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Delocalization and limiting spectral distribution of Erdős-Rényi graphs with constant expected degree

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Abstract

For fixed $\lambda > 0$, it is known that Erdős-Rényi graphs $\{G(n, \lambda/n), n \in \mathbb{N}\}$, with edge-weights $1/\sqrt{\lambda}$, have a limiting spectral distribution, ν_λ. As $\lambda \to \infty$, $\{\nu_\lambda\}$ converges to the semicircle distribution. For large λ, we find an orthonormal eigenvector basis of $G(n, \lambda/n)$ where most of the eigenvectors have small infinity norms as $n \to \infty$, providing a variant of an eigenvector delocalization result of Tran, Vu, and Wang (2013).

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1 Introduction

The spectral theory of graphs is important since many principal invariants of graphs are essentially related with their spectra. On the other hand, powerful tools used to investigate the spectrum of random matrices have been developed following the seminal work by Wigner [20]. In this paper, we study a class of random matrices related to graphs, namely the adjacency matrices of Erdős-Rényi random graphs.

Let $G(n, p)$ be the Erdős-Rényi random graph with n vertices and connection probability p. More precisely, letting $M_{n,p}$ denote the adjacency matrix of $G(n, p)$, for $i > j$ we independently set,

$$M_{n,p}(i,j) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p, \end{cases}$$

and $M_{n,p}(i,j) = M_{n,p}(j,i)$ if $i < j$. Also, the graph has no loops, so $M_{n,p}(i,i) = 0$ for all i. Note that $M_{n,p}$ is symmetric so its spectrum is real.

Recently, many outstanding results have been shown under the condition (with $p = p_n$)

$$\lim_{n\to\infty} np = \infty,$$

in other words, under the condition that $G(n, p)$ has an expected degree, np, diverging with n. Under this condition, the spectral distribution of the scaled Erdős-Rényi ensemble

$$\frac{1}{\sqrt{np(1-p)}} M_{n,p}\ , \quad n \in \mathbb{N}$$

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weakly converges to the standard semicircle distribution [19]. Moreover, a local semicircle law holds [11]. Also, remarkably, all the l^2-normalized eigenvectors "delocalize" in term of their l^∞-norm [11, 19].

The situation is different if the expected degree is fixed. If, for all n, we impose that $p = \lambda/n$ for some fixed $\lambda > 0$, convergence to the semicircle law and delocalization do not hold [2, 4, 21]. Let $\nu_{n,\lambda}$ be the empirical spectral distribution of the scaled random adjacency matrix

$$\frac{1}{\sqrt{\lambda}} M_{n,\lambda/n} ,$$

which is defined as

$$\nu_{n,\lambda} := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$$

where $\{x_i\}_{i=1}^n$ are the eigenvalues of $\lambda^{-1/2} M_{n,\lambda/n}$ and δ is a Dirac delta distribution. As shown in [2, 4, 21], $\nu_{n,\lambda}$ almost surely has a deterministic limiting distribution ν_λ as λ goes to infinity; however, it is an open problem to find an explicit form for ν_λ, or even to give a characterization of its decomposition into pure-point, absolutely-continuous, and singular-continuous parts [7]. In [2], Bauer and Golinelli analyzed ν_λ using the moment method; we use the moment asymptotics given by their work as a starting point for this study. A numerical simulation is also given in [2], and one can see that the numerical approximation of ν_λ there, simulates the semicircle distribution as λ increases.

Theorem 1.1. *For each $\lambda > 0$, let $\nu_{n,\lambda}$ be the empirical spectral distribution of $\frac{1}{\sqrt{\lambda}} M_{n,\lambda/n}$. Let*

$$\nu_\lambda := \lim_{n\to\infty} \nu_{n,\lambda} \quad \textit{where the limit is in the weak sense.}$$

Then, as λ goes to infinity, ν_λ converges weakly to the standard semicircle distribution ρ_{sc} where

$$\rho_{sc}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{\{|x|\le 2\}}(dx).$$

It was recently pointed out to us that the above result was proved in [10], nevertheless we provide two independent proofs of this fact since they are both different from the proof given in [10]. These proofs are provided also for the sake of completeness, since the above result will play a crucial role in the proof of our main result, Theorem 1.3.

Let us also remark that while the semicircle convergence results of [11, 19] look similar to the above, there is a difference in the "order of limits": suppose $\{\lambda_m\}$ is an expected degree sequence such that $\lim_{m\to\infty} \lambda_m = \infty$. In [11, 19], a limiting "diagonal" spectral distribution sequence is considered,

$$\lim_{n\to\infty} \nu_{n,\lambda_n},$$

whereas we are interested in the limit of limiting distributions $\{\nu_{\lambda_m}\}$,

$$\lim_{m\to\infty} \nu_{\lambda_m} = \lim_{m\to\infty} \lim_{n\to\infty} \nu_{n,\lambda_m}.$$

In addition to results about the spectral distribution, another natural question is whether the l^2-normalized eigenvectors of $M_{n,\lambda/n}$ localize or delocalize. This question was raised, for example, by Dekel et al. [8]:

Problem 1.2 (Question 2 of [8]).

(i) *Is it true that, almost surely, every unit eigenvector* u *of* $G(n,p)$ *has* $\|u\|_\infty = o(1)$*?*

(ii) *Further, can we show that, almost surely,* $\|u\|_\infty = n^{-\frac{1}{2}+o(1)}$*?*

If the answer to (i) is positive, we say that the unit eigenvectors **delocalize**. Tao and Vu [18] showed that (i) and (ii) hold when $p = 1/2$, which is of course independent of n. However, if $p = \lambda/n$, it is easy to see that $G(n,p)$ almost surely has $\mathbf{O}(n)$ isolated vertices which persist in the limit. Thus, almost surely there exist at least $\mathbf{O}(n)$ eigenvectors such that their infinity norms are asymptotically 1, so delocalization fails.

One can, however, obtain a weak form of delocalization as follows. For any $\epsilon > 0$, one can choose n and λ large enough so that most of the vectors in some l^2-normalized orthonormal basis have an infinity norm smaller than ϵ. We need some notation in order to state this result more precisely. For any symmetric $n \times n$ matrix H, the eigenvalues of H are denoted by $\{\Lambda_i(H)\}_{i=1}^n$. Without loss of generality, we suppose

$$\Lambda_1(H) \le \Lambda_2(H) \le \cdots \le \Lambda_n(H)$$

throughout this paper. Since H is symmetric, H has an orthonormal basis $\{u_i(H)\}_{i=1}^n$ such that $u_i(H)$ is a unit eigenvector corresponding to $\Lambda_i(H)$.

Theorem 1.3. *Let* $\epsilon > 0$. *Using the above notation, define a subset* $U(n,\lambda,\epsilon)$ *of* $\{1,2,\cdots,n\}$ *as follows,*

$$U(n,\lambda,\epsilon) := \{i \in \{1,2,\cdots,n\} : \|u_i(M_{n,\lambda/n})\|_\infty < \epsilon\}.$$

Then, there exists an orthonormal basis $\{u_i(M_{n,\lambda/n})\}_{i=1}^n$ *satisfying*

$$\liminf_{\lambda\to\infty} \liminf_{n\to\infty} \frac{|U(n,\lambda,\epsilon)|}{n} = 1 \quad \textit{almost surely.}$$

The strategy and main tools for proving the above are provided by Theorem 1.16 in [19] which we restate here for the reader's convenience.

Theorem 1.4 (Theorem 1.16 in [19]). *Assume that the expected degree depends on* n*, i.e.,* $\lambda = \lambda_n$. *Let* $M_n := M_{n,\lambda_n/n}$. *Suppose*

$$\lim_{n\to\infty} \frac{\lambda_n}{\log n} = \infty. \tag{1.1}$$

Then there exists, a.s., an orthonormal eigenvector basis $\{u_i(M_n) : i = 1,2,\cdots,n\}$ *such that*

$$\|u_i(M_n)\|_\infty = o(1)$$

for $1 \le i \le n$.

In fact, we also get a "diagonalized convergence" result as a corollary to Theorem 1.3. The corollary should be viewed as a variant of the above Theorem 1.4. While the conclusion of the corollary is weaker than that of Theorem 1.4, the assumptions also allow for a broader class of sequences $\{\lambda_n\}$. This is one benefit of a priori considering the limiting behavior as two separate limits instead of one single diagonalized limit.

Corollary 1.5. *Let* $\lambda = \lambda_n$ *depend on* n *and set* $M_n := M_{n,\lambda_n/n}$. *Also, suppose* $\lim_{n\to\infty} \lambda_n = \infty$. *Let* $\epsilon > 0$*, and using the above notation, define* $U'(n,\epsilon)$ *by*

$$U'(n,\epsilon) := \{i \in \{1,2,\cdots,n\} : \|u_i(M_n)\|_\infty < \epsilon\}.$$

Then, there exists a.s. an orthonormal eigenvector basis $\{u_i(M_n) : i = 1,2,\cdots,n\}$ *such that*

$$\liminf_{n\to\infty} \frac{|U'(n,\epsilon)|}{n} = 1\ .$$

The outline of the rest of this paper is as follows. In the next section (Section 2), we give two proofs of Theorem 1.1 using respectively the moment method and the Stieltjes transform method. Section 3 is devoted to the proofs of Theorem 1.3 and Corollary 1.5.

2 Convergence to the semicircle distribution

As a preliminary to the two proofs, let us recall that the limiting distribution ν_λ exists [2, 4, 21]. In particular, [4] argues this via showing that the sequence of random graphs $\{G(n,\lambda/n)\}_{n\in\mathbb{N}}$ converges, in the Benjamini-Schramm topology on rooted graphs, to a Galton-Watson tree with offspring distribution Pois(λ) (Poisson with intensity λ). This fact will be useful to us in our second proof. Let us begin, however, with the classical moment method.

2.1 Moment method proof

Fix $\lambda > 0$ and suppose $n \geq \lambda$. Let m_{ij} be the (i,j) element of $M_{n,\lambda/n}$. A standard calculation in random matrix theory gives

$$\mathbb{E}\left\langle \nu_{n,\lambda}, x^k \right\rangle = \frac{1}{n\lambda^{k/2}}\mathbb{E}\left[\mathrm{Tr} M^k_{n,\lambda/n}\right] = \frac{1}{n\lambda^{k/2}} \sum_{1\leq i_1,\cdots,i_k\leq n} \mathbb{E}\left[m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1}\right]. \tag{2.1}$$

We first obtain an asymptotic formula for $\mathbb{E}\left\langle \nu_{n,\lambda}, x^k\right\rangle$ using the method and terminology of [2]. If a k-tuple $(i_1, i_2, \cdots, i_k)$ satisfies $i_1 \neq i_2$, $i_2 \neq i_3$, $\cdots$, $i_{k-1} \neq i_k$ and $i_k \neq i_1$, it is said to be **admissible**. Non-admissible k-tuples do not contribute to the sum (2.1) since $M_{n,\lambda/n}$ has vanishing diagonal entries. For each positive integer $j \leq k$, define W_j as the set of admissible k-tuple $(i_1, i_2, \cdots, i_k)$ satisfying $|\{i_1, i_2, \cdots, i_k\}| = j$. The set W of all admissible k-tuples is

$$W := \bigcup_{1\leq j\leq k} W_j.$$

A k-tuple $(i_1, i_2, \cdots, i_k)$ is called **normalized** if it is admissible and $i_j > 1$ implies that there exist $j' < j$ such that $i_{j'} = i_j - 1$. Let N_j be the set of normalized k-tuples $(i_1, i_2, \cdots, i_k)$ such that $\{i_1, i_2, \cdots, i_k\} = \{1, 2, \cdots, j\}$. For $j \leq n$, Per(j, n) is defined to be the set of injective maps from $\{1, 2, \cdots, j\}$ to $\{1, 2, \cdots, n\}$. It is observed that, there is a one to one correspondence between W_j and $\{(\omega, \sigma)|\omega \in N_j \text{ and } \sigma \in \mathrm{Per}(j, n)\}$. The set N of all normalized k-tuples is expressed as

$$N := \bigcup_{1\leq j\leq k} N_j.$$

In Eq. (2.1), $m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1}$ can be identified with a closed walk along the graph given by the adjacency matrix $M_{n,\lambda/n}$. That is to say, $m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1}$ corresponds with the closed walk $i_1i_2\cdots i_ki_1$ ("closed" means that it ends where it started). Let the sets of distinct edges and distinct vertices in the closed walk $i_1i_2\cdots i_ki_1$ corresponding to k-tuple $\omega = (i_1, i_2, \cdots, i_k)$ be denoted by $E(\omega)$ and $V(\omega)$, respectively. We denote an edge e connecting the vertices with indices i_j and i_{j+1} by $e = i_ji_{j+1}$. Since

$$m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1} = 1$$

if and only if $m_e = 1$ for all $e \in E(\omega)$,

$$\begin{aligned}\frac{1}{n\lambda^{k/2}} \sum_{1\leq i_1,\cdots,i_k\leq n} \mathbb{E}\left[m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1}\right] &= \frac{1}{n\lambda^{k/2}} \sum_{\omega\in W} \left(\frac{\lambda}{n}\right)^{|E(\omega)|} \\ &= \frac{1}{n\lambda^{k/2}} \sum_{\omega\in N} \left(\frac{\lambda}{n}\right)^{|E(\omega)|} |\mathrm{Per}(|V(\omega)|, n)| \end{aligned} \tag{2.2}$$

The moment method proof of Theorem 1.1 will follow from Lemma 2.1 and Lemma 2.3 below.

Lemma 2.1. *For every positive integer* m*,*

$$\lim_{n\to\infty} \mathbb{E}\left\langle \nu_{n,\lambda}, x^k \right\rangle = \begin{cases} 0 & k = 2m-1 \\ \frac{1}{m+1}\binom{2m}{m} + \mathbf{O}(\lambda^{-1}) & k = 2m \end{cases} \tag{2.3}$$

Proof. Let $\omega = (i_1, i_2, \cdots, i_k) \in N$ and set $G(\omega)$ as the graph consisting of edges $E(\omega)$ and vertices $V(\omega)$. We have $|E(\omega)| \geq |V(\omega)| - 1$ since the graph $G(\omega)$ is connected. On the other hand, it is clear that in order to survive in the limit as $n \to \infty$ in (2.2), one must have $|V(\omega)| = |E(\omega)| + 1$ because for any positive integer j

$$\lim_{n\to\infty} \frac{|\mathrm{Per}(j,n)|}{n^j} = 1\ .$$

In particular this implies that $G(\omega)$ must be a tree (rooted at 1).

Henceforth assume $|V(\omega)| = |E(\omega)| + 1$. Then, $i_1 i_2 \cdots i_k i_1$ is a closed walk on a tree and so the multiplicity of every edge in the closed walk $i_1 i_2 \cdots i_k i_1$ is even. Thus, $2|E(\omega)| \leq k$. Let a_l be the number of normalized k-tuples ω such that $|E(\omega)| = l$ and $|V(\omega)| = |E(\omega)| + 1$. In particular, if k is odd, $a_l = 0$ for all $1 \leq l \leq k$ which proves the case $k = 2m-1$ in (2.3). The $k = 2m$ portion of (2.3)) follows from (2.4).

$$\lim_{n\to\infty} \frac{1}{n\lambda^{k/2}} \sum_{\omega\in N} \left(\frac{\lambda}{n}\right)^{|E(\omega)|} |\mathrm{Per}(|V(\omega)|, n)| = \frac{1}{\lambda^{k/2}} \sum_{l=1}^{\lfloor k/2 \rfloor} a_l \lambda^l. \tag{2.4}$$

When $k = 2m$, it is clear that a_m is precisely the Catalan number, C_m, since the multiplicity of every edge in the closed walk $i_1 i_2 \cdots i_k i_1$ is exactly 2. □

Remark 2.2. More precisely, when $k = 2m$, one can easily check that

$$\lim_{n\to\infty} \mathbb{E}\left\langle \nu_{n,\lambda}, x^k \right\rangle = \frac{1}{m+1}\binom{2m}{m} + \sum_{l=1}^{m-1} a_l \lambda^{l-m}.$$

Lemma 2.3.

$$\lim_{n\to\infty} \mathbb{E}\left\langle \nu_{n,\lambda}, x^k \right\rangle = \left\langle \nu_\lambda, x^k \right\rangle \tag{2.5}$$

Proof. By Theorem 1 and Example 2 in [4] (see also [13, Thm 1.1]), $\nu_{n,\lambda}$ converges weakly to ν_λ as $n \to \infty$. Thus, $\lim_{n\to\infty} \mathbb{E}\langle \nu_{n,\lambda}, f\rangle = \langle \nu_\lambda, f\rangle$ for any bounded continuous f, by dominated convergence. The lemma follows from a standard truncation argument. It is enough to consider the case for k even because ν_λ is symmetric (e.g., [13, Thm 1.1]). For $M > 1$, define even functions g_M with $g_M(x) = g_M(-x)$ by

$$g_M(x) = \begin{cases} 1 & 0 \leq x \leq M \\ 0 & x \geq M+1 \\ -x + M + 1 & M < x < M+1 \end{cases}$$

so that

$$|\mathbb{E}\left\langle \nu_{n,\lambda}, x^{2m} \right\rangle - \mathbb{E}\left\langle \nu_{n,\lambda}, x^{2m} g_M \right\rangle| \leq \mathbb{E}\left\langle \nu_{n,\lambda}, x^{2m} \mathbf{1}_{|x|>M} \right\rangle \leq \frac{\mathbb{E}\left\langle \nu_{n,\lambda}, x^{4m} \right\rangle}{M^{2m}}$$

Using the moment bound (2.3), take $n \to \infty$ then $M \to \infty$ to obtain (2.5). □

Recall that ρ_{sc} is the standard semicircle distribution. It is easy to see that

$$\lim_{\lambda\to\infty} \left\langle \nu_\lambda, x^k \right\rangle = \left\langle \rho_{sc}, x^k \right\rangle .$$

Since ρ_{sc} has bounded support, its moments characterize it uniquely, which implies that ν_λ converges weakly to ρ_{sc} (See Theorem 30.2 in [3]).

2.2 Stieltjes transform proof

For later use, recall from [15, pg. 225] the notion of a spectral measure ν_ϕ, of a self-adjoint operator A, associated to a unit vector e_ϕ. Such a probability measure, ν_ϕ, can be defined by finding the unique measure satisfying

$$\int_{\mathbb{R}} f(x)\nu_\phi(dx) = \langle e_\phi, f(A)e_\phi\rangle$$

for all bounded, continuous f.

Using spectral theory and exchangeability, [4] argued that the mean of the random measure $\nu_{n,\lambda}$ can be regarded as the expected spectral measure at vertex 1 (or any other fixed vertex) of the Erdős-Rényi graph $G(n,\lambda/n)$ (with weights $1/\sqrt{\lambda}$ on the edges). Moreover, the limiting deterministic measure ν_λ is the expected spectral measure associated to the root of a Galton-Watson tree with offspring distribution Pois(λ) and weights $1/\sqrt{\lambda}$, which is the limit of $\{G(n,\lambda/n)\}$ with weighted edges in the Benjamini-Schramm topology (see also [5, 13]). The adjacency operator $\frac{1}{\sqrt{\lambda}}M_\infty^{(\lambda)}$ of the limiting graph is self-adjoint ([13, Lemma 5.2]) and its resolvent $R^{(\lambda)}$ is well-defined. Letting ϕ denote the root of the tree and e_ϕ denote the root vector, i.e. a Kronecker-delta function at the root, define the random variable

$$R_{\phi,\phi}^{(\lambda)}(z) := \left\langle e_\phi, \left(\frac{1}{\sqrt{\lambda}}M_\infty^{(\lambda)} - zI\right)^{-1} e_\phi \right\rangle$$

where the domain of z is $\mathbb{C}\backslash\mathbb{R}$.

Let S_λ be the Stieltjes transform of the limiting distribution ν_λ. According to [4, Thm 2],

$$R_{\phi,\phi}^{(\lambda)}(z) \overset{d}{=} -\left[\frac{1}{z+\frac{1}{\lambda}\sum_{k=1}^{\mathrm{Pois}(\lambda)} R_{k,k}^{(\lambda)}(z)}\right] \tag{2.6}$$

where $(R_{k,k}^{(\lambda)}(z))_{k\in\mathbb{N}}$ is an i.i.d. sequence with the same distribution as $R_{\phi,\phi}^{(\lambda)}(z)$ and Pois(λ) is a Poisson random variable independent from $(R_{k,k}^{(\lambda)}(z))_{k\in\mathbb{N}}$. Thus,

$$S_\lambda(z) = \mathbb{E}R_{\phi,\phi}^{(\lambda)}(z).$$

The strategy of the proof is to show that

$$S(z) := \lim_{\lambda\to\infty} S_\lambda(z)$$

exists for all $z\in\mathbb{C}\backslash\mathbb{R}$ and satisfies the self-consistent equation,

$$S(z) = -\frac{1}{z+S(z)} \tag{2.7}$$

implying that $S(z) = -\frac{1}{2}(z - \sqrt{z^2-4})$ by choosing the solution of (2.7) such that the imaginary parts of $S(z)$ and z are the same. By the Stieltjes inversion formula, ν_λ converges weakly to ρ_{sc}, the standard semicircle law, as $\lambda \to \infty$.

Let us now carry out the above strategy. Define Y_λ and f_λ as follows.

$$Y_\lambda := \frac{1}{\lambda} \sum_{k=1}^{\mathrm{Pois}(\lambda)} R_{k,k}^{(\lambda)}(z) \text{ and } f_\lambda(\theta) := \mathbb{E} \exp\left(i\theta R_{\phi,\phi}^{(\lambda)}(z)\right)$$

so that

$$\begin{aligned}
\mathbb{E}\exp(i\theta Y_\lambda) &= \mathbb{E}\big[\big\{f_\lambda(\tfrac{\theta}{\lambda})\big\}^{\mathrm{Pois}(\lambda)}\big] = \mathbb{E}\big[\exp[\mathrm{Pois}(\lambda)\log f_\lambda(\tfrac{\theta}{\lambda})]\big] \\
&= \exp\big[\lambda\big(e^{\log f_\lambda(\frac{\theta}{\lambda})} - 1\big)\big] = \exp\big[\lambda\big(f_\lambda(\tfrac{\theta}{\lambda}) - 1\big)\big] \\
&= \exp\big[\lambda\big(\tfrac{i\theta}{\lambda}\mathbb{E}R_{\phi,\phi}^{(\lambda)}(z) + o(\tfrac{1}{\lambda})\big)\big] \qquad (2.8)
\end{aligned}$$

where $o(\frac{1}{\lambda})$ depends on θ. The last equality in (2.8) comes from the Taylor expansion of the characteristic function f which is possible since we have the a.s. bound

$$\left|R_{\phi,\phi}^{(\lambda)}(z)\right| \le \int \left|\frac{1}{x-z}\right| d\nu_\phi \le \frac{1}{|\mathrm{Im}(z)|}. \qquad (2.9)$$

Choose a subsequence $\lambda_n \to \infty$ such that a limit $S(z)$ exists. Eq. (2.8) tells us that

$$\frac{1}{\lambda_n} \sum_{k=1}^{\mathrm{Pois}(\lambda_n)} R_{k,k}^{(\lambda_n)}(z) \xrightarrow{pr} S(z) \text{ as } n \to \infty,$$

by convergence of the characteristic functions of $\{Y_{\lambda_n}, \lambda_n > 0\}$, and the fact that the limit is a constant. Next, suppose without loss of generality that $z \in \mathbb{C}^+$. Then

$$\Im(S(z) + z) \ge \Im(z) > 0$$

which implies $S(z) \neq z$. By the continuous mapping theorem,

$$-\frac{1}{z + \frac{1}{\lambda_n}\sum_{k=1}^{\mathrm{Pois}(\lambda_n)} R_{k,k}^{(\lambda_n)}(z)} \xrightarrow{pr} -\frac{1}{z + S(z)}.$$

By (2.6), the left-hand side above has the same distribution as $R_{\phi,\phi}^{(\lambda_n)}(z)$ which by (2.9) is bounded for any fixed $z \in \mathbb{C}\backslash\mathbb{R}$. Thus

$$\lim_{n\to\infty} S_{\lambda_n}(z) = \lim_{n\to\infty} \mathbb{E}R_{\phi,\phi}^{(\lambda_n)}(z) = -\frac{1}{z + S(z)}.$$

Therefore $S(z)$ satisfies (2.7) and must be the Stieltjes transform of the semicircle law. The proof follows since the measures $\{\nu_\lambda\}$ are tight, while the above argument shows that there is a unique limit point.

3 Delocalization

Recall that $\{\Lambda_i(H)\}_{i=1}^n$ and $\{u_i(H)\}_{i=1}^n$ denote the eigenvalues and eigenvectors of a symmetric $n \times n$ matrix H, respectively. We begin with several lemmas, the first of which is Eq. (5.8) in [12]. We state the version from [18, Lemma 41]:

Lemma 3.1 (Lemma 41 in [18])**.**
Let

$$H = \begin{pmatrix} a & X^T \\ X & \tilde{H} \end{pmatrix}$$

be an $n \times n$ *symmetric matrix for some* $a \in \mathbb{R}$ *and* $X \in \mathbb{R}^{n-1}$*, and let* $\begin{pmatrix} x \\ v \end{pmatrix}$ *be the unit eigenvector with eigenvalue* $\Lambda_i(H)$ *where* $x \in \mathbb{R}$ *and* $v \in \mathbb{R}^{n-1}$*. Assume that none of the eigenvalues of* $\tilde{H}$ *are equal to* $\Lambda_i(H)$*. Then,*

$$|x|^2 = \frac{1}{1 + \sum_{\alpha=1}^{n-1} (\Lambda_\alpha(\tilde{H}) - \Lambda_i(H))^{-2} |\langle u_\alpha(\tilde{H}), X \rangle|^2}$$

where $\langle \cdot, \cdot \rangle$ *denotes the inner product between vectors.*

The second lemma is a consequence of Talagrand's inequality that was proved in Lemma 68 of [18]. We state the version from [19, Lemma 3.4]:

Lemma 3.2 (Lemma 3.4 in [19])**.** *Let* $Y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ *be a random vector whose coordinates are i.i.d. centered random variables which are a.s. bounded in absolute value by 1 and have variance* σ^2*. Let* $\mathcal{H}$ *be a subspace of dimension* k *and* $\pi_{\mathcal{H}}$ *the orthogonal projection onto* $\mathcal{H}$*. Then,*

$$\mathbb{P}\left(\Big| \, \|\pi_{\mathcal{H}}(Y)\| - \sigma\sqrt{k} \, \Big| \geq t \right) \leq 10 \exp(-t^2/4)$$

where $\|\cdot\|$ *is the Euclidean norm.*

Remark 3.3. In the above lemma, if the coordinates of Y are Gaussian, then similar inequalities are simply obtained. First, $\|\pi_{\mathcal{H}}(Y)\|$ has the same distribution as $\sum_{i=1}^k y_i^2$ by rotation invariance. Since $\sum_{i=1}^k y_i^2$ has a chi-square distribution, one can then use (7.50) in [14] or alternatively use the Chernoff bound.

Let N_n be a symmetric $n \times n$ matrix whose upper triangular elements are independent standard normal variables $N(i,j)$. Note that even though the perturbed matrix elements are unbounded, we have that

$$\mathbb{P}[|N(i,j)| > \sqrt{n}] \leq Ce^{-\frac{n}{4}}.$$

As $\sum_n n^2 e^{-\frac{n}{4}} < \infty$, by Borel-Cantelli we have that $|N(i,j)| \leq \sqrt{n}$ a.s. for all n large enough and all $1 \leq i, j, \leq n$. This will allow us to use Lemma 3.2 later on.

Assume that N_n is also independent from $M_{n,\lambda/n}$. Let $\{\delta(n)\}_{n \in \mathbb{N}}$ be a sequence of positive numbers satisfying

$$\delta(n) = o(n^{-1/2}). \tag{3.1}$$

Denote the scaled adjacency matrix and a perturbed version of it as follows:

$$A_{n,\lambda} := \frac{1}{\sqrt{\lambda}} M_{n,\lambda/n} \,,$$
$$B_{n,\lambda} := A_{n,\lambda} + \delta(n) N_n \,.$$

The reason for introducing the perturbed matrix is that it almost surely has a simple spectrum (see [17, Exercise 1.3.10]):

$$\Lambda_1(B_{n,\lambda}) < \Lambda_2(B_{n,\lambda}) < \cdots < \Lambda_n(B_{n,\lambda}) \text{ almost surely.} \tag{3.2}$$

Write $B_{n,\lambda}$ in the following matrix form:

$$B_{n,\lambda} = \begin{pmatrix} a & X^T \\ X & \tilde{B}_{n,\lambda} \end{pmatrix} \text{ where } a \in \mathbb{R} \text{ and } X \in \mathbb{R}^{n-1}. \tag{3.3}$$

Then,

$$\{\Lambda_i(B_{n,\lambda}) : i = 1, 2, \cdots, n\} \cap \{\Lambda_\alpha(\tilde{B}_{n,\lambda}) : \alpha = 1, 2, \cdots, n-1\} = \emptyset \text{ almost surely} \tag{3.4}$$

by (3.2) and the Cauchy interlacing principle. Note that (3.4) allows us to use Lemma 3.1. Our third preliminary lemma bounds the effect of the above perturbation on infinity norms of eigenvectors:

Lemma 3.4 (Lemma 3.1 in [19]).
Recall that $B_{n,\lambda}$ is defined as the perturbation $A_{n,\lambda}+\delta(n)N_n$. There exists an orthonormal basis of eigenvectors $\{u_i(A_{n,\lambda})\}_{i=1}^n$ such that, for every $1 \le i \le n$,

$$\|u_i(A_{n,\lambda})\|_\infty \le \|u_i(B_{n,\lambda})\|_\infty + \alpha(n) \tag{3.5}$$

where $\alpha(n) \to 0$ as $n \to \infty$, and $\alpha(n)$ can be chosen to be arbitrarily small depending only on $\delta(n)$.

Henceforth assume $u_i(A_{n,\lambda}) = u_i(M_{n,\lambda/n})$ for all i and n and that the orthonormal basis $\{u_i(A_{n,\lambda})\}_{i=1}^n$ satisfies (3.5).

Lemma 3.5. *Let $\tilde{\mu}_{n,\lambda}$ be the empirical spectral distribution of $\tilde{B}_{n,\lambda}$. Then, for $a < b$, and $\delta(n)$ satisfying* (3.1)*,*

$$\limsup_{\lambda\to\infty}\limsup_{n\to\infty} |\nu_\lambda([a,b]) - \tilde{\mu}_{n,\lambda}([a,b])| = 0, \text{ almost surely.}$$

The above lemma follows simply from Theorem 1.1 and Weyl's inequality; however, for completeness, we provide an explicit proof in Appendix A.1.

3.1 Proof of Theorem 1.3

By Lemma 3.4, it is sufficient to show that the conclusion of Theorem 1.3 holds when $u_i(M_{n,\lambda/n})$ is replaced with $u_i(B_{n,\lambda})$. Let $\tilde{U}$ be defined by

$$\tilde{U}(n,\lambda,\epsilon) := \{i \in \{1,2,\cdots,n\} : \|u_i(B_{n,\lambda})\|_\infty < \epsilon/2\}.$$

Our goal is to prove

$$\liminf_{\lambda\to\infty}\liminf_{n\to\infty} \frac{|\tilde{U}(n,\lambda,\epsilon)|}{n} = 1 \quad \text{almost surely.}$$

For this, it suffices to show

$$\limsup_{\lambda\to\infty}\limsup_{n\to\infty} \|u_i(B_{n,\lambda})\|_\infty < \frac{\epsilon}{2} \quad \text{almost surely,} \tag{3.6}$$

uniformly for all $\Lambda_i(B_{n,\lambda}) \in [-2,2]$, since by Lemma 3.5

$$\liminf_{\lambda\to\infty}\liminf_{n\to\infty} \frac{|\{i \in \{1,2,\cdots,n\} : \Lambda_i(B_{n,\lambda}) \in [-2,2]\}|}{n} = 1 \quad \text{almost surely.} \tag{3.7}$$

By (3.4), we can apply Lemma 3.1 to get

$$|x|^2 = \frac{1}{1+\sum_{\alpha=1}^{n-1}(\Lambda_\alpha(\tilde{B}_{n,\lambda}) - \Lambda_i(B_{n,\lambda}))^{-2}\left|\left\langle u_\alpha(\tilde{B}_{n,\lambda}), X\right\rangle\right|^2} \tag{3.8}$$

where $x = u_i(1)$ is the first coordinate of $u_i(B_{n,\lambda})$. A similar bound holds for any other coordinate $u_i(k)$ of $u_i(B_{n,\lambda})$ by replacing $\tilde{B}_{n,\lambda}$ with an appropriate submatrix. Thus, we will see that it suffices to find an upper bound of $|x|^2$, with high enough probability, in order to get an upper bound for $\|u_i(B_{n,\lambda})\|_\infty$, uniformly in i with high probability.

Let Q be a positive integer and set

$$l := 4/Q. \tag{3.9}$$

Choose Q large enough so that $Q \geq 5$ and

$$\frac{1}{1 + 1/(\pi\sqrt{3l})} < \frac{\epsilon^2}{4}. \tag{3.10}$$

We fix this value of Q (thus fixing l) henceforth and note that they only depend on ϵ.

Partition the interval $[-2, 2]$ into $\{[a_q, a_{q+1}]\}_{q=1}^{Q}$ so that $a_1 = -2$, $a_{Q+1} = 2$ and $a_{q+1} - a_q = l$ for every q. Define a subset J of $\{1, 2, \cdots, n-1\}$ as

$$J(n, q) := \{\alpha \in \{1, 2, \cdots, n-1\} : \Lambda_\alpha(\tilde{B}_{n,\lambda}) \in [a_q, a_{q+1}]\}$$

and define $\mathcal{H} = \mathcal{H}(n, q) := \mathrm{span}_{\alpha \in J(n,q)}\{u_\alpha(\tilde{B}_{n,\lambda})\}$. Suppose that $\Lambda_i(B_{n,\lambda}) \in [-2, 2]$ so that there is a q_i such that $\Lambda_i(B_{n,\lambda}) \in [a_{q_i}, a_{q_i+1}]$. For this q_i,

$$J(n, q_i) \subset \{\alpha \in \{1, 2, \cdots, n-1\} : |\Lambda_\alpha(\tilde{B}_{n,\lambda}) - \Lambda_i(B_{n,\lambda})| \leq l\}.$$

If $\pi_{\mathcal{H}}$ denotes the orthogonal projection onto $\mathcal{H}$, then we have for $\mathcal{H}_i = \mathcal{H}(n, q_i)$,

$$\begin{aligned} \sum_{\alpha=1}^{n-1} (\Lambda_\alpha(\tilde{B}_{n,\lambda}) - \Lambda_i(B_{n,\lambda}))^{-2} \left|\left\langle u_\alpha(\tilde{B}_{n,\lambda}), X \right\rangle\right|^2 &\geq \frac{1}{l^2} \sum_{\alpha \in J(n,q_i)} \left|\left\langle u_\alpha(\tilde{B}_{n,\lambda}), X \right\rangle\right|^2 \\ &= \frac{1}{l^2} \|\pi_{\mathcal{H}_i}(X)\|^2 \\ &\geq \frac{1}{l^2} \min_{q \in \{1,2,\cdots,Q\}} \|\pi_{\mathcal{H}(n,q)}(X)\|^2 \end{aligned} \tag{3.11}$$

Note that our choice of q_i depends on $B_{n,\lambda}$ and thus it is random (with its randomness inherited from $B_{n,\lambda}$). The same is true for the random subspace $\mathcal{H}_i$. On the other hand, for fixed q, $\mathcal{H}(n, q)$ is a random subspace depending only on $\tilde{B}_{n,\lambda}$, by definition, thus it is independent of X.

Now, define a random vector Y from the vector X which is as in (3.3)

$$Y := X - \frac{\sqrt{\lambda}}{n}\mathbf{1}(n-1) \;\text{ where }\; \mathbf{1}(n) = (1, 1, \cdots, 1) \in \mathbb{R}^n.$$

Let $\mathcal{H}'$ be the orthogonal complement of $\mathrm{span}\{\mathbf{1}(n-1)\}$. Then, for generic q and $\mathcal{H} = \mathcal{H}(n, q)$,

$$\|\pi_{\mathcal{H}}(X)\| \geq \|\pi_{\mathcal{H}\cap\mathcal{H}'}(X)\| = \|\pi_{\mathcal{H}\cap\mathcal{H}'}(Y)\|.$$

Observe in particular, that $\dim(\mathcal{H} \cap \mathcal{H}') \geq \dim(\mathcal{H}) - 1$. Since $\tilde{B}_{n,\lambda}$ is independent of Y, Lemma 3.2 can be applied with

$$t = t(n) = \sqrt{\sqrt{n} \cdot \log n}$$

after conditioning on $\tilde{B}_{n,\lambda}$, and also after normalizing Y so that $\sigma = 1$. Thus with probability at least $1 - 10\exp(-(\sqrt{n} \cdot \log n)/4)$,

$$\left\|\pi_{\mathcal{H}\cap\mathcal{H}'}\left(\left(\frac{1 - \lambda/n}{n} + \delta^2\right)^{-\frac{1}{2}} \cdot Y\right)\right\| \geq \sqrt{|J(n, q)| - 1} - \sqrt{\sqrt{n} \cdot \log n}. \tag{3.12}$$

The Borel-Cantelli lemma implies that the inequality (3.12) holds almost surely for large n and for every subspace $\mathcal{H}(n, q)$ with $q \in [1, Q]$, so in particular it holds for $\mathcal{H}(n, q_i)$. Plugging (3.12) into (3.11), and recalling that $\delta(n) = o(n^{-1/2})$, we have, almost surely,

$$\liminf_{n\to\infty} \sum_{\alpha=1}^{n-1} (\Lambda_\alpha(\tilde{B}_{n,\lambda}) - \Lambda_i(B_{n,\lambda}))^{-2} \left|\left\langle u_\alpha(\tilde{B}_{n,\lambda}), X \right\rangle\right|^2 \geq \liminf_{n\to\infty} \min_{q \in \{1,2,\cdots,Q\}} \frac{|J(n, q)|}{n \cdot l^2} \tag{3.13}$$

Recall that $\tilde{\mu}_{n,\lambda}$ is the empirical spectral distribution of $\tilde{B}_{n,\lambda}$. Fix $q \in \{1, 2, \cdots, Q\}$ and note that

$$\frac{|J(n,q)|}{n-1} = \tilde{\mu}_{n,\lambda}([a_q, a_{q+1}]). \tag{3.14}$$

Applying Theorem 1.1 and Lemma 3.5 to

$$\begin{aligned}\tilde{\mu}_{n,\lambda}([a_q, a_{q+1}]) \geq\ & \rho_{sc}([a_q, a_{q+1}]) - |\rho_{sc}([a_q, a_{q+1}]) - \nu_\lambda([a_q, a_{q+1}])| \\ & - |\nu_\lambda([a_q, a_{q+1}]) - \tilde{\mu}_{n,\lambda}([a_q, a_{q+1}])|\end{aligned} \tag{3.15}$$

and using the calculation in Appendix A.2, we have almost surely,

$$\begin{aligned}\liminf_{\lambda\to\infty} \liminf_{n\to\infty} \min_{q\in\{1,2,\cdots,Q\}} \frac{|J(n,q)|}{n-1} &\geq \min_{q\in\{1,2,\cdots,Q\}} \rho_{sc}\big([a_q, a_{q+1}]\big) \\ &\geq \frac{l^{3/2}}{\pi\sqrt{3}}\,.\end{aligned} \tag{3.16}$$

Combining (3.8), (3.10), (3.13) and (3.16) we get that $|x| < \epsilon/2$ almost surely for large n and large λ under the assumption $\Lambda_i(B_{n,\lambda}) \in [-2, 2]$.

Finally, recall that a relation similar to (3.8) holds for any other coordinate $u_i(k)$ of $u_i(B_{n,\lambda})$ and so using a union bound over $k \in \{1, \ldots, n\}$, and noting that $\sum_n 10n \exp(-(\sqrt{n} \cdot \log n)/4) < \infty$ (in order to invoke the Borel-Cantelli lemma for a union of probabilities), we obtain (3.6). This completes the proof.

3.2 Proof of Corollary 1.5

From now on, we let the expected degree depend on n, i.e., $\lambda = \lambda_n$. Recall that in contrast to Theorem 1.4 where the growth condition (1.1) is required, we consider the more general case where

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Recall that $M_n := M_{n,\lambda_n/n}$. Also, let $\nu_n := \nu_{n,\lambda_n}$. According to Theorem 1.3 in [19], the empirical spectral measure ν_n weakly converges to the standard semicircle distribution ρ_{sc} as n goes to infinity. We can use the same argument as in the proof of Theorem 1.3 up until (3.14). After that, set $\mu_n := \mu_{n,\lambda_n}$ and $\tilde{\mu}_n := \tilde{\mu}_{n,\lambda_n}$ and use the following inequality instead of (3.15):

$$\tilde{\mu}_n([a_q, a_{q+1}]) \geq\ \rho_{sc}([a_q, a_{q+1}]) - |\rho_{sc}([a_q, a_{q+1}]) - \tilde{\mu}_n([a_q, a_{q+1}])|\,.$$

By the absolute continuity of ρ_{sc}, and the argument in Appendix A.1, we have

$$\limsup_{n\to\infty} |\rho_{sc}([a_q, a_{q+1}]) - \tilde{\mu}_n([a_q, a_{q+1}])| = 0.$$

Consequently,

$$\liminf_{n\to\infty} \min_{q\in\{1,2,\cdots,Q\}} \frac{|J(n,q)|}{n \cdot l^2} \geq \min_{q\in\{1,2,\cdots,Q\}} \frac{\rho_{sc}\big([a_q, a_{q+1}]\big)}{l^2} \geq \frac{1}{\pi\sqrt{3l}}\,.$$

Since $\liminf_{n\to\infty} \mu_n([-2, 2]) = 1$, the result follows.

While Corollary 1.5 has the advantage of holding without any growth rate condition on λ_n, it has the drawback that it give no information about the infinity norms of eigenvectors corresponding to the eigenvalues outside of $[-2, 2]$. Note that $[-2, 2]$ corresponds to the support of the standard semicircle law.

A Some additional tools

A.1 Proof of Lemma 3.5

With some abuse of notation, write $\tilde{B}_{n,\lambda} = A_{n-1,\lambda} + \delta(n)N_{n-1}$. Then, Weyl's theorem implies

$$|\Lambda_i(\tilde{B}_{n,\lambda}) - \Lambda_i(A_{n-1,\lambda})| \le \delta(n)\|N_{n-1}\|_{\text{op}} = O(\delta(n)\sqrt{n}) \text{ for all } 1 \le i \le n$$

since $\{(n)^{-1/2}N_n\}_{n\in\mathbb{N}}$ is a Wigner ensemble with moments of all order. Using (3.1), Weyl's inequality, and the Cauchy interlacing theorem, there is a sequence $\lim_{n\to\infty}\zeta_n = 0$ such that

$$\nu_{n-1,\lambda}([a+\zeta_n, b-\zeta_n]) \le \tilde{\mu}_{n,\lambda}([a,b]) \le \nu_{n-1,\lambda}([a-\zeta_n, b+\zeta_n]).$$

Note that

$$\limsup_{n\to\infty} \nu_{n,\lambda}([a-\zeta_n, b+\zeta_n]) \le \lim_{\xi\downarrow 0}\limsup_{n\to\infty} \nu_{n,\lambda}([a-\xi, b+\xi]).$$

For fixed $\xi > 0$, choose continuous functions f_ξ and g_ξ which converge pointwisely to $\mathbf{1}_{[a,b]}$, as $\xi \to \infty$, and which satisfy

$$0 \le f_\xi \le \mathbf{1}_{[a-\xi,b+\xi]} \le g_\xi \le 1.$$

Then, almost surely,

$$\int f_\xi \, d\nu_\lambda = \lim_{n\to\infty}\int f_\xi \, d\nu_{n,\lambda} \le \limsup_{n\to\infty}\nu_{n,\lambda}([a-\xi, b+\xi]) \le \lim_{n\to\infty}\int g_\xi \, d\nu_{n,\lambda} = \int g_\xi \, d\nu_\lambda \ .$$

We deduce that

$$\limsup_{n\to\infty} |\nu_\lambda([a,b]) - \tilde{\mu}_{n,\lambda}([a,b])| = \nu_\lambda(\{a\}) + \nu_\lambda(\{b\}) \ .$$

Finally by Theorem 1.1, both $\nu_\lambda(\{a\})$ and $\nu_\lambda(\{b\})$ go to 0 as $\lambda \to \infty$.

A.2 A simple bound for the semicircle edge

Recall from (3.9) that $l = 4/Q$. Here we will show that when $[-2,2]$ is partitioned into Q equal parts,

$$\min_{q\in\{1,2,\cdots,Q\}} \rho_{sc}\big([a_q, a_{q+1}]\big) \ge \frac{l^{3/2}}{\pi\sqrt{3}}.$$

Observe that

$$\min_{q\in\{1,2,\cdots,Q\}} \rho_{sc}\big([a_q, a_{q+1}]\big) = \frac{1}{2\pi}\int_{2-l}^{2} \sqrt{4-x^2}\, dx.$$

Since $4 - x^2 = (2+x)(2-x)$, we have for $l < 1$,

$$\frac{1}{2\pi}\int_{2-l}^{2} \sqrt{4-x^2}\, dx \ge \frac{\sqrt{3}}{2\pi}\int_0^l \sqrt{x}\, dx = \frac{l^{3/2}}{\pi\sqrt{3}}.$$

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