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Absolute continuity of complex martingales and of solutions to complex smoothing equations

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Abstract

Let X be a $\mathbb{C}$-valued random variable with the property that

$$X \text{ has the same law as } \sum_{j\geq 1} T_j X_j$$

where X_j are i.i.d. copies of X, which are independent of the (given) $\mathbb{C}$-valued random variables $(T_j)_{j\geq 1}$. We provide a simple criterion for the absolute continuity of the law of X that requires, besides the known conditions for the existence of X, only finiteness of the first and second moment of N - the number of nonzero weights T_j. Our criterion applies in particular to Biggins' martingale with complex parameter.

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1 Introduction

In a variety of models coming from theoretical computer science, applied probability, economics or statistical physics, quantities of interest exhibit asymptotic fluctuations that do not have a normal or α-stable distribution. In many cases, the limiting law μ can be characterized as a fixed point of a mapping $\mathcal{S}$ of the form

$$\mathcal{S}(\mu) \;=\; \text{Law of } \Big(\sum_{j\geq 1} T_j X_j\Big), \tag{1.1}$$

where X_j are i.i.d. complex-valued random variables with law μ and independent of the given complex variables $(T_j)_{j\geq 1}$. See [12] and references therein for a list of examples.

The fixed point property $\mu = \mathcal{S}(\mu)$ then may and shall be used to analyze properties of μ. Let us stress at this early point that $\mathcal{S}$ usually has multiple fixed points, which have to be analyzed by different methods. They can roughly be classified by a parameter α: The first class of fixed points are mixtures of α-stable laws, while the second class of fixed points appears only for $\alpha \geq 1$. Fixed points of the second class are limits of martingales in an associated weighted branching process. Under an additional very mild assumption, fixed points from the second class are integrable.

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In this note, we will study absolute continuity of fixed points of $\mathcal{S}$ and take advantage of the classification described above, which was recently given in [12]. This simplifies essentially the approach. For fixed points from the first class, absolute continuity can be proved along similar lines as for infinitely divisble laws. For fixed points from the second class, we apply Fourier analytic methods and then integrability allows us to work with derivatives of the characteristic function.

History of the problem and related works

In the theory of branching processes, it is a well-established technique to analyze decay rates of the derivative of the characteristic function ϕ in order to prove absolute continuity of martingale limits; see [2, 4, 9, 13]. The idea is to prove that $\phi'(t)$ decays at infinity at least like $t^{-1/2-\varepsilon}$ for a positive ε, which is sufficient to prove that ϕ' is square-integrable, which in turn provides an L^2 density for the size-biased law of the random variable in question. In the complex setting considered here, one has to provide more refined estimates: Namely, if X is a complex-valued random variable, we define its characteristic function as if X were a $\mathbb{R}^2$-valued random vector. Then one has to prove that $\phi' \in L^2(\mathbb{R}^2)$. For this to hold, we need a twice as fast decay at infinity as for real-valued random variables.

Alternatively, we may consider $I(K) = \int_{|t|\le K} |\phi'(t)|^2 \, dt$ and prove, by a kind of recurrence argument, that $I(K)$ is bounded independently of K. However, again the dimension makes a difference and neither of previously suggested arguments works directly for $\mathbb{R}^2$.

Absolute continuity of solutions to (1.1) has been considered before; mainly in the setup where T_j and X_j are *nonnegative* real-valued random variables, see [2, 4, 9]. The most general conditions for absolute continuity of nonnegative solutions are provided in [9].

The real- and complex-valued setup considered in our work has been treated recently also in [8] within the framework of systems of fixed point equations, but under stronger assumptions than imposed here. For instance negative moments of $(T_j)_{j\ge1}$ are required which is not natural for (1.1), while integrability of $(T_j)_{j\ge1}$ is. The approach in [8] is different, for it does not take into account a-priori knowledge such as the classification of fixed points described above. Absolute continuity of a specific complex-valued model was also studied in [3].

We continue in Section 2 with a precise description of the setup and the set of fixed points of $\mathcal{S}$. Then we state our results and describe several examples that motivated our study. The proofs are given in Section 3.

2 Statement of results

2.1 Solutions to complex smoothing equations

Let $(T_j)_{j\ge1}$ be complex-valued random variables, satisfying

$$N \ := \ \#\{j \,:\, T_j \neq 0\} = \max\{j \,:\, T_j \neq 0\} < \infty \quad \mathbb{P}\text{-a.s.}$$

Let X be a complex random variable with law μ such that $\mathcal{S}(\mu) = \mu$. Then

$$X \ \stackrel{\text{law}}{=} \ \sum_{j=1}^{N} T_j X_j, \tag{2.1}$$

where $\stackrel{\text{law}}{=}$ means same law. Upon introducing the function

$$m(s) \ := \ \mathbb{E}\Big[\sum_{j\ge1} |T_j|^s\Big],$$

consider the following conditions.

$$m(0) = \mathbb{E}[N] > 1. \tag{A1}$$

$$m(\alpha) = 1 \text{ for some } \alpha > 0. \tag{A2}$$

Under (A2), $W_1 := \sum_{j=1}^N |T_j|^\alpha$ defines a mean one random variable. Consider further

$$m'(\alpha) := \mathbb{E}\Big[\sum_{j=1}^N |T_j|^\alpha \log|T_j|\Big] \in (-\infty, 0) \text{ and } \mathbb{E}\Big[W_1 \log_+ W_1\Big] < \infty. \tag{A3}$$

Let $U \subset \mathbb{C}$ be the smallest closed multiplicative subgroup generated by the support of $(T_j)_{j\geq 1}$.

If the assumptions (A1)–(A3) (plus an additional technical assumption if $\alpha = 1$) are satisfied, then [12, Theorem 1.2] describes all solutions to (2.1). Namely, there exists a nonnegative random variable W with unit mean and a $\mathbb{C}$-valued random variable Z (both are described in detail below) such that if X satisfies (2.1), then

$$X \stackrel{\text{law}}{=} Y_W + xZ, \tag{2.2}$$

where $x \in \mathbb{C}$ and $(Y_t)_{t\geq 0}$ is a complex-valued Lévy process with the invariance property

$$uY_t \stackrel{\text{law}}{=} Y_{|u|^\alpha t} \qquad \text{for all } u \in U, t > 0, \tag{2.3}$$

and $(Y_t)_{t\geq 0}$ is independent of (W, Z). Note that $Y_t \equiv 0$ is a valid choice. If $(Y_t)_{t\geq 0}$ is nontrivial, it holds $\mathbb{E}\big[|Y_W|^\alpha\big] = \infty$, see [12, Remark 1.4].

2.1.1 Martingales and the weighted branching process

To give a description of W and Z, let us define a weighted branching process as follows: Let $\mathbb{V} = \bigcup_{n=0}^\infty \mathbb{N}^n$ denote the infinite tree with Harris-Ulam labelling and root $\emptyset$. For each $v \in \mathbb{V}$, we denote by $|v|$ its generation. To each $v \in \mathbb{V}$, we attach an independent copy $(T_1(v), T_2(v), \dots)$ of $(T_j)_{j\geq 1}$ and define the weighted branching process by

$$L(\emptyset) := 1, \quad L(vi) := T_i(v)L(v),$$

where vi denotes concatenation: if $v = v_1 \cdots v_k$, then $vi = v_1 \cdots v_k i$.

Then $W := \lim_{n\to\infty} W_n := \lim_{n\to\infty} \sum_{|v|=n} |L(v)|^\alpha$ with $\mathbb{E}[W] = 1$. Here, (A2) implies that W_n is a martingale and (A3) guarantees its convergence in L^1 by Biggins' theorem, see [10].

$Z = 0$ unless $\mathbb{E}\big[\sum_{j=1}^N T_j\big] = 1$ and $\alpha \geq 1$. If these requirements are satisfied, then $Z_n := \sum_{|v|=n} L(v)$ defines a $\mathbb{C}$-valued martingale with mean one. Whenever this martingale converges a.s., we set $Z := \lim_{n\to\infty} Z_n$; and define $Z := 0$ if the convergence fails. Note that convergence of the martingale Z_n is not guaranteed by the assumptions (A1) – (A3) considered so far. That is, the representation (2.2) holds, but in order to determine whether or not $Z = 0$, we need additional assumptions. To proceed, we have to distinguish three cases.

If $\alpha = 1$, then $1 = m(1) = \mathbb{E}[\sum_{j=1}^N |T_j|] = \mathbb{E}[\sum_{j=1}^N T_j]$ implies $U \subset \mathbb{R}_+$, hence $Z_n = W_n$. Since $W_n \to W$ a.s and in L^1 under (A1) – (A3), it holds $Z = W$. Continuity properties of the nonnegative random variable W have been studied in [2, 9].

If $\alpha \geq 2$, under mild conditions, Z cannot be absolutely continuous. More precisely, under (A1)–(A3), [12, Proposition 1.1] shows that Z_n converges to a nonzero limit if and only if $Z = Z_n = z$ a.s. for a deterministic $z \in \mathbb{C} \setminus \{0\}$. In addition, [7, Proposition 2.2] gives conditions under which even the convergence to a nonzero deterministic limit fails for $\alpha = 2$.

Hence, the interesting case is $1 < \alpha < 2$. In this case, it follows from [12, Proposition 1.1] that Z_n is uniformly integrable, hence $Z_n \to Z$ in L^1, whenever Z_n converges at all. This is why we will require the following assumption in our results:

$$\lim_{n\to\infty} Z_n \text{ exists a.s. and in } L^1. \tag{Z1}$$

If (A1)-(A2) hold then a sufficient condition for Z_n to converge a.s. and in L^p for all $p < \alpha$ is $\alpha \in (1, 2)$ and

$$m'(\alpha) \le 0 \ \text{ and } \ \mathbb{E}\Big[\big|Z_1\big|^{\alpha} \log_+^{2+\epsilon} |Z_1|\Big] \text{ for some } \epsilon > 0, \tag{A4}$$

see [7, Theorem 2.1].

To summarize, under (A1)–(A3), we have to investigate absolute continuity of the law of Y_W for $\alpha \in (0, 2]$ (there is only the trivial process $Y_t \equiv 0$ if $\alpha > 2$) and absolute continuity of the law of Z for $\alpha \in (1, 2)$, for Z is trivial or deterministic if $\alpha \notin (1, 2)$.

2.2 Results

When studying absolute continuity of Z, we may focus on the case $1 < \alpha < 2$ by the above discussion. We can further assume that $\mathbb{P}(N = 0) = 0$, i.e., the weighted branching process survives with probability 1. Namely, if there were a positive probability $q \in (0, 1)$ of extinction, then any fixed point μ of $\mathcal{S}$ can be decomposed as $\mu = q\delta_0 + (1-q)\mu^*$, where μ^* is a fixed point of a mapping $\mathcal{S}^*$ associated with the weighted branching process consisting of individuals with infinite lines of descent, conditioned on non-extinction. See [2, p. 742] for the details of this construction, which extends to our complex setting.

Finally, to avoid trivial cases, we have to assume that

$$\mathbb{P}\Big(\sum_{j=1}^{N} T_j = 1\Big) \ < \ 1, \tag{Z2}$$

since otherwise, $Z = Z_n = 1$ a.s.

Theorem 2.1. *Suppose $N > 0$ a.s., (A1)-(A2) with $\alpha \in (1, 2)$, (Z1)-(Z2) and $U \not\subseteq \mathbb{R}$ together with*

$$\mathbb{E}\big[N^2\big] < \infty \ \textit{ and } \ \mathbb{E}\Big[N \sum_{j=1}^{N} \log_+ |T_j|\Big] < \infty. \tag{C1}$$

Then the law of Z is absolutely continuous.
If $U \subset \mathbb{R}$, then (C1) can be replaced with the assumption $\mathbb{E}[N] < \infty$.

Recall that U is the smallest closed multiplicative subgroup generated by the support of $(T_j)_{j\ge1}$. Hence $U \subset \mathbb{R}$ entails $\operatorname{supp}(Z) \subset \mathbb{R}$. As mentioned before, (A4) is a mild sufficient condition for (Z1). If higher order moment conditions on Z and N are satisfied, one can prove further smoothness properties of the Fourier transform of Z, see Remark 3.6.

Concerning $(Y_t)_{t\ge0}$, standard arguments yield the following continuity result:

Proposition 2.2. *Suppose (A1)-(A2) with $\alpha \in (0, 2]$. Suppose that $(Y_t)_{t\ge0}$ is a nondegenerate complex-valued Lévy process satisfying (2.3) and that there is no U-invariant $\mathbb{R}$-linear subspace of $\mathbb{C}$. Then for each $t > 0$, the law of Y_t is absolutely continuous.*

Combining both results and using that $(Y_t)_{t\ge0}$ is independent of (W, Z) in the representation (2.2), we have:

Corollary 2.3. *Suppose (A1)-(A4), (C1), (Z2) and that there is no U-invariant $\mathbb{R}$-linear subspace of $\mathbb{C}$. Then the law of any nontrivial solution to (2.1) is absolutely continouous.*

2.3 Examples

Biggins' martingale with complex parameter

A branching random walk is defined as follows. An ancestor at the origin produces offspring which is displaced on $\mathbb{R}$ according to a point process. Each new particle then produces again offspring independently of all other particles according to the same law. Denote the positions of the n-th generation particles by $(S(v))_{|v|=n}$ and suppose that for some $\lambda \in \mathbb{C}$,

$$\mathfrak{m}(\lambda) \ := \ \mathbb{E}\Big[\sum_{|v|=1} e^{-\lambda S(v)} \Big]$$

exists and is nonzero. Then

$$\mathcal{W}_n(\lambda) \ := \ \mathfrak{m}(\lambda)^{-n} \sum_{|v|=n} e^{-\lambda S(v)}$$

defines a $\mathbb{C}$-valued martingale that coincides with Z_n upon identifying

$$T_j = \mathfrak{m}(\lambda)^{-1} e^{-\lambda S(j)}.$$

These complex martingales were studied in [1] to analyze the frequencies of particles with a certain speed in the branching random walk. See [1, 7] for conditions ensuring (Z1) and explicit examples.

Cyclic Pólya urns

A cyclic Pólya urn consists of balls of b different types. Each time a ball of type m is drawn, it is placed back into the urn together with a ball of type $m+1 \mod b$. If $b \geq 7$, the asymptotic fluctuations of the proportion of balls of a given type are described in terms of a complex random variable X with finite variance that satisfies

$$X \overset{\text{law}}{=} U^{\zeta} X_1 + \zeta(1-U)^{\zeta} X_2,$$

where $\zeta = \exp(\mathrm{i}\frac{2\pi}{b})$ is a b-th root of unity and X_1, X_2 are i.i.d. copies of X which are independent of U, which is a uniform $[0,1]$-random variable; see e.g. [6].

We show how our result applies. With $T_1 = U^{\zeta}$ and $T_2 = \zeta(1-U)^{\zeta}$, the Assumptions (A1)–(A4) and (Z1) are readily checked. The real part $\Re(\zeta)$ of ζ is greater than 1/2 iff $b \geq 7$, thus $\alpha = 1/\Re(\zeta) \in (1,2)$ as soon as $b \geq 7$. Since the solution of interest has a second moment, it has to be $X = xZ$ for some $x \in \mathbb{C}$. The set $\mathcal{Z} := \operatorname{supp}(Z)$ has to satisfy

$$u^{\zeta}\mathcal{Z} + \zeta(1-u)^{\zeta}\mathcal{Z} \ \subset \ \mathcal{Z} \qquad \text{for all } u \in [0,1]$$

which yields that $\mathcal{Z} \not\subseteq \mathbb{R}$. Hence Theorem 2.1 applies and shows that X has a density.

3 Proofs

3.1 An equation for characteristic functions

In order to obtain an equation for the characteristic function of X satisfying (2.1), we first note that its characteristic function is defined as if X were a $\mathbb{R}^2$-valued random vector. That is, we define for $\xi = \xi_1 + \mathrm{i}\xi_2$, $X = X_{(1)} + \mathrm{i}X_{(2)}$

$$\phi_X(\xi) \ := \ \mathbb{E}\Big[\exp\big(-\mathrm{i}\big[\xi_1 X_{(1)} + \xi_2 X_{(2)}\big]\big)\Big] \ = \ \mathbb{E}\Big[\exp\big(-\mathrm{i}\frac{1}{2}\big[\xi\bar{X} + \bar{\xi}X\big]\big)\Big] \tag{3.1}$$

where $\bar{\xi} = \xi_1 - \mathrm{i}\xi_2$ denotes the complex conjugate. We deduce that the characteristic function of TX, $T \in \mathbb{C}$, is then given by

$$\phi_{TX}(\xi) \;=\; \mathbb{E}\Big[\exp\big(-\mathrm{i}\frac{1}{2}\big[\xi\cdot\overline{TX}+\bar{\xi}\cdot TX\big]\big)\Big] \;=\; \mathbb{E}\Big[\exp\big(-\mathrm{i}\frac{1}{2}\big[\bar{T}\xi\cdot\bar{X}+\overline{\bar{T}\xi}\cdot X\big]\big)\Big] \;=\; \phi_X(\bar{T}\xi).$$

Using this, Eq. (2.1) gives rise to an equation for the characteristic function ϕ_X, namely

$$\phi_X(\xi) \;=\; \mathbb{E}\Big[\prod_{j=1}^{N}\phi_X(\bar{T}_j\xi)\Big]. \tag{3.2}$$

The above identity will be used later to study properties of ϕ and its derivatives.

3.2 Proof of Propositon 2.2

Let X be a random vector in $\mathbb{R}^d$ with characteristic function ϕ_X. Then (the law of) X is called *full*, if for all $v \neq 0$ in $\mathbb{R}^d$, $\langle v, X\rangle$ is not a point mass. A complex-valued random variable X is full, if it is full upon identifying $\mathbb{C} \simeq \mathbb{R}^2$. If X is full, then there is $\epsilon > 0$ such that $|\phi_X(\xi)| < 1$ for all $0 < |\xi| < \epsilon$, see [11, Lemma 1.3.15].

Proof of Proposition 2.2. If there is no U-invariant $\mathbb{R}$-linear subspace, then the invariance property (2.3) yields that the support of Y_t is also not contained in a proper linear subspace of $\mathbb{C}$, hence Y_t is full. By (A1) and (A2), the function m is not constant, hence there is $u \in U$ with $|u| \neq 1$. Then, using that Y_t is infinitely divisible, Eq. (2.3) yields that Y_t is *operator semistable* (see [11, Definition 7.1.2]). By [11, Theorem 7.1.15], a full operator semistable law has a density with respect to Lebesgue measure. $\square$

3.3 Proof of Theorem 2.1

Lemma 3.1. *Assume* (A1)*,* (A2) *and* (Z1)*. Then* $\mathrm{supp}(Z)$ *is closed under multiplication.*

Proof. Mutatis mutandis, this is proved along the same lines as [2, Theorem 2]. $\square$

In the following, we restrict our attention to the case where Z is properly $\mathbb{C}$-valued, *i.e.*, $\mathrm{supp}(Z) \not\subseteq \mathbb{R}$. This automatically excludes the case $\mathbb{P}(\sum_{j=1}^N T_j = 1) = 1$. The simpler case $\mathrm{supp}(Z) \subset \mathbb{R}$ requires only minor modifications. If $\mathrm{supp}(Z) \not\subseteq \mathbb{R}$, then Lemma 3.1 yields that $\mathrm{supp}(Z)$ is not contained in any affine $\mathbb{R}$-linear subspace of $\mathbb{C}$, hence Z is full.

From now on, ϕ is the characteristic function ϕ_Z of Z.

Lemma 3.2. *Assume* (A1)*,* (A2) *and* (Z1)*, as well as* $N \geq 1$ *a.s. and* $\mathbb{E}[N] < \infty$*. Then* $\ell \;:=\; \limsup_{|\xi|\to\infty}|\phi(\xi)| = 0$*.*

Proof. By the same arguments as in [9, Lemma 3.1 (i)], $\ell \in \{0, 1\}$. As the next step, we prove that $|\phi(\xi)| < 1$ for all $\xi \neq 0$.

Since Z is full, [11, Lemma 1.3.15] yields that there is $\eta > 0$ such that $|\phi(\xi)| < 1$ for all $0 < |\xi| < \eta$. Suppose

$$R := \inf\big\{r > 0 \,:\, \exists\,\xi \text{ with } |\xi| = r \text{ s.t. } |\phi(\xi)| = 1\big\} < \infty.$$

Then choose ξ^* with $|\xi^*| = R$ and $|\phi(\xi^*)| = 1$. Taking absolute values on both sides of Eq. (3.2) yields

$$1 \leq |\phi(\xi^*)| \leq \mathbb{E}|\phi(\bar{T}_1\xi^*)| \leq 1,$$

thus $|\phi(\bar{T}_1\xi^*)| = 1$ a.s. This would require $|T_1| \geq 1$ a.s. At the same time, since $N \geq 1$ a.s. and $\mathbb{E}N > 1$, we have that $\mathbb{P}(|T_2| > 0) > 0$. Together, this would give $m(s) \geq \mathbb{E}\big[|T_1|^s + |T_2|^s\big] > 1$ for all $s > 0$; which contradicts $m(\alpha) = 1$. Hence $R = \infty$.

It remains to prove $\ell < 1$. By (A2) and the branching property, $\mathbb{E}\sum_{|v|=n}|L(v)|^\alpha = 1$ for all $n \in \mathbb{N}$, which yields that the expected number of summands exceeding 1 has to be smaller than one. In addition, $\mathbb{E}[\#\{v : |v| = n\}] = (\mathbb{E}[N])^n < \infty$ gives that we can choose δ and n such that

$$\mathcal{L} \ := \ \{v : |v| = n,\ \delta \le |L(v)| \le 1\}$$

satisfies $1 < \mathbb{E}[\#\mathcal{L}] < \infty$. Hence, for the moment generating function $\kappa(s) := \mathbb{E}\big[s^{\#\mathcal{L}}\big]$ it holds $s - \kappa(s) > 0$ for all $s \in (\eta, 1)$, where η is the unique root of $\kappa(s) - s = 0$ on the interval $[0, 1)$.

Suppose $\ell = 1$. By the previous step, for sufficiently small $\epsilon > 0$, there are $0 < t_1 < t_2$ with $t_1 < \delta t_2$ s.t. $|\phi(\xi)| < 1 - \epsilon$ for all $t_1 < |\xi| < t_2$, while there is ξ^* with $|\xi^*| = t_2$ s.t. $|\phi(\xi^*)| = 1 - \epsilon$. By iterating Eq. (3.2), we obtain

$$1 - \epsilon = |\phi(\xi^*)| \ \le \ \mathbb{E}\Big[\prod_{v \in \mathcal{L}} \big|\phi\big(\overline{L(v)}\xi^*\big)\big|\Big] \le \mathbb{E}\big[(1-\epsilon)^{\#\mathcal{L}}\big] \ = \ \kappa(1-\epsilon),$$

which contradicts $s > \kappa(s)$ for all $s \in (\eta, 1)$. □

Derivatives of the characteristic function

Note that ϕ is differentiable as soon as $\mathbb{E}\big[|Z|\big] < \infty$. To proceed further, we will consider the complex derivatives $\partial_{\bar\xi}\phi(\xi)$ and $\partial_\xi\phi(\xi)$, which are obtained by considering ξ and $\bar\xi$ as independent variables when differentiating, i.e. $\partial_\xi(\bar\xi) = 0$ and vice versa. See [5, pp. 22-23] for a list of properties of the complex derivatives.

Recall from (3.1) that the characteristic function ϕ of Z is given by

$$\phi(\xi) \ = \ \mathbb{E}\Big[\exp\big(-\mathrm{i}\frac{1}{2}(\xi\bar Z + \bar\xi Z)\big)\Big].$$

Hence

$$\partial_\xi\phi(\xi) \ = \ \mathbb{E}\Big[-\frac{\mathrm{i}}{2}\bar Z\exp\big(-\mathrm{i}\frac{1}{2}(\xi\bar Z + \bar\xi Z)\big)\Big],$$
$$\partial_{\bar\xi}\phi(\xi) \ = \ \mathbb{E}\Big[-\frac{\mathrm{i}}{2}Z\exp\big(-\mathrm{i}\frac{1}{2}(\xi\bar Z + \bar\xi Z)\big)\Big].$$

because $\partial_\xi(\xi z) = z$, $\partial_{\bar\xi}(\xi z) = 0$ for $z \in \mathbb{C}$. Therefore, by the chain rule for complex differentiation (see [5, p. 23])

$$\partial_\xi\phi(\bar T\xi) = \bar T(\partial_\xi\phi)(\bar T\xi), \qquad \partial_{\bar\xi}\phi(\bar T\xi) = T(\partial_{\bar\xi}\phi)(\bar T\xi). \tag{3.3}$$

As the first step, we are going to prove decay rates for both derivatives.

Decay rates

Lemma 3.3. *Suppose $N > 0$ a.s., (A1)-(A2) with $\alpha \in (1, 2)$, (Z1) and $\mathbb{E}[N] < \infty$. Then there is a finite constant C such that*

$$|\partial_\xi\phi(\xi)| \le C(1+|\xi|)^{-1} \quad \textit{and} \quad |\partial_{\bar\xi}\phi(\xi)| \le C(1+|\xi|)^{-1} \quad \textit{for all } \xi \in \mathbb{C}. \tag{3.4}$$

Remark 3.4. If $\mathrm{supp}(Z) \subset \mathbb{R}$ it follows that $\partial_\xi\phi(\xi), \partial_{\bar\xi}\phi(\xi)$ are square-integrable w.r.t. Lebesgue measure on $\mathbb{R}$.

Proof. We will prove the estimate for $\partial_{\bar\xi}\phi$. The proof for $\partial_\xi\phi$ is completely analogous, up to replacing T_j by $\bar T_j$. Define $g(\xi) := \partial_{\bar\xi}\phi(\xi)$. Then, differentiating both sides of Eq. (3.2) and using (3.3)

$$g(\xi) = \mathbb{E}\Big[\sum_{j=1}^N T_j g(\bar T_j\xi)\prod_{i \ne j}\phi(\bar T_i\xi)\Big]. \tag{3.5}$$

Note that the right hand side is finite by using that $m(1) < \infty$ and that g is bounded by $\mathbb{E}|Z| < \infty$. By Lemma 3.2, for every ε, there is t_ε such that $|\phi(\xi)| < \varepsilon$ for every $|\xi| > t_\varepsilon$. Given $\delta > 0$ let

$$N_\delta = \sum_{j=1}^{N} \mathbf{1}\{|T_j| > \delta\}.$$

If $N_\delta \geq 1$ and $|\xi| > t_\varepsilon \delta^{-1}$ then for all $1 \leq j \leq N$,

$$\prod_{i \neq j} |\phi(T_i \xi)| \leq \varepsilon^{N_\delta - 1} \tag{3.6}$$

and hence

$$|g(\xi)| \leq \mathbb{E}\Big[\varepsilon^{(N_\delta-1)_+} \sum_{j=1}^{N} |T_j||g(\bar{T}_j\xi)|\Big] \qquad \text{for } |\xi| > t_\varepsilon \delta^{-1}. \tag{3.7}$$

Here $(x)_+ = \max\{0, x\}$. Define a complex valued random variable B by setting

$$\mathbb{E}h(B) = q_{\varepsilon,\delta}^{-1} \mathbb{E}\Big[\varepsilon^{(N_\delta-1)_+} \sum_{j=1}^{N} |T_j| h(\bar{T}_j)\Big], \tag{3.8}$$

for any bounded, measurable function h, where $q_{\varepsilon,\delta} = \mathbb{E}\Big[\varepsilon^{(N_\delta-1)_+} \sum_{j=1}^{N} |T_j|\Big]$. If $\delta \to 0$ then $N_\delta \to N \geq 1$ and thus, using that $m(1) < \infty$ and monotone convergence

$$\lim_{\delta \to 0} q_{\varepsilon,\delta} \;=\; \mathbb{E}\Big[\varepsilon^{N-1} \sum_{j=1}^{N} |T_j|\Big]. \tag{3.9}$$

Moreover,

$$\begin{aligned} q_{\varepsilon,\delta} \mathbb{E}\big[|B|^{-1}\big] \;&=\; \mathbb{E}\Big[\varepsilon^{(N_\delta-1)_+} \sum_{j=1}^{N} |T_j||\bar{T}_j|^{-1}\Big] \\ &=\; \mathbb{E}\Big[N\varepsilon^{(N_\delta-1)_+}\Big] \overset{\delta \to 0}{\to} \mathbb{E}\Big[N\varepsilon^{N-1}\Big] \end{aligned} \tag{3.10}$$

when $\delta \to 0$, using that $\mathbb{E}\big[N\big] < \infty$ by assumption. Hence, by Eq.s (3.9) and (3.10), we can choose δ and ε small enough such that $q_{\varepsilon,\delta} < 1$ and $q_{\varepsilon,\delta}\mathbb{E}\big[|B|^{-1}\big] < 1$. Recall that we assume throughout that $\mathbb{P}(N = 0) = 0$ to avoid an atom at zero.

From now on, δ and ε are fixed and we write $p := q_{\varepsilon,\delta} < 1$. By (3.7), it holds for all $|\xi| \geq t_\varepsilon \delta^{-1}$ that

$$|g(\xi)| \leq p\mathbb{E}\big[|g(B\xi)|\big],$$

and we have that $p\mathbb{E}\big[|B|^{-1}\big] < 1$. Recalling that $|g|$ is bounded by $\mathbb{E}\big[|Z|\big]$, we can apply a Gronwall-type Lemma [9, Lemma 3.2] to the real-valued function

$$g^* : \mathbb{R}_+ \to \mathbb{R}_+, \qquad g^*(t) \;:=\; \max\{|g(\xi)| \,:\, |\xi| = t\}$$

to conclude that $g^*(t) = O(t^{-1})$. The assertion follows. □

Lemma 3.5. *Suppose $N > 0$ a.s.,* (A1)-(A2) *with $\alpha \in (1, 2)$,* (Z1) *and* (C1)*. Then $\partial_\xi \phi$ and $\partial_{\bar{\xi}}$ are square-integrable (w.r.t. Lebesgue measure on $\mathbb{C}$).*

Proof. As before, we focus on $g(\xi) = \partial_{\bar{\xi}}\phi(\xi)$. By taking squares in Eq. (3.7) and applying Jensen's inequality to the discrete probability measure $\sum_{j=1}^{N} \frac{1}{N}\delta\{|T_j||g(\bar{T}_j\xi)|\}$, we obtain

$$\begin{aligned} |g(\xi)|^2 \;&\le\; \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N^2 \Big(\frac{1}{N}\sum_{j=1}^{N} |T_j||g(\bar{T}_j\xi)|\Big)^2\Big] \\ &\le\; \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} \big(|T_j||g(\bar{T}_j\xi)|\big)^2\Big], \end{aligned} \tag{3.11}$$

and this estimate is valid for all ξ with $|\xi| \ge t_\epsilon \delta^{-1}$. Using the decay properties of g provided by Lemma 3.3, we have that the right hand side in (3.11) is bounded by

$$\mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} \big(|T_j| C(1+|T_j||\xi|)^{-1}\big)^2\Big] \;\le\; \frac{C}{|\xi|^2}\mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N^2\Big],$$

which is finite due to (C1). Defining

$$I(K) \;:=\; \int_{|\xi|\le K} |g(\xi)|^2 \, d\xi$$

and using the change-of-variables formula (on $\mathbb{C}$), we have with $U := t_\epsilon \delta^{-1}$

$$I(K) \;\le\; I(U) + \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} I\big(|T_j|K\big)\Big] \tag{3.12}$$

Now choose ϵ and δ small such that

$$\gamma := \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N^2\Big] < 1.$$

This is possible since $N_\delta \to N$ a.s. for $\delta \to 0$, $\mathbb{P}(N > 1) > 0$ and $\mathbb{E}\big[N^2\big] < \infty$. Recall that

$$\beta := \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} \log_+ |T_j|\Big] < \infty$$

by assumption. The remainder of the proof relies on the following claim.

Claim: For all $m \in \mathbb{N}$,

$$I(K) \;\le\; \sum_{n=0}^{m} \gamma^n I(U) + m\gamma^{m-1}\beta C + \gamma^m C \log_+ K,$$

where $C < \infty$ is the constant factor in the bound on $g(\xi) = \partial_{\bar{\xi}}\phi(\xi)$ provided by Lemma 3.3.

If the claim holds, then $I(K) \;\le\; \frac{I(U)}{1-\gamma} < \infty$ for all K, which proves that $g : \mathbb{C} \to \mathbb{C}$ is in L^2 and thus finishes the proof.

Proof of the Claim: We proceed by induction over $m \in \mathbb{N}$. For $m = 0$, we use the bound $|g(\xi)|^2 \le C(1 + |\xi|)^{-2}$, provided by Lemma 3.3 to estimate

$$I(K) \le \int_{|\xi|\le 1} C \, d\xi + \int_{1<|\xi|\le K} \frac{C}{|\xi|^2} \, d\xi \;=\; C\pi + \int_0^{2\pi}\int_1^K \frac{C}{r^2} r \, dr \, d\varphi \;=\; C\pi + 2\pi C \log_+ K,$$

where we used polar coordinates in the second expression for the integrals. After making U and C larger if necessary, we have

$$I(K) \;\le\; I(U) + C\log_+ K.$$

Suppose the claim holds for $m \in \mathbb{N}$. This means $I(K) \le a + b\log_+ K$ with the values

$$a = \sum_{n=0}^{m} \gamma^n I(U) + m\gamma^{m-1}\beta C, \qquad b = \gamma^m C.$$

Using Eq. (3.12) to iterate, we obtain

$$\begin{aligned} I(K) \;&\le\; I(U) + \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} \big(a + b\log_+ |T_j| + b\log_+ K\big)\Big] \\ &=\; I(U) + \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N^2 \big(a + b\log_+ K\big)\Big] + \mathbb{E}\Big[\varepsilon^{2(N_\delta-1)_+} N \sum_{j=1}^{N} b\log_+ |T_j|\Big] \\ &=\; I(U) + \gamma a + \gamma b \log_+ K + \beta b \\ &=\; I(U) + \gamma \sum_{n=0}^{m} \gamma^n I(U) + m\gamma^m \beta C + \gamma^{m+1} C \log_+ K + \beta\gamma^m C \end{aligned}$$

which proves the claim. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Writing $\xi = \xi_1 + \mathrm{i}\xi_2$ and using

$$\partial_{\xi_1}\phi = \partial_\xi \phi + \partial_{\bar\xi}\phi, \qquad \partial_{\xi_2}\phi = \mathrm{i}\Big(\partial_\xi\phi - \partial_{\bar\xi}\phi\Big)$$

(see [5, (1.1.2)]) we have obtained the square-integrability of $\partial_{\xi_1}\phi$ and $\partial_{\xi_2}\phi$.

For $j = 1, 2$, $(\partial_{\xi_j}\phi(\xi))d\xi$ defines a tempered distribution [14, VI.2.(4')]. By the Plancherel theorem [14, VI.2.(19)], its Fourier inverse

$$\mathcal{F}^{-1}\big(\partial_{\xi_j}\phi(\xi)d\xi\big) \;=:\; f_j(z)dz$$

is a tempered distribution defined with square-integrable function f_j. On the other hand,

$$\mathcal{F}^{-1}\big(\partial_{\xi_j}\phi(\xi)d\xi\big) \;=\; -\,\mathrm{i}z_j \mathcal{F}^{-1}\big(\phi(\xi)d\xi\big) \qquad \Big(z = z_1 + \mathrm{i}z_2\Big)$$

by [14, VI.2.(18)]. But $\mathcal{F}^{-1}\big(\phi(\xi)d\xi\big)$ is nothing but the tempered distribution given by $\mathbb{P}(Z \in dz)$ (in the sense of [14, VI.2.(4)]), this can be seen as in [14, VI.2.(11)]. Hence

$$f_j(z)dz \;=\; -\,\mathrm{i}z_j \mathbb{P}(Z \in dz).$$

This shows that for $j = 1, 2$, $-\mathrm{i}z_j\mathbb{P}\big(Z \in d(z_1, z_2)\big)$ has a square-integrable density f_j on $\mathbb{C}$. We decompose $\mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2$ into the disjoint union of sets

$$C_1 = \{z = (z_1, z_2) \,:\, |z_2| < z_1\}, \qquad C_2 = \{z \,:\, |z_1| \le z_2, z_2 \ne 0\},$$

$C_3 = -C_1$ and $C_4 = -C_2$. On $C_1 \cup C_3$, $\mathbb{P}(Z \in dz)$ has a density given by $(-\mathrm{i}z_1)^{-1} f_1(z)$, while on $C_2 \cup C_4$, a density for $\mathbb{P}(Z \in dz)$ is given by $(-\mathrm{i}z_2)^{-1} f_2(z)$.

Therefore $\mathbb{P}(Z \in dz) = \mathbb{P}(Z = 0)\delta_0 + \nu$, where ν has a density. Then it holds that $\mathbb{P}(Z = 0) = \limsup_{|\xi|\to\infty} |\phi(\xi)| = 0$ in view of Lemma 3.2 and so $\mathbb{P}(Z \in dz)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{C}$. □

Remark 3.6. If $m(2) < \infty$, $\mathbb{E}[N^2] < \infty$ and $\mathbb{E}|Z|^2 < \infty$, then $h(\xi) := \partial_{\bar\xi}^{(2)}\phi(\xi)$ is in $L^{1+\varepsilon}$ for any $\varepsilon > 0$, namely $h(\xi) = O(|\xi|^{-2})$.
In a similar way, for all $k \in \mathbb{N}$, $k > 2$ the following holds: $m(k) < \infty$, $\mathbb{E}[N^k] < \infty$ and $\mathbb{E}|Z|^k < \infty$ imply that $\partial_{\bar\xi}^{(k)}\phi(\xi) = O(|\xi|^{-k})$. Hence the density f of $\mathbb{P}(Z \in dz)$ belongs to $C^{k-3}(\mathbb{C} \setminus \{0\})$ and derivatives of f of order for $k-2$ exist in a weak sense on $\mathbb{C} \setminus \{0\}$.

Proof of Remark 3.6. Firstly, $\mathbb{E}|Z|^2 < \infty$ guarantees the existence of $h(\xi)$ and that h is bounded. By the convexity of m, the finiteness of both $m(0) = \mathbb{E}N < \mathbb{E}[N^2]$ and $m(2)$ yields that $m(1) < \infty$. Hence the assumptions of Lemma 3.3 are satisfied and we obtain the bound $|g(\xi)| = |\partial_\xi \phi(\xi)| \le C(1 + |\xi|)^{-1}$. Taking derivatives on both sides of Eq. 3.5, we have

$$h(\xi) = \mathbb{E}\Big[\sum_{j=1}^{N} T_j^2 h(\bar{T}_j\xi) \prod_{i\neq j} \phi(\bar{T}_i\xi) + 2 \sum_{1\le i<j\le N} T_i T_j g(\bar{T}_i\xi) g(\bar{T}_j\xi) \prod_{k\neq i,j} \phi(\bar{T}_k\xi)\Big].$$

Using the weaker estimate $|g(\bar{T}_i\xi)| \le C|T_j|^{-1}|\xi|^{-1}$, we deduce

$$|h(\xi)| \ \le\ \mathbb{E}\Big[\sum_{j=1}^{N} |T_j|^2 |h(\bar{T}_j\xi)| \prod_{i\neq j} |\phi(\bar{T}_i\xi)|\Big] + 2C\mathbb{E}\big[N^2\big]|\xi|^{-2}. \tag{3.13}$$

Now one can proceed as in the proof of Lemma 3.3, defining a complex random variable B such that for any test function f

$$\mathbb{E}f(B) \ =\ p^{-1}\mathbb{E}\Big[\varepsilon^{(N_\delta-1)_+} \sum_{j=1}^{N} |T_j|^2 f(\bar{T}_j)\Big]$$

with the normalization constant $p < 1$. Then $p\mathbb{E}\big[|B|^{-2}\big] \ \le\ \mathbb{E}\big[\varepsilon^{(N_\delta-1)_+}N\big] \ <\ 1$ for ε sufficiently small, and

$$|h(\xi)| \ \le\ p\mathbb{E}\big[h(B\xi)\big] + C'|\xi|^{-2}.$$

This is indeed sufficient to proceed as in [9, Lemma 3.2] to conclude that $|h(\xi)| = O(|\xi^{-2})$.

This estimate can then be used in a similar way to produce bounds for $\partial_\xi^{(3)}\phi(\xi)$, and so on. □

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