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Uniqueness of solution to scalar BSDEs with $L\exp\left(\mu\sqrt{2\log(1+L)}\right)$-integrable terminal values*

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Abstract

In [5], the existence of the solution is proved for a scalar linearly growing backward stochastic differential equation (BSDE) if the terminal value is $L\exp\left(\mu\sqrt{2\log(1+L)}\right)$-integrable with the positive parameter μ being bigger than a critical value μ_0. In this note, we give the uniqueness result for the preceding BSDE.

Keywords: backward stochastic differential equation; $L\exp(\mu\sqrt{2\log(1+L)})$ integrability; uniqueness.
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1 Introduction

Let $\{W_t, t \geq 0\}$ be a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t, t \geq 0\}$ its natural filtration augmented by all $\mathbb{P}$-null sets of $\mathcal{F}$. Let us fix a nonnegative real number $T > 0$. The σ-field of predictable subsets of $\Omega \times [0,T]$ is denoted by $\mathcal{P}$.

For any real $p \geq 1$, denote by L^p the set of all $\mathcal{F}_T$-measurable random variables η such that $E|\eta|^p < \infty$, by $\mathcal{S}^p$ the set of (equivalent classes of) all real-valued, adapted and càdlàg processes $\{Y_t, 0 \leq t \leq T\}$ such that

$$||Y||_{\mathcal{S}^p} := \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^p\right]^{1/p} < +\infty,$$

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by $\mathcal{L}^p$ the set of (equivalent classes of) all real-valued adapted processes $\{Y_t, 0 \le t \le T\}$ such that

$$||Y||_{\mathcal{L}^p} := \mathbb{E}\left[\int_0^T |Y_t|^p \, dt\right]^{1/p} < +\infty,$$

and by $\mathcal{M}^p$ the set of (equivalent classes of) all predictable processes $\{Z_t, 0 \le t \le T\}$ with values in $\mathbb{R}^{1\times d}$ such that

$$||Z||_{\mathcal{M}^p} := \mathbb{E}\left[\left(\int_0^T |Z_t|^2 \, dt\right)^{p/2}\right]^{1/p} < +\infty.$$

Consider the following Backward Stochastic Differential Equation (BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{1.1}$$

Here, f (hereafter called the generator) is a real valued random function defined on the set $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$, measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1\times d})$, and continuous in the last two variables with the following linear growth:

$$|f(s, y, z) - f(s, 0, 0)| \le \beta|y| + \gamma|z|, \quad (s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$$

with $f_0 := f(\cdot, 0, 0) \in \mathcal{L}^1, \beta \ge 0$ and $\gamma > 0$. ξ is a real $\mathcal{F}_T$-measurable random variable, and hereafter called the terminal condition or terminal value.

Definition 1.1. *By a solution to BSDE (1.1) we mean a pair $\{(Y_t, Z_t), 0 \le t \le T\}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$ such that $\mathbb{P}$-a.s., $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2(0,T)$ and $t \mapsto f(t, Y_t, Z_t)$ is integrable, and $\mathbb{P}$-a.s. (Y, Z) verifies (1.1).*

By BSDE (ξ, f), we mean the BSDE with generator f and terminal condition ξ.

It is well known that for $(\xi, f_0) \in L^p \times \mathcal{L}^p$ (with $p > 1$), BSDE (1.1) admits a unique adapted solution (y, z) in the space $\mathcal{S}^p \times \mathcal{M}^p$ if the generator f is uniformly Lipschitz in the pair of unknown variables. See e.g. [7, 4, 1] for more details. For $(\xi, f_0) \in L^1 \times \mathcal{L}^1$, one needs to restrict the generator f to grow sub-linearly with respect to z, i.e., with some $q \in [0, 1)$,

$$|f(t, y, z) - f_0(t)| \le \beta|y| + \gamma|z|^q, \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$$

to have for BSDE (1.1) a unique adapted solution (see [1]) if the generator f is uniformly Lipschitz in the pair of unknown variables.

In [5], the existence of the solution is given for a scalar linearly growing BSDE (1.1) if the terminal value is $L \exp\left(\mu\sqrt{2\log(1+L)}\right)$-integrable with the positive parameter μ being greater than a critical value $\mu_0 = \gamma\sqrt{T}$, and the preceding integrability of the terminal value for a positive parameter μ less than critical value μ_0 is shown to be not sufficient for the existence of a solution. It is well-known that the continuity of the generator is not sufficient for the uniqueness of the solution, and extra regularity assumptions like Lipschitz or monotone conditions are required to address the uniqueness issue. In this note, we give the uniqueness result for the preceding BSDE under the preceding integrability of the terminal value for $\mu > \mu_0$.

We first establish some interesting properties of the function

$$\psi(x, \mu) = x \exp\left(\mu\sqrt{2\log(1+x)}\right).$$

We observe that the obtained solution Y in [5] has the nice property that $\psi(|Y|, a)$ belongs for some $a > 0$ to the class (D) of processes X for which the family X_τ, τ running all stopping times less or equal to T, is uniformly integrable. This property is used to prove the uniqueness of the solution by dividing the whole interval $[0, T]$ into a finite number of sufficiently small subintervals.

2 Uniqueness

Define the function ψ:

$$\psi(x, \mu) := x \exp\left(\mu\sqrt{2\log(1+x)}\right), \quad (x, \mu) \in [0, +\infty) \times (0, +\infty).$$

In what follows, we occasionally write $\psi_\mu(\cdot)$ for $\psi(\cdot, \mu)$ for simplicity.

The following two lemmas can be found in Hu and Tang [5].

Lemma 2.1. *For any $x \in \mathbb{R}$ and $y \geq 0$, we have*

$$e^x y \leq e^{\frac{x^2}{2\mu^2}} + e^{2\mu^2}\psi(y, \mu). \tag{2.1}$$

Lemma 2.2. *Let $\mu > \gamma\sqrt{T}$. For any d-dimensional adapted process q with $|q_t| \leq \gamma$ almost surely, for $t \in [0, T]$,*

$$\mathbb{E}\left[e^{\frac{1}{2\mu^2}|\int_t^T q_s dW_s|^2} \,\Big|\, \mathcal{F}_t\right] \leq \frac{1}{\sqrt{1 - \frac{\gamma^2}{\mu^2}(T-t)}}. \tag{2.2}$$

Proposition 2.3. *We have the following assertions on ψ:*

(i) For $\mu > 0$, $\psi(\cdot, \mu)$ is convex.

(ii) For $c > 1$, we have $\psi_\mu(cx) \leq \psi_\mu(c)\psi_\mu(x)$, for all $x \geq 0$.

(iii) For any triple (a, b, c) with $a > 0, b > 0$ and $c > 0$, we have

$$\psi(\psi(x, a), b) \leq e^{\frac{ab^2}{c}}\psi(x, a+b+c).$$

Proof. The first assertion has been shown in [5]. It remains to show the Assertions (ii) and (iii).

We prove Assertion (ii).

$$\begin{aligned}
\psi_\mu(cx) &= cx\exp\left(\mu\sqrt{2\log(1+cx)}\right) \\
&\leq cx\exp\left(\mu\sqrt{2\log[(1+c)(1+x)]}\right) \\
&= cx\exp\left(\mu\sqrt{2\log(1+c) + 2\log(1+x)}\right) \\
&\leq cx\exp\left(\mu\sqrt{2\log(1+c)} + \mu\sqrt{2\log(1+x)}\right) \\
&= \psi_\mu(c)\psi_\mu(x).
\end{aligned}$$

We now prove Assertion (iii). We have

$$\begin{aligned}
&(\psi_b \circ \psi_a)(x) \\
=\ & \psi_a(x)\exp\left(b\sqrt{2\log(1+\psi_a(x))}\right) \\
=\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log\left(1+xe^{a\sqrt{2\log(1+x)}}\right)}\right) \\
\leq\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log\left((1+x)e^{a\sqrt{2\log(1+x)}}\right)}\right) \\
=\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log(1+x) + 2a\sqrt{2\log(1+x)}}\right).
\end{aligned}$$

In view of the following elementary inequality:

$$2a\sqrt{2\log(1+x)} \leq \frac{a^2b^2}{c^2} + \frac{2c^2}{b^2}\log(1+x),$$

we have

$$\begin{aligned}
&(\psi_b \circ \psi_a)(x) \\
\leq\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log(1+x) + \frac{a^2b^2}{c^2} + \frac{2c^2}{b^2}\log(1+x)}\right) \\
\leq\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log(1+x)} + b\sqrt{\frac{a^2b^2}{c^2}} + b\sqrt{\frac{2c^2}{b^2}\log(1+x)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\psi_b \circ \psi_a)(x) \\
\leq\ & x\exp\left(a\sqrt{2\log(1+x)}\right)\exp\left(b\sqrt{2\log(1+x)} + \frac{ab^2}{c} + c\sqrt{2\log(1+x)}\right) \\
\leq\ & xe^{\frac{ab^2}{c}}\exp\left((a+b+c)\sqrt{2\log(1+x)}\right).
\end{aligned}$$

□

Consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s, \tag{2.3}$$

where f satisfies

$$|f(s,y,z) - f_0(s,0,0)| \leq \beta|y| + \gamma|z|, \tag{2.4}$$

with $f_0 := f(\cdot,0,0) \in \mathcal{L}^1, \beta \geq 0$ and $\gamma > 0$.

Theorem 2.4. *Let f be a generator which is continuous with respect to (y,z) and verifies inequality (2.4), and let ξ be a terminal condition. Let us suppose that there exists $\mu > \gamma\sqrt{T}$ such that $\psi(|\xi| + \int_0^T|f_0(t)|\,dt, \mu) \in L^1(\Omega, \mathbb{P})$. Then BSDE (2.3) admits a solution (Y,Z) such that*

$$|Y_t| \quad \leq \quad \frac{1}{\sqrt{1-\frac{\gamma^2}{\mu^2}(T-t)}}e^{\beta(T-t)} + e^{2\mu^2+\beta(T-t)}\,\mathbb{E}\left[\psi_\mu\left(|\xi| + \int_t^T|f_0(s)|\,ds\right)\,\Big|\,\mathcal{F}_t\right].$$

Furthermore, there exists $a > 0$ such that $\psi(Y,a)$ belongs to the class (D).

Proof. Let us fix $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$. Set

$$\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p, \quad f_0^{n,p} := f_0^+ \wedge n - f_0^- \wedge p, \quad f^{n,p} := f - f_0 + f_0^{n,p}.$$

As the terminal value $\xi^{n,p}$ and $f^{n,p}(\cdot,0,0)$ are bounded (hence square-integrable) and $f^{n,p}$ is a continuous generator with a linear growth, in view of the existence result in [6], the BSDE $(\xi^{n,p}, f^{n,p})$ has a (unique) minimal solution $(Y^{n,p}, Z^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$. Set

$$\bar{f}^{n,p}(s,y,z) = |f_0^{n,p}(s)| + \beta y + \gamma|z|, \quad (s,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}.$$

In view of Pardoux and Peng [7], the BSDE $(|\xi^{n,p}|, \bar{f}^{n,p})$ has a unique solution $(\bar{Y}^{n,p}, \bar{Z}^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$. By the comparison theorem, it is easy to see that $\bar{Y}^{n,p}$ is non-negative. Hence $(\bar{Y}^{n,p}, \bar{Z}^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$ satisfies also the BSDE $(|\xi^{n,p}|, \widetilde{f}^{n,p})$ with

$$\widetilde{f}^{n,p}(s,y,z) = |f_0^{n,p}(s)| + \beta|y| + \gamma|z|, \quad (s,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}.$$

By the comparison theorem once again,

$$|Y_t^{n,p}| \le \bar{Y}_t^{n,p}.$$

Letting $q_s^{n,p} = \gamma \ \mathrm{sgn}(Z_s^{n,p})$ (where the sign function $\mathrm{sgn}(x) = \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(x)$) and

$$d\mathbb{P}_{q^{n,p}} = \exp\{\int_0^T q_s^{n,p} dW_s - \frac{1}{2}\int_0^T |q_s^{n,p}|^2 ds\} d\mathbb{P},$$

we obtain,

$$\begin{aligned}
|Y_t^{n,p}| &\le \bar{Y}_t^{n,p} \\
&= \mathbb{E}_{q^{n,p}}\left[e^{\beta(T-t)}|\xi^{n,p}| + \int_t^T e^{\beta(s-t)}|f_0^{n,p}(s)|\,ds \Big| \mathcal{F}_t \right] \\
&\le e^{\beta(T-t)}\mathbb{E}_{q^{n,p}}\left[|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \Big| \mathcal{F}_t \right] \\
&\le \frac{1}{\sqrt{1-\frac{\gamma^2}{\mu^2}(T-t)}} e^{\beta(T-t)} + e^{2\mu^2+\beta(T-t)}\ \mathbb{E}\left[\psi_\mu\left(|\xi| + \int_t^T |f_0(s)|\,ds \right) \Big| \mathcal{F}_t \right].
\end{aligned}$$

Here in the last inequality, we have used both Lemmas 2.1 and 2.2. Since $Y^{n,p}$ is nondecreasing in n and non-increasing in p, then by the localization method in [2], there is some $Z \in L^2(0,T;\mathbb{R}^{1\times d})$ almost surely such that $(Y := \inf_p \sup_n Y^{n,p}, Z)$ is an adapted solution. Therefore, for $a > 0$, using Jensen's inequality and the convexity of $\psi_a(\cdot) := \psi(\cdot, a)$ together with Assertion (ii) of Proposition 2.3, we have

$$\begin{aligned}
\psi_a(|Y_t^{n,p}|) &\le \psi_a\left(e^{\beta(T-t)}\mathbb{E}_{q^{n,p}}\left[|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \Big| \mathcal{F}_t \right] \right) \\
&\le \psi_a\left(e^{\beta(T-t)}\right)\psi_a\left(\mathbb{E}_{q^{n,p}}\left[|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \ \Big| \mathcal{F}_t \right] \right) \\
&\le \psi_a\left(e^{\beta(T-t)}\right)\mathbb{E}_{q^{n,p}}\left[\psi_a\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \right) \Big| \mathcal{F}_t \right] \\
&\le \psi_a\left(e^{\beta(T-t)}\right)\mathbb{E}\left[\exp\left(\int_t^T q_s^{n,p}\,dW_s \right)\psi_a\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \right) \Big| \mathcal{F}_t \right].
\end{aligned}$$

For $b > \gamma\sqrt{T}$, applying Lemma 2.1, we have

$$\begin{aligned}
\psi_a(|Y_t^{n,p}|) &\le \psi_a\left(e^{\beta(T-t)}\right)\mathbb{E}\left[\exp\left(\int_t^T q_s^{n,p}\,dW_s \right)\psi_a\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \right) \Big| \mathcal{F}_t \right] \\
&\le \psi_a\left(e^{\beta(T-t)}\right)\left(\mathbb{E}\left[\exp\left(\frac{1}{2b^2}\left(\int_t^T q_s^{n,p}\,dW_s \right)^2 \right) \Big| \mathcal{F}_t \right] \right. \\
&\qquad \left. + e^{2b^2}\mathbb{E}\left[\psi_b \circ \psi_a\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \right) \Big| \mathcal{F}_t \right] \right).
\end{aligned}$$

Using Lemma 2.2 and Assertion (iii) of Proposition 2.3, we have for any $c > 0$,

$$\begin{aligned}
&\psi_a(|Y_t^{n,p}|) \\
&\le \psi_a\left(e^{\beta(T-t)}\right)\left(\frac{1}{\sqrt{1-\frac{\gamma^2}{b^2}(T-t)}} + e^{2b^2+\frac{ab^2}{c}}\mathbb{E}\left[\psi_{a+b+c}\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds \right) \Big| \mathcal{F}_t \right] \right).
\end{aligned}$$

For $\mu > \gamma\sqrt{T}$, we can choose $a > 0, b > \gamma\sqrt{T}$, and $c > 0$ such that $a + b + c = \mu$. Then, we have

$$\begin{aligned}
&\psi_a(|Y_t^{n,p}|)\\
\leq\ & \psi_a\left(e^{\beta(T-t)}\right)\left(\frac{1}{\sqrt{1-\frac{\gamma^2}{b^2}(T-t)}} + e^{2b^2+\frac{ab^2}{c}}\,\mathbb{E}\left[\psi_\mu\left(|\xi^{n,p}| + \int_t^T |f_0^{n,p}(s)|\,ds\right)\Big|\mathcal{F}_t\right]\right)\\
\leq\ & \psi_a\left(e^{\beta(T-t)}\right)\left(\frac{1}{\sqrt{1-\frac{\gamma^2}{b^2}(T-t)}} + e^{2b^2+\frac{ab^2}{c}}\,\mathbb{E}\left[\psi_\mu\left(|\xi| + \int_t^T |f_0(s)|\,ds\right)\Big|\mathcal{F}_t\right]\right).
\end{aligned}$$

Letting first $n \to \infty$ and then $p \to \infty$, we have

$$\begin{aligned}
&\psi_a(|Y_t|)\\
\leq\ & \psi_a\left(e^{\beta(T-t)}\right)\left(\frac{1}{\sqrt{1-\frac{\gamma^2}{b^2}(T-t)}} + e^{2b^2+\frac{ab^2}{c}}\,\mathbb{E}\left[\psi_\mu\left(|\xi| + \int_t^T |f_0(s)|\,ds\right)\Big|\mathcal{F}_t\right]\right)\\
\leq\ & \psi_a\left(e^{\beta T}\right)\left(\frac{1}{\sqrt{1-\frac{\gamma^2 T}{b^2}}} + e^{2b^2+\frac{ab^2}{c}}\,\mathbb{E}\left[\psi_\mu\left(|\xi| + \int_0^T |f_0(s)|\,ds\right)\Big|\mathcal{F}_t\right]\right).
\end{aligned}$$

Consequently, we have $\psi_a(|Y|)$ belongs to the class (D). □

Now we state our main result of this note.

Theorem 2.5. *Assume that the generator f is uniformly Lipschitz in (y, z), i.e., there are $\beta > 0$ and $\gamma > 0$ such that for all $(y^i, z^i) \in R \times R^{1\times d}$, $i = 1, 2$, we have*

$$|f(t, y^1, z^1) - f(t, y^2, z^2)| \leq \beta|y^1 - y^2| + \gamma|z^1 - z^2|. \tag{2.5}$$

Furthermore, assume that there exists $\mu > \gamma\sqrt{T}$ such that $\psi(|\xi| + \int_0^T |f(t, 0, 0)|\,dt, \mu) \in L^1(\Omega, P)$. Then, BSDE (2.3) admits a unique solution (Y, Z) such that $\psi(Y, a)$ belongs to the class (D) for some $a > 0$.

Proof. The existence of an adapted solution has been proved in the preceding theorem. It remains to prove the uniqueness.

For $i = 1, 2$, let (Y^i, Z^i) be a solution of BSDE (2.3) such that $\psi_{a^i}(Y^i)$ belongs to the class (D) for some $a^i > 0$. Define

$$a := a^1 \wedge a^2, \quad \delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2.$$

Then both $\psi_a(Y^1)$ and $\psi_a(Y^2)$ are in the class (D), since $\psi(x, \mu)$ is nondecreasing in μ, and the pair $(\delta Y, \delta Z)$ satisfies the following equation

$$\delta Y_t = \int_t^T [f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)]\,ds - \int_t^T \delta Z_s\,dW_s, \quad t \in [0, T]. \tag{2.6}$$

By a standard linearization we see that there exists an adapted pair of processes (u, v) such that $|u_s| \leq \beta, |v_s| \leq \gamma$, and $f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2) = u_s\delta Y_s + \delta Z_s v_s$.

We define the stopping times

$$\tau_n := \inf\{t \geq 0 : |Y_t^1| + |Y_t^2| \geq n\} \wedge T, \quad n = 1, 2, \cdots,$$

with the convention that $\inf \emptyset = \infty$. Since $(\delta Y, \delta Z)$ satisfies the linear BSDE

$$\delta Y_t = \int_t^T (u_s\delta Y_s + \delta Z_s v_s)\,ds - \int_t^T \delta Z_s\,dW_s, \quad t \in [0, T],$$

we have the following formula

$$\delta Y_{t\wedge\tau_n} = \mathbb{E}\left[e^{\int_{t\wedge\tau_n}^{\tau_n} u_s\, ds + \int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle - \frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n} |v_s|^2 ds} \delta Y_{\tau_n} \,\middle|\, \mathcal{F}_t \right].$$

Therefore,

$$\begin{aligned} |\delta Y_{t\wedge\tau_n}| &\leq \mathbb{E}\left[e^{\int_{t\wedge\tau_n}^{\tau_n} u_s\, ds + \int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle} |\delta Y_{\tau_n}| \,\middle|\, \mathcal{F}_t \right] \\ &\leq e^{\beta T}\mathbb{E}\left[e^{\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle} |\delta Y_{\tau_n}| \,\middle|\, \mathcal{F}_t \right]. \end{aligned} \tag{2.7}$$

Now we show that the family of random variables $e^{\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle}|\delta Y_{\tau_n}|$ is uniformly integrable. For this note that, thanks to Lemma 2.1,

$$e^{\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle}|\delta Y_{\tau_n}| \leq e^{\frac{1}{2a^2}\left(\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle\right)^2} + e^{2a^2}\psi_a(|\delta Y_{\tau_n}|). \tag{2.8}$$

For $t \in [T - \frac{a^2}{4\gamma^2}, T]$, we have from Lemma 2.2,

$$\mathbb{E}\left[\left| e^{\frac{1}{2a^2}\left(\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle\right)^2} \right|^2 \right] = \mathbb{E}\left[e^{\frac{1}{a^2}\left(\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle\right)^2} \right] \leq \frac{1}{\sqrt{1 - \frac{2\gamma^2}{a^2}(T-t)}} \leq \sqrt{2},$$

and, thus, the family of random variables $e^{\frac{1}{2a^2}\left(\int_{t\wedge\tau_n}^{\tau_n} \langle v_s, dW_s\rangle\right)^2}$ is uniformly integrable.

On the other hand, since ψ_a is nondecreasing and convex, we have thanks to Proposition 2.3 (ii)

$$\begin{aligned} \psi_a(|\delta Y_{\tau_n}|) &\leq \psi_a(|Y^1_{\tau_n}| + |Y^2_{\tau_n}|) = \psi_a(\frac{1}{2}\times 2|Y^1_{\tau_n}| + \frac{1}{2}\times 2|Y^2_{\tau_n}|) \\ &\leq \frac{1}{2}\psi_a(2|Y^1_{\tau_n}|) + \frac{1}{2}\psi_a(2|Y^2_{\tau_n}|) \leq \frac{1}{2}\psi_a(2)[\psi_a(|Y^1_{\tau_n}|) + \psi_a(|Y^2_{\tau_n}|)]. \end{aligned}$$

From (2.8) it now follows that, for $t \in [T - \frac{a^2}{4\gamma^2}, T]$, the family of random variables $e^{\int_{t\wedge\tau_n}^{\tau_n} \langle v_s\, dW_s\rangle}|\delta Y_{\tau_n}|$ is uniformly integrable.

Finally, letting $n \to \infty$ in inequality (2.7), we have $\delta Y = 0$ on the interval $[T - \frac{a^2}{4\gamma^2}, T]$. It is then clear that $\delta Z = 0$ on $[T - \frac{a^2}{4\gamma^2}, T]$. The uniqueness of the solution is obtained on the interval $[T - \frac{a^2}{4\gamma^2}, T]$. In an identical way, we have the uniqueness of the solution on the interval $[T - \frac{a^2}{2\gamma^2}, T - \frac{a^2}{4\gamma^2}]$. By a finite number of steps, we cover in this way the whole interval $[0, T]$, and we conclude the uniqueness of the solution on the interval $[0, T]$. □

Remark 2.6. The uniformly Lipschitz condition of Theorem 2.5 can be relaxed into the following monotone one:

$$\begin{aligned} (y_1 - y_2)[f(t, y^1, z) - f(t, y^2, z)] &\leq \beta|y^1 - y^2|^2, \\ |f(t, y, z^1) - f(t, y, z^2)| &\leq \gamma|z^1 - z^2|. \end{aligned}$$

In fact, in view of (2.6), applying Tanaka's formula to compute $e^{\beta t}|\delta Y_t|$, we have

$$\begin{aligned} & e^{\beta t}|\delta Y_t| + 2\int_t^T d\Lambda_s \qquad (2.9) \\ = & \int_t^T \left[-\beta e^{\beta s}|\delta Y_s| + e^{\beta s}\mathrm{sgn}(Y_s)\left[f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s) \right] \right] ds \\ & - \int_t^T e^{\beta s}\mathrm{sgn}(Y_s)\delta Z_s\, dW_s \end{aligned}$$

where Λ is the local time of the semi-martingale δY at the origin. By a conventional linearization, we see that there exists an adapted process v such that $|v_s| \le \gamma$, and $f(s, Y_s^2, Z_s^1) - f(s, Y_s^2, Z_s^2) = \delta Z_s v_s$. Define

$$d\mathbb{Q} := e^{\int_0^T \langle v_s, dW_s\rangle - \frac{1}{2}\int_0^T |v_s|^2 ds} d\mathbb{P}.$$

Then the process $\widetilde{W}_t := W_t - \int_0^t v_s\, ds, t \in [0,T]$ is a Brownian motion under the new probability $\mathbb{Q}$. We have from (2.9) the following

$$\begin{aligned} & e^{\beta(t\wedge\tau_n)}|\delta Y_{t\wedge\tau_n}| + \int_{t\wedge\tau_n}^{\tau_n} e^{\beta s}\mathrm{sgn}(Y_s)\delta Z_s\, d\widetilde{W}_s \\ \le\ & e^{\beta\tau_n}|\delta Y_{\tau_n}| + \int_{t\wedge\tau_n}^{\tau_n} \left[-\beta e^{\beta s}|\delta Y_s| + e^{\beta s}\mathrm{sgn}(Y_s)\left[f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^1)\right]\right] ds \\ \le\ & e^{\beta\tau_n}|\delta Y_{\tau_n}|. \end{aligned}$$

Taking the expectation conditional at $\mathcal{F}_t$ under $\mathbb{Q}$, we deduce that

$$\begin{aligned} e^{\beta(t\wedge\tau_n)}|\delta Y_{t\wedge\tau_n}| &\le \mathbb{E}^{\mathbb{Q}}\left[e^{\beta\tau_n}|\delta Y_{\tau_n}| \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{\int_{t\wedge\tau_n}^{\tau_n}\langle v_s, dW_s\rangle - \frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}|v_s|^2 ds} e^{\beta\tau_n}|\delta Y_{\tau_n}| \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

The rest of the proof is identical to that of the preceding theorem.

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