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A spectral decomposition for the block counting process and the fixation line of the beta(3,1)-coalescent

Martin Möhle*

Abstract

A spectral decomposition for the generator of the block counting process of the $\beta(3,1)$-coalescent is provided. This decomposition is strongly related to Riordan matrices and particular Fuss–Catalan numbers. The result is applied to obtain formulas for the distribution function and the moments of the absorption time of the $\beta(3,1)$-coalescent restricted to a sample of size n. We also provide the analog spectral decomposition for the fixation line of the $\beta(3,1)$-coalescent. The main tools in the proofs are generating functions and Siegmund duality. Generalizations to the $\beta(a,1)$-coalescent with parameter $a \in (0,\infty)$ are discussed leading to fractional differential or integral equations.

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1 Introduction and main results

Coalescents with multiple collisions, independently introduced by Pitman [17] and Sagitov [20], are Markovian processes $\Pi = (\Pi_t)_{t\geq 0}$ with state space $\mathcal{P}$, the set of partitions of $\mathbb{N} := \{1,2,\ldots\}$. Each coalescent with multiple collisions is characterized by a finite measure Λ on the unit interval $[0,1]$. These processes are hence also called Λ-coalescents. The most prominent example is the Kingman coalescent [9], where Λ is the Dirac measure at 0. Another important example is the Bolthausen–Sznitman coalescent [1], where Λ is uniformly distributed on $[0,1]$. The Bolthausen–Sznitman coalescent obviously belongs to the class of beta coalescents, where $\Lambda = \beta(a,b)$ is the beta distribution with parameters $a,b \in (0,\infty)$ having density $x \mapsto (B(a,b))^{-1}x^{a-1}(1-x)^{b-1}$, $x \in (0,1)$, with respect to Lebesgue measure on $(0,1)$. Here $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$ denotes the beta function.

For $t \geq 0$ let N_t denote the number of blocks of Π_t. The process $(N_t)_{t\geq 0}$ is called the block counting process of Π. In [16] a spectral decomposition for the generator of the block counting process of the Bolthausen–Sznitman coalescent is provided. Kukla and Pitters [11] provide a spectral decomposition for the generator of the (partition-valued) Bolthausen–Sznitman coalescent restricted to a sample of size $n \in \mathbb{N}$. For other beta coalescents explicit spectral decompositions have been unknown so far.

*Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany.
E-mail: martin.moehle@uni-tuebingen.de

We focus on the particular beta coalescent with $\Lambda = \beta(3,1)$ the beta distribution with parameters $a = 3$ and $b = 1$ having density $x \mapsto 3x^2$, $x \in (0,1)$. One reason why we focus on this particular beta coalescent is the fact that its block counting process moves from state $i \in \mathbb{N}$ with $i \geq 2$ to any state $j \in \mathbb{N}$ with $j < i$ with equal probability $1/(i-1)$ not depending on j. The generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ of the block counting process of the $\beta(3,1)$-coalescent has entries (see, for example, [4, Eq. (2.6)] with $a = 3$ and $b = 1$)

$$q_{ij} \;=\; \begin{cases} \dfrac{3}{i+1} & \text{if } j < i, \\ -\dfrac{3(i-1)}{(i+1)} & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases} \tag{1.1}$$

Let $q_i := -q_{ii} = 3(i-1)/(i+1)$ denote the total rates, $i \in \mathbb{N}$. In order to state the results the following definition from [21] is useful.

Definition 1.1 (Riordan matrix). *A lower left triangular matrix $R = (r_{ij})_{i,j\in\mathbb{N}}$ is called a Riordan matrix if for every $j \in \mathbb{N}$ the jth vertical generating function $r_j(z) := \sum_{i=j}^\infty r_{ij}z^i$ has the form $r_j(z) = f(z)(g(z))^j$ for some functions f and g of the form $f(z) = 1 + f_1 z + f_2 z^2 + \cdots$ and $g(z) = z + g_2 z^2 + g_3 z^3 + \cdots$ defined in some neighborhood of 0.*

In this case we write $R = (f(z), g(z))$ or $R = (f, g)$. Riordan matrices are closed under the usual matrix multiplication and $R = (f, g)$ has inverse $R^{-1} = (1/(f \circ g^{-1}), g^{-1})$. A typical example for a Riordan matrix is (see [21, Example (A)]) the Pascal matrix $R = (r_{ij})_{i,j\in\mathbb{N}}$ with binomial entries $r_{ij} := \binom{i}{j}$, in which case it is easily checked that $R = (f, g)$ with $f(z) := 1/(1-z)$ and $g(z) := z/(1-z)$. Since $g^{-1}(z) = z/(z+1)$ it follows that R has inverse $L := R^{-1} = (1/(z+1), z/(z+1))$. Thus, $l_j(z) := \sum_{i=j}^\infty l_{ij}z^i = (1/(z+1))(z/(z+1))^j = z^j/(z+1)^{j+1}$ and Taylor expansion of $l_j(z)$ shows that L has entries $l_{ij} = (-1)^{i-j}\binom{i}{j}$. Further examples of Riordan matrices are provided in [21] and [22]. Riordan matrices bridge several disciplines in mathematics. They are highly useful for calculating combinatorial sums [22]. Their group structure makes them as well appealing to the algebraic community.

Our main result (Theorem 1.2 below) provides an explicit spectral decomposition for the generator Q of the block counting process of the $\beta(3,1)$-coalescent. The result is remarkable, since the $\beta(3,1)$-coalescent seems to be the only beta coalescent different from the Bolthausen–Sznitman coalescent where such an explicit spectral decomposition is available. For further information on this topic we refer the reader to Section 2 where the delicate question on extensions to other beta coalescents is discussed. Moreover, Theorem 1.2 sheds some new light on particular Fuss–Catalan numbers and generalized Stirling numbers as explained in the remarks after the theorem.

In the following Γ denotes the gamma function. We furthermore use the notation $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The proof of the following theorem and of all other results are provided in Section 3.

Theorem 1.2. (Spectral decomposition of the generator of the block counting process) *The generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ of the block counting process of the $\beta(3,1)$-coalescent has spectral decomposition $Q = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ii} = -3(i-1)/(i+1)$, $i \in \mathbb{N}$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are the lower left triangular Riordan matrices*

$$R = \left(\frac{1}{\sqrt{1-z}}, \frac{z}{\sqrt{1-z}}\right) \quad \textit{and} \quad L = \left(\sqrt{1+\frac{z^2}{4}} - \frac{z}{2}, z\left(\sqrt{1+\frac{z^2}{4}} - \frac{z}{2}\right)\right) \tag{1.2}$$

having entries

$$r_{ij} \;=\; (-1)^{i-j}\binom{-\frac{j+1}{2}}{i-j} \;=\; \binom{i-\frac{j+1}{2}}{i-j} \;=\; \frac{\Gamma(i-\frac{j-1}{2})}{\Gamma(\frac{j+1}{2})\Gamma(i-j+1)}, \qquad i \geq j, \tag{1.3}$$

and

$$l_{ij} \;=\; \frac{j+1}{2}(-1)^{i-j}\frac{\Gamma(\frac{i+1}{2})}{\Gamma(j-\frac{i}{2}+\frac{3}{2})\Gamma(i-j+1)}, \qquad i \geq j, \tag{1.4}$$

with the convention that $l_{ij} = 0$ *if* $j - \frac{i}{2} + \frac{3}{2} \in -\mathbb{N}_0$*.*

Remark 1.3. (i) Eqs. (1.3) and (1.4) are useful to compute the entries of R and L numerically. One obtains

$$R = \begin{pmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 & \frac{3}{2} & 1 &&&&& \\ 1 & \frac{15}{8} & 2 & 1 &&&& \\ 1 & \frac{35}{16} & 3 & \frac{5}{2} & 1 &&& \\ 1 & \frac{315}{128} & 4 & \frac{35}{8} & 3 & 1 && \\ 1 & \frac{693}{256} & 5 & \frac{105}{16} & 6 & \frac{7}{2} & 1 & \\ \vdots &&&&&&& \ddots \end{pmatrix}$$

and

$$L = \begin{pmatrix} 1 &&&&&&& \\ -1 & 1 &&&&&& \\ \frac{1}{2} & -\frac{3}{2} & 1 &&&&& \\ -\frac{1}{8} & \frac{9}{8} & -2 & 1 &&&& \\ 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 &&& \\ \frac{1}{128} & \frac{15}{128} & -\frac{5}{4} & \frac{25}{8} & -3 & 1 && \\ 0 & 0 & \frac{1}{2} & -\frac{5}{2} & \frac{9}{2} & -\frac{7}{2} & 1 & \\ \vdots &&&&&&& \ddots \end{pmatrix}.$$

We do not have an intuitive explanation for the fact that R has non-negative entries.

(ii) (Relations to the Fuss–Catalan numbers) For $n \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ the Fuss–Catalan numbers $c_n(\alpha, \beta)$ are defined (see, for example, Mlotkowski [14] or Riordan [19, p. 148 or p. 168]) via $c_0(\alpha, \beta) := 1$ and

$$c_n(\alpha,\beta) \;:=\; \frac{\beta}{n!}\prod_{i=1}^{n-1}(\alpha n + \beta - i) \;=\; \frac{\beta}{n}\binom{\alpha n + \beta - 1}{n-1}, \qquad n \in \mathbb{N}, \alpha, \beta \in \mathbb{R}. \tag{1.5}$$

It is readily checked that the entries r_{ij} and l_{ij}, $i, j \in \mathbb{N}$ with $i \geq j$, are related to the Fuss–Catalan numbers via $r_{ij} = c_{i-j}(1, \frac{j+1}{2}) = \frac{j+1}{i+1}c_{i-j}(\frac{1}{2}, \frac{i+1}{2})$ and $l_{ij} = (-1)^{i-j}c_{i-j}(\frac{1}{2}, \frac{j+1}{2}) = \frac{j+1}{i+1}c_{i-j}(1, -\frac{i+1}{2})$.

(iii) (Relations to generalized Stirling numbers) Let $S(i, j; \alpha, \beta, r)$ denote the generalized Stirling numbers in the notation of Hsu and Shiue [7]. Using the recursion for these numbers (see, for example [7, Theorem 1]) a straightforward induction on i shows that for the particular case $\alpha = 2\beta$ and $r = 0$ the Stirling number $S(i, j; 2\beta, \beta, 0)$ is related to the gamma function via

$$S(i,j;2\beta,\beta,0) \;=\; \left(-\frac{\beta}{2}\right)^{i-j}\frac{\Gamma(2i-j)}{\Gamma(j)\Gamma(i-j+1)} \;=\; (-2\beta)^{i-j}\frac{\Gamma(i-\frac{j-1}{2})\Gamma(i-\frac{j}{2})}{\Gamma(\frac{j}{2})\Gamma(\frac{j+1}{2})\Gamma(i-j+1)},$$

$i \geq j$, where the last equality holds by Legendre's duplication formula $\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2}) = 2^{1-x}\sqrt{\pi}\Gamma(x)$. Applying this formula with $\beta = -1/2$ and $\beta = 1/2$ it follows from (1.3) that

$$r_{ij} \;=\; \frac{S(i,j;-1,-\frac{1}{2},0)}{[\frac{j}{2}]_{i-j}} \;=\; \frac{(-1)^{i-j}S(i,j;1,\frac{1}{2},0)}{[\frac{j}{2}]_{i-j}}, \qquad i \geq j, \tag{1.6}$$

where $[x]_0 := 1$ and $[x]_n := x(x+1)\cdots(x+n-1)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, denote the ascending factorials.

Let us provide some applications of Theorem 1.2. For $n \in \mathbb{N}$ let $\Pi^{(n)} = (\Pi_t^{(n)})_{t\geq 0}$ denote the coalescent restricted to a sample of size n (n-coalescent) and let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, $t \geq 0$. We are interested in $\tau_n := \inf\{t > 0 : N_t^{(n)} = 1\}$, the absorption time of $\Pi^{(n)}$. In the biological context τ_n is called the time back to most recent common ancestor. For the $\beta(3,1)$-coalescent it has been recently shown [15, Proposition 3.4] that τ_n has the convolution representation

$$\tau_n \;\stackrel{d}{=}\; E_n + \sum_{k=2}^{n-1} \xi_k E_k, \qquad n \in \{2,3,\ldots\}, \tag{1.7}$$

where $E_2, E_3, \ldots$ are independent and E_k is exponentially distributed with parameter q_k, and $\xi_2, \xi_3, \ldots$ are independent Bernoulli random variables and independent of $E_2, E_3, \ldots$ with $\mathbb{E}(\xi_k) = 1/k$, $k \in \{2,3,\ldots\}$. Formula (1.7) is intuitively clear by interpreting E_k as the sojourn time of the block counting process in state k and $\{\xi_k = 1\}$ as the event that the jump chain of the block counting process ever visits state $k \in \{2,\ldots,n\}$ when started from state n. The independence of $(E_k)_k$ and $(\xi_k)_k$ and the fact that ξ_k does not depend on the initial state n are however particular for the $\beta(3,b)$-coalescent with parameter $b \in (0,\infty)$ and related to the property that for this particular class of coalescents the block counting process has constant hitting probabilities. For more details we refer the reader to the proof of [15, Proposition 3.4]. From (1.7) one may derive formulas for the distribution function of τ_n. The detail computations are however not very amusing. Instead, we proceed as follows. Theorem 1.2 implies that the transition matrix $P(t) = (p_{ij}(t))_{i,j\in\mathbb{N}} = e^{tQ}$ of the block counting process has spectral decomposition $P(t) = Re^{tD}L$. Thus, Theorem 1.2 immediately yields

$$\begin{aligned} \mathbb{P}(\tau_n \leq t) \;&=\; p_{n1}(t) \;=\; \sum_{k=1}^{n} r_{nk}e^{-q_k t}l_{k1} \;=\; 1 + \sum_{k=2}^{n} r_{nk}l_{k1}e^{-q_k t}\\ &=\; 1 - \sum_{k=2}^{n} (-1)^k \frac{\Gamma(n-\frac{k-1}{2})}{\Gamma(n-k+1)\Gamma(\frac{5-k}{2})\Gamma(k)} e^{-\frac{3(k-1)}{k+1}t}. \end{aligned} \tag{1.8}$$

Consequently, τ_n has moments

$$\begin{aligned} \mathbb{E}(\tau_n^j) \;&=\; \int_0^\infty jt^{j-1}\mathbb{P}(\tau_n > t)\,\mathrm{d}t \;=\; -\sum_{k=2}^{n} r_{nk}l_{k1}\int_0^\infty jt^{j-1}e^{-q_k t}\,\mathrm{d}t\\ &=\; -\sum_{k=2}^{n} r_{nk}l_{k1}\frac{j!}{q_k^j} \;=\; \sum_{k=2}^{n}(-1)^k \frac{\Gamma(n-\frac{k-1}{2})}{\Gamma(n-k+1)\Gamma(\frac{5-k}{2})\Gamma(k)} \frac{j!}{\big(\frac{3(k-1)}{k+1}\big)^j}, \quad j \in \mathbb{N}. \end{aligned}$$

From (1.7) and the central limit theorem it follows that $(3\tau_n - \log n)/\sqrt{2\log n}$ is asymptotically standard normal distributed, in agreement with Table 2 of [5].

The total tree length of the $\beta(3,1)$-n-coalescent has a convolution representation (see [15, Proposition 3.5]) similar to the representation (1.7) for τ_n. Theorem 1.2 seems to be not directly useful to derive formulas for the distribution function or the moments of the total tree length, since these functionals cannot be expressed easily in terms of the transition matrices $P(t)$, $t \geq 0$.

We now turn to the fixation line of the $\beta(3,1)$-coalescent. For $n \in \mathbb{N}$ and $t \geq 0$ define

$$L_t^{(n)} \ := \ \sup\{k \in \mathbb{N} : N_t^{(k)} \leq n\} \tag{1.9}$$

and set $L_t := L_t^{(1)}$ for convenience. The process $(L_t)_{t \geq 0}$ is called the fixation line of the coalescent. It is easily seen from (1.9) and well known (see [4, Theorem 2.9] or [6, Lemma 2.4]) that the block counting process is Siegmund dual (see [23]) to the fixation line, i.e. $\mathbb{P}(N_t^{(i)} \leq j) = \mathbb{P}(L_t^{(j)} \geq i)$, $i, j \in \mathbb{N}$, $t \geq 0$. Moreover, (1.9) implies that $N_t^{(n)} = \inf\{k \in \mathbb{N} : L_t^{(k)} \geq n\}$, $n \in \mathbb{N}$, $t \geq 0$. An alternative definition of the fixation line is based on the lookdown construction of the coalescent going back to Donnelly and Kurtz [2, 3]. This more involved definition is provided in [6, p. 3010] for the Λ-coalescent and in [4, Section 1] for general exchangeable coalescents. In this construction each individual is equipped with a level being a positive integer. The construction is such that when L_t reaches state n, all individuals at time t having level less than or equal to n are offspring of a single individual (the individual at time 0 having level 1), an event called fixation in genetics. For our purposes (and in many cases) it suffices to work with (1.9). The precise definition via the lookdown construction is therefore omitted here.

A spectral decomposition for the generator of the fixation line of the Bolthausen–Sznitman coalescent is provided in [10, Theorem 3.1]. Theorem 1.4 below is the analog result for the $\beta(3,1)$-coalescent and can be viewed as the Siegmund dual counterpart of Theorem 1.2.

The generator $G = (g_{ij})_{i,j \in \mathbb{N}}$ of the fixation line of the $\beta(3,1)$-coalescent has entries (see, for example, [4, Eq. (2.10)])

$$g_{ij} \ = \ \begin{cases} \dfrac{3i}{(j+1)(j+2)} & \text{if } i < j, \\ -\dfrac{3i}{i+2} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{1.10}$$

Note that the total rates $g_i := -g_{ii} = 3i/(i+2)$, $i \in \mathbb{N}$, of the fixation line are related to those of the block counting process via $g_i = q_{i+1}$, $i \in \mathbb{N}$. The latter equality holds for all exchangeable coalescents (see, for example, [4, Proposition 2.5]) and is essentially a consequence of the Siegmund duality relations $q_{j,\leq i} = g_{i,\geq j}$, $i, j \in \mathbb{N}$, for the generator entries (see, for example, [4, Eq. (5.1)]). Choosing $j := i+1$ in these relations yields $-q_{i+1} = q_{i+1,\leq i} = g_{i,\geq i+1} = -g_i$, hence $g_i = q_{i+1}$ for all $i \in \mathbb{N}$.

Theorem 1.4. (Spectral decomposition of the generator of the fixation line) *The generator $G = (g_{ij})_{i,j \in \mathbb{N}}$ of the fixation line of the $\beta(3,1)$-coalescent has spectral decomposition $G = \tilde{R}\tilde{D}\tilde{L}$, where $\tilde{D} = (\tilde{d}_{ij})_{i,j \in \mathbb{N}}$ is the diagonal matrix with entries $\tilde{d}_{ii} = -3i/(i+2)$, $i \in \mathbb{N}$, and $\tilde{R} = (\tilde{r}_{ij})_{i,j \in \mathbb{N}}$ and $\tilde{L} = (\tilde{l}_{ij})_{i,j \in \mathbb{N}}$ are upper right triangular matrices with entries*

$$\tilde{r}_{ij} \ = \ \frac{i}{2}(-1)^{j-i} \frac{\Gamma(\frac{j}{2})}{\Gamma(i - \frac{j}{2} + 1)\Gamma(j-i+1)}, \qquad i \leq j, \tag{1.11}$$

with the convention that $\tilde{r}_{ij} = 0$ if $i - \frac{j}{2} + 1 \in -\mathbb{N}_0$, and

$$\tilde{l}_{ij} \ = \ (-1)^{j-i}\binom{-\frac{i}{2}}{j-i} \ = \ \binom{j - \frac{i}{2} - 1}{j-i} \ = \ \frac{\Gamma(j - \frac{i}{2})}{\Gamma(\frac{i}{2})\Gamma(j-i+1)}, \qquad i \leq j. \tag{1.12}$$

Remark 1.5. (i) Computations based on (1.11) and (1.12) yield

$$\tilde{R} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{8} & 0 & -\frac{1}{128} & 0 & \frac{1}{1024} & \cdots \\ & 1 & -1 & \frac{1}{2} & -\frac{1}{8} & 0 & \frac{1}{128} & \\ & & 1 & -\frac{3}{2} & \frac{9}{8} & -\frac{1}{2} & \frac{15}{128} & \\ & & & 1 & -2 & 2 & -\frac{5}{4} & \\ & & & & 1 & -\frac{5}{2} & \frac{25}{8} & \\ & & & & & 1 & -3 & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{pmatrix}$$

and

$$\tilde{L} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{8} & \frac{5}{16} & \frac{35}{128} & \frac{63}{256} & \frac{231}{1024} & \cdots \\ & 1 & 1 & 1 & 1 & 1 & 1 & \\ & & 1 & \frac{3}{2} & \frac{15}{8} & \frac{35}{16} & \frac{315}{128} & \\ & & & 1 & 2 & 3 & 4 & \\ & & & & 1 & \frac{5}{2} & \frac{35}{8} & \\ & & & & & 1 & 3 & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{pmatrix}.$$

In contrast to the spectral decomposition for the block counting process, here the matrix $\tilde{L}$ has non-negative entries. Again we do not have an intuitive explanation for this fact.

(ii) It is readily seen that $\tilde{r}_{ij}$ and $\tilde{l}_{ij}$ are related to the Fuss–Catalan numbers $c_n(\alpha,\beta)$ defined in (1.5) via $\tilde{r}_{ij} = c_{j-i}(1/2,-i/2) = (i/j)c_{j-i}(1,-j/2)$ and $\tilde{l}_{ij} = c_{j-i}(1,i/2) = (i/j)c_{j-i}(1/2,j/2)$, $i,j\in\mathbb{N}$, $i\le j$.

(iii) For every $i\in\mathbb{N}$ the horizontal generating functions $\tilde{r}_i(z) := \sum_{j=i}^\infty \tilde{r}_{ij}z^j$ and $\tilde{l}_i(z) := \sum_{j=i}^\infty \tilde{l}_{ij}z^j$, $|z|<1$, are provided in (3.9) and (3.10), implying that the transposed matrices $\tilde{R}^\top$ and $\tilde{L}^\top$ are both particular Riordan matrices $\tilde{R}^\top = (1, z(\sqrt{1+z^2/4}-z/2))$ and $\tilde{L}^\top = (1, z/\sqrt{1-z})$ in the notation introduced after Definition 1.1.

(iv) From Theorem 1.2 and Theorem 1.4 it follows that

$$r_{ij} \ = \ \tilde{l}_{j+1,i+1} \quad \text{and} \quad l_{ij} \ = \ \tilde{r}_{j+1,i+1}, \qquad i,j\in\mathbb{N}. \tag{1.13}$$

The relations (1.13) are special for the $\beta(3,1)$-coalescent and do not hold for arbitrary Λ-coalescents, in particular not for the Bolthausen–Sznitman coalescent, which is easily seen by comparing the spectral decomposition for the generator of the block counting process [16, Theorem 1.1] and for the generator of the fixation line [10, Theorem 3.1] of the Bolthausen–Sznitman coalescent. The general Siegmund duality relations between R and $\tilde{L}$ or, alternatively, between L and $\tilde{R}$, are slightly more involved, see (3.11) for the details.

2 On extensions to the beta(a,1)-coalescent

In this section generalizations to the $\beta(a,1)$-coalescent with parameter $a\in(0,\infty)$ are discussed. For the $\beta(a,1)$-coalescent with parameter $a\in(0,\infty)$ the block counting process has rates (see, for example, [4, Eq. (2.6)])

$$q_{ij} \ = \ a\frac{\Gamma(i+1)}{\Gamma(i+a-1)}\frac{\Gamma(i-j+a-1)}{\Gamma(i-j+2)}, \qquad 1\le j<i. \tag{2.1}$$

The total rate $q_i := \sum_{j=1}^{i-1} q_{ij}$ therefore simplifies to

$$q_i = a\frac{\Gamma(i+1)}{\Gamma(i+a-1)}\sum_{k=1}^{i-1}\frac{\Gamma(k+a-1)}{\Gamma(k+2)} = \begin{cases} \dfrac{a}{2-a}\left(\dfrac{\Gamma(a)\Gamma(i+1)}{\Gamma(i+a-1)}-1\right) & \text{if } a\in(0,\infty)\setminus\{2\},\\ 2(h_i-1) & \text{if } a=2, \end{cases}$$

where $h_i := \sum_{j=1}^{i} 1/j$ denotes the ith harmonic number, $i \in \mathbb{N}$. Note that

$$q_i - q_j \;=\; \begin{cases} \dfrac{\Gamma(a+1)}{2-a}\left(\dfrac{\Gamma(i+1)}{\Gamma(i+a-1)}-\dfrac{\Gamma(j+1)}{\Gamma(j+a-1)}\right) & \text{if } a\in(0,\infty)\setminus\{2\},\\ 2(h_i-h_j) & \text{if } a=2. \end{cases} \tag{2.2}$$

Similarly, the fixation line has rates (see, for example, [4, Eq. (2.10)])

$$g_{ij} \;=\; ai\frac{\Gamma(j+1)}{\Gamma(j+a)}\frac{\Gamma(j-i-1+a)}{\Gamma(j-i+2)}, \qquad 1\le i<j, \tag{2.3}$$

and total rates

$$g_i \;=\; \sum_{j=i+1}^{\infty} g_{ij} \;=\; \begin{cases} \dfrac{a}{2-a}\left(\dfrac{\Gamma(a)\Gamma(i+2)}{\Gamma(i+a)}-1\right) & \text{if } a\in(0,\infty)\setminus\{2\},\\ 2(h_{i+1}-1) & \text{if } a=2. \end{cases} \tag{2.4}$$

For $a \in (0,\infty)$ and $|z| < 1$ define

$$\phi(z) \;:=\; \frac{2-a}{\Gamma(a)}\sum_{n=1}^{\infty}\frac{\Gamma(n+a-1)}{\Gamma(n+2)}z^n \;=\; \begin{cases} 1+\dfrac{(1-z)^{2-a}-(1-z)}{(1-a)z} & \text{if } a\in(0,\infty)\setminus\{1\},\\ 1+\dfrac{(1-z)\log(1-z)}{z} & \text{if } a=1. \end{cases} \tag{2.5}$$

For $a \in (0,2)$ the function ϕ is the probability generating function of a random variable η with distribution

$$\mathbb{P}(\eta=n) \;=\; \frac{2-a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{\Gamma(n+2)}, \qquad n\in\mathbb{N}.$$

For $a \in [2,\infty)$ there is no analog probabilistic interpretation for ϕ, however we can still work with ϕ. Note that $\phi(z) = 0$ for $a = 2$. The following lemma provides a partial answer towards the spectral decomposition $Q = RDL$ of the generator Q of the block counting process of the $\beta(a,1)$-coalescent.

In order to state the result let us briefly recall fractional integrals and derivatives. For a function $f : [0,1) \to \mathbb{R}$ the Riemann–Liouville fractional integral of order $\alpha \in (0,\infty)$ is defined by (see, for example, [8, p. 69, Eq. (2.1.1)])

$$(I^\alpha f)(x) \;:=\; \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t, \qquad x\in[0,1),$$

provided that the integral on the right hand side exists. The Riemann–Liouville fractional derivative of order $\alpha \in [0,\infty)$ is defined by

$$(D^\alpha f)(x) \;:=\; \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I^{n-\alpha}f)(x), \qquad x\in[0,1),$$

where $n := \lfloor\alpha\rfloor + 1$, provided that the expression on the right hand side exists. For $\alpha \in (-\infty,0)$ we also write $(D^\alpha f)(x) := (I^{-\alpha}f)(x)$. We also use the notation $D_x^\alpha(f(x)) := (D^\alpha f)(x)$, $\alpha \in \mathbb{R}$, $x \in [0,1)$. For general information on fractional calculus theory we refer the reader to the books of Kilbas, Srivastava and Trujillo [8], Miller and Ross [13] and Podlubny [18].

Lemma 2.1. *Let $a \in (0,\infty)$ and let $R = (r_{ij})_{i,j\in\mathbb{N}}$ denote the matrix in the spectral decomposition $Q = RDL$ of the generator Q of the block counting process of the $\beta(a,1)$-coalescent. Then for every $j \in \mathbb{N}$ the vertical generating function $r_j(z) := \sum_{i=j}^\infty r_{ij}z^i$ is a solution to the equation*

$$(1-\phi(z))r_j(z) \;=\; \frac{\Gamma(j+1)}{\Gamma(j+a-1)} D_z^{a-2}\big(z^{a-2}r_j(z)\big), \qquad z\in[0,1), \tag{2.6}$$

where $\phi(z)$ is defined via (2.5).

Remark 2.2. For $a=1$ Eq. (2.6) is of the form $(1-\phi(z))r_j(z) \;=\; j\int_0^z r_j(t)/t\,\mathrm{d}t$. Taking the derivative with respect to z yields the differential equation $(1-\phi(z))r_j'(z)-\phi'(z)r_j(z) = r_j(z)/z$, which was solved in [16, Eq. (2.7)]. For $a=3$ Eq. (2.6) reduces to the differential equation (3.3) with solution (3.4). For $a=2$ Eq. (2.6) leads to the uninformative equation $r_j(z)=r_j(z)$. For integer $a\in\{4,5,\ldots\}$ one may be able to solve the differential equation (2.6) of order $a-2$ for r_j, however the solution may turn out to have a rather complicated form. For non-integer parameter a it might be possible to solve the truly fractional equation (2.6) by applying fractional calculus theory; see for example Kilbas, Srivastava and Trujillo [8], Miller and Ross [13] or Podlubny [18]. We leave the solution of (2.6) for arbitrary $a\in(0,\infty)\setminus\{1,2,3\}$ as an open problem.

For completeness we finally provide the dual analog of Lemma 2.1 for the fixation line.

Lemma 2.3. *Let $a \in (0,\infty)$ and let $\tilde{L} = (\tilde{l}_{ij})_{i,j\in\mathbb{N}}$ denote the matrix in the spectral decomposition $G=\tilde{R}\tilde{D}\tilde{L}$ of the generator G of the fixation line of the $\beta(a,1)$-coalescent. Then for every $i\in\mathbb{N}$ the horizontal generating function $\tilde{l}_i(z) := \sum_{j=i}^\infty \tilde{l}_{ij}z^j$ is a solution to the equation*

$$\tilde{l}_i(z)+z(1-\phi(z))\tilde{l}_i'(z) \;=\; \frac{\Gamma(i+2)}{\Gamma(i+a)} D_z^{a-1}\big(z^{a-1}\tilde{l}_i(z)\big), \qquad z\in[0,1), \tag{2.7}$$

where $\phi(z)$ is defined via (2.5).

Remark 2.4. Again, to the best of the authors knowledge, explicit solutions of (2.7) are only known for $a=1$ (see [10]) and $a=3$ (see the proof of Theorem 1.4). For $a=2$ Eq. (2.7) degenerates to the uninformative equation $\tilde{l}_i(z)+z\tilde{l}_i'(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z\tilde{l}_i(z))$.

Remark 2.5. In this final remark we provide some further information explaining why the parameter values $a=1$ and $a=3$ are rather particular. For $a\in\{1,3\}$ the matrix $R=(r_{ij})_{i,j\in\mathbb{N}}$ of the spectral decomposition $Q=RDL$ of the generator Q of the block counting process of the $\beta(a,1)$-coalescent has entries

$$r_{ij} \;=\; \frac{S(i,j;-1,\frac{1-a}{a+1},0)}{[\frac{2}{a+1}j]_{i-j}}, \qquad i,j\in\mathbb{N}, \tag{2.8}$$

where $S(i,j;\alpha,\beta,r)$ are the generalized Stirling numbers as defined in Hsu and Shiue [7] and $[x]_0 := 1$ and $[x]_n := x(x+1)\cdots(x+n-1)$, $n\in\mathbb{N}$, denote the ascending factorials. For $a=3$ (2.8) reduces to (1.6) and for $a=1$ (2.8) holds by [16, Theorem 1.1] and the fact that the generalized Stirling numbers $S(i,j;-1,0,0)$ coincide with the standard absolute Stirling numbers of the first kind.

Note that (2.8) even holds for the limiting case $a\to 0$ (Kingman coalescent), which follows from the formula for r_{ij} for the Kingman coalescent provided in the appendix of [16] and from $S(i,j;-1,1,0) = \frac{i!}{j!}\binom{i-1}{j-1}$ (Lah numbers).

However, (2.8) does not hold for $a\in(0,\infty)\setminus\{1,3\}$. More precisely, for $a\in(0,\infty)\setminus\{1,3\}$, the first indices i and j where (2.8) fails to hold are $i=6$ and $j=2$, which is seen as follows. Using the general recursion (3.1) for the entries r_{ij} it follows that, for the $\beta(a,1)$-coalescent,

$$r_{62} \;=\; \frac{3(2a^4+31a^3+204a^2+755a+1200)}{2(a+5)(a^3+12a^2+59a+168)}, \qquad a\in(0,\infty).$$

On the other hand, using the recursion (see Theorem 1 of [7]) for the generalized Stirling numbers one obtains

$$\frac{S(6,2;-1,\frac{1-a}{a+1},0)}{[\frac{4}{a+1}]_4} \;=\; \frac{9a^2+53a+75}{(a+5)(3a+7)}.$$

Taking the difference yields

$$r_{62}-\frac{S(6,2;-1,\frac{1-a}{a+1},0)}{[\frac{4}{a+1}]_4} \;=\; \frac{(a+1)a(1-a)(a-3)}{2(a+5)(a+7)(3a+7)(a^2+5a+24)}, \tag{2.9}$$

which differs from 0 for $a\in(0,\infty)\setminus\{1,3\}$. The latter expression vanishes for $a=0$ (Kingman coalescent), $a=1$ (Bolthausen–Sznitman coalescent) and $a=3$ ($\beta(3,1)$-coalescent), explaining why these three cases are particular. Note that (2.9) has an additional negative root at $a=-1$. One may analyse this additional case in more detail, however the parameter value $a=-1$ does not seem to have any meaning in the context of coalescent theory.

3 Proofs

We first provide the proofs of Theorems 1.2 and 1.4. Both proofs are based on generating functions. We additionally present an alternative short proof of Theorem 1.4 based on Siegmund duality. We start with the proof of Theorem 1.2, since this proof turns out to be slightly less technical than the (first) proof of Theorem 1.4.

Proof. (of Theorem 1.2) As in [16] it follows that the entries r_{ij} of R satisfy for each $j\in\mathbb{N}$ the recursion $r_{jj}=1$ and

$$r_{ij} \;=\; \frac{1}{q_i-q_j}\sum_{k=j}^{i-1} q_{ik}r_{kj}, \qquad i\in\{j+1,j+2,\ldots\}. \tag{3.1}$$

Plugging in $q_{ik}=3/(i+1)$, $1\le k<i$, and $q_i-q_j=\dfrac{3(i-1)}{i+1}-\dfrac{3(j-1)}{j+1}=\dfrac{6(i-j)}{(i+1)(j+1)}$ it follows that

$$r_{ij} \;=\; \frac{j+1}{2(i-j)}\sum_{k=j}^{i-1} r_{kj}, \qquad i\in\{j+1,j+2,\ldots\}. \tag{3.2}$$

In order to solve this recursion we proceed similar as in the proof of Theorem 1.1 of [16]. For $j\in\mathbb{N}$ define the generating function $r_j(z):=\sum_{i=j}^\infty r_{ij}z^i$, $|z|<1$, and consider the modified generating function $f_j(z):=\sum_{i=j}^\infty (i-j)r_{ij}z^i$, $|z|<1$. By (3.2),

$$f_j(z) \;=\; \sum_{i=j+1}^\infty \frac{j+1}{2}\sum_{k=j}^{i-1} r_{kj}z^i \;=\; \frac{j+1}{2}\sum_{k=j}^\infty r_{kj}z^k\sum_{i=k+1}^\infty z^{i-k} \;=\; \frac{j+1}{2}\frac{z}{1-z}r_j(z).$$

On the other hand $f_j(z)=\sum_{i=j}^\infty ir_{ij}z^i-j\sum_{i=j}^\infty r_{ij}z^i=zr_j'(z)-jr_j(z)$. Thus, $zr_j'(z)-jr_j(z)=((j+1)/2)(z/(1-z))r_j(z)$ or, equivalently,

$$r_j'(z) \;=\; \left(\frac{j+1}{2}\frac{1}{1-z}+\frac{j}{z}\right)r_j(z), \qquad 0<|z|<1. \tag{3.3}$$

The solution of this first order homogeneous differential equation with initial conditions $r_j(0)=\cdots=r_j^{(j-1)}(0)=0$ and $r_j^{(j)}(0)=j!$, where $r_j^{(i)}$ denotes the ith derivative of r_j, is

$$r_j(z) \;=\; \frac{z^j}{(1-z)^{\frac{j+1}{2}}} \;=\; \frac{1}{\sqrt{1-z}}\left(\frac{z}{\sqrt{1-z}}\right)^j, \qquad j\in\mathbb{N}, |z|<1, \tag{3.4}$$

showing that R is the Riordan matrix $R=(1/\sqrt{1-z}, z/\sqrt{1-z})$ in the notation introduced after Definition 1.1. The function r_j has Taylor expansion $r_j(z)=z^j\sum_{k=0}^\infty\binom{-\frac{j+1}{2}}{k}(-z)^k=\sum_{i=j}^\infty\binom{-\frac{j+1}{2}}{i-j}(-1)^{i-j}z^i$. For $i\ge j$ the coefficient r_{ij} in front of z^i in the Taylor expansion of r_j is hence given by (1.3).

Let us now turn to $L:=R^{-1}$. We have $(z,z^2,\ldots)R=(r_1(z),r_2(z),\ldots)$. Multiplying with L it follows that $(z,z^2,\ldots)=(r_1(z),r_2(z),\ldots)L$. Thus, $z^j=\sum_{i=j}^\infty r_i(z)l_{ij}=\sum_{i=j}^\infty z^i(1-z)^{-(i+1)/2}l_{ij}=(1-z)^{-1/2}\sum_{i=j}^\infty(z(1-z)^{-1/2})^i l_{ij}$, or, equivalently,

$$l_j\left(\frac{z}{\sqrt{1-z}}\right) := \sum_{i=j}^\infty l_{ij}\left(\frac{z}{\sqrt{1-z}}\right)^i \;=\; z^j\sqrt{1-z}, \qquad |z|<1.$$

Substituting $u:=z/\sqrt{1-z}$ or $z=u(\sqrt{1+u^2/4}-u/2)$ it follows that

$$l_j(u) \;=\; \sum_{i=j}^\infty l_{ij}u^i \;=\; \frac{z^{j+1}}{u} \;=\; u^j\left(\sqrt{1+\frac{u^2}{4}}-\frac{u}{2}\right)^{j+1}, \qquad j\in\mathbb{N}, |u|<1, \tag{3.5}$$

showing that L is the Riordan matrix $L=(\sqrt{1+z^2/2}-z/2, z(\sqrt{1-z^2/2}-z/2))$. Binomial expansion leads to

$$\begin{aligned} l_j(u) &= u^j\sum_{r=0}^{j+1}\binom{j+1}{r}\left(-\frac{u}{2}\right)^{j+1-r}\left(1+\frac{u^2}{4}\right)^{\frac{r}{2}} \\ &= u^j\sum_{r=0}^{j+1}\binom{j+1}{r}\left(-\frac{u}{2}\right)^{j+1-r}\sum_{l=0}^\infty\binom{\frac{r}{2}}{l}\left(\frac{u}{2}\right)^{2l}. \end{aligned}$$

For $i\ge j$ the coefficient l_{ij} in front of u^i in the Taylor expansion of $l_j(u)$ is hence given by (choose $r=2j+2l-i+1$ above)

$$\begin{aligned} l_{ij} &= \left(-\frac{1}{2}\right)^{i-j}\sum_{l=0}^\infty\binom{j+1}{2j+2l-i+1}\binom{j+l-\frac{i-1}{2}}{l} \\ &= \frac{j+1}{2}(-1)^{i-j}\frac{\Gamma(\frac{i+1}{2})}{\Gamma(j-\frac{i}{2}+\frac{3}{2})\Gamma(i-j+1)} \end{aligned}$$

by Lemma 4.1 in the appendix (applied with $u:=j+1$ and $v:=2j-i+1$), with the usual convention that $1/\Gamma(z)=0$ for $z\in-\mathbb{N}_0$, and, hence, $l_{ij}=0$ if $j-\frac{i}{2}+\frac{3}{2}\in-\mathbb{N}_0$. Thus, (1.4) is established. $\square$

Remark 3.1. For the Bolthausen–Sznitman coalescent the corresponding generating functions r_j and l_j, $j\in\mathbb{N}$, are given by (see [16, Eqs. (2.7) and (2.9)])

$$r_j(z) \;=\; \frac{z}{1-z}(-\log(1-z))^{j-1} \quad\text{and}\quad l_j(z) \;=\; ze^{-z}(1-e^{-z})^{j-1}, \qquad j\in\mathbb{N}, |z|<1,$$

showing that the matrices R and L of the spectral decomposition $Q=RDL$ of the generator Q of the block counting process of the Bolthausen–Sznitman coalescent are Riordan matrices of the form

$$R \;=\; \left(\frac{z}{(1-z)(-\log(1-z))}, -\log(1-z)\right) \quad\text{and}\quad L \;=\; \left(\frac{ze^{-z}}{1-e^{-z}}, 1-e^{-z}\right).$$

Proof. (of Theorem 1.4) Two proofs of Theorem 1.4 are provided. The first proof is self-contained and again based on generating functions. The second proof is rather short and exploits Theorem 1.2 and the fact that the block counting process is Siegmund dual to the fixation line.

Proof 1. We follow the first proof of Theorem 3.1 of [10]. Let $\tilde{D} = (\tilde{d}_{ij})_{i,j\in\mathbb{N}}$ be the diagonal matrix with entries $\tilde{d}_{ii} := -g_i = g_{ii}$, $i \in \mathbb{N}$, and let $\tilde{R} = (\tilde{r}_{ij})_{i,j\in\mathbb{N}}$ be the upper right triangular matrix with entries defined for each $j \in \mathbb{N}$ recursively via $\tilde{r}_{jj} := 1$ and $\tilde{r}_{ij} := (g_i - g_j)^{-1} \sum_{k=i+1}^{j} g_{ik}\tilde{r}_{kj}$ for $i \in \{j-1, j-2, \ldots, 1\}$. Since $g_{ii} = -g_i$, $i \in \mathbb{N}$, we conclude that $\tilde{r}_{ij}g_{jj} = \sum_{k=i}^{j} g_{ik}\tilde{r}_{kj}$, Thus, the entries of $\tilde{R}$ are defined such that $\tilde{R}\tilde{D} = G\tilde{R}$. Define $\tilde{L} := \tilde{R}^{-1}$. Then, the spectral decomposition $G = \tilde{R}\tilde{D}\tilde{L}$ holds. Moreover, $\tilde{D}\tilde{L} = \tilde{L}G$ and, hence, $g_{ii}\tilde{l}_{ij} = \sum_{k=i}^{j} \tilde{l}_{ik}g_{kj}$, $i, j \in \mathbb{N}$. Since $g_{ii} = -g_i$, $i \in \mathbb{N}$, we obtain for each $i \in \mathbb{N}$ the recursion $\tilde{l}_{ii} = 1$ and

$$\tilde{l}_{ij} \;=\; \frac{1}{g_j - g_i}\sum_{k=i}^{j-1} \tilde{l}_{ik}g_{kj}, \qquad j \in \{i+1, i+2, \ldots\}. \tag{3.6}$$

For $i \in \mathbb{N}$ define the generating function $\tilde{l}_i(z) := \sum_{j=i}^{\infty} \tilde{l}_{ij}z^j$, $|z| < 1$, and consider the modified generating function $f_i(z) := \sum_{j=i}^{\infty}(j+1)(j+2)(g_j - g_i)\tilde{l}_{ij}z^j$, $|z| < 1$. From $g_j - g_i \;= 6(j-i)/((i+2)(j+2))$ we conclude that

$$\begin{aligned}
f_i(z) \;&=\; \frac{6}{i+2}\sum_{j=i}^{\infty}(j+1)(j-i)\tilde{l}_{ij}z^j \;=\; \frac{6}{i+2}\left(\sum_{j=i}^{\infty}(j+1)j\tilde{l}_{ij}z^j - i\sum_{j=i}^{\infty}(j+1)\tilde{l}_{ij}z^j\right)\\
&=\; \frac{6}{i+2}\left(z\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2\big(z\tilde{l}_i(z)\big) - i\frac{\mathrm{d}}{\mathrm{d}z}\big(z\tilde{l}_i(z)\big)\right)\\
&=\; \frac{6}{i+2}\big(z^2\tilde{l}_i''(z) + (2-i)z\tilde{l}_i'(z) - i\tilde{l}_i(z)\big).
\end{aligned} \tag{3.7}$$

On the other hand, by the recursion (3.6), we obtain

$$\begin{aligned}
f_i(z) \;&=\; \sum_{j=i+1}^{\infty}(j+1)(j+2)(g_j - g_i)\tilde{l}_{ij}z^j \;=\; \sum_{j=i+1}^{\infty}(j+1)(j+2)\sum_{k=i}^{j-1}\tilde{l}_{ik}g_{kj}z^j\\
&=\; \sum_{k=i}^{\infty}\tilde{l}_{ik}\sum_{j=k+1}^{\infty}\underbrace{(j+1)(j+2)g_{kj}}_{=3k}z^j\\
&=\; 3\sum_{k=i}^{\infty}k\tilde{l}_{ik}\sum_{j=k+1}^{\infty}z^j \;=\; 3\sum_{k=i}^{\infty}k\tilde{l}_{ik}\frac{z^{k+1}}{1-z} \;=\; \frac{3z^2}{1-z}\tilde{l}_i'(z).
\end{aligned} \tag{3.8}$$

Since (3.7) and (3.8) coincide, we obtain

$$\frac{z^2\tilde{l}_i'(z)}{1-z} \;=\; \frac{2}{i+2}\big(z^2\tilde{l}_i''(z) + (2-i)z\tilde{l}_i'(z) - i\tilde{l}_i(z)\big)$$

or, after some straightforward manipulation,

$$2z^2\tilde{l}_i''(z) + \frac{z}{1-z}(4 - 2i - 6z + iz)\tilde{l}_i'(z) - 2i\tilde{l}_i(z) \;=\; 0.$$

The solution of this homogeneous second order differential equation with initial conditions $\tilde{l}_i(0) = \cdots = \tilde{l}_i^{(i-1)}(0) = 0$ and $\tilde{l}_i^{(i)}(0) = i!$, where $\tilde{l}_i^{(j)}$ denotes the jth derivative of $\tilde{l}_i$, is

$$\tilde{l}_i(z) \;=\; \frac{z^i}{(1-z)^{\frac{i}{2}}}, \qquad i \in \mathbb{N}, |z| < 1. \tag{3.9}$$

The function $\tilde{l}_i$ has Taylor expansion $\tilde{l}_i(z) = z^i \sum_{k=0}^\infty \binom{-\frac{i}{2}}{k}(-z)^k = \sum_{j=i}^\infty \binom{-\frac{i}{2}}{j-i}(-1)^{j-i}z^j$. The coefficient $\tilde{l}_{ij}$ in front of z^j in the Taylor expansion of $\tilde{l}_i$ is hence given by (1.12).

Let us now turn to $\tilde{R} = \tilde{L}^{-1}$. We have $\tilde{L}(z, z^2, \ldots)^\top = (\tilde{l}_1(z), \tilde{l}_2(z), \ldots)^T$. Multiplying from the left with $\tilde{R}$ it follows that $(z, z^2, \ldots)^\top = \tilde{R}(\tilde{l}_1(z), \tilde{l}_2(z), \ldots)^\top$. Thus, $z^i = \sum_{j=i}^\infty \tilde{r}_{ij}\tilde{l}_j(z) = \sum_{j=i}^\infty \tilde{r}_{ij} z^j (1-z)^{-j/2}$. Substituting $u = z/\sqrt{1-z}$ or $z = u(\sqrt{1+u^2/4} - u/2)$ it follows that

$$\tilde{r}_i(u) \;:=\; \sum_{j=i}^\infty r_{ij}u^j \;=\; z^i \;=\; u^i\left(\sqrt{1+\frac{u^2}{4}} - \frac{u}{2}\right)^i, \qquad i \in \mathbb{N}, |u| < 1, \tag{3.10}$$

and binomial expansion leads to

$$\tilde{r}_i(u) \;=\; u^i \sum_{r=0}^i \binom{i}{r}\left(-\frac{u}{2}\right)^{i-r}\left(1+\frac{u^2}{4}\right)^{\frac{r}{2}} \;=\; u^i \sum_{r=0}^i \binom{i}{r}\left(-\frac{u}{2}\right)^{i-r} \sum_{l=0}^\infty \binom{\frac{r}{2}}{l}\left(\frac{u}{2}\right)^{2l}.$$

For $i \le j$ the coefficient $\tilde{r}_{ij}$ in front of u^j in the Taylor expansion of $\tilde{r}_i(u)$ is hence given by (choose $r = 2i + 2l - j$ above)

$$\tilde{r}_{ij} \;=\; \left(-\frac{1}{2}\right)^{j-i} \sum_{l=0}^\infty \binom{i}{2i+2l-j}\binom{i+l-\frac{j}{2}}{l} \;=\; \frac{i}{2}(-1)^{j-i}\frac{\Gamma(\frac{j}{2})}{\Gamma(i-\frac{j}{2}+1)\Gamma(j-i+1)}$$

by Lemma 4.1 in the appendix (applied with $u := i$ and $v := 2i - j$), with the usual convention that $1/\Gamma(z) = 0$ for $z \in -\mathbb{N}_0$, and hence $\tilde{r}_{ij} = 0$ if $i - \frac{j}{2} + 1 \in -\mathbb{N}_0$. Thus, (1.11) is established. □

Proof 2. (via duality) Exploiting the fact that the block counting process is Siegmund dual to the fixation line it follows as in the second proof of Theorem 3.1 of [10] that

$$\tilde{r}_{ij} \;=\; -\sum_{k=1}^i l_{j+1,k} \quad\text{and}\quad \tilde{l}_{ij} \;=\; r_{j+1,i+1} - r_{j,i+1}, \qquad i, j \in \mathbb{N}. \tag{3.11}$$

Plugging in the formulas (1.3) and (1.4) for r_{ij} and l_{ij} known from Theorem 1.2, we obtain

$$\tilde{r}_{ij} = -\sum_{k=1}^j \frac{k+1}{2}(-1)^{j+1-k}\frac{\Gamma(\frac{j}{2}+1)}{\Gamma(k-\frac{j}{2}+1)\Gamma(j-k+2)} = \frac{i}{2}(-1)^{j-i}\frac{\Gamma(\frac{j}{2})}{\Gamma(i-\frac{j}{2}+1)\Gamma(j-i+1)}$$

and

$$\tilde{l}_{ij} \;=\; \frac{\Gamma(j-\frac{i}{2}+1)}{\Gamma(\frac{i}{2}+1)\Gamma(j-i+1)} - \frac{\Gamma(j-\frac{i}{2})}{\Gamma(\frac{i}{2}+1)\Gamma(j-i)} \;=\; \frac{\Gamma(j-\frac{i}{2})}{\Gamma(\frac{i}{2})\Gamma(j-i+1)}.$$

□

Remark 3.2. For the Bolthausen–Sznitman coalescent the corresponding generating functions $\tilde{r}_i$ and $\tilde{l}_i$, $i \in \mathbb{N}$, are given by (see [10, p. 954]) $\tilde{r}_i(z) = (1-e^{-z})^i$ and $\tilde{l}_i(z) = (-\log(1-z))^i$, $i \in \mathbb{N}$, $|z| < 1$, showing that the transposed matrices $\tilde{R}^\top$ and $\tilde{L}^\top$ of the spectral decomposition $G = \tilde{R}\tilde{D}\tilde{L}$ of the generator G of the fixation line of the Bolthausen–Sznitman coalescent are Riordan matrices of the form $\tilde{R}^\top = (1, 1-e^{-z})$ and $\tilde{L}^\top = (1, -\log(1-z))$.

We finish this section with the proofs of Lemma 2.1 and Lemma 2.3.

Proof. (of Lemma 2.1) Fix $j \in \mathbb{N}$. Eq. (2.6) holds for $a = 2$, since $\phi(z) = 0$ for $a = 2$. We can therefore assume that $a \in (0,\infty) \setminus \{2\}$. Consider the modified generating function

$$g_j(z) \;:=\; \frac{2-a}{\Gamma(a+1)} \sum_{i=j}^\infty \frac{\Gamma(i+a-1)}{\Gamma(i+1)}(q_i - q_j) r_{ij} z^i, \qquad |z| < 1.$$

The recursion (3.1) holds for any exchangeable coalescent as long as the total rates q_i, $i \in \mathbb{N}$, are pairwise distinct. Thus, by the recursion (3.1) and by (2.1),

$$\begin{aligned} g_j(z) &= \frac{2-a}{\Gamma(a+1)} \sum_{i=j+1}^{\infty} \frac{\Gamma(i+a-1)}{\Gamma(i+1)} \sum_{k=j}^{i-1} q_{ik} r_{kj} z^i \\ &= \frac{2-a}{\Gamma(a+1)} \sum_{k=j}^{\infty} r_{kj} z^k \sum_{i=k+1}^{\infty} \frac{\Gamma(i+a-1)}{\Gamma(i+1)} q_{ik} z^{i-k} \\ &= \frac{2-a}{\Gamma(a)} \sum_{k=j}^{\infty} r_{kj} z^k \sum_{i=k+1}^{\infty} \frac{\Gamma(i-k+a-1)}{\Gamma(i-k+2)} z^{i-k} \;=\; r_j(z)\phi(z). \end{aligned} \tag{3.12}$$

On the other hand, it follows from (2.2) that

$$\begin{aligned} g_j(z) &= \sum_{i=j}^{\infty} \left(1 - \frac{\Gamma(j+1)}{\Gamma(j+a-1)} \frac{\Gamma(i+a-1)}{\Gamma(i+1)}\right) r_{ij} z^i \\ &= r_j(z) - \frac{\Gamma(j+1)}{\Gamma(j+a-1)} \sum_{i=j}^{\infty} \frac{\Gamma(i+a-1)}{\Gamma(i+1)} r_{ij} z^i. \end{aligned}$$

Applying the fractional calculus formula for monomials (see, for example, Kilbas, Srivastava and Trujillo [8, p. 71, Eqs. (2.1.16) and (2.1.17)])

$$D_z^{\beta}(z^q) \;=\; \frac{\Gamma(q+1)}{\Gamma(q-\beta+1)} z^{q-\beta}, \qquad q, q-\beta > -1,$$

with $\beta := a-2$ and $q := i+a-2$ (> -1) it follows that

$$\begin{aligned} g_j(z) &= r_j(z) - \frac{\Gamma(j+1)}{\Gamma(j+a-1)} D_z^{a-2}\Big(\sum_{i=j}^{\infty} r_{ij} z^{i+a-2}\Big) \\ &= r_j(z) - \frac{\Gamma(j+1)}{\Gamma(j+a-1)} D_z^{a-2}\big(z^{a-2} r_j(z)\big). \end{aligned} \tag{3.13}$$

Since both expressions (3.12) and (3.13) are equal the result follows. □

Proof. (of Lemma 2.3) Fix $i \in \mathbb{N}$. Eq. (2.7) clearly holds for $a = 2$. We can therefore assume that $a \neq 2$. By the recursion (3.6) and by (2.3),

$$\begin{aligned} h_i(z) &:= \frac{2-a}{\Gamma(a+1)} \sum_{j=i}^{\infty} \frac{\Gamma(j+a)}{\Gamma(j+1)} (g_j - g_i) \tilde{l}_{ij} z^j \;=\; \frac{2-a}{\Gamma(a+1)} \sum_{j=i}^{\infty} \frac{\Gamma(j+a)}{\Gamma(j+1)} \sum_{k=i}^{j-1} \tilde{l}_{ik} g_{kj} z^j \\ &= \frac{2-a}{\Gamma(a)} \sum_{k=i}^{\infty} k \tilde{l}_{ik} z^k \sum_{j=k+1}^{\infty} \frac{\Gamma(j-k+a-1)}{\Gamma(j-k+2)} z^{j-k} \;=\; z\tilde{l}_i'(z)\phi(z) \end{aligned}$$

with ϕ as defined in (2.5). On the other hand, it follows from (2.4) that

$$\begin{aligned} h_i(z) &= \sum_{j=i}^{\infty} \frac{\Gamma(j+a)}{\Gamma(j+1)} \left(\frac{\Gamma(j+2)}{\Gamma(j+a)} - \frac{\Gamma(i+2)}{\Gamma(i+a)}\right) \tilde{l}_{ij} z^j \\ &= \sum_{j=i}^{\infty} (j+1) \tilde{l}_{ij} z^j - \frac{\Gamma(i+2)}{\Gamma(i+a)} \sum_{j=i}^{\infty} \frac{\Gamma(j+a)}{\Gamma(j+1)} \tilde{l}_{ij} z^j \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\big(z\tilde{l}_i(z)\big) - \frac{\Gamma(i+2)}{\Gamma(i+a)} D_z^{a-1}\Big(\sum_{j=i}^{\infty} \tilde{l}_{ij} z^{j+a-1}\Big) \\ &= \tilde{l}_i(z) + z\tilde{l}_i'(z) - \frac{\Gamma(i+2)}{\Gamma(i+a)} D_z^{a-1}\big(z^{a-1}\tilde{l}_i(z)\big), \end{aligned}$$

which leads to the desired equation for $\tilde{l}_i$. □

4 Appendix

With the usual convention that $1/\Gamma(z) = 0$ for $z \in -\mathbb{N}_0$, the binomial coefficient $\binom{u}{v} = \Gamma(u+1)/(\Gamma(v+1)\Gamma(u-v+1)) \in \mathbb{R}$ is defined for all $u, v \in \mathbb{R}$ with $u \notin -\mathbb{N}$. For nonnegative integer $v \in \mathbb{N}_0$ the binomial coefficient $\binom{u}{v} = u(u-1)\cdots(u-v+1)/v!$ is even defined for all $u \in \mathbb{R}$. The following technical lemma is used in the proofs of Theorems 1.2 and 1.4.

Lemma 4.1. *For all* $u \in \mathbb{R} \setminus -\mathbb{N}$ *and all* $v \in \mathbb{R}$ *with* $v < 2u$*,*

$$\sum_{n=0}^{\infty} \binom{u}{2n+v} \binom{n+\frac{v}{2}}{n} \;=\; u 2^{u-v-1} \frac{\Gamma(u-\frac{v}{2})}{\Gamma(\frac{v}{2}+1)\Gamma(u-v+1)},$$

with the usual convention that $1/\Gamma(z) = 0$ *for* $z \in -\mathbb{N}_0$*.*

Remark 4.2. For $v = 0$ the formula reduces to $\sum_{n=0}^{\infty} \binom{u}{2n} = 2^{u-1}$, $u \in (0, \infty)$.

Proof. For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$ let $[x]_n$ denote the rising Pochhammer symbol, i.e. $[x]_0 := 1$ and $[x]_n := x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$. Using Legendre's duplication formula $\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2}) = 2^{1-x}\sqrt{\pi}\Gamma(x)$ a somewhat cumbersome but straightforward computation shows that, for $|z| < 1$,

$$\sum_{n=0}^{\infty} \binom{u}{2n+v} \binom{n+\frac{v}{2}}{n} z^n \;=\; \binom{u}{v} {}_2F_1\Big(\frac{v-u}{2}, \frac{v-u+1}{2}; \frac{v+1}{2}; z\Big),$$

where ${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{[a]_n [b]_n}{[c]_n} \frac{z^n}{n!}$, $|z| < 1$, denotes the hypergeometric function. Applying the Gauß formula ${}_2F_1(a, b; c; 1) = \Gamma(c)\Gamma(c-a-b)/(\Gamma(c-a)\Gamma(c-b))$, $c > a+b$, with $a := (v-u)/2$, $b := (v-u+1)/2$ and $c := (v+1)/2$ leads to

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{u}{2n+v} \binom{n+\frac{v}{2}}{n} \;&=\; \binom{u}{v} {}_2F_1\Big(\frac{v-u}{2}, \frac{v-u+1}{2}; \frac{v+1}{2}; 1\Big)\\ &=\; \binom{u}{v} \frac{\Gamma(\frac{v+1}{2})\Gamma(u-\frac{v}{2})}{\Gamma(\frac{u+1}{2})\Gamma(\frac{u}{2})} \;=\; u 2^{u-v-1} \frac{\Gamma(u-\frac{v}{2})}{\Gamma(\frac{v}{2}+1)\Gamma(u-v+1)},\end{aligned}$$

where the last equality follows again from Legendre's duplication formula. □

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