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Explicit formula for the density of local times of Markov jump processes

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Abstract

In this note we show a simple formula for the joint density of local times, last exit tree and cycling numbers of continuous-time Markov chains on finite graphs, which involves the modified Bessel function of the first type.

Keywords: Markov jump process; density of local times; last exit trees; cycling numbers; modified Bessel function.
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1 Introduction

Let $\mathcal{G} = (V, E, \sim)$ be an unoriented, connected finite graph with no multiple edges, equipped with positive edge conductances $\{W_e,\, e \in E\}$. For any $i \in V$, let $W_i := \sum_{j\sim i} W_{ij}$.

Consider the associated Markov jump process $(X_t)_{t\geq 0}$ on V, that is the continuous-time discrete-space random walk which jumps from a vertex $i \in V$ to a neighbor j at rate $W_{ij} = W_{ji}$, i.e. with generator

$$\mathcal{L}f(i) = \sum_{j\in V: j\sim i} W_{ij}(f(j) - f(i)), \text{ for any } i \in V.$$

Let $\vec{E} = \{ij : \{i, j\} \in E\}$ be the set of directed edges, where each undirected edge in E is replaced by two directed edges with opposite directions. For any oriented spanning tree $\vec{T}$, we call its root the unique site from which no edge goes out. We denote by $\delta_i(j) = \mathbf{1}\{i = j\}$ the Kronecker delta.

Let $\mathcal{I}$ be the set of currents on the graph, i.e.

$$\mathcal{I} = \{a \in \mathbb{Z}^{\vec{E}} \,:\, a_{ji} = -a_{ij},\, i,\, j\, \in V\}.$$

For any $a \in \mathbb{Z}^{\vec{E}}$ and $i \in V$, we let $a_i = \sum_{j\sim i} a_{ij}$. If $a \in \mathcal{I}$, then a_i can be interpreted as the divergence of a at site i.

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For any $k \in \mathbb{N}^{\vec{E}}$, let $a(k) \in \mathcal{I}$ be defined by $a(k)_{ij} = k_{ij} - k_{ji}$. For any $a \in \mathcal{I}$ and any oriented spanning tree T of $\mathcal{G}$, let $\tilde{a} \in \mathbb{Z}^E$ be defined by

$$\tilde{a}_{ij} = a_{ij} - \mathbf{1}\{ij \in \vec{T}\} + \mathbf{1}\{ji \in \vec{T}\},\ ij \in E.$$

Set similarly $\tilde{a}_i = \sum_{j\sim i} \tilde{a}_{ij}$ for $i \in V$.

For any $\sigma > 0$ and any right-continuous path $x = (x(t))_{t\geq 0}$, let us define $\ell(x,\sigma) \in \mathbb{R}_+^V$ as the vector of local times at time σ, that is,

$$\ell(x,\sigma)_i = \int_0^\sigma \mathbf{1}\{x(s) = i\}\, ds,\ i \in V.$$

Let us also define $k(x,\sigma) = (k_{ij}(x,\sigma))_{(i,j)\in\vec{E}}$ the vector of oriented crossings up to time σ, that is,

$$k_{ij}(x,\sigma) = |\{t \leq \sigma : x(t-) = i, x(t) = j\}|. \tag{1.1}$$

Let $\vec{T}(x,\sigma)$ be the last-exit tree of the path x on the interval $[0,\sigma]$, that is, the collection of directed edges taken by path x for the last departures from all vertices visited in that time interval except the endpoint $x(\sigma)$. In other words, $(i,j) \in \vec{T}(x,\sigma)$ iff there exists $t \in (0,\sigma]$ such that $(x(t-), x(t)) = (i,j)$ and $x(s) \neq i$ for every $s \geq t$. One can easily check that $\vec{T}(x,\sigma)$ is indeed a tree.

For any $\nu \in \mathbb{R}$, the modified Bessel function of the first kind is defined by

$$I_\nu(z) = \sum_{k=0}^\infty \frac{1}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu},\ z \in \mathbb{R}. \tag{1.2}$$

Recall that $I_\nu(z) = I_{-\nu}(z)$. Therefore, for any $a \in \mathcal{I}$, any unoriented edge $e = \{i,j\} \in E$ and $z \in \mathbb{R}$, we have that $I_{\tilde{a}_{ij}}(z) = I_{\tilde{a}_{ji}}(z)$, which we denote by $I_{\tilde{a}_e}(z)$.

Define the simplex

$$\mathcal{L}_\sigma := \left\{\ell \in (0,\infty)^V : \sum_{i\in V} \ell_i = \sigma\right\}, \quad \sigma > 0\,, \tag{1.3}$$

and the Lebesgue measure m_σ on $\mathcal{L}_\sigma$.

The main result of this note is the following.

Theorem 1.1. *Let i_0, $i_1 \in V$, $\sigma > 0$, $A \subseteq \mathcal{L}_\sigma$ Lebesgue-measurable, let $\vec{T}$ be an oriented spanning tree of the graph with root i_1, and let $a \in \mathcal{I}$ be such that $a_i = \delta_{i_0}(i) - \delta_{i_1}(i)$ for all $i \in V$. Then*

$$\begin{aligned}&\mathbb{P}_{i_0}\Big(a(k(X,\sigma)) = a,\, \ell(X,\sigma) \in A,\, \vec{T}(X,\sigma) = \vec{T}\Big)\\ &= \int_A e^{-\sum_{i\in V} W_i\ell_i}\Big(\prod_{\{i,j\}\in E} I_{\tilde{a}_{ij}}\Big(2W_{ij}\sqrt{\ell_i\ell_j}\Big)\Big)\Big(\prod_{ij\in\vec{T}} W_{ij}\Big)\Big(\prod_{i\in V}\ell_i^{\tilde{a}_i/2}\Big) m_\sigma(d\ell)\,.\end{aligned}$$

Remark 1.2. It is easy to extend Theorem 1.1 to the case where σ is a stopping time

$$\sigma = \sigma_h^b = \inf\{t \geq 0 :\ \ell(X,t)_b > h\}, \quad h > 0,\, b \in V,$$

in which case the density is replaced by

$$\begin{aligned}&\mathbb{P}_{i_0}\Big(a(k(X,\sigma)) = a,\, \ell(X,\sigma)_{V\setminus\{b\}} \in (\ell, \ell + d\ell),\, \vec{T}(X,\sigma) = \vec{T}\Big)\\ &= \mathbf{1}_{\{\ell_b = h\}} e^{-\sum_{i\in V} W_i\ell_i} \prod_{\{i,j\}\in E} I_{\tilde{a}_{ij}}\Big(2W_{ij}\sqrt{\ell_i\ell_j}\Big) \prod_{ij\in\vec{T}} W_{ij} \prod_{i\in V} \ell_i^{\tilde{a}_i/2} \prod_{i\in V\setminus\{b\}} d\ell_i\,,\end{aligned} \tag{1.4}$$

since we impose $\ell_b = h$.

In the case where $\mathcal{G} = \mathbb{T}$ is a finite tree, further simplifications can be made. Indeed, consider the Markov jump process at time $\sigma_h^{i_0}$: if $\ell(X, \sigma_h^{i_0})_i > 0$ for all $i \in V$, then the only possible last exit tree $\vec{T}(X, \sigma_h^{i_0})$ is the tree $\mathbb{T}$ itself, oriented towards i_0. Moreover, $a(k(X, \sigma_h^{i_0})) = \mathbf{0}$. Hence $\mathbb{P}_{i_0}(\ell(X, \sigma_h^{i_0})_{V \setminus \{i_0\}} \in (\ell, \ell + d\ell))$ equals

$$\mathbf{1}_{\{\ell_{i_0}=h\}} e^{-\sum_{i\in V} W_i \ell_i} \Big(\prod_{\{i,j\}\in E} W_{ij} I_1 \left(2W_{ij} \sqrt{\ell_i \ell_j} \right) \Big) \Big(\prod_{ij\in\vec{T}} \sqrt{\frac{\ell_j}{\ell_i}} \Big) \prod_{i\in V\setminus\{i_0\}} d\ell_i \,. \tag{1.5}$$

This is the Markov jump process analogue of the second Ray-Knight Theorem [9] that relates the local times of Brownian motion on $\mathbb{R}$ at time σ_h^0 to zero-dimensional squared Bessel process. We give a more precise statement (and proof) in Section 3, Corollary 3.1.

Remark 1.3. One could obtain a formula for the density of the local times alone by summing the formula obtained in Theorem 1.1 over all possible spanning trees and cycling numbers. Note that for any fixed oriented spanning tree $\vec{T}$, any arbitrary choice of cycling numbers on $\vec{E} \setminus \vec{T}$ can be uniquely extended to cycling numbers on the whole graph via a linear map.

Explicit formulas for the joint density of local times of continous-time Markov chains were already proposed, see for instance [1, 2, 4, 7, 6, 10, 11]. In [7], the author obtained non rigorously asymptotic formulae of some Markov paths. Brydges, van der Hofstad and König [2] provide an abstract and rather involved formula for the density of the local times of Markov jump processes. Our result shows that, once we consider the local times together with the last-exit tree and the cycling numbers, one obtains a simple and tractable formula that could be used in practice. It would be interesting to understand if there is a direct link between the two apparently different formulae. Let us also mention that an extensive literature survey is provided in [2].

Merkl, Rolles and Tarrès proposed in [10] a formula for the joint density of the oriented edge crossings, local times and last-exit tree for the Vertex-Reinforced Jump Process on a general graph, whose counterpart in the context of continuous-time Markov chains is simple and stated in Proposition 2.1 below.

Le Jan independently obtained in Theorem 4.1 [6], in the context of loop soups $\mathcal{L}_1$ with intensity 1, an expression for the joint density of the cycling numbers and local time, which also involves the first modified Bessel function. We can deduce that result from the construction of those loop soups by Wilson's algorithm, in the following manner.

Let us first quickly recall that algorithm: we order all the sites of our finite graph $V = \{j_1, \ldots, j_{|V|}\}$, and we assume that the walk is transient with cemetery Δ. We start a loop-erased Markov chain $\{\eta_1\}$ starting from j_1 and ending at Δ, where $\{\eta_1\}$ denotes the set of vertices visited by this self-avoiding path. Then, from the next vertex in $V \setminus \{\eta_1\}$ we start a loop-erased Markov chain $\{\eta_2\}$ ending in $\{\eta_1\} \cup \{\Delta\}$, and so on. The union of all η_i is a spanning tree, whose leaves are starting sites of the successive loop-erased chains.

Given a fixed spanning tree T, we can easily obtain a formula similar to the one in Theorem 1.1 for the joint density of the succession of Markov chains starting successively at all leaves of T with respect to the order on sites given above and killed at cemetery Δ. Now the loop soup extracted from that succession of Markov chains by Wilson's algorithm has the same local time at all sites (see for instance Chapter 8 [5]), its cycling numbers are $k = \tilde{a}$ after extraction of the spanning tree; $\tilde{a}$ satisfies $\tilde{a}_i = 0$ for all $i \in V$, so that the term $\prod_{i\in V} \ell_i^{\tilde{a}_i/2}$ in the density is 1. Summing $\prod_{\{i,j\}\in T} W_{ij}$ over all spanning trees of G yields a determinant by matrix-tree theorem, which enables to deduce Theorem 4.1 [6].

2 Proof of Theorem 1.1

We first show the following Proposition 2.1. Its proof relies on an argument similar to the proof of Theorem 1.6 in [10]; the technique for determining the cardinality of the set of paths with given last exit tree is from Lemma 6 by Keane and Rolles in [3].

Proposition 2.1. *Let i_0, $i_1 \in V$, $\sigma > 0$, $A \subseteq \mathcal{L}_\sigma$ measurable, let $\vec{T}$ be an oriented spanning tree of the graph with root i_1, and let $k \in \mathbb{N}^{\vec{E}}$ be such that $a(k)_i = \delta_{i_0}(i) - \delta_{i_1}(i)$ for all $i \in V$. Then*

$$\begin{aligned}&\mathbb{P}_{i_0}\Big(k(X,\sigma) = k,\, \ell(X,\sigma) \in A,\, \vec{T}(X,\sigma) = \vec{T}\Big)\\ &= \int_A e^{-\sum_{i\in V} W_i\ell_i}\Big(\prod_{ij\in\vec{E}} \frac{(W_{ij}\ell_i)^{k_{ij}}}{k_{ij}!}\Big)\Big(\prod_{ij\in\vec{T}} \frac{k_{ij}}{\ell_i}\Big) m_\sigma(d\ell)\,. \end{aligned} \tag{2.1}$$

Proof. It follows from a simple argument (similar but simpler than Lemma 1 in [3]) that, for any $k \in \mathbb{N}^{\vec{E}}$ such that $a(k) = \delta_{i_0} - \delta_{i_1}$, there exists a path from i_0 to i_1 realizing the edge crossings prescribed by k. Consider first adding to the event on the l.h.s. of (2.1) the additional requirement that $(X_t)_{0\le t\le\sigma}$ takes such a given path $\gamma = \{\gamma_0 = i_0, \gamma_1, ..., \gamma_{n-1}, \gamma_n = i_1\}$. The probability turns out to be independent of such choice and is equal to

$$\int_A \prod_{i\in V}\Big(W_i^{k_i} e^{-W_i\ell_i} \prod_{j\sim i}\Big(\frac{W_{ij}}{W_i}\Big)^{k_{ij}}\Big)\mathcal{V}(k,\ell,i_1) m_\sigma(d\ell)\,, \tag{2.2}$$

where $\mathcal{V}(k,\ell,i_1)$ denotes the volume factor associated to the choice of jump times out of each vertex within its given local time:

$$\mathcal{V}(k,\ell,i_1) := \Big(\prod_{i\neq i_1} \frac{\ell_i^{k_i-1}}{(k_i-1)!}\Big)\Big(\frac{\ell_{i_1}^{k_{i_1}}}{k_{i_1}!}\Big)\,. \tag{2.3}$$

It remains to count the number of all possible paths γ that start at i_0 and end at i_1, while respecting the fixed last exit tree $\vec{T}$ and edge crossing numbers $k(x,\sigma)$. This number equals the number of relative orders of exiting each vertex and follows a multinomial distribution:

$$\prod_{i\in V} \frac{(k_i - \mathbf{1}\{i\neq i_1\})!}{\prod_{j\sim i}(k_{ij} - \mathbf{1}\{ij\in\vec{T}\})!}\,. \tag{2.4}$$

Multiplying (2.4) to (2.2), and simplifying, yields the proposition. □

Let us now prove Theorem 1.1. Let $a \in \mathcal{I}$ be such that $a_i = \delta_{i_0}(i) - \delta_{i_1}(i)$ for all $i \in V$. For each unoriented edge $e \in E$, let us choose a unique orientation $\vec{e} = ij$ with $e = \{i,j\}$ such that $a_{ij} \geq 0$, and let $E^+ = \{\vec{e} :\ e \in E\}$.

In order to compute the probability considered in the statement of the theorem, we need to sum all the contributions from Proposition 2.1 for all $k \in \mathbb{N}^{\vec{E}}$ such that $a(k) = a$. For each $ij \in E^+$, we sum over all $k_{ji} \geq 0$ and k_{ij} is determined by $k_{ij} = k_{ji} + a_{ij} \geq k_{ji}$. Therefore, using Proposition 2.1, recalling that $\ell_i > 0$ for every $i \in V$ and joining the contributions from ij and ji for each $ij \in E^+$, we have

$$\begin{aligned}&\mathbb{P}_{i_0}\Big(a(k(X,\sigma)) = a,\, \ell(X,\sigma) \in A,\, \vec{T}(X,\sigma) = \vec{T}\Big)\\ &= \int_A e^{-\sum_{i\in V} W_i\ell_i} \sum_{(k_{ji})\in\mathbb{N}^{E^+}} \Big(\prod_{ij\in E^+} \frac{(W_{ij}\ell_j)^{k_{ji}}}{k_{ji}!}\frac{(W_{ij}\ell_i)^{k_{ij}}}{k_{ij}!} \prod_{ij\in\vec{T}} \frac{k_{ij}}{\ell_i}\Big) m_\sigma(d\ell)\\ &= \int_A e^{-\sum_{i\in V} W_i\ell_i} \prod_{ij\in E^+}\Big(\sum_{k_{ji}\geq \mathbf{1}\{ji\in\vec{T}\}} \frac{W_{ij}^{k_{ij}+k_{ji}}\ell_i^{k_{ij}-\mathbf{1}\{ij\in\vec{T}\}}\ell_j^{k_{ji}-\mathbf{1}\{ji\in\vec{T}\}}}{(k_{ij}-\mathbf{1}\{ij\in\vec{T}\})!(k_{ji}-\mathbf{1}\{ji\in\vec{T}\})!}\Big) m_\sigma(d\ell)\,.\end{aligned}$$

In the second equality we use that, if $ji \in \vec{T}$, then the summand is 0 iff $k_{ji} = 0$.

Let $k'_{ji} = k_{ji} - \mathbf{1}\{ji \in \vec{T}\}$. Then

$$\begin{aligned} k_{ij} - \mathbf{1}\{ij \in \vec{T}\} &= k'_{ji} + \tilde{a}_{ij} \\ k_{ij} + k_{ji} &= 2k'_{ji} + \tilde{a}_{ij} + \mathbf{1}\{\{i,j\} \in T\} \end{aligned}$$

where T is the unoriented tree associated to $\vec{T}$, so that

$$\begin{aligned} &\sum_{k_{ji} \geq \mathbf{1}\{ji \in \vec{T}\}} \frac{W_{ij}^{k_{ij}+k_{ji}} \ell_i^{k_{ij}-\mathbf{1}\{ij\in\vec{T}\}} \ell_j^{k_{ji}-\mathbf{1}\{ji\in\vec{T}\}}}{(k_{ij} - \mathbf{1}\{ij \in \vec{T}\})!(k_{ji} - \mathbf{1}\{ji \in \vec{T}\})!} \\ &= \sum_{k'_{ji} \geq 0} \frac{W_{ij}^{2k'_{ji}+\tilde{a}_{ij}+\mathbf{1}\{\{i,j\}\in T\}} \ell_i^{k'_{ji}+\tilde{a}_{ij}} \ell_j^{k'_{ji}}}{(k'_{ji} + \tilde{a}_{ij})!(k'_{ji})!} \\ &= W_{ij}^{\mathbf{1}\{\{i,j\}\in T\}} \ell_i^{\tilde{a}_{ij}/2} \ell_j^{\tilde{a}_{ji}/2} \sum_{k'_{ji} \geq 0} \frac{(W_{ij}\sqrt{\ell_i \ell_j})^{2k'_{ji}+\tilde{a}_{ij}}}{(k'_{ji} + \tilde{a}_{ij})!(k'_{ji})!} \\ &= W_{ij}^{\mathbf{1}\{\{i,j\}\in T\}} \ell_i^{\tilde{a}_{ij}/2} \ell_j^{\tilde{a}_{ji}/2} I_{\tilde{a}_{ij}}(2W_{ij}\sqrt{\ell_i \ell_j}) \,. \end{aligned}$$

We conclude the proof by the observation that

$$\prod_{ij \in E^+} \ell_i^{\tilde{a}_{ij}/2} \ell_j^{\tilde{a}_{ji}/2} = \prod_{i \in V} \prod_{j:\{i,j\} \in E} \ell_i^{\tilde{a}_{ij}/2} = \prod_{i \in V} \ell_i^{\tilde{a}_i/2},$$

as can be seen by splitting the product over positively and negatively oriented edges.

3 Link with Ray-Knight theorems

In this section, we derive Ray-Knight identities as a simple corollary of our results. The statements are similar to those of Theorem 4.1 in [2], which are a generalization of results from [8].

In [8], the proof of (a part of) these statements uses the Brownian Ray-Knight Theorems, and in [2] the proof is done by applying an involved formula for the density of the local times of Markov jump processes, giving rise to a rather long proof. Our proof is a simple application of Theorem 1.1, which becomes very simple on trees, as highlighted in Remark 1.2. Recall that we defined

$$\sigma_h^b = \inf\{t \geq 0 : \ell(X,t)_b > h\}.$$

Corollary 3.1 (Ray-Knight Theorem for continuous-time simple random walk)**.** *Let* $\mathcal{G} = \mathbb{Z}$*, let* $W_{ij} = 1$ *for any* $\{i,j\} \in E$*, let* $b \in \mathbb{Z}^+ \setminus \{0\}$ *and let* $h > 0$*. Let us consider* X *the simple random walk on* $\mathbb{Z}$*, started at* 0*.*

(i) *The process* $(\ell(X,\sigma_h^b)_{b-x})_{x=0,\dots,b}$ *is a time-homogeneous discrete-time Markov chain on* $(0,\infty)$*, starting at* h *and with transition density given by*

$$f(h_1,h_2) = e^{-h_1-h_2} I_0(2\sqrt{h_1 h_2}), \qquad h_1,h_2 > 0.$$

(ii) *The processes* $(\ell(X,\sigma_h^b)_{b+x})_{x\geq 0}$ *and* $(\ell(X,\sigma_h^b)_{-x})_{x\geq 0}$ *are time-homogeneous discrete-time Markov chains on* $[0,\infty)$*, with transition probability given by*

$$P(h_1,dh_2) = e^{-h_1}\delta_0(dh_2) + e^{-h_1-h_2}\sqrt{\frac{h_1}{h_2}} I_1(2\sqrt{h_1 h_2})dh_2, \qquad h_1,h_2 \geq 0.$$

(iii) *The Markov chains above are independent.*

Proof. The statement (iii) is a simple consequence of the Markovian structure.

To prove (i), one can proceed by applying the formula of Theorem 1.1 inductively, for $x \in \{0, \dots, b-1\}$, on graphs consisting of a single edge $\{b-x-1, b-x\}$, started at $b-x-1$ until the stopping time $\sigma_{h_1}^{b-x}$. In that case, $\tilde{a}_{(b-x-1,b-x)} = 0$ and the density in (1.4) becomes

$$\mathbf{1}_{\{\ell_{b-x}=h_1\}} e^{-h_1-h_2} I_0\left(2\sqrt{h_1 h_2}\right),$$

which proves (i).

Let us now prove (ii). We will write the proof for $(\ell(X, \sigma_h^b)_{b+x})_{x\geq 0}$, but this trivially translates to $(\ell(X, \sigma_h^b)_{-x})_{x\geq 0}$.

Note that, if $\ell_{b+x} = h_1$, for some $x \geq 0$, and if the walk starts and ends on the left of $b+x$, then the walker could jump or not jump to $b+x+1$. The walker does not jump to $b+x+1$ with probability e^{-h_1}. If the walker jumps to $b+x+1$, then we can apply the formula of Theorem 1.1 on the graph consisting of the single edge $\{b+x, b+x+1\}$, with starting and ending point $b+x$, until the stopping time $\sigma_{h_1}^{b+x}$. In this case, $\tilde{a}_{(b+x,b+x+1)} = 1$, $\tilde{a}_{b+x} = -\tilde{a}_{b+x+1} = 1$ and the density in (1.5) becomes

$$\mathbf{1}_{\{\ell_{b+x}=h_1\}} e^{-h_1-h_2} I_1\left(2\sqrt{h_1 h_2}\right)\sqrt{\frac{h_1}{h_2}},$$

which proves (ii). □

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