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On the supremum of products of symmetric stable processes

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Abstract

We study the asymptotics, for small and large values, of the supremum of a product of symmetric stable processes. We show in particular that the lower tail exponent remains the same as for only one process, possibly up to some logarithmic terms. The proof relies on a path construction of stable bridges using last sign changes.

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1 Introduction

For $n \in \mathbb{N}$, let $(Z^{(i)}, 1 \leq i \leq n)$ be independent symmetric α-stable Lévy processes with $\alpha \in (0, 2]$. In this short note, we are interested in the study of the random variable

$$\mathcal{S}_n = \sup_{0 \leq u \leq 1} \prod_{i=1}^n Z_u^{(i)}.$$

Except when $n = 1$, in which case the double Laplace transform of $\mathcal{S}_1$ is classically given by fluctuation theory (see for instance Bertoin [4, p.174]), it does not seem evident to compute explicitly the law of $\mathcal{S}_n$, and we shall rather study its asymptotics $\mathbb{P}(\mathcal{S}_n \geq x)$ as $x \to +\infty$ and $\mathbb{P}(\mathcal{S}_n \leq \varepsilon)$ as $\varepsilon \to 0$.

Most of the paper is devoted to the computation of the limit as $\varepsilon \to 0$, which is known as a lower tail problem, see for instance [9]. By scaling, this problem is equivalent to a persistence problem (see the surveys [1, 6]) and amounts to the study of the first entrance time of the n-dimensional stable process $(Z^{(i)}, 1 \leq i \leq n)$ into the "hyperbolic" domain $\mathcal{H}_n = \{(z_1, \ldots, z_n) \in \mathbb{R}^n,\ \prod_{i=1}^n z_i \geq 1\}$:

$$\mathbb{P}(\mathcal{S}_n \leq \varepsilon) = \mathbb{P}\left(R_n > \frac{1}{\varepsilon^{\frac{\alpha}{n}}}\right) \qquad \text{where} \qquad R_n = \inf\left\{u \geq 0,\ \prod_{i=1}^n Z_u^{(i)} \geq 1\right\}.$$

There are several papers in the literature dealing with entrance and exit times of symmetric stable processes, mainly for three families of domains : cones and wedges (Bañuelos and Bogdan [2], Méndez-Hernández [11]), parabolic domains (Bañuelos and Bogdan [3]) and unbounded convex domains (Méndez-Hernández [10]). Here, since the

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domain $\mathcal{H}_n$ is non-connected, not much is known regarding R_n and we shall tackle the problem directly by working with $\mathcal{S}_n$.

In the following, for any real functions f and g we will use the standard notation $f(x) \asymp g(x)$ as $x \to +\infty$ to express the fact that there exist two positive finite constants κ_1 and κ_2 such that $\kappa_1 f(x) \le g(x) \le \kappa_2 f(x)$ as $x \to +\infty$.

We start with the Brownian case, i.e. $\alpha = 2$.

Theorem 1.1. *Let $(Z^{(i)}, 1 \le i \le n)$ be independent Brownian motions.*

1. *Large deviations : there is the asymptotics*

$$\mathbb{P}\left(\sup_{0\le u\le 1} \prod_{i=1}^n Z_u^{(i)} \ge x\right) \asymp x^{-\frac{1}{n}} \exp\left(-\frac{n}{2} x^{\frac{2}{n}}\right) \qquad (x \to +\infty)$$

2. *Lower tail probability : there exist two constants $0 < \kappa_1 \le \kappa_2 < +\infty$ such that*

$$\kappa_1\, \varepsilon \le \mathbb{P}\left(\sup_{0\le u\le 1} \prod_{i=1}^n Z_u^{(i)} \le \varepsilon\right) \le \kappa_2\, \varepsilon\, |\ln(\varepsilon)|^n \qquad (\varepsilon \to 0)$$

In the non-Gaussian stable case, the situation is different.

Theorem 1.2. *Let $(Z^{(i)}, 1 \le i \le n)$ be independent symmetric α-stable Lévy processes with $\alpha \in (0, 2)$.*

1. *Large deviations : there is the asymptotics*

$$\mathbb{P}\left(\sup_{0\le u\le 1} \prod_{i=1}^n Z_u^{(i)} \ge x\right) \asymp \frac{(\ln(x))^{n-1}}{x^\alpha} \qquad (x \to +\infty)$$

2. *Lower tail probability : there exist two constants $0 < \kappa_1 \le \kappa_2 < +\infty$ such that*

$$\kappa_1\, \varepsilon^{\alpha/2} \le \mathbb{P}\left(\sup_{0\le u\le 1} \prod_{i=1}^n Z_u^{(i)} \le \varepsilon\right) \le \kappa_2\, \varepsilon^{\alpha/2}\, |\ln(\varepsilon)| \qquad (\varepsilon \to 0)$$

The main part of the proof deals with the computation of an upper bound for the lower tail probabilities. As can be seen, the exponent is the same as for only one process, possibly up to logarithmic terms. Therefore, a simple approach would be to try to bound the quantity $\mathcal{S}_n$ by $\prod_{i=1}^n Z_{\theta_1}^{(i)}$ where θ_1 is the value at which one of the Lévy processes, say $Z^{(n)}$, reach its maximum on $[0, 1]$. This yields of course two main difficulties.

1. First, the product of the other processes $\prod_{i=1}^{n-1} Z_{\theta_1}^{(i)}$ might not be positive. This can be however easily circumvented thanks to Slepian's inequality, since the processes are symmetric.
2. The second difficulty is less obvious and is due to the arcsine law for stable processes. There is a high probability that θ_1 will be close to 0, hence, although $Z_{\theta_1}^{(n)}$ will be large, the remaining product $\prod_{i=1}^{n-1} Z_{\theta_1}^{(i)}$ will also be close to zero, thus not providing us with a good upper bound.

The general idea of the proof will be to decompose the path of the processes $(Z^{(i)})$ at some last passage times and then use a time reversal argument, so as to find a value not to close to the origin at which $Z^{(n)}$ is large enough.

The presence of logarithmic terms in the upper bound of the lower tail probability in the Brownian case is due to some additive phenomenons. Indeed, as suggested above, our upper bound will involve expressions of the form $\sup_{0\le u\le 1} Z_u^{(n)} \prod_{i=1}^{n-1} |Z_1^{(i)}|$. Then, recalling the estimates (see Bertoin [4, p.219]):

$$\mathbb{P}(|Z_1| \le \varepsilon) \underset{\varepsilon\to 0}{\sim} k\,\varepsilon \qquad \text{and} \qquad \mathbb{P}\left(\sup_{0\le u\le 1} Z_u \le \varepsilon\right) \underset{\varepsilon\to 0}{\sim} c\,\varepsilon^{\frac{\alpha}{2}} \tag{1.1}$$

for some positive constants k and c, we see that when $\alpha < 2$, the second asymptotics will be the leading one, while for $\alpha = 2$, they will be of the same order, and some compensations will appear, see Lemma 2.1 below.

The outline of the paper is as follows : the large deviation results are proved in Section 2, the lower tail probabilities in Section 3, and finally Section 4 provides the proof of an intermediary lemma.

2 Large deviations

The proof of the large deviations result relies on the symmetry of the processes $(Z^{(i)})$, and on the fact that the asymptotics of both random variables $|Z_1|$ and $\sup_{0\le u\le 1} Z_u$ are similar. Indeed, on the one hand, the lower bound is easily given by the use of symmetry:

$$\mathbb{P}(\mathcal{S}_n \ge x) \ge \mathbb{P}\left(\prod_{i=1}^n Z_1^{(i)} \ge x\right) = \frac{1}{2}\mathbb{P}\left(\prod_{i=1}^n |Z_1^{(i)}| \ge x\right).$$

On the other hand, still by symmetry,

$$\begin{aligned}
\mathbb{P}(\mathcal{S}_n \ge x) &\le \mathbb{P}\left(\sup_{0\le u\le 1} Z_u^{(n)} \sup_{0\le s\le 1} \prod_{i=1}^{n-1} Z_s^{(i)} \ge x\right) + \mathbb{P}\left(\inf_{0\le u\le 1} Z_u^{(n)} \inf_{0\le s\le 1} \prod_{i=1}^{n-1} Z_s^{(i)} \ge x\right)\\
&= 2\,\mathbb{P}\left(\sup_{0\le u\le 1} Z_u^{(n)} \sup_{0\le s\le 1} \prod_{i=1}^{n-1} Z_s^{(i)} \ge x\right)\\
&\le 2^n\,\mathbb{P}\left(\prod_{i=1}^{n} \sup_{0\le u\le 1} Z_u^{(i)} \ge x\right) \qquad \text{(by iteration).}
\end{aligned}$$

It remains thus to compute the involved quantities in both cases.

• In the Brownian case, since $\sup_{0\le u\le 1} Z_u \overset{\text{(law)}}{=} |Z_1|$, we deduce that the asymptotics of $\mathcal{S}_n$ is given by that of $\prod_{i=1}^n |Z_1^{(i)}|$. Its Mellin transform reads, for $\nu > -1$:

$$\mathbb{E}\left[\prod_{i=1}^n |Z_1^{(i)}|^\nu\right] = \left(\frac{2^\nu}{\pi}\right)^{\frac{n}{2}} \left(\Gamma\left(\frac{1+\nu}{2}\right)\right)^n. \tag{2.1}$$

The converse mapping theorem, see Janson [8, Theorem 6.1], yields :

$$\mathbb{P}\left(\prod_{i=1}^n |Z_1^{(i)}| \in dx\right)/dx \underset{x\to+\infty}{\sim} \kappa\, x^{\frac{1}{n}-1}\, e^{-\frac{n}{2}x^{\frac{2}{n}}}$$

for some positive constant κ. The result then follows by integration, using the asymptotics of the incomplete Gamma function.

• Next, when $\alpha \in (0,2)$, it is known from Bertoin [4, p.221] that there exists $k > 0$ such that

$$\mathbb{P}(|Z_1| \geq x) \underset{x\to+\infty}{\sim} \frac{2k}{x^\alpha} \qquad \text{and} \qquad \mathbb{P}\left(\sup_{0\leq u\leq 1} Z_u \geq x\right) \underset{x\to+\infty}{\sim} \frac{k}{x^\alpha}. \tag{2.2}$$

Point 1. of Theorem 1.2 is then consequence of the following lemma (see for instance Lemma 2.2 in Profeta-Simon [12]):

Lemma 2.1. *Let X and Y be two independent positive random variables satisfying the asymptotics :*

$$\mathbb{P}(X \geq z) \underset{z\to+\infty}{\asymp} \frac{(\ln(z))^n}{z^\nu} \qquad \textit{and} \qquad \mathbb{P}(Y \geq z) \underset{z\to+\infty}{\asymp} \frac{(\ln(z))^p}{z^\mu}$$

where $n, p \in \mathbb{N}$ and ν, μ are positive constants such that $0 < \nu \leq \mu$. Then it holds :

$$\mathbb{P}(XY \geq z) \underset{z\to+\infty}{\asymp} \begin{cases} z^{-\nu}(\ln(z))^n & \text{if } \nu < \mu \\ z^{-\nu}(\ln(z))^{n+p+1} & \text{if } \nu = \mu. \end{cases}$$

□

3 Lower tail probabilities

We now turn our attention to the lower tail estimates and start with some notations. Let X be a symmetric stable process. We denote by $\mathbb{P}_x$ the probability measure of X when started from $x \in \mathbb{R}$, with the usual convention that $\mathbb{P} = \mathbb{P}_0$. Let T_0 be the first time that X takes a negative value :

$$T_0 = \inf\{t \geq 0, X_t \leq 0\}.$$

We recall from Bertoin [4, p.219] that since X is symmetric, there exists $c > 0$ such that

$$\mathbb{P}_1\left(T_0 \geq t\right) \underset{t\to+\infty}{\sim} \frac{c}{\sqrt{t}}. \tag{3.1}$$

Finally, let us introduce the last change of sign of X before time $t > 0$:

$$g_t = \sup\left\{0 \leq u \leq t,\ X_u X_{u-} \leq 0\right\}.$$

This random time will be the key to the computation of the lower tail probabilities.

Remark 3.1. In the following, when applying the Markov property, $\widehat{X}$ will always denote an independent copy of X. Besides, we shall use the notations c and κ to denote positive constants that may change from line to line.

We first show that the asymptotics of the distribution of g_1 is similar to that of the arcsine law.

Lemma 3.2. *There exists a positive constant c such that*

$$\mathbb{P}(g_1 \in dr)/dr \underset{r\to 0}{\sim} \frac{c}{\sqrt{r}}.$$

Proof. We first have, using the symmetry of X and applying the Markov property with $r \in (0,1)$:

$$\mathbb{P}(g_1 \leq r) = \mathbb{E}\left[\widehat{\mathbb{P}}_{|X_r|}\left(\widehat{T}_0 \geq 1-r\right)\right].$$

By scaling, this is further equal to

$$\mathbb{P}(g_1 \leq r) = \mathbb{E}\left[\widehat{\mathbb{P}}_1\left(\widehat{T}_0 \geq \frac{1-r}{r|X_1|^\alpha}\right)\right].$$

Recall now from Doney-Savov [7] that under $\mathbb{P}_1$, the random variable T_0 admits a continuous density h satisfying $h(z) \underset{z\to+\infty}{\sim} \kappa\, z^{-3/2}$ for some constant $\kappa > 0$. Therefore, differentiating, we deduce that

$$\mathbb{P}(g_1 \in dr)/dr = \frac{1}{r^2}\,\mathbb{E}\left[\frac{1}{|X_1|^\alpha} h\left(\frac{1-r}{r|X_1|^\alpha}\right)\right] \underset{r\to 0}{\sim} \frac{\kappa}{\sqrt{r}}\,\mathbb{E}\left[|X_1|^{\frac{\alpha}{2}}\right]$$

which is the announced result. □

3.1 Lower bound for the lower tail probabilities

Observe first that by scaling

$$\begin{aligned}\mathbb{P}(\mathcal{S}_n \le \varepsilon) &= \mathbb{P}\left(\sup_{u\in[0,\varepsilon^{-\alpha/n}]} \prod_{i=1}^n Z_u^{(i)} \le 1\right)\\ &\ge \mathbb{P}\left(\sup_{u\in[0,\varepsilon^{-\alpha/n}]} \prod_{i=1}^n Z_u^{(i)} \le 1,\ \prod_{i=1}^n Z_1^{(i)} \le 0,\ \sup_{1\le i\le n} g^{(i)}_{\varepsilon^{-\frac{\alpha}{n}}} \le 1\right)\\ &= \mathbb{P}\left(\sup_{u\in[0,1]} \prod_{i=1}^n Z_u^{(i)} \le 1,\ \prod_{i=1}^n Z_1^{(i)} \le 0,\ \sup_{1\le i\le n} g^{(i)}_{\varepsilon^{-\frac{\alpha}{n}}} \le 1\right)\end{aligned}$$

where the last equality follows from the fact that, by definition of the $(g^{(i)})$, the product $\prod_{i=1}^n Z^{(i)}$ remains negative after time 1. Applying the Markov property at time 1 and then the scaling property, we obtain :

$$\mathbb{P}(\mathcal{S}_n \le \varepsilon) \ge \mathbb{E}\left[\prod_{i=1}^n \widehat{\mathbb{P}}_{|Z_1^{(i)}|}\left(\widehat{T}_0^{(i)} \ge \frac{1}{\varepsilon^{\frac{\alpha}{n}}} - 1\right)\mathbb{1}_A\right] \ge \mathbb{E}\left[\prod_{i=1}^n \widehat{\mathbb{P}}_1\left(\widehat{T}_0^{(i)} \ge \frac{1}{\varepsilon^{\frac{\alpha}{n}}|Z_1^{(i)}|^\alpha}\right)\mathbb{1}_A\right] \tag{3.2}$$

where $A = \left\{\sup_{u\in[0,1]} \prod_{i=1}^n Z_u^{(i)} \le 1,\ \prod_{i=1}^n Z_1^{(i)} \le 0\right\}$. From (3.1), there exists $\kappa > 0$ such that for $\delta > 0$ small enough

$$\widehat{\mathbb{P}}_1\left(\widehat{T}_0^{(i)} \ge \frac{1}{\varepsilon^{\frac{\alpha}{n}}|Z_1^{(i)}|^\alpha}\right)\mathbb{1}_{\{\varepsilon^{\frac{\alpha}{n}}|Z_1^{(i)}|^\alpha \le \delta\}} \ge \kappa\,\varepsilon^{\frac{\alpha}{2n}}|Z_1^{(i)}|^{\frac{\alpha}{2}}\,\mathbb{1}_{\{\varepsilon^{\frac{\alpha}{n}}|Z_1^{(i)}|^\alpha \le \delta\}}.$$

Plugging this inequality in (3.2), we deduce that

$$\mathbb{P}(\mathcal{S}_n \le \varepsilon) \ge \kappa^n \varepsilon^{\frac{\alpha}{2}}\,\mathbb{E}\left[\prod_{i=1}^n |Z_1^{(i)}|^{\frac{\alpha}{2}}\mathbb{1}_{\{\varepsilon^{\frac{\alpha}{n}}|Z_1^{(i)}|^\alpha \le \delta\}}\mathbb{1}_A\right] \underset{\varepsilon\to 0}{\sim} \kappa^n\varepsilon^{\frac{\alpha}{2}}\,\mathbb{E}\left[\prod_{i=1}^n |Z_1^{(i)}|^{\frac{\alpha}{2}}\mathbb{1}_A\right]$$

which gives the lower bound.

3.2 Upper bound for the lower tail probabilities

We assume in this section that $n \ge 2$, since the bounds are known to hold for $n = 1$. Using the fact that the processes $(Z^{(i)})$ all have the same law, we first write :

$$\mathbb{P}(\mathcal{S}_n \le \varepsilon) = n\,\mathbb{P}\left(\sup_{0\le u\le 1} \prod_{i=1}^n Z_u^{(i)} \le \varepsilon,\ g_1^{(n)} \ge \sup_{1\le i\le n-1} g_1^{(i)}\right).$$

where we have used that $g_1^{(i)} \ne g_1^{(j)}$ a.s. for $i \ne j$ since the $(g^{(i)})$ are independent random variables having a density. To simplify the notation, we shall remove the superscript $^{(n)}$ and denote

$$X = Z^{(n)}, \qquad g_1 = g_1^{(n)} \qquad \text{and} \qquad \xi_t = \sup_{1\le i\le n-1} g_t^{(i)}.$$

This yields

$$\begin{aligned}\mathbb{P}\left(\mathcal{S}_n \le \varepsilon\right) &= n\,\mathbb{P}\left(\sup_{0\le u\le 1} X_u \prod_{i=1}^{n-1} Z_u^{(i)} \le \varepsilon,\ g_1 \ge \xi_1\right)\\ &\le n\,\mathbb{P}\left(\sup_{0\le u< g_1} X_u \prod_{i=1}^{n-1} Z_u^{(i)} \le \varepsilon,\ g_1 \ge \xi_1\right)\\ &= 2n\,\mathbb{P}\left(\sup_{0\le u<1} X_{ug_1} \prod_{i=1}^{n-1} Z_{ug_1}^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_{g_1}^{(i)} \ge 0,\ g_1 \ge \xi_1\right)\end{aligned}$$

where the last equality follows by symmetry since the time of the last change of sign of X and of $-X$ is the same. By scaling, we further obtain

$$\mathbb{P}\left(\mathcal{S}_n \le \varepsilon\right) \le 2n \int_0^1 \mathbb{P}\left(\sup_{0\le u<1} \frac{X_{ur}}{r^{1/\alpha}} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_u^{(i)} \le \varepsilon, \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0, 1 \ge \xi_{\frac{1}{r}} \,\middle|\, g_1 = r\right) \mathbb{P}(g_1 \in dr). \tag{3.3}$$

We set $X^{(x,t,y)}$ for the α-stable bridge of length t starting from x and ending at y. Notice that when X is a Brownian motion, then g_1 coincides with the last zero of X before time 1, so that $X_{g_1} = 0$ a.s. and it is well-known that the process $\left(\frac{X_{ug_1}}{\sqrt{g_1}},\ 0 \le u \le 1\right)$ is a standard Brownian bridge, independent of g_1, see Bertoin-Pitman [5]. We shall extend this result to the stable case in the following lemma, whose proof is postponed to the end of the paper.

Lemma 3.3. *We set by convention $X_{0-} = X_0$. Conditionally on the event $\left\{\frac{X_{g_1-}}{g_1^{1/\alpha}} = a\right\}$, the process*

$$\left(\frac{X_{ug_1-}}{g_1^{1/\alpha}},\ 0 \le u \le 1\right)$$

is independent of g_1 and has the same law as the stable bridge $\left(X_{u-}^{(0,1,a)}, 0 \le u \le 1\right)$.

Remark 3.4. When dealing with stable Lévy processes Z, there exist several similar results in the literature, for instance replacing g_1 by the last zero of Z before time 1 (assuming $\alpha > 1$, see Bertoin [4, p.230, Theorem 12]), or replacing g_1 by the last time that Z equals its minimum before time 1 (see Bertoin [4, p.230, Proposition 16]).

Let us denote by $\rho(da, dr)$ the law of the pair $\left(g_1^{-1/\alpha} X_{g_1-},\ g_1\right)$. Since the $(Z^{(i)})$ are quasi-left continuous and independent of X, we deduce from Lemma 3.3 that

$$\begin{aligned}\mathbb{P}\left(\mathcal{S}_n \le \varepsilon\right) &\le 2n \int_0^{+\infty}\int_0^1 \mathbb{P}\left(\sup_{0\le u\le 1} X_{u-}^{(0,1,a)} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_u^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0,\ 1 \ge \xi_{\frac{1}{r}}\right) \rho(da, dr)\\ &= 2n \int_0^{+\infty}\int_0^1 \mathbb{P}\left(\sup_{0\le u\le 1} X_u^{(a,1,0)} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0,\ 1 \ge \xi_{\frac{1}{r}}\right) \rho(da, dr)\end{aligned} \tag{3.4}$$

where the equality follows from the time-reversal property of stable bridges. We shall now decompose the right-hand side of this inequality according as $\{a \le 1\}$ or $\{a > 1\}$.

3.2.1 The case $\{a \le 1\}$

We start with the term giving the main contribution :

$$I_n(\varepsilon) := \int_0^1\int_0^1 \mathbb{P}\left(\sup_{0\le u\le 1} X_u^{(a,1,0)} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0,\ 1 \ge \xi_{\frac{1}{r}}\right) \rho(da, dr).$$

Let us denote by p_t the density of the random variable X_t, and recall that it is even, and decreasing on $(0,+\infty)$. Taking the supremum only on $[0,\frac{1}{2}]$, and using the change of measure formula for the stable bridge (see [4, p.229]), we get

$$\begin{aligned} I_n(\varepsilon) &\leq \int_0^1\int_0^1 \mathbb{P}\left(\sup_{0\leq u\leq 1/2} X_u^{(a,1,0)} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \leq \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right)\rho(da,dr) \\ &= \int_0^1\int_0^1 \mathbb{E}\left[\frac{p_{\frac{1}{2}}(a+X_{\frac{1}{2}})}{p_1(a)} \mathbb{1}_{\left\{\sup\limits_{0\leq u\leq 1/2}(a+X_u) r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)}\leq\varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)}\geq 0,\ 1\geq\xi_{\frac{1}{r}}\right\}}\right]\rho(da,dr) \\ &\leq \frac{p_{\frac{1}{2}}(0)}{p_1(1)} \int_0^1\int_0^1 \mathbb{P}\left(\sup_{0\leq u\leq 1/2}(a+X_u) r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \leq \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right)\rho(da,dr). \end{aligned} \tag{3.5}$$

We now study the integrand in (3.5). Recall that X admits the representation $(B_{\tau_u}, u\geq 0)$ where B is a standard Brownian motion and τ a stable subordinator with index $\frac{\alpha}{2}$ independent of B. Let us consider the conditional expectation :

$$\mathbb{P}\left(\sup_{0\leq u\leq 1/2}(a+B_{\eta_u}) r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} \omega_{1-u}^{(i)} \leq \varepsilon \,\middle|\, \begin{array}{ll} Z^{(i)} = \omega^{(i)}, & \tau = \eta \\ 1\leq i\leq n-1 & \end{array}\right)$$

where η and $(\omega^{(i)}, 1\leq i\leq n-1)$ are some fixed càdlàg paths. We apply Slepian's lemma with the Gaussian processes

$$U_u = a\prod_{i=1}^{n-1}\omega_{1-u}^{(i)} + B_{\eta_u}\prod_{i=1}^{n-1}\omega_{1-u}^{(i)} \qquad \text{and} \qquad V_u = a\prod_{i=1}^{n-1}\omega_{1-u}^{(i)} + B_{\eta_u}\prod_{i=1}^{n-1}|\omega_{1-u}^{(i)}|$$

which satisfy for every $0\leq u\leq s\leq \frac{1}{2}$,

$$\mathbb{E}[U_u] = \mathbb{E}[V_u], \qquad \mathbb{E}\left[U_u^2\right] = \mathbb{E}\left[V_u^2\right] \quad \text{and} \quad \mathbb{E}[U_uU_s] \leq \mathbb{E}[V_uV_s].$$

This yields, using the tower property of conditional expectations :

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq u\leq 1/2}(a+X_u) r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \leq \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right) \\ &\qquad \leq \mathbb{P}\left(\sup_{0\leq u\leq 1/2}\left(a\prod_{i=1}^{n-1} Z_{1-u}^{(i)} + X_u\prod_{i=1}^{n-1}|Z_{1-u}^{(i)}|\right) r^{\frac{n}{\alpha}} \leq \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right) \\ &\qquad = \mathbb{P}\left(\sup_{0\leq u\leq 1/2}\left(a\prod_{i=1}^{n-1} Z_{1-u}^{(i)} + X_u\prod_{i=1}^{n-1}|Z_{1-u}^{(i)}|\right) r^{\frac{n}{\alpha}} \leq \varepsilon,\ \varepsilon\geq a r^{\frac{n}{\alpha}}\prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right) \end{aligned}$$

where the equality follows from the fact that the additional condition $ar^{\frac{n}{\alpha}}\prod_{i=1}^{n-1} Z_1^{(i)}\leq\varepsilon$ already appears (implicitly) in the first condition by taking $u=0$. Then, denoting $\theta_{\frac{1}{2}} = \operatorname*{Argmax}_{0\leq u\leq 1/2} X_u$, we may replace the supremum by its value at $\theta_{\frac{1}{2}}$ to get the bound

$$\mathbb{P}\left(\left(a\prod_{i=1}^{n-1} Z_{1-\theta_{\frac{1}{2}}}^{(i)} + X_{\theta_{\frac{1}{2}}}\prod_{i=1}^{n-1}|Z_{1-\theta_{\frac{1}{2}}}^{(i)}|\right) r^{\frac{n}{\alpha}} \leq \varepsilon,\ \varepsilon\geq a\, r^{\frac{n}{\alpha}}\prod_{i=1}^{n-1} Z_1^{(i)} \geq 0,\ 1\geq \xi_{\frac{1}{r}}\right). \tag{3.6}$$

We further decompose this expression according to the sign of $\prod_{i=1}^{n-1} Z_{1-\theta_{\frac{1}{2}}}^{(i)}$.

1. When $\prod_{i=1}^{n-1} Z^{(i)}_{1-\theta_{\frac12}} \geq 0$, the expression (3.6) is smaller than

$$\mathbb{P}\left(X_{\theta_{\frac12}} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_{1-\theta_{\frac12}}| \leq \varepsilon,\ 1 \geq \xi_{\frac1r}\right) =: \Psi_n(r,\varepsilon). \tag{3.7}$$

2. When $\prod_{i=1}^{n-1} Z^{(i)}_{1-\theta_{\frac12}} \leq 0$, the situation is slightly more complex. Plugging the second condition in the first one, we have

$$\begin{aligned}
&\mathbb{P}\left((X_{\theta_{\frac12}} - a) r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_{1-\theta_{\frac12}}| \leq \varepsilon,\ \varepsilon \geq a\, r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z^{(i)}_1 \geq 0,\ 1 \geq \xi_{\frac1r}\right) \\
&\quad \leq \mathbb{P}\left(X_{\theta_{\frac12}} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_{1-\theta_{\frac12}}| \leq \varepsilon \left(1 + \frac{\prod_{i=1}^{n-1} |Z^{(i)}_{1-\theta_{\frac12}}|}{\prod_{i=1}^{n-1} Z^{(i)}_1}\right), \varepsilon \geq a r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z^{(i)}_1 \geq 0, 1 \geq \xi_{\frac1r}\right) \\
&\quad \leq \Psi_n(r, 2\varepsilon) + \mathbb{P}\left(X_{\theta_{\frac12}} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_1| \leq 2\varepsilon,\ 1 \geq \xi_{\frac1r}\right) =: \Psi_n(r,2\varepsilon) + \Phi_n(r,2\varepsilon)
\end{aligned}$$

where we have used in the last line the inequality : $1_{\{x \leq \varepsilon(a+b)\}} \leq 1_{\{x \leq 2a\varepsilon\}} + 1_{\{x \leq 2b\varepsilon\}}$ for $a, b \geq 0$. Going back to (3.5), we have thus proven that

$$I_n(\varepsilon) \leq \frac{p_{\frac12}(0)}{p_1(1)} \int_0^1 (2\Psi_n(r,2\varepsilon) + \Phi_n(r,2\varepsilon)) \mathbb{P}(g_1 \in dr)$$

and it remains to study the asymptotics of the right-hand side.

We start with $\Psi_n(r, 2\varepsilon)$ which will give the main contribution. From Lemma 3.2, we may choose $\delta \in (0,1)$ small enough such that

$$\forall r \leq \delta, \qquad \mathbb{P}(g_1 \in dr)/dr \leq \frac{c}{\sqrt{r}} \tag{3.8}$$

for some constant $c > 0$. On the one hand, when $r \geq \delta$, we obtain by scaling and using that $\theta_{\frac12} \leq \frac12$:

$$\begin{aligned}
\int_\delta^1 \Psi_n(r,2\varepsilon)\mathbb{P}(g_1 \in dr) &\leq \mathbb{P}\left(X_{\theta_{\frac12}} \delta^{n/\alpha} \left(1 - \theta_{\frac12}\right)^{\frac{n-1}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_1| \leq 2\varepsilon\right) \\
&\leq \mathbb{P}\left(X_{\theta_{\frac12}} \delta^{n/\alpha} \left(\frac12\right)^{\frac{n-1}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_1| \leq 2\varepsilon\right) \underset{\varepsilon \to 0}{\asymp} \begin{cases} \kappa\, \varepsilon^{\frac{\alpha}{2}} & \text{if } \alpha \in (0,2), \\ \kappa\, \varepsilon\, |\ln(\varepsilon)|^{n-1} & \text{if } \alpha = 2 \end{cases}
\end{aligned}$$

where the asymptotics follow from (1.1) and Lemma 2.1. On the other hand, when $r \leq \delta$, we deduce from the Markov property at time 1 and the scaling property that :

$$\Psi_n(r,2\varepsilon) = \mathbb{E}\left[\prod_{i=1}^{n-1} \widehat{\mathbb{P}}^{(i)}_{|Z^{(i)}_1|}\left(\widehat{T}^{(i)}_0 \geq \frac1r - 1\right) \mathbb{1}_A\right] \leq \mathbb{E}\left[\prod_{i=1}^{n-1} \widehat{\mathbb{P}}^{(i)}_1 \left(|Z^{(i)}_1|^\alpha \widehat{T}^{(i)}_0 \geq \frac{1-\delta}{r}\right) \mathbb{1}_A\right]$$

where $A = \left\{X_{\theta_{\frac12}} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z^{(i)}_{1-\theta_{\frac12}}| \leq 2\varepsilon\right\}$. Recall next that by the independent increments property of $Z^{(i)}$, for any fixed $u \geq 0$, the process $Y^{(i)} = (Z^{(i)}_{t+u} - Z^{(i)}_u, t \geq 0)$ is an α-stable Lévy process independent from Z_u. This yields the identity $Z^{(i)}_1 = Z^{(i)}_u + Y^{(i)}_{1-u}$ where on the right-hand side, Z_u and Y_{1-u} are independent. Replacing u by the independent random time $1-\theta_{\frac{1}{2}}$ and using the classic inequality $|x+y|^\alpha \leq 2(|x|^\alpha+|y|^\alpha)$ (since $\alpha\in(0,2]$) we deduce that

$$\Psi_n(r,2\varepsilon) \leq \mathbb{E}\left[\prod_{i=1}^{n-1} \widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_{1-\theta_{\frac{1}{2}}}|^\alpha + |Y^{(i)}_{\theta_{\frac{1}{2}}}|^\alpha\right)\widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r}\right)\mathbb{1}_A\right].$$

The scaling property of the processes $(Z^{(i)})$ and $(Y^{(i)})$ as well as the fact that $\theta_{\frac{1}{2}}\in[0,\frac{1}{2}]$ then yield

$$\Psi_n(r,2\varepsilon) \leq \mathbb{E}\left[\prod_{i=1}^{n-1} \widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha + |Y^{(i)}_1|^\alpha\right)\widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r}\right)\mathbb{1}_{B_{n-1}}\right] \tag{3.9}$$

where, for $1\leq k\leq n-1$:

$$B_k = \left\{X_{\theta_{\frac{1}{2}}}\, r^{\frac{n}{\alpha}}\left(\frac{1}{2}\right)^{\frac{n-1}{\alpha}}\prod_{i=1}^{k}|Z^{(i)}_1| \leq 2\varepsilon\right\}.$$

Next, using the inequalities

$$\widehat{\mathbb{P}}_1^{(i)}\left(|Z^{(i)}_1|^\alpha \geq \frac{1-\delta}{2r}\right) \leq \widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha + |Y^{(i)}_1|^\alpha\right) \geq \frac{1-\delta}{2r}\right) \leq 2\,\widehat{\mathbb{P}}_1^{(i)}\left(|Z^{(i)}_1|^\alpha \geq \frac{1-\delta}{4r}\right)$$

since $Y^{(i)}_1 \overset{\text{(law)}}{=} Z^{(i)}_1$, we deduce from (2.2) that when $\alpha\in(0,2)$,

$$\widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha + |Y^{(i)}_1|^\alpha\right) \geq \frac{1-\delta}{2r}\right) \underset{r\to 0}{\asymp} r.$$

Therefore, from Lemma 2.1 and the asymptotics (3.1), we may choose δ small enough such that

$$\forall r\leq\delta, \qquad \mathbb{E}\left[\widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha\vee 1 + |Y^{(i)}_1|^\alpha\right)\widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r}\right)\right] \leq \kappa\sqrt{r} \tag{3.10}$$

for some positive constant κ (independent of i), and where $a\vee b=\max(a,b)$. Note that (3.10) is also valid for $\alpha=2$ since Brownian motion admits moments of all order. We shall now proceed by iteration starting from (3.9).

1. If $|Z^{(n-1)}_1|\geq 1$, then, we may remove $|Z^{(n-1)}_1|$ from the product in B_{n-1}, and deduce from (3.10) that $\Psi_n(r,2\varepsilon)$ is smaller than

$$\kappa\sqrt{r}\,\mathbb{E}\left[\prod_{i=1}^{n-2} \widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha + |Y^{(i)}_1|^\alpha\right)\widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r}\right)\mathbb{1}_{B_{n-2}}\right].$$

2. If $|Z^{(n-1)}_1|\leq 1$, then, we may replace $|Z^{(n-1)}_1|$ by 1 in the first product in (3.9), and deduce, still from (3.10), that $\Psi_n(r,2\varepsilon)$ is smaller than

$$\kappa\sqrt{r}\,\mathbb{E}\left[\prod_{i=1}^{n-2} \widehat{\mathbb{P}}_1^{(i)}\left(\left(|Z^{(i)}_1|^\alpha + |Y^{(i)}_1|^\alpha\right)\widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r}\right)\mathbb{1}_{B_{n-1}}\right].$$

Iterating the procedure, we obtain that $\Psi_n(r,2\varepsilon)$ may be bounded by a sum of 2^{n-1} terms:

$$\Psi_n(r,2\varepsilon) \leq \kappa\, r^{\frac{n-1}{2}} \sum_{\Delta\subset\{1,\dots,n-1\}} \mathbb{P}\left(X_{\theta_{\frac{1}{2}}}\, r^{\frac{n}{\alpha}}\left(\frac{1}{2}\right)^{\frac{n-1}{\alpha}}\prod_{i\in\Delta}|Z^{(i)}_1| \leq 2\varepsilon\right)$$

where the sum is taken over all the subsets of $\{1, 2, \ldots, n-1\}$ (including the empty set). The change of variable $\varepsilon x = r^{\frac{n}{\alpha}}$ and the estimate (3.8) yield then the upper bound

$$\int_0^\delta \Psi_n(r, 2\varepsilon)\mathbb{P}(g_1 \in dr) \leq \kappa\varepsilon^{\frac{\alpha}{2}} \sum_{\Delta \subset \{1,\ldots,n-1\}} \int_0^{\frac{\delta^{n/\alpha}}{\varepsilon}} \mathbb{P}\left(X_{\theta_{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{n-1}{\alpha}} \prod_{i\in\Delta} |Z_1^{(i)}| x \leq 2 \right) x^{\frac{\alpha}{2}-1} dx \tag{3.11}$$

and it remains to study the asymptotics of the integrands. From Lemma 2.1 and (1.1), we deduce that

1. when $\alpha \in (0, 2)$ all the terms have the same contribution :

$$\mathbb{P}\left(X_{\theta_{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{n-1}{\alpha}} \prod_{i\in\Delta} |Z_1^{(i)}| x \leq 2 \right) \underset{x\to+\infty}{\asymp} c\left(\frac{1}{x}\right)^{\alpha/2}$$

2. while, for $\alpha = 2$, they depend on the cardinality of Δ :

$$\mathbb{P}\left(X_{\theta_{\frac{1}{2}}} \left(\frac{1}{2}\right)^{\frac{n-1}{\alpha}} \prod_{i\in\Delta} |Z_1^{(i)}| x \leq 2 \right) \underset{x\to+\infty}{\asymp} c\,\frac{(\ln(x))^{|\Delta|}}{x}.$$

Plugging these expressions in (3.11) finally gives the announced upper bound.

The study of the asymptotics of $\Phi_n(r, 2\varepsilon)$ follows the same pattern of proof, except that we do not need to introduce the random variables $(Y_1^{(i)})$. Indeed, when $r \geq \delta$, we get the same asymptotics bound while for $r \leq \delta$ we obtain, applying the Markov property:

$$\Phi_n(r, 2\varepsilon) \leq \mathbb{E}\left[\prod_{i=1}^{n-1} \widehat{\mathbb{P}}_1^{(i)} \left(|Z_1^{(i)}|^\alpha \widehat{T}_0^{(i)} \geq \frac{1-\delta}{r} \right) \mathbb{1}_{C_{n-1}} \right]$$

where, for $1 \leq k \leq n-1$,

$$C_k = \left\{ X_{\theta_{\frac{1}{2}}} r^{\frac{n}{\alpha}} \prod_{i=1}^{k} |Z_1^{(i)}| \leq 2\varepsilon \right\}.$$

We now follow the same steps as for $\Psi_n(r, 2\varepsilon)$:

1. If $|Z_1^{(n-1)}| \geq 1$, then, we may remove $|Z_1^{(n-1)}|$ from the product in C_{n-1}, and deduce from (3.10) that $\Phi_n(r, 2\varepsilon)$ is smaller than

$$\kappa\sqrt{r}\,\mathbb{E}\left[\prod_{i=1}^{n-2} \widehat{\mathbb{P}}_1^{(i)} \left(|Z_1^{(i)}|^\alpha \widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r} \right) \mathbb{1}_{C_{n-2}} \right].$$

2. If $|Z_1^{(n-1)}| \leq 1$, then, we may apply directly (3.1) to deduce that $\Phi_n(r, 2\varepsilon)$ is smaller than

$$\kappa\sqrt{r}\,\mathbb{E}\left[|Z_1^{(n-1)}|^{\frac{\alpha}{2}} \prod_{i=1}^{n-2} \widehat{\mathbb{P}}_1^{(i)} \left(|Z_1^{(i)}|^\alpha \widehat{T}_0^{(i)} \geq \frac{1-\delta}{2r} \right) \mathbb{1}_{C_{n-1}} \right].$$

Iterating the procedure, we obtain as before that $\Phi_n(r, 2\varepsilon)$ may be bounded by a sum of 2^{n-1} terms:

$$\int_0^\delta \Phi_n(r, 2\varepsilon)\mathbb{P}(g_1 \in dr) \leq \kappa\varepsilon^{\frac{\alpha}{2}} \sum_{\Delta \subset \{1,\ldots,n-1\}} \int_0^{\frac{\delta^{n/\alpha}}{\varepsilon}} \mathbb{E}\left[\prod_{i\in\Delta} |Z_1^{(i)}|^{\frac{\alpha}{2}} \mathbb{1}_{\left\{ X_{\theta_{\frac{1}{2}}} \prod_{i\in\Delta} |Z_1^{(i)}| x \leq 2 \right\}} \right] x^{\frac{\alpha}{2}-1} dx \tag{3.12}$$

and, as $\varepsilon \to 0$, all the terms on the right-hand side have the same asymptotics : $\varepsilon^{\frac{\alpha}{2}} |\ln(\varepsilon)|$.

3.2.2 The case $\{a > 1\}$

Starting back from (3.4), we first bound the supremum by its value at $u = 0$:

$$\int_1^{+\infty}\int_0^1 \mathbb{P}\left(\sup_{0\le u\le 1} X_u^{(a,1,0)} r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_{1-u}^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0,\ 1 \ge \xi_{\frac{1}{r}}\right)\rho(da,dr)$$
$$\le \int_1^{+\infty}\int_0^1 \mathbb{P}\left(ar^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} Z_1^{(i)} \le \varepsilon,\ \prod_{i=1}^{n-1} Z_1^{(i)} \ge 0,\ 1 \ge \xi_{\frac{1}{r}}\right)\rho(da,dr)$$
$$\le \int_0^1 \mathbb{P}\left(r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z_1^{(i)}| \le \varepsilon,\ 1 \ge \xi_{\frac{1}{r}}\right)\mathbb{P}\left(g_1 \in dr\right).$$

The study of this last expression will be similar to that of $\Phi_n(r, 2\varepsilon)$, replacing $X_{\theta_{\frac{1}{2}}}$ by 1. Indeed, on the one hand, we first deduce from Lemma 2.1, taking δ small enough as before, that :

$$\int_\delta^1 \mathbb{P}\left(r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z_1^{(i)}| \le \varepsilon,\ 1 \ge \xi_{\frac{1}{r}}\right)\mathbb{P}\left(g_1 \in dr\right) \le \mathbb{P}\left(\delta^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z_1^{(i)}| \le \varepsilon\right) \underset{\varepsilon\to 0}{\asymp} \kappa\,\varepsilon\,|\ln(\varepsilon)|^{n-2}.$$

On the other hand, for $r \le \delta$, we deduce, as for (3.12), that

$$\int_0^\delta \mathbb{P}\left(r^{\frac{n}{\alpha}} \prod_{i=1}^{n-1} |Z_1^{(i)}| \le \varepsilon,\ 1 \ge \xi_{\frac{1}{r}}\right)\mathbb{P}\left(g_1 \in dr\right)$$
$$\le \kappa\,\varepsilon^{\frac{\alpha}{2}} \sum_{\Delta\subset\{1,\dots,n-1\}} \int_0^{\frac{\delta^{n/\alpha}}{\varepsilon}} \mathbb{E}\left[\prod_{i\in\Delta} |Z_1^{(i)}|^{\frac{\alpha}{2}} \mathbb{1}_{\left\{\prod_{i\in\Delta}|Z_1^{(i)}|x\le 1\right\}}\right] x^{\frac{\alpha}{2}-1}dx.$$

When $\varepsilon \to 0$, all the integrals on the right-hand side are finite, hence we obtain the asymptotics $\varepsilon^{\frac{\alpha}{2}}$ which is negligible. □

4 Proof of Lemma 3.3

Proof. This lemma being classical for Brownian motion (see Bertoin-Pitman [5]), we assume that $\alpha \in (0, 2)$. Let $0 < s \le t \le 1$ and take F a positive functional. Let us denote by $f(y; z, r)$ the probability density function of (X_{T_0}, T_0) when $X_0 = y$. By symmetry and time reversal, we first have

$$\mathbb{E}\left[F\left(\frac{X_{u-}}{g_1^{1/\alpha}}, s \le u \le g_1\right)1_{\{g_1\ge t\}}\right]$$
$$= 2\int_0^{+\infty} \mathbb{E}^{(y,1,0)}\left[F\left(\frac{X_u}{(1-T_0)^{1/\alpha}}, T_0 \le u \le 1-s\right)1_{\{T_0\le 1-t\}}\right]p_1(y)dy.$$

The change of measure formula for the stable bridge as well as the Markov property then yield

$$2\int_0^{+\infty} \mathbb{E}_y\left[p_s(X_{1-s})F\left(\frac{X_u}{(1-T_0)^{1/\alpha}}, T_0 \le u \le 1-s\right)1_{\{T_0\le 1-t\}}\right]dy$$
$$= 2\int_0^{+\infty}\int_{-\infty}^0\int_0^{1-t} \mathbb{E}_z\left[p_s(X_{1-s-r})F\left(\frac{X_u}{(1-r)^{1/\alpha}}, 0 \le u \le 1-s-r\right)\right]f(y; z, r)\,dydzdr$$

Next, by scaling and using that $t^{1/\alpha}p_t(z) = p_1\left(\frac{z}{t^{1/\alpha}}\right)$,

$$\begin{aligned}
&2\int_0^{+\infty}\int_{-\infty}^0\int_0^{1-t} \mathbb{E}_{\frac{z}{(1-r)^{1/\alpha}}}\left[p_s\left((1-r)^{1/\alpha}X_{1-\frac{s}{1-r}}\right)F\left(X_{\frac{u}{1-r}}, 0\le u\le 1-s-r\right)\right] \\
&\quad\times f(y;z,r)\,dydzdr \\
&\quad = 2\int_0^{+\infty}\int_{-\infty}^0\int_0^{1-t}\mathbb{E}_a\left[p_{\frac{s}{1-r}}\left(X_{1-\frac{s}{1-r}}\right)F\left(X_u, 0\le u\le 1-\frac{s}{1-r}\right)\right] \\
&\quad\times f(y;a(1-r)^{1/\alpha},r)\,dydadr \\
&\quad = 2\int_0^{+\infty}\int_{-\infty}^0\int_0^{1-t}\mathbb{E}^{(a,1,0)}\left[F\left(X_u, 0\le u\le 1-\frac{s}{1-r}\right)\right]p_1(a)f(y;a(1-r)^{1/\alpha},r)\,dydadr \\
&\quad = 2\int_{-\infty}^0 da\,\mathbb{E}^{(0,1,a)}\left[F\left(X_{u^-}, \frac{s}{1-r}\le u\le 1\right)\right]p_1(a)\int_0^{+\infty}\int_0^{1-t} f(y;a(1-r)^{1/\alpha},r)\,dydr
\end{aligned}$$

where in the second line we have used the change of variable $z = a(1-r)^{1/\alpha}$, in the third line the change of measure formula for the stable bridge and in the last line the time reversal property of the stable bridge. Letting $s\to 0$, we finally deduce that

$$\begin{aligned}
&\mathbb{E}\left[F\left(\frac{X_{u-}}{g_1^{1/\alpha}}, 0\le u\le g_1\right)1_{\{g_1\ge t\}}\right] \\
&\qquad = 2\int_{-\infty}^0 da\,\mathbb{E}^{(0,1,a)}\left[F\left(X_{u-}, 0\le u\le 1\right)\right]p_1(a)\int_0^{+\infty}\int_0^{1-t} f(y;a(1-r)^{1/\alpha},r)\,dydr
\end{aligned}$$

which proves Lemma 3.3. □

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