Fluctuations for block spin Ising models

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Abstract

We analyze the high temperature fluctuations of the magnetization of the so-called Ising block model. This model was recently introduced by Berthet et al. in [2]. We prove a Central Limit Theorems (CLT) for the magnetization in the high temperature regime. At the same time we show that this CLT breaks down at a line of critical temperatures. At this line we prove a non-standard CLT for the magnetization.

1 Introduction

In a recent paper Berthet, Rigollet and Srivastavaz studied a block version of the Curie-Weiss-Ising model [2]. This model is inspired by extensive studies of block models in the recent past, see e.g. [1], [4], [5], [17], [22]. On the other hand, similar models were considered a bit earlier in the statistical mechanics literature, see [6], [7], [14], [15], [16]. To define our model, we partition the set $\{1, \ldots, N\}$ for N even into a set $S \subset \{1, \ldots, N\}$ with $|S| = \frac{N}{2}$ and its complement S^c. This segmentation induces a partitioning of the binary hypercube $\{-1, +1\}^N, N \in \mathbb{N}$, the state space of the *Ising block model*. For $\beta > 0$ and $0 \le \alpha \le \beta$ the model we will consider is defined by the Hamiltonian

$$H_{N,\alpha,\beta,S}(\sigma) := -\frac{\beta}{2N} \sum_{i \sim j} \sigma_i \sigma_j - \frac{\alpha}{2N} \sum_{i \nsim j} \sigma_i \sigma_j, \quad \sigma \in \{-1, +1\}^N.$$

Here we write $i \sim j$, if either $i, j \in S$ or $i, j \in S^c$ and $i \nsim j$ otherwise. This Hamiltonian induces a Gibbs measure

$$\mu_{N,\alpha,\beta}(\sigma) = \mu_{N,\alpha,\beta,S}(\sigma) := \frac{e^{-H_{N,\alpha,\beta}(\sigma)}}{\sum_{\sigma'} e^{-H_{N,\alpha,\beta}(\sigma')}} =: \frac{e^{-H_{N,\alpha,\beta}(\sigma)}}{Z_{N,\alpha,\beta}}. \tag{1.1}$$

A closely related version of this model has been investigated in [19]. However, the couplings in [19] between the blocks have the same strength of interaction as the couplings within a block. We were informed that a more general version of the model will be studied in [20]. Similar to the Curie-Weiss model the Ising block model has an order parameter: the vector of block magnetizations, $m := m^N := (m_1^N, m_2^N)$, where

$$m_1 := m_1^N := m_1(\sigma) := \frac{2}{N} \sum_{i \in S} \sigma_i \quad \text{and} \quad m_2 := m_2^N := m_2(\sigma) := \frac{2}{N} \sum_{i \notin S} \sigma_i.$$

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Indeed, the Hamiltonian is handily rewritten as

$$H_{N,\alpha,\beta,S}(\sigma) = -\frac{N}{2}\left(\frac{1}{2}\alpha m_1 m_2 + \beta\frac{1}{4}m_1^2 + \frac{1}{4}\beta m_2^2\right).$$

This observation is not only a convenient way to analyze the block spin Ising model, it also makes m an obvious choice to describe its behaviour and its phase transitions. To characterize them, recall that with the above notation for $\alpha = \beta$ one reobtains the Curie-Weiss or mean-field Ising model at inverse temperature β, i.e. the model on $\{-1, +1\}^N$ given by $H_{CW}(\sigma) = \frac{1}{2N}\sum_{i,j}\sigma_i\sigma_j$ and Gibbs measure $\mu_{N,\beta}^{CW}(\sigma) = \frac{e^{-\beta H_{CW}(\sigma)}}{Z_{N,\beta}^{CW}}$. Also, recall ([11]) that the Curie-Weiss model undergoes a phase transition at $\beta = 1$. This phase transition can be described by saying that the distribution of the parameter $\overline{m} = \frac{1}{N}\sum_i\sigma_i$ (also called the magnetization) weakly converges to the Dirac measure in 0, δ_0, if $\beta \leq 1$ while it converges to the mixture $\frac{1}{2}(\delta_{m^+(\beta)} + \delta_{-m^+(\beta)})$, if $\beta > 1$. Here $m^+(\beta)$ is the largest solution of the so-called Curie-Weiss equation $z = \tanh(\beta z)$. A similar result was proven for the block spin Ising model in [2] (the authors also allow for negative values of α). There the authors (implicitly) show

Theorem 1.1. cf. [2, Proposition 1] In the above assume that $0 \leq \alpha < \beta$ and denote by $\rho_{N,\alpha,\beta}$ the distribution of m under the Gibbs measure $\mu_{N,\alpha,\beta}$. Then

- If $\beta + \alpha \leq 2$, then $\rho_{N,\alpha,\beta}$ weakly converges to the Dirac measure in $(0, 0)$.
- If $\beta + \alpha > 2$ and $\alpha = 0$ then $\rho_{N,\alpha,\beta}$ weakly converges to the mixture of Dirac measures $\frac{1}{4}\sum_{s_1,s_2\in\{-,+\}}\delta_{(s_1 m^+(\beta/2), s_2 m^+(\beta/2))}$.
- If $\beta + \alpha > 2$ and $\alpha > 0$ then $\rho_{N,\alpha,\beta}$ weakly converges to the mixture of Dirac measures $\frac{1}{2}(\delta_{(m^+(\frac{\alpha+\beta}{2}),m^+(\frac{\alpha+\beta}{2}))} + \delta_{(-m^+(\frac{\alpha+\beta}{2}),-m^+(\frac{\alpha+\beta}{2}))})$.

Theorem 1.1 is the trigger for another question: In the Curie-Weiss model the phase transition can also be observed on the level of fluctuations of the magnetization: As is shown in [13], [12] or in [11, Theorems V.9.4 and V.9.5] or [10], for $\beta < 1$ the parameter $\sqrt{N}\overline{m}$ obeys a standard CLT with expectation 0 and variance $\frac{1}{1-\beta}$, while for $\beta = 1$ one has to scale differently: Then $N^{1/4}\overline{m}$ converges in distribution to a random variable that has Lebesgue density proportional to $\exp(-\frac{1}{12}x^4)$. Our question in this note is, whether a similar behaviour can be observed for the block spin Ising model and how the limit distribution depends on the relation between α and β. To answer this question we show

Theorem 1.2. For the block spin Ising model assume that $0 \leq \alpha < \beta$ and that $\beta + \alpha < 2$. Then, $\sqrt{N}m$ converges in distribution to a 2-dimensional Gaussian random variable with expectation 0 and covariance matrix $\Sigma = s^2\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ with $s^2 = \frac{8-4\beta}{(2-\beta)^2-\alpha^2}$ and $r = \frac{\alpha}{2-\beta}$.

Remark 1.3. It is well known that in the standard Curie-Weiss a CLT also holds in the presence of an external field [12], i.e. with a Hamiltonian of the form $H_{CW}(\sigma) = \frac{1}{2N}\sum_{i,j}\sigma_i\sigma_j + h\sum_i\sigma_i$ with $h > 0$. We are firmly convinced that a similar result is true for our model as well. However, we did not try to prove it. Finally, one might ask, whether – in the spirit of [10] – Stein's method may be applied to our situation as well. We consider this a more challenging question, because a multi-dimensional version of Stein's method would be needed. We may consider this problem in a different paper.

On the other hand, if $\beta + \alpha = 2$ the fluctuations are no longer Gaussian

Theorem 1.4. For the block spin Ising model assume that $0 \leq \alpha < \beta$ and that $\beta + \alpha = 2$. Then, $N^{\frac{1}{4}}m_1$ converges in distribution to a probability measure ρ on $\mathbb{R}$. The measure ρ is absolutely continuous with Lebesgue-density $g(x) = \exp(-\frac{1}{12}x^4)/Z$, where Z is a normalizing constant. The difference between m_1 and m_2 multiplied by $\sqrt{N}$, i.e. $\tilde{m}_1 - \tilde{m}_2 := \sqrt{N}(m_1 - m_2)$, however, is asymptotically Gaussian with mean 0 and variance $\frac{2}{2-(\beta-\alpha)}$.

We will show Theorems 1.2 and 1.4 in Section 3 using a Hubbard-Stratonovich transformation for an appropriate function of m. Before, in Section 2, however, we will give an alternative proof of Theorem 1.1 using the theory of large deviations. This is not only interesting in its own right, but also provides a way to derive limit theorems in more complicated settings, see e.g [21].

2 An LDP for the vector of block magnetizations

Differing from the line of arguments in [2], Theorem 1.1 can also be shown by proving a large deviation principle (LDP) for $0 \leq \alpha \leq \beta$. To this end, we will slightly change our variables and consider the vector $v = (v_1, v_2)$ with $v_1 := \frac{1}{2}m_1$ and $v_2 := \frac{1}{2}m_2$. We show

Theorem 2.1. *For every $S \subset \{1, \ldots, N\}$ with $|S| = \frac{N}{2}$ the vector v obeys a principle of large deviations (LDP) under the Gibbs measure $\mu_{N,\alpha,\beta} := \mu_{N,\alpha,\beta,S}$, with speed N and rate function $J_v(x) := \sup_{y \in \mathbb{R}^2}[F_v(y) - J(y)] - [F_v(x) - J(x)]$. Here $F_v : \mathbb{R}^2 \to \mathbb{R}$ is defined by*

$$F_v(x) := \frac{1}{2} \left(\beta x_1^2 + \beta x_2^2 + 2\alpha x_1 x_2 \right) \tag{2.1}$$

and for $x \in \mathbb{R}^2$

$$J(x) := \sup_{t \in \mathbb{R}^2} \left[\langle t, x \rangle - \frac{1}{2} \log \cosh(t_1) - \frac{1}{2} \log \cosh(t_2) \right].$$

This implies that the convergence in Theorem 1.1 (for $0 \leq \alpha \leq \beta$) is exponentially fast.

Proof. We will prove this theorem in two steps, first we show the LDP, then, how one derives Theorem 1.1 from it.

Step 1:

First, note that the case $\alpha = 0$ is trivial. Then, the system consists of two independent Curie-Weiss models on $\frac{N}{2}$ spins at temperature β. The LDP for the magnetization in each of the systems is known (cf. e.g. [11]) and transferring these LDPs to the vector v (with independent components) is trivial. We will thus assume that $\alpha > 0$.

Let us consider the moment generating function of the vector v. To this end let $t = (t_1, t_2) \in \mathbb{R}^2$. Then the moment generating function of v in t is given by

$$\mathbb{E} \exp(N \langle t, v \rangle) = \cosh(t_1)^{\frac{N}{2}} \cosh(t_2)^{\frac{N}{2}},$$

where here $\mathbb{E}$ denotes the expectation with respect to the a priori measure $\frac{1}{2}(\delta_{-1} + \delta_{+1})$. This readily yields $\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \exp(N \langle t, v \rangle) = \frac{1}{2} \log \cosh(t_1) + \frac{1}{2} \log \cosh(t_2)$. As the right hand side of this expression is finite and differentiable on all of $\mathbb{R}^2$, by the Gärtner-Ellis Theorem [8, Theorem 2.3.6] this computation implies an LDP for v under the uniform distribution with speed N and rate function

$$J(x) := \sup_{t \in \mathbb{R}^2} \left[\langle t, x \rangle - \frac{1}{2} \log \cosh(t_1) - \frac{1}{2} \log \cosh(t_2) \right] = \frac{1}{2} I(2x_1) + \frac{1}{2} I(2x_2)$$

for $x \in \mathbb{R}^2$. Here $I(x) := \frac{1}{2}(1+x)\log(1+x) + \frac{1}{2}(1-x)\log(1-x)$. Now the Hamiltonian $H_{N,\alpha,\beta,S}(\sigma)$ of our model can also be rewritten in terms of v:

$$H_{N,\alpha,\beta,S}(\sigma) = -\frac{N}{2} \left(\beta v_1^2(\sigma) + \beta v_2^2(\sigma) + 2\alpha v_1(\sigma) v_2(\sigma) \right).$$

This fact, together with the above LDP and the exponential form of the Gibbs measure and the LDP for integrals of exponential functions (see e.g. [9, Theorem III.17] – a direct

consequence of Varadhan's Lemma [8, Theorem 4.3.1] – implies that the distribution of v under $\mu_{N,\alpha,\beta,S}$ satisfies an LDP with speed N and rate function

$$J_v(x) := \sup_{y \in \mathbb{R}^2} [F_v(y) - J(y)] - [F_v(x) - J(x)],$$

where $F_v : \mathbb{R}^2 \to \mathbb{R}$ is given by (2.1). A change of variables yields the desired LDP.

Step 2:

If M denotes the set of minima of J_v and $B_\varepsilon(M) := \bigcup_{y \in M} B_\varepsilon(y)$ (where $B_\varepsilon(y)$ are open balls of radius $\varepsilon > 0$ centered around y) we obtain from the upper bound of the LDP that

$$\mathbb{P}(v \notin B_\varepsilon(M)) \leq \exp\left(-\frac{N}{2} \inf_{x \in B_\varepsilon^c(M)} J_v(x)\right)$$

for N large enough. The $\inf$ on the right hand side of the inequality is positive. In this sense, v concentrates in the minima of J_v exponentially fast. By a change of variables, again, this implies that m concentrates exponentially fast in the (global) minima of J_m defined by

$$J_m(x) := \sup_{y \in \mathbb{R}^2} [F_m(y) - \tilde{J}(y)] - [F_m(x) - \tilde{J}(x)],$$

$$\tilde{J}(x) := \frac{1}{2}I(x_1) + \frac{1}{2}I(x_2) \quad \text{and} \quad F_m(x) := \frac{1}{2}\left(\beta\frac{1}{4}x_1^2 + \beta\frac{1}{4}x_2^2 + \frac{1}{2}\alpha x_1 x_2\right).$$

The minima of J_m are the maxima of $F_m(x) - \tilde{J}(x)$. These satisfy $\nabla(F_m - \tilde{J})(x) = 0$, i.e.

$$\frac{1}{2}\beta x_1 + \frac{1}{2}\alpha x_2 = \operatorname{artanh}(x_1) \quad \text{and} \quad \frac{1}{2}\beta x_2 + \frac{1}{2}\alpha x_1 = \operatorname{artanh}(x_2). \tag{2.2}$$

Note that the vector $(0,0)$ is always a solution to this system of equations and hence a critical point of $F_m - \tilde{J}$.

We start with the high temperature regime, i.e. we consider $\beta + \alpha < 2$. By an easy calculation we find that the Hesse matrix of $F_m(x,y) - \tilde{J}(x,y)$ is given by

$$\frac{1}{2}\begin{pmatrix} \frac{1}{2}\beta - \frac{1}{1-x^2} & \frac{1}{2}\alpha \\ \frac{1}{2}\alpha & \frac{1}{2}\beta - \frac{1}{1-y^2} \end{pmatrix}.$$

Hence, the Hesse matrix in the point $(0,0)$ is negative definite, i.e. $(0,0)$ is a local maximum of $F_m(x) - \tilde{J}(x)$, if $0 \leq \alpha \leq \beta < 2$ and $\left(1 - \frac{1}{2}\beta\right)^2 - \frac{1}{4}\alpha^2 > 0$. This is true, if $\alpha + \beta < 2$.

Next, we will see that, in this case, the point $(0,0)$ is the only solution to the system of equations in (2.2), and hence the global maximum of $F_m(x) - \tilde{J}(x)$. To this end, we rewrite the equations in (2.2) as

$$x_2 = \frac{2}{\alpha}\left(\operatorname{artanh}(x_1) - \frac{1}{2}\beta x_1\right) \quad \text{and} \quad x_1 = \frac{2}{\alpha}\left(\operatorname{artanh}(x_2) - \frac{1}{2}\beta x_2\right).$$

Hence, for $|x| < 1$ and $f(x) := \frac{2}{\alpha}\left(\operatorname{artanh}(x) - \frac{1}{2}\beta x\right)$ we have

$$x_1 = f(x_2), \quad x_2 = f(x_1) \quad \text{resp.} \quad x_1 = f^2(x_1), \quad x_2 = f^2(x_2).$$

This means we are looking for the fixed points of f^2. We note that for $0 \leq \beta \leq 2$, we have for all $|x| < 1$

$$f'(x) = \frac{2}{\alpha}\left(\frac{1}{1-x^2} - \frac{1}{2}\beta\right) > 0$$

and hence f is invertible. The fixed points of f^2 are thus the same as the fixed points of $(f^{-1})^2$ resp. of (f^{-1}). We see that f^{-1} is a strong contraction for $\alpha + \beta < 2$ from $(f^{-1})' = \frac{1}{f'}$ and for all $y \in (-1, 1)$

$$\frac{1}{f'(y)} = \frac{\alpha}{2} \frac{1}{\left(\frac{1}{1-y^2} - \frac{1}{2}\beta\right)} \leq \frac{\alpha}{2} \frac{1}{\left(1 - \frac{1}{2}\beta\right)} \leq 1 - \varepsilon,$$

for $\varepsilon > 0$ small enough. Thus, by Banach's fixed point theorem, there is a single fixed point, which has to be equal to zero.

Next consider the critical line $\beta + \alpha = 2$. Here the arguments are almost the same. The only difference is, that now $\frac{1}{f'(y)}\big|_{y=0} = 1$, while $\frac{1}{f'(y)} < 1$ for all other y. Hence f^{-1} is a weak contraction for $\alpha + \beta = 2$. However, the magnetizations m_1 and m_2 live on the compact interval $[-1, 1]$, such that we can again conclude that f^{-1} has the unique fixed point 0.

Now we consider $\alpha + \beta > 2$. In this case $(0, 0)$ is still a solution to (2.2). However, in this case, it is either a saddle point or a local minimum of $F_m - \tilde{J}$, because it is not a maximum. Indeed, choose $x = y$, i.e.

$$F_m(x, x) - \tilde{J}(x, x) = \frac{1}{2}\left(\frac{\beta + \alpha}{2}\right) x^2 - I(x).$$

From the one dimensional Curie-Weiss model we know that for $\alpha + \beta > 2$ the maximum at attained away from zero. Hence, $(0, 0)$ is not a maximum of $F_m - \tilde{J}$. Recalling that $\alpha > 0$, we see directly from the definition of F_m and $\tilde{J}$ that a point (x, y) can only be a maximum, if x and y have the same sign. We will see that f and thus f^2 has exactly one positive fixed point m^* and one negative fixed point $-m^*$. This again is shown using a fixed point argument for f^{-1}. Note that now $\frac{1}{f'(0)} > 1$. However, the function $y \mapsto \frac{1}{f'(y)}$ is always non-negative on $[0, 1]$, it is decreasing, and depends continuously on y and by the intermediate value theorem there is y_0 such that $\frac{1}{f'(y)} \leq 1$ for all $y \in [y_0, 1]$. Thus, f^{-1} restricted to $[y_0, 1]$ is a weakly contracting self-map and therefore has a unique fixed point. But this fixed point of f^{-1} is the fixed point of f and is easily checked to satisfy

$$\tanh\left(\frac{\alpha + \beta}{2} m^*\right) = m^*.$$

This proves the claim. □

Remark 2.2. In the spirit of [19], one can also allow for other sizes of S, i.e. for $0 < \gamma < 1$ we can consider sets $S \subset \{1, \ldots, N\}$ with $|S| = \gamma N$. Here, we assume, for simplicity, γN to be an integer. In this case, the Hamiltonian $H_{N,\alpha,\beta,S}$ is the same as in (1.1) and the magnetizations are $m_1(\sigma) = \frac{1}{\gamma N}\sum_{i \in S}\sigma_i$, and $m_2(\sigma) := \frac{1}{(1-\gamma)N}\sum_{i \notin S}\sigma_i$. The Hamiltonian can then be rewritten as $H_{N,\alpha,\beta,S}(\sigma) = -\frac{N}{2}\left(2\gamma(1-\gamma)\alpha m_1 m_2 + \beta\gamma^2 m_1^2 + (1-\gamma)^2\beta m_2^2\right)$. We believe that a result analogue to Theorem 1.1 can be shown by generalizing the large deviation techniques in the proof of Theorem 2.1. In the same spirit as in Theorem 1.2 and Theorem 1.4, one can also show a Central Limit Theorem for this generalized setting. The technical problems are, however, more demanding. We will return to these questions in a later publication.

3 Proof of Theorem 1.2 and 1.4

The proofs of Theorems 1.2 and 1.4 rely on the same idea. We will first prove limit theorems for two other parameters, that are closely related to m_1 and m_2. To this end

we introduce the random variables

$$w_1 := w_1(\sigma) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i \quad \text{and} \quad w_2 := w_2(\sigma) := \frac{1}{N} \left(\sum_{i \in S} \sigma_i - \sum_{i \notin S} \sigma_i \right)$$

and the corresponding standardized versions

$$\tilde{w}_1 := \sqrt{N} w_1 \quad \text{and} \quad \tilde{w}_2 := \sqrt{N} w_2.$$

Note that $m_1 = w_1 + w_2$ and $m_2 = w_1 - w_2$ and thus limit theorems for $w = (w_1, w_2)$ will imply limit theorems for m and vice versa.

Again, note that the Hamiltonian $H_{N,\alpha,\beta,S}$ can also be rewritten in terms of the variables w_1 and w_2 resp. in terms of $\tilde{w}_1$ and $\tilde{w}_2$ as

$$H_{N,\alpha,\beta,S}(\sigma) = -\frac{N}{2} \left(2\alpha \frac{1}{4} m_1 m_2 + \beta \frac{1}{4} m_1^2 + \beta \frac{1}{4} m_2^2 \right) = -\frac{1}{4}((\alpha+\beta)\tilde{w}_1{}^2 + (\beta-\alpha)\tilde{w}_2{}^2).$$

Next we will show a Central Limit Theorem for the vector $\tilde{w} := (\tilde{w}_1, \tilde{w}_2)$ in the high temperature region $0 \le \alpha < \beta < 2$ and $\alpha + \beta < 2$.

Lemma 3.1. *Assume that $0 \le \alpha < \beta$ and $\beta + \alpha < 2$. Then, as $N \to \infty$, under the Gibbs measure $\mu_{\alpha,\beta,S}$ the vector $\tilde{w}$ converges to a 2-dimensional Gaussian distribution with expectation 0 and covariance matrix $\Sigma = \begin{pmatrix} \frac{1}{1-\frac{\alpha+\beta}{2}} & 0 \\ 0 & \frac{1}{1-\frac{\beta-\alpha}{2}} \end{pmatrix}$.*

Proof. Our principal strategy consists of computing a suitable Hubbard-Stratonovich transform of our measure of interest (as e.g. in [18]) and expanding it. To this end, let $\mathcal{N}(0,C)$ denote a two-dimensional Gaussian distribution with expectation 0 and covariance matrix C given by $C = \begin{pmatrix} \frac{2}{\beta+\alpha} & 0 \\ 0 & \frac{2}{\beta-\alpha} \end{pmatrix}$. We will now compute the density of $\chi_{N,\alpha,\beta} := \mu_{N,\alpha,\beta}(\tilde{w})^{-1} * \mathcal{N}(0,C)$: Let A be a Borel subset of $\mathbb{R}^2$ and let $\varrho_{N,\beta}(\sigma)$ denote the density of $\mu_{N,\alpha,\beta}$. Then

$$\chi_{N,\alpha,\beta}(A) = (\tilde{w})^{-1} * \mathcal{N}(0,C)(A) = \sum_{\sigma \in \{-1,1\}^N} \mathcal{N}(0,C)(A - \tilde{w}) \mu_{N,\alpha,\beta}(\sigma)$$

$$= K_1 \sum_{\sigma \in \{-1,1\}^N} \int_{A-\sqrt{\tilde{w}}} \exp\left(-\frac{1}{2} \left(\frac{\alpha+\beta}{2} x^2 + \frac{\beta-\alpha}{2} y^2 \right) \right) \varrho_{N,\beta}(\sigma) dx dy$$

$$= K_1 \sum_{\sigma \in \{-1,1\}^N} \int_{A} \exp\left(-\frac{1}{2} \left(\frac{\alpha+\beta}{2} (x-\tilde{w}_1)^2 + \frac{\beta-\alpha}{2} (y-\tilde{w}_2)^2 \right) \right) \varrho_{N,\beta}(\sigma) dx dy$$

$$= K_2 \sum_{\sigma \in \{-1,1\}^N} \int_{A} \exp\left(-\frac{1}{2} \left(\frac{\alpha+\beta}{2} (x-\tilde{w}_1)^2 + \frac{\beta-\alpha}{2} (y-\tilde{w}_2)^2 \right) \right)$$

$$\times \exp\left(\frac{1}{4} \left((\alpha+\beta)\tilde{w}_1{}^2 + (\beta-\alpha)\tilde{w}_2{}^2 \right) \right) dx dy$$

$$= K_2 \sum_{\sigma \in \{-1,1\}^N} \int_{A} \exp\left(-\frac{\alpha+\beta}{4} x^2 - \frac{\beta-\alpha}{4} y^2 \right) \exp\left(\frac{\alpha+\beta}{2} x\tilde{w}_1 + \frac{\beta-\alpha}{2} y\tilde{w}_2 \right) dx dy$$

$$= K_3 \int_{A} \exp\left(-\frac{\alpha+\beta}{4} x^2 - \frac{\beta-\alpha}{4} y^2 \right) \exp\left(\frac{N}{2} \log \cosh\left(\frac{1}{\sqrt{N}} \left(\frac{\alpha+\beta}{2} x + \frac{\beta-\alpha}{2} y \right) \right) \right.$$

$$\left. + \frac{N}{2} \log \cosh\left(\frac{1}{\sqrt{N}} \left(\frac{\alpha+\beta}{2} x - \frac{\beta-\alpha}{2} y \right) \right) \right) dx dy$$

Here, we used $K_1 := \frac{1}{2\pi\sqrt{\det C}}$, $K_2 := \frac{1}{2\pi\sqrt{\det C}Z_{N,\alpha,\beta}}$ and $K_3 := K_2 2^N$. Denote by

$$
\begin{aligned}
\Phi(x,y) :=& \Phi_{N,\alpha,\beta,S}(x,y) \\
:=& \frac{1}{4}(\alpha+\beta)x^2 + \frac{1}{4}(\beta-\alpha)y^2 - \frac{N}{2}\log\cosh\left(\frac{1}{\sqrt{N}}\left(\frac{\alpha+\beta}{2}x + \frac{\beta-\alpha}{2}y\right)\right) \\
& - \frac{N}{2}\log\cosh\left(\frac{1}{\sqrt{N}}\left(\frac{\alpha+\beta}{2}x - \frac{\beta-\alpha}{2}y\right)\right).
\end{aligned}
$$

Now recall the second order Taylor expansion $\log\cosh(x) = \frac{1}{2}x^2 + \mathcal{O}(x^4)$. Thus

$$
\begin{aligned}
\Phi(x,y) =& \frac{1}{4}(\alpha+\beta)x^2 + \frac{1}{4}(\beta-\alpha)y^2 - \frac{1}{4}\left(\frac{(\alpha+\beta)^2}{4}x^2 + \frac{\beta^2-\alpha^2}{2}xy + \frac{(\beta-\alpha)^2}{4}y^2\right) \\
& - \frac{1}{4}\left(\frac{(\alpha+\beta)^2}{4}x^2 - \frac{\beta^2-\alpha^2}{2}xy + \frac{(\beta-\alpha)^2}{4}y^2\right) + \mathcal{O}(N^{-1}), \\
=& \frac{x^2}{2}\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right) + \frac{y^2}{2}\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right) + \mathcal{O}(N^{-1}),
\end{aligned}
$$

where the constant in the $\mathcal{O}(N^{-1})$-term depends on x and y. However, the convergence is uniform on compact subsets of $\mathbb{R}^2$. Thus

$$
\begin{aligned}
\chi_{N,\alpha,\beta}(A) &= K_3\int_A e^{-\Phi(x,y)}dx\,dy \\
&= K_3\int_A \exp\left[-\frac{x^2}{2}\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right) - \frac{y^2}{2}\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right) + \mathcal{O}(N^{-1})\right]dx\,dy
\end{aligned}
$$

and the convergence in the $\mathcal{O}(N^{-1})$-term is uniform on compact subsets of $\mathbb{R}^2$.

To turn this into a weak convergence statement, we need to control integrals over unbounded sets as well, in particular, we need to treat the case $A = \mathbb{R}^2$ to see that K_3 converges to $\frac{1}{2\pi\sqrt{\det\Sigma'}}$ for

$$
\Sigma' := \begin{pmatrix} \left[\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right]^{-1} & 0 \\ 0 & \left[\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right]^{-1} \end{pmatrix}. \tag{3.1}
$$

Hence, for any measurable set $A \subset \mathbb{R}^2$ we write

$$
\begin{aligned}
\int_A \Psi(x,y)dx\,dy \\
= \int_{A\cap B(0,R)} \Psi(x,y)dx\,dy + \int_{A\cap B(0,R)^c\cap B(0,r\sqrt{N})} \Psi(x,y)dx\,dy + \int_{A\cap B(0,r\sqrt{N})^c} \Psi(x,y)dx\,dy,
\end{aligned} \tag{3.2}
$$

where we set

$$
\Psi(x,y) := e^{-\Phi(x,y)}.
$$

Here for any $l > 0$ we denote by $B(0,l)$ the ball in $\mathbb{R}^2$ with center in 0 and radius l. Further, we consider numbers $R > 0$ and $r > 0$ and we will send R to ∞ and consider r sufficiently small. We will refer to the summands on the right hand of (3.2) as inner region, intermediate region and outer region, respectively. The goal is to see that the inner region contributes all mass to the integral as $R \to \infty$.

As already marked above for fixed $R > 0$

$$\lim_{N \to \infty} \int_{A \cap B(0,R)} \Psi(x,y) dx\, dy$$
$$= \int_{A \cap B(0,R)} \exp\left[-\frac{1}{2}x^2\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right) - \frac{1}{2}y^2\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right)\right] dx\, dy.$$

Next, we treat the outer region. Let us rewrite the exponent in this case as

$$\Phi(x,y) = N\tilde{\Phi}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right),$$

where

$$\tilde{\Phi}(x,y) := \frac{1}{4}(\alpha+\beta)x^2 + \frac{1}{4}(\beta-\alpha)y^2$$
$$- \frac{1}{2}\log\cosh\left(\frac{\alpha+\beta}{2}x + \frac{\beta-\alpha}{2}y\right) - \frac{1}{2}\log\cosh\left(\frac{\alpha+\beta}{2}x - \frac{\beta-\alpha}{2}y\right).$$

Analyzing $\tilde{\Phi}$ we see that it becomes minimal only if $\nabla\tilde{\Phi} = 0$ and

$$\nabla\tilde{\Phi}(x,y) = \begin{pmatrix} \frac{1}{2}c_1 x - \frac{c_1}{4}\tanh(\frac{c_1 x + c_2 y}{2}) - \frac{c_1}{4}\tanh(\frac{c_1 x - c_2 y}{2}) \\ \frac{1}{2}c_2 y - \frac{c_2}{4}\tanh(\frac{c_1 x + c_2 y}{2}) + \frac{c_2}{4}\tanh(\frac{c_1 x - c_2 y}{2}) \end{pmatrix},$$

where we abbreviate $c_1 := \alpha + \beta$ and $c_2 := \beta - \alpha$. This means, we aim to solve

$$x = \frac{1}{2}\tanh\left(\frac{c_1 x + c_2 y}{2}\right) + \frac{1}{2}\tanh\left(\frac{c_1 x - c_2 y}{2}\right)$$
$$y = \frac{1}{2}\tanh\left(\frac{c_1 x + c_2 y}{2}\right) - \frac{1}{2}\tanh\left(\frac{c_1 x - c_2 y}{2}\right).$$

This is done in the spirit of the arguments in Section 2. Indeed, denoting by

$$G(x,y) := \begin{pmatrix} \frac{1}{2}\tanh(\frac{c_1 x + c_2 y}{2}) + \frac{1}{2}\tanh(\frac{c_1 x - c_2 y}{2}) \\ \frac{1}{2}\tanh(\frac{c_1 x + c_2 y}{2}) - \frac{1}{2}\tanh(\frac{c_1 x - c_2 y}{2}) \end{pmatrix},$$

we see that its Jacobian is given by

$$J_G(x,y) = \begin{pmatrix} \frac{c_1}{4}\left(\frac{1}{\cosh^2(\frac{c_1 x + c_2 y}{2})} + \frac{1}{\cosh^2(\frac{c_1 x - c_2 y}{2})}\right) & \frac{c_2}{4}\left(\frac{1}{\cosh^2(\frac{c_1 x + c_2 y}{2})} - \frac{1}{\cosh^2(\frac{c_1 x - c_2 y}{2})}\right) \\ \frac{c_1}{4}\left(\frac{1}{\cosh^2(\frac{c_1 x + c_2 y}{2})} - \frac{1}{\cosh^2(\frac{c_1 x - c_2 y}{2})}\right) & \frac{c_2}{4}\left(\frac{1}{\cosh^2(\frac{c_1 x + c_2 y}{2})} + \frac{1}{\cosh^2(\frac{c_1 x - c_2 y}{2})}\right) \end{pmatrix}.$$

This means that $||J_G(x,y)||_1 \leq \max\left(\frac{c_1}{2}, \frac{c_2}{2}\right) = \frac{c_1}{2} < 1$, where $\|\cdot\|_1$ denotes the maximum absolute column sum of a matrix. Thus G is a (strict) contraction with fixed point $(0,0)$ and hence $\tilde{\Phi}$ is minimal in $(0,0)$. Therefore, for every $r > 0$.

$$\inf_{(x,y) \notin B(0,r)} \tilde{\Phi}(x,y) > 0.$$

This implies that

$$\lim_{N \to \infty} \int_{A \cap B(0,r\sqrt{N})^c} \Psi(x,y) dx\, dy = \lim_{N \to \infty} \int_{A \cap B(0,r\sqrt{N})^c} e^{-N\tilde{\Phi}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right)} = 0$$

for any $r > 0$. Therefore, the outer region is asymptotically negligible.

Let us turn to the intermediate region. Here we take again a Taylor expansion of the $\log\cosh$ on an interval $[-z_0, z_0]$, $z_0 > 0$, around the origin to first order with a Lagrange

bound on the remainder: $\log\cosh(z) = \frac{z^2}{2} + Cz^4$ with a constant C that depends on z_0. This yields for $\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}} \in B(0,r)$

$$N\tilde{\Phi}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right) = \frac{x^2}{2}\left(\frac{\alpha+\beta}{2}\right) + \frac{y^2}{2}\left(\frac{\beta-\alpha}{2}\right)$$
$$- \frac{N}{4}\left(\frac{(\alpha+\beta)x}{2\sqrt{N}} + \frac{(\beta-\alpha)y}{2\sqrt{N}}\right)^2 - \frac{N}{4}\left(\frac{(\alpha+\beta)x}{2\sqrt{N}} - \frac{(\beta-\alpha)y}{2\sqrt{N}}\right)^2$$
$$- C_r N\left[\left(\frac{(\alpha+\beta)x}{2\sqrt{N}} + \frac{(\beta-\alpha)y}{2\sqrt{N}}\right)^4 + \left(\frac{(\alpha+\beta)x}{2\sqrt{N}} - \frac{(\beta-\alpha)y}{2\sqrt{N}}\right)^4\right]$$
$$\geq \frac{x^2}{2}\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right) + \frac{y^2}{2}\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right) - 2C_r N\left(\frac{|x|+|y|}{\sqrt{N}}\right)^4,$$

where we used $\frac{\alpha+\beta}{2} \leq 1$. However, on $A_R^r := A \cap B(0,R)^c \cap B(0,r\sqrt{N})$ we can estimate the last line by

$$2C_r N\left(\frac{|x|+|y|}{\sqrt{N}}\right)^4 = 2C_r(|x|+|y|)^2\left(\frac{|x|+|y|}{\sqrt{N}}\right)^2 \leq 8C_r(x^2+y^2)r^2 =: \tilde{C}_r r^2(x^2+y^2)$$

for $\tilde{C}_r$ uniformly on $B(0, r\sqrt{N})$. Note that $\tilde{C}_r r^2$ depends continuously on r and converges to 0 as $r \to 0$. In particular, if r is small enough we have that $\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right) - \tilde{C}_r r^2 > 0$ as well as $\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right) - \tilde{C}_r r^2 > 0$. But for this choice of r and $\tilde{C}_r$ we arrive at

$$\int_{A_R^r} \Psi(x,y)\,dx\,dy$$
$$\leq \int_{A_R^r} \exp\left[-\frac{x^2}{2}\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2 - C_r r^2\right) - \frac{y^2}{2}\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2 - C_r r^2\right)\right] dxdy$$

and the right hand side is an integrable function. Thus for $R \to \infty$ the right hand side as well as the left hand side converges to 0.

Putting the estimates together, we have seen that $\chi_{N,\alpha,\beta}$ converges weakly to the 2-dimensional Gaussian distribution with expectation 0 and covariance matrix Σ', where Σ' is given in (3.1). This weak convergence is equivalent to the convergence of the characteristic functions.

Computing the characteristic functions of the Gaussian distribution involved in the above proof, we have therefore shown that the characteristic function of $\tilde{w}$ in the point $t = (t_1, t_2) \in \mathbb{R}^2$ satisfies

$$\lim_{N\to\infty} \mathbb{E}(e^{it\tilde{w}}) e^{-\frac{1}{2}\left(\frac{2}{\alpha+\beta}\right)t_1^2 - \frac{1}{2}\left(\frac{2}{\beta-\alpha}\right)t_2^2} = e^{-\frac{1}{2}t_1^2\left[\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right]^{-1} - \frac{1}{2}t_2^2\left[\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right]^{-1}}.$$

This implies that

$$\lim_{N\to\infty} \mathbb{E}(e^{it\tilde{w}}) = e^{\frac{1}{2}\left(\frac{2}{\alpha+\beta}\right)t_1^2 + \frac{1}{2}\left(\frac{2}{\beta-\alpha}\right)t_2^2} e^{-\frac{1}{2}t_1^2\left[\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2}\right)^2\right]^{-1} - \frac{1}{2}t_2^2\left[\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right]^{-1}}$$
$$= e^{-\frac{1}{2}t_1^2\left(\frac{1}{1 - \frac{\alpha+\beta}{2}}\right) - \frac{1}{2}t_2^2\left(\frac{1}{1 - \frac{\beta-\alpha}{2}}\right)}.$$

Turning this into a weak convergence statement again, we obtain

$$(\tilde{w}_1, \tilde{w}_2) \xrightarrow{N\to\infty} \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{pmatrix} \frac{1}{1 - \frac{\alpha+\beta}{2}} & 0 \\ 0 & \frac{1}{1 - \frac{\beta-\alpha}{2}} \end{pmatrix}$$

in distribution. $\qquad\square$

Proof of Theorem 1.2. The proof of Theorem 1.2 is straightforward from the above lemma. As observed we have that $m_1 = w_1 + w_2$ and $m_2 = w_1 - w_2$, thus $\sqrt{N}m_1 = \sqrt{N}(w_1 + w_2) = \tilde{w}_1 + \tilde{w}_2$ as well as $\sqrt{N}m_2 = \sqrt{N}(w_1 - w_2) = \tilde{w}_1 - \tilde{w}_2$. Thus Lemma 3.1 gives that m_1 and m_2 are asymptotically normal with mean 0 and variance

$$\lim_{N \to \infty} \mathbb{V}(\sqrt{N}m_1) = \frac{1}{1 - \frac{\beta + \alpha}{2}} + \frac{1}{1 - \frac{\beta - \alpha}{2}} = \frac{2}{2 - \beta - \alpha} + \frac{2}{2 - \beta + \alpha} = \frac{4(2 - \beta)}{(2 - \beta)^2 - \alpha^2}.$$

Moreover, the same considerations together with Lemma 3.1 show that their covariance is given by

$$\lim_{N \to \infty} \text{Cov}(\sqrt{N}m_1, \sqrt{N}m_2) = \mathbb{V}\tilde{w}_1 - \mathbb{V}\tilde{w}_2 = \frac{4\alpha}{(2 - \beta)^2 - \alpha^2} = \frac{4(2 - \beta)}{(2 - \beta)^2 - \alpha^2}\frac{\alpha}{2 - \beta}$$

as proposed. □

On the other hand, the proof Lemma 3.1 also inspires the proof of Theorem 1.4. Indeed, redoing the computations there shows that for $\alpha + \beta = 2$ the quadratic term in the first component of $\chi_{N,\alpha,\beta}$ cancels. To this end we have to rescale $\tilde{w}_1$ to make the second term in the Taylor expansion of $\log \cosh$ appear (as a matter of fact this is very similar, to what happens in the Curie-Weiss model at its critical temperature $\beta = 1$).

Proof of Theorem 1.4. As motivated above we will now consider the vector $\hat{w} = (\hat{w}_1, \hat{w}_2)$ consisting of the components

$$\hat{w}_1 := N^{1/4}w_1 \qquad \text{and} \qquad \hat{w}_2 := \sqrt{N}w_2 = \tilde{w}_2.$$

This time we will convolute the distribution of $\hat{w}$ under the Gibbs measure $\mu_{N,\alpha,\beta,S}$ with a two-dimensional Gaussian distribution $\mathcal{N}(0, \hat{C})$, where $\hat{C} = \begin{pmatrix} \frac{1}{\sqrt{N}} & 0 \\ 0 & \frac{2}{\beta - \alpha} \end{pmatrix}$ (note that this is well defined since $\beta > \alpha$). Computing the density of $\hat{\chi}_{N,\alpha,\beta} := \mu_{N,\alpha,\beta}(\hat{w})^{-1} * \mathcal{N}(0, \hat{C})$ as in the proof of Lemma 3.1 we obtain building on the fact that now $\alpha + \beta = 2$

$$\hat{\chi}_{N,\alpha,\beta}(A)$$
$$= \hat{K}_1 \sum_{\sigma \in \{-1,1\}^N} \int_A \exp\left(-\frac{1}{2}(\sqrt{N}(x - \hat{w}_1)^2 + \frac{\beta - \alpha}{2}(y - \hat{w}_2)^2)\right) \varrho_{N,\alpha,\beta}(\sigma) dx dy$$
$$= \hat{K}_2 \sum_{\sigma \in \{-1,1\}^N} \int_A \exp\left(-\frac{1}{2}(\sqrt{N}(x - \hat{w}_1)^2 + \frac{\beta - \alpha}{2}(y - \hat{w}_2)^2)\right)$$
$$\times \exp\left(\frac{1}{4}(2\sqrt{N}\hat{w}_1^2 + (\beta - \alpha)\tilde{w}_2^2)\right) dx dy$$
$$= \hat{K}_2 \sum_{\sigma \in \{-1,1\}^N} \int_A \exp\left(-\frac{\sqrt{N}}{2}x^2 - \frac{1}{4}(\beta - \alpha)y^2\right) \exp\left(\sqrt{N}x\hat{w}_1 + \frac{1}{2}(\beta - \alpha)y\tilde{w}_2\right) dx dy$$
$$= \hat{K}_3 \int_A \exp\left(-\frac{\sqrt{N}}{2}x^2 - \frac{1}{4}(\beta - \alpha)y^2\right) \exp\left(\frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} + \frac{\beta - \alpha}{2\sqrt{N}}y\right)\right.$$
$$\left. + \frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} - \frac{\beta - \alpha}{2\sqrt{N}}y\right)\right) dx dy$$

with the normalizing constants $\hat{K}_1, \hat{K}_2$, and $\hat{K}_3$ chosen similarly to K_1, K_2, and K_3 in the proof of Lemma 3.1. Now we expand the $\log \cosh$ to fourth order: $\log\cosh(z) = \frac{z^2}{2} - \frac{1}{12}z^4 + \mathcal{O}(z^6)$. We thus see that the x^2 terms in the exponent cancel and so do the

xy-terms (fortunately). For fixed x and y only the x^4 is of vanishing order. The y^2 terms are treated as in the proof of Lemma 3.1. We thus see that

$$-\frac{\sqrt{N}}{2}x^2 - \frac{1}{4}(\beta-\alpha)y^2 + \frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} + \frac{\beta-\alpha}{2\sqrt{N}}y\right) + \frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} - \frac{\beta-\alpha}{2\sqrt{N}}y\right)$$
$$= -\frac{1}{12}x^4 - \frac{1}{2}y^2\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right) + \mathcal{O}(N^{-\frac{1}{2}})$$

with a $\mathcal{O}(N^{-\frac{1}{2}})$ term that depends on x and y. To conclude the convergence of $\hat{\chi}_{N,\alpha,\beta}(A)$ we now proceed as in the proof of Lemma 3.1. Here we will only sketch the differences, because many steps are very similar. The exact steps are left to the reader. The main differences to the above proof of Lemma 3.1 is that the inner region is again $B(0,R)$, while the intermediate region now is the rectangle $[-,N^{\frac{1}{4}}r] \times [-r\sqrt{N}, r\sqrt{N}]$ to take into account that the Taylor expansion gives another order for x than for y. Correspondingly in the intermediate region the integrable function that dominates

$$\exp\left[-\frac{\sqrt{N}}{2}x^2 - \frac{1}{4}(\beta-\alpha)y^2\right]$$
$$\times \exp\left[\frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} + \frac{\beta-\alpha}{2\sqrt{N}}y\right) + \frac{N}{2}\log\cosh\left(\frac{x}{N^{1/4}} - \frac{\beta-\alpha}{2\sqrt{N}}y\right)\right]$$

is given by $\exp(-d_1 x^4 - d_2 y^2)$ for suitable constants $d_1, d_2 > 0$. With these changes we see that $\hat{\chi}_{N,\alpha,\beta}(A)$ converges to a 2-dimensional distribution with density proportional to

$$\exp\left(-\frac{1}{12}x^4 - \frac{1}{2}y^2\left(\frac{\beta-\alpha}{2} - \left(\frac{\beta-\alpha}{2}\right)^2\right)\right)$$

with respect to the 2-dimensional Lebesgue measure. However, from here we see that $\hat{w}_1$ converges in distribution to a random variable with density proportional to $e^{-\frac{1}{12}x^4}$ since the Gaussian measure we convoluted the first coordinate of $\hat{w}$ with converges to 0 in probability. Moreover, the same computation as in Lemma 3.1 shows that $\hat{w}_2 = \tilde{w}_2$ converges to a normal distribution with mean 0 and variance $\frac{2}{2-(\beta-\alpha)}$.

However, the latter convergence implies that $N^{\frac{1}{4}}w_2$ converges to 0 in probability. Thus $N^{\frac{1}{4}}m_1 = N^{\frac{1}{4}}w_1 + N^{\frac{1}{4}}w_2$ also converges in distribution to a random variable with density proportional to $e^{-\frac{1}{12}x^4}$ (see e.g. [3, Theorem 3.1]). $\qquad\square$

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