in $\Delta_v U$, the vertices of $\Delta_v U$ form a circuit in $\mathbb{Z}^2$, which clearly corresponds to a right-most path γ. Proposition 2.7 furnishes a corresponding interface ∂, and the pair (γ, ∂) has the desired properties, i.e., we do not even need to use the inductive hypothesis here.

Thus the *claim* is settled for all U as above with $|U| \geq 2$. We remark that Figure 2 in [8] is a helpful visual accompaniment to this discussion. We also remark that, in fact, the lemma in question could not be true when $|U| = 1$, as there are no edges in U to form a path with. Nonetheless, if U consists of a single vertex x, the curve $\partial D(x)$ is naturally associated to an interface ∂ of length four which naturally corresponds to a right-most path of length eight. All other aspects of the lemma (barring containment in U) hold for this pair, and we treat this case as exceptional, as it does not affect our arguments.

It remains to discuss the case that $\mathbf{C}_\infty \setminus U$ consists of a unique infinite connected component, and a non-zero collection of finite connected components $\Lambda_1, \dots, \Lambda_m$. Each Λ_j has the property that $\mathbf{C}_\infty \setminus \Lambda_j$ is an infinite connected graph, and thus the above argument applies: there are pairs (γ_j, ∂_j) for each Λ_j satisfying (1) and (2) above. Together with (γ, ∂) obtained from running the above argument on $U \cup \bigcup_{j=1}^m \Lambda_j$, these pairs of circuits and interfaces have all properties needed, which proves Lemma 2.10. □

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