

A Brownian optimal switching problem under incomplete information

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Abstract

In this paper we study an incomplete information optimal switching problem in which the manager only has access to noisy observations of the underlying Brownian motion $\{W_t\}_{t \geq 0}$. The manager can, at a fixed cost, switch between having the production facility open or closed and must find the optimal management strategy using only the noisy observations. Using the theory of linear stochastic filtering, we reduce the incomplete information problem to a full information problem, show that the value function is non-decreasing with the amount of information available, and that the value function of the incomplete information problem converges to the value function of the corresponding full information problem as the noise in the observed process tends to 0. Our approach is deterministic and relies on the PDE-representation of the value function.

Keywords: optimal switching problem; stochastic filtering; incomplete information.

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1 Introduction

In this paper we study the two mode optimal switching problem (OSP) under incomplete information. The manager can choose to be either in state 0 or state 1. In state 1 the production facility is open and one continuously collects the stochastic revenue X_t . In state 0, the facility is closed and generates no revenue. The task of the manager is to choose when to open/close the facility in order to generate maximum profit. If the process X_t is fully observed and there is no cost of switching between ‘open’ and ‘closed’, it is intuitively clear that the manager should let the facility be open when $X_t \geq 0$ and closed when $X_t < 0$. However, when costs are associated to opening or closing, it may be more profitable to leave the plant open for a while even if $X_t < 0$ (or, similarly, leave it closed even if $X_t \geq 0$). Furthermore, if the process X_t cannot be fully observed, the manager has to base her decisions on incomplete information of the underlying process X_t .

The problem of optimal switching under full information has been widely studied during the last decades and mainly three different approaches have turned out to be useful, two based on stochastic techniques and one on deterministic ditto. Firstly, Snell envelopes was used by [DHP10] in combination with a verification theorem to prove the existence of a unique value function. Secondly, the optimal switching problem can

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be tackled using reflected backward stochastic differential equations (BSDEs) and we refer to [DHP10], [HT07], [HZ10], and the references therein for more on this reflected BSDE approach. For the connection between Snell envelopes and BSDEs we refer to [EKPPQ97]. Thirdly, the optimal switching problem has been studied using deterministic methods based on systems of variational inequalities. More on this deterministic approach to the OSP can be found in [EH09], [HM12], [LuNO12], [TY93] and the references therein. When dealing with the incomplete information optimal switching problem (IOSP) considerably less work seems to be done and the author is only aware of the paper [LiNO14], in which the authors formulate and develop numerical methods for this type of problem. Under incomplete information and when using the formalism of stochastic filtering as in [LiNO14], the IOSP is in general not analytically tractable and numerical methods seem inevitable. However, in this paper we derive some analytical results concerning a simplistic model of the IOSP introduced in [LiNO14]. To be more specific, we study the IOSP in a setting similar to that of [F78], in which the related problem of incomplete information optimal stopping is studied. We prove that the value function is convex, non-decreasing in the amount of available information and that it converges to the value function of the standard OSP as the noise in the observation tends to 0. We emphasize that, in the current situation, the analytical approach of this paper is mainly conceptual while the numerical method developed in [LiNO14] deals with more general IOSPs and is readily available for applications.

The rest of this paper is organized as follows. In section 2 we formulate the IOSP, state precise assumptions and present some preliminaries. Section 3 contains the main results while section 4 is devoted to proving these. In section 5 we give a numerical example of the problem studied in this paper. Finally, we end with section 6 containing conclusions and some future lines of research.

2 Problem formulation and preliminaries

2.1 Full information optimal switching problems

We begin by briefly outlining the standard OSP under full information. Let $X = \{X_t\}_{t \geq 0}$ be a p -dimensional stochastic process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad X_t = x$$

and denote the infinitesimal generator of X by \mathcal{L} . Let $\mathcal{Q} := \{0, 1, \dots, d\}$ be the finite set of available states. A management strategy is a combination of a non-decreasing sequence of stopping times $\{\tau_k\}_{k \geq 0}$, where, at time τ_k , the manager decides to switch production from its current mode to another one, and a sequence of indicators $\{\gamma_k\}_{k \geq 0}$, taking values in \mathcal{Q} , indicating the mode to which the production is switched. At τ_k the production is switched from mode γ_{k-1} to γ_k . The cost of switching from state i to state j at time s is denoted $c_{ij}(s, X_s)$. A strategy $(\{\tau_k\}_{k \geq 0}, \{\gamma_k\}_{k \geq 0})$ can be represented by the simple function

$$\mu_s = \sum_{i \geq 1} \gamma_i \chi_{(\tau_i, \tau_{i+1}]}(s) + \gamma_0 \chi_{[\tau_0, \tau_1]}(s)$$

indicating the current state of the facility. We will throughout the paper alternate between these two notations without further notice. A strategy is said to be *admissible* if

$$\lim_{n \rightarrow \infty} \tau_n = T \quad \mathbb{P} - a.s$$

and *finite* if

$$\mathbb{P}(\{\omega : \tau_n(\omega) < T \text{ for all } n \geq 0\}) = 0.$$

More clearly, a strategy is finite if it allows the manager to make only a finite number of switches during $[0, T]$. If the facility is in state i at time s , the generated revenue per unit time is $\psi_i(s, X_s)$. Hence, when the production is run under a strategy μ over a finite horizon $[t, T]$, the expected total profit is

$$J(t, x, \mu) = E \left[\left(\int_t^T \psi_{\mu_s}(s, X_s) ds - \sum_{t \leq \tau_k \leq T} c_{\gamma_{k-1}, \gamma_k}(\tau_k, X_{\tau_k}) \right) \right].$$

The task in the OSP is to find the value function

$$v(t, x) = \sup_{\mu \in \mathcal{A}_{t,i}^X} J(t, x, \mu) \tag{2.1}$$

where $\mathcal{A}_{t,i}^X$ denotes the set of admissible strategies adapted to the filtration generated by X which are in state i at time t .

Under sufficient conditions on the drift b and diffusion σ of X , the payoff functions ψ_i and switching costs c_{ij} , and the so called “no-loop condition”, the following theorems can be proven, see [DHP10].

Theorem 2.1. *The vector of value functions $(v_1(t, x), \dots, v_d(t, x))$ solves the system of variational inequalities*

$$\min \left\{ v_i - \max_{i \neq j} (v_j(t, x) - c_{ij}(t, x)), -\partial_t v_i(t, x) - \mathcal{L}v_i(t, x) - \psi_i(t, x) \right\} = 0, \\ v_i(T, x) = 0,$$

in the viscosity sense.¹ Furthermore, $(v_1(t, x), \dots, v_d(t, x))$ is the unique solution satisfying the polynomial growth condition

$$|v_i(t, x)| \leq C(1 + |x|^\eta),$$

$i \in \mathcal{Q}$, for some $\eta \geq 1$.

Theorem 2.2. *There exists a finite strategy $\mu^* \in \mathcal{A}_{t,i}^X$ such that $J(t, x, \mu^*) \geq J(t, x, \mu)$ for any $\mu \in \mathcal{A}_{t,i}^X$.*

For future reference we introduce the regions C_i and S_i , $i \in \mathcal{Q}$, defined as

$$C_i = \{(t, x) \in [0, T] \times \mathbb{R}^p : v_i(t, x) > v_j(t, x) - c_{ij}(t, x)\}, \\ S_i = \{(t, x) \in [0, T] \times \mathbb{R}^p : v_i(t, x) = \max_{j \in \mathcal{Q} \setminus \{i\}} \{v_j(t, x) - c_{ij}(t, x)\}\}.$$

The sets S_i and C_i are usually referred to as “Switching regions” and “Continuation regions”, respectively. It is optimal to switch from region i when the underlying process (t, X_t) hits the switching region S_i , see [DHP10].

2.2 Incomplete information optimal switching problems

In contrast to the OSP outlined above, the manager in an IIOSP only has access to incomplete information about the underlying process X , information acquired through a fully observable X -dependent process Y . In particular, the manager can only observe the process Y , solution to the stochastic differential equation

$$dY_s = h(X_s)ds + dU_s, \quad Y_t = y,$$

where U is a Brownian motion independent of W , and she must base her decisions solely on the information contained in \mathcal{F}^Y , the σ -algebra generated by Y . This lack of information prevents us from directly applying results concerning standard OSP.

¹For the definition of viscosity solutions see [CIL92].

In the theory of stochastic filtering, and we refer to [BC09] for details concerning this, the main goal is to compute conditional expectations $\mathbb{E}[\phi(X_t)|\mathcal{F}_t^Y]$ for suitably chosen test functions $\phi: \mathbb{R}^p \rightarrow \mathbb{R}$. The solution to the stochastic filtering problem is the distribution of the random variable X_t conditional on the σ -algebra \mathcal{F}_t^Y . We let π_t denote this distribution and hence

$$\mathbb{E}[\phi(X_t)|\mathcal{F}_t^Y] = \int_{\mathbb{R}^p} \phi(x)\pi_t(dx) \triangleq \pi_t(\phi). \tag{2.2}$$

Based on the solution π_t of the stochastic filtering problem we define, following [LiNO14], the expected total profit when the production is run under an \mathcal{F}_t^Y -adapted strategy $\mu^Y = (\{\tau_k^Y\}_{k \geq 0}, \{\gamma_k^Y\}_{k \geq 0})$, over a finite horizon $[t, T]$, to be

$$\begin{aligned} \tilde{J}(t, \hat{\pi}, \mu^Y) &= \mathbb{E} \left[\int_t^T \pi_s(\psi_{\mu_s^Y}(s, \cdot)) ds - \sum_{t \leq \tau_n^Y \leq T} \pi_{\tau_k^Y}(c_{\gamma_{n-1}^Y, \gamma_n^Y}(\tau_k^Y, \cdot)) \middle| X_t \sim \hat{\pi} \right] \\ &= \mathbb{E} \left[\left(\int_t^T \mathbb{E}[\psi_{\mu_s^Y}(s, X_s)|\mathcal{F}_s^Y] ds \right. \right. \\ &\quad \left. \left. - \sum_{t \leq \tau_n^Y \leq T} \mathbb{E}[c_{\gamma_{n-1}^Y, \gamma_n^Y}(\tau_k^Y, X_{\tau_k^Y})|\mathcal{F}_{\tau_k^Y}^Y] \right) \middle| X_t \sim \hat{\pi} \right], \end{aligned}$$

where $X_t \sim \hat{\pi}$ is used to indicate that the distribution of X_t conditional on \mathcal{F}_t^Y is $\hat{\pi}$. Let $\mathcal{A}_{t,i}^Y, t \in [0, T], i \in \mathcal{Q}$, denote the set of \mathcal{F}_t^Y -adapted strategies such that $\tau_1 \geq t$ and $\gamma_0 = i$ and recall the notation introduced in (2.2). Given $t \in [0, T]$ and a probability measure $\hat{\pi}$, we define $v_i(t, \hat{\pi})$, the value function associated with the IOSP, as

$$v_i(t, \hat{\pi}) = \sup_{\mu^Y \in \mathcal{A}_{t,i}^Y} \tilde{J}(t, \hat{\pi}, \mu^Y). \tag{2.3}$$

The function $v_i(t, \hat{\pi})$ stands for the optimal expected profit if, at time t , the production is in mode i and the probability distribution of the unknown X_t is $\hat{\pi}$.

2.3 Assumptions

We now turn to the specific setup studied in this paper. We let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space on which we define two independent Brownian motions (W_t, \mathcal{F}_t) and $(\bar{W}_t, \mathcal{F}_t)$. The notation W^x is used to indicate that the Brownian motion is started from the point x at time $t = 0$. The assumption that the initial observation is made at time $t = 0$ is made w.l.o.g., see Remark 3.1 of [LiNO14]. We let W^x be the underlying stochastic process and let, for an arbitrary but fixed $\epsilon > 0$, the process ξ^ϵ defined by,

$$d\xi_t^\epsilon = W_t^x dt + \epsilon d\bar{W}_t, \quad \xi_0^\epsilon = x,$$

represent noisy observations of W . We denote by $\{\mathcal{F}_t^\epsilon\}_{t \geq 0}$ the filtration generated by ξ^ϵ . We stress that $\mathcal{F}_t \not\subset \mathcal{F}_t^\epsilon$ and consequently the value of W_t^x is not known based on the information in \mathcal{F}_t^ϵ . The manager can choose between having the production facility ‘open’ (state 1) or ‘closed’ (state 0), i.e., $\mathcal{Q} = \{0, 1\}$, and the corresponding payoff functions are

$$\psi_0(x) = 0, \quad \psi_1(x) = x. \tag{2.4}$$

The manager can only observe the process ξ^ϵ and the decision to open/close the production at time t must hence be made based solely on the information contained in \mathcal{F}_t^ϵ . Concerning the cost of switching we assume that

$$c_{01} \text{ and } c_{10} \text{ are fixed positive constants.} \tag{2.5}$$

The solution to the IIOSP outlined above is the value function $v_i^\epsilon(t, \hat{\pi})$, $i \in \mathcal{Q}$, defined as

$$v_i^\epsilon(t, \hat{\pi}) = \sup_{\mu \in \mathcal{A}_{t,i}^{\xi,\epsilon}} \mathbb{E} \left[\int_t^T \mathbb{E} [W_s^x \mathcal{I}_{\{\mu_s=1\}} | \mathcal{F}_s^\epsilon] ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| W_t^x \sim \hat{\pi} \right], \quad (2.6)$$

where $\hat{\pi}$ is a probability measure and $\mathcal{A}_{t,i}^{\xi,\epsilon}$ denotes the set of admissible strategies adapted to the filtration \mathcal{F}_t^ϵ which are in state i at time t . We denote by $m_t^{x,\epsilon} = \mathbb{E} [W_t^x | \mathcal{F}_t^\epsilon]$ and $\theta_t^{x,\epsilon} = \mathbb{E} [(m_t^\epsilon - W_t^x)^2]$ the conditional mean and conditional variance of W_t^x , respectively. Using this notation and the fact that μ is by definition an \mathcal{F}^ϵ -measurable strategy (2.6) can be simplified to read

$$v_i^\epsilon(t, \hat{\pi}) = \sup_{\mu \in \mathcal{A}_{t,i}^{\xi,\epsilon}} \mathbb{E} \left[\int_t^T m_s^{x,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| W_t^x \sim \hat{\pi} \right]$$

As with (2.3), the function $v_i^\epsilon(t, \hat{\pi})$ can be interpreted as the optimal expected profit if, at time t , the facility is in state i and, given the available observations, the probability distribution of W_t^x is $\hat{\pi}$. For future reference we also introduce the value function $v_i^0(t, \hat{x})$ defined as

$$v_i^0(t, \hat{x}) = \sup_{\mu \in \mathcal{A}_{t,i}^{W^x}} \mathbb{E} \left[\int_t^T W_s^x \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| W_t^x = \hat{x} \right],$$

i.e., the value function of the standard OSP corresponding to the assumptions above.

3 Main results

To simplify the statements of the main results, we first state an immediate consequence of Theorem 4.1, further explained in the bulk of the paper. In particular, Proposition 3.1 reduces the a priori infinite dimensionality of the IIOSP.

Proposition 3.1. *The stochastic probability measure π_t of W_t^x conditional on \mathcal{F}_t^ϵ is fully characterized by its conditional mean $m_t^{x,\epsilon} = \mathbb{E} [W_t^x | \mathcal{F}_t^\epsilon]$ and a time-dependent deterministic function θ_t .*

With this proposition in mind, it is clear that the value function $v^\epsilon(t, \hat{\pi})$ can be expressed as a function $v^\epsilon(t, m) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$. With slight abuse of notation we will write

$$v_i^\epsilon(t, \hat{\pi}) = v_i^\epsilon(t, m) = \sup_{\mu \in \mathcal{A}_{t,i}^{\xi,\epsilon}} \mathbb{E} \left[\int_t^T m_s^{x,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| m_t^{x,\epsilon} = m \right] \quad (3.1)$$

for $m = \int_{\mathbb{R}} x d\hat{\pi}(dx) = \mathbb{E} [W_t^x | \mathcal{F}_t^\epsilon]$. We are now ready to state the results.

Theorem 3.2 (Monotonicity in ϵ). *For any ϵ_1, ϵ_2 such that $0 \leq \epsilon_1 \leq \epsilon_2$ we have*

$$v_i^{\epsilon_2}(t, m) \leq v_i^{\epsilon_1}(t, m).$$

In words, Theorem 3.2 states that the value of information is positive, i.e., decreasing the noise in the observation process ξ increases the optimal expected profit. The intuitive reason for this result is that more accurate observations simplify the managers task of optimally controlling the production facility. When the noise in ξ tends to 0, more and more information becomes available and the task of the manager starts to resemble that under full information. Consequently, the expected optimal profit under incomplete information should tend to the expected optimal profit under full information. This intuition is confirmed by the following theorem.

Theorem 3.3 (Convergence of v^ϵ to v^0). As $\epsilon \searrow 0$, $v^\epsilon(t, m) \rightarrow v^0(t, m)$ such that

$$0 \leq v^0(t, m) - v^\epsilon(t, m) \leq (T - t) \left(\frac{4}{\pi} \log(2) - \frac{2}{\pi} \right)^{1/2} \sqrt{\epsilon}$$

4 Proof of the main results

The outline of this section is as follows. Firstly, we state some results from the theory of stochastic filtering which will constitute a base for the proofs of the results in Section 3. We then show that the IIOSP stated above can be reduced to a full information OSP and apply standard results, in particular the results presented in Section 2, to prove Theorem 3.2 and Theorem 3.3.

4.1 Stochastic filtering

Results concerning standard OSP are in general not directly applicable to the IIOSP due to the lack of information. The main idea of this paper is to reduce the IIOSP to a standard OSP to which the results of Section 2 can be applied. This reduction is made possible by the following well-known results from the theory of linear stochastic filtering. Proofs of the below statements can be found in [LS78] (Theorem 10.3 and Theorems 7.12 and 7.16).

Theorem 4.1. *The distribution of W_t^x conditional on \mathcal{F}_t^ϵ is Gaussian. Furthermore, the conditional mean $m_t^{x,\epsilon} = \mathbb{E}[W_t^x | \mathcal{F}_t^\epsilon]$ and variance $\theta_t^\epsilon = \mathbb{E}[(m_t^{x,\epsilon} - W_t)^2]$ are the unique pair of processes satisfying*

$$dm_t^{x,\epsilon} = \frac{\theta_t^\epsilon}{\epsilon^2} (d\xi_t^\epsilon - m_t^{x,\epsilon} dt), \quad m_0^\epsilon = x, \tag{4.1}$$

$$\frac{d\theta}{dt} = 1 - \left(\frac{\theta_t^\epsilon}{\epsilon} \right)^2, \quad \theta_0^\epsilon = 0. \tag{4.2}$$

Theorem 4.2. *The random process $N_t = (N_t, \mathcal{F}_t^\epsilon)$, $0 \leq t \leq T$, with*

$$N_t = \xi_t^\epsilon - \int_0^t m_s^\epsilon ds,$$

is a Wiener process and the filtration \mathcal{F}^N generated by N coincides with that generated by ξ^ϵ , i.e., $\mathcal{F}_t^\epsilon = \mathcal{F}_t^N$ for all $t \in [0, T]$.

Note that the starting point $m_0 = \mathbb{E}[W_0^x]$ is given by the starting point of the underlying process W^x . Since the distribution of W_t^x conditional on \mathcal{F}_t^ϵ is Gaussian it is fully characterized by its mean m_t and variance θ_t . Proposition 3.1 follows immediately since θ_t is given by the ordinary differential equation (4.2).

4.2 Reduction to the complete information case

Before proving the main results, we reduce the problem to a complete information setting and prove convexity of the value function.

Proposition 4.3. *Fix $\epsilon > 0$ and assume (2.4) and (2.5). Let $X_s^{m,t,\epsilon}$ be the solution to*

$$dX_s^{m,t,\epsilon} = \tanh\left(\frac{s}{\epsilon}\right) dW_s, \quad X_t^{m,t,\epsilon} = m, \tag{4.3}$$

and let

$$v_i(t, m) = \sup_{\mu \in \mathcal{A}_i^X} \mathbb{E} \left[\int_t^T X_s^{m,t,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \right]$$

be the value function of the full information OSP (2.1). Then, $v_i^\epsilon(t, m) = v_i(t, m)$.

Proof. Recall that

$$v_i^\epsilon(t, m) = \sup_{\mu \in \mathcal{A}_{t,i}^\epsilon} \mathbb{E} \left[\int_t^T m_s^{x,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| m_t^{x,\epsilon} = m \right]$$

where $m_t^{x,\epsilon} = \mathbb{E}[W_t^x | \mathcal{F}_t^\epsilon]$. By Theorem 4.1 the dynamics of $m_t^{x,\epsilon}$ is given by (4.1) which, after solving (4.2) and inserting the solution $\theta_t^\epsilon = \epsilon \tanh(\frac{t}{\epsilon})$, reads

$$dm_t^\epsilon = \frac{1}{\epsilon} \tanh(\frac{t}{\epsilon}) d\xi_t^\epsilon - \frac{1}{\epsilon} \tanh(\frac{t}{\epsilon}) m_t^\epsilon dt. \tag{4.4}$$

Since the value function (3.1) is an expected value we can replace the underlying process (and corresponding set of strategies) in our optimal switching problem with any other process having the same distributional properties without changing its value. With this in mind, we introduce the innovations process

$$N_t = \xi_t^\epsilon - \int_0^t m_s^\epsilon ds \tag{4.5}$$

and put $dR_t = \frac{1}{\epsilon} dN_t$. Then, by (4.5) the process ξ^ϵ admits a representation of the form

$$d\xi_t^\epsilon = m_t^{x,\epsilon} dt + \epsilon dR_t.$$

and by Theorem 4.2 the process $(R_t, \mathcal{F}_t^\epsilon)$ is a Brownian motion. Furthermore, the σ -algebra \mathcal{F}_t^R generated by R_t coincides with \mathcal{F}_t^ϵ for all $t \geq 0$. The above together with (4.4) and the initial condition $m_t^{x,\epsilon} = m$ gives that

$$m_s^\epsilon = m + \int_t^s \tanh(\frac{r}{\epsilon}) dR_r.$$

Recall the underlying Brownian motion (W_t, \mathcal{F}_t) and the dynamics (4.3) of $X^{m,t,\epsilon}$,

$$X_s^{m,t,\epsilon} = m + \int_t^s \tanh(\frac{r}{\epsilon}) dW_r.$$

The distributional properties of the Brownian motions $(R_t, \mathcal{F}_t^\epsilon)$ and (W_t, \mathcal{F}_t) coincide and hence, since $\mathcal{F}_t^R = \mathcal{F}_t^\epsilon$ for all $t \geq 0$, it follows that

$$\begin{aligned} v_i^\epsilon(t, m) &= \sup_{\mu \in \mathcal{A}_i^\epsilon} \mathbb{E} \left[\int_t^T m_s^{x,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| m_t^{x,\epsilon} = m \right] \\ &= \sup_{\mu \in \mathcal{A}_i^R} \mathbb{E} \left[\int_t^T m_s^{x,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \middle| m_t^{x,\epsilon} = m \right] \\ &= \sup_{\mu \in \mathcal{A}_i} \mathbb{E} \left[\int_t^T X_s^{m,t,\epsilon} \mathcal{I}_{\{\mu_s=1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_{n-1}} \mu_{\tau_n}} \right] = v_i(t, m) \end{aligned} \tag{4.6}$$

and the proof is complete. □

Remark 4.4. Note that although the process $m_t^{x,\epsilon}$ is not explicitly observable, its path is completely determined by the deterministic function θ_t and the observed process ξ^ϵ . Hence, we can w.l.o.g. consider $m^{x,\epsilon}$ as being the observed process rather than ξ^ϵ .

Lemma 4.5. For any $\epsilon > 0$, the value function $v_i^\epsilon(t, m)$, $i \in \mathcal{Q}$, is convex in m .

Proof. Recall the characterization of $v_i^\epsilon(t, m)$ given in Proposition 4.3. Since the dynamics of $X^{m,t,\epsilon}$ is independent of its current value, the resulting process is linear w.r.t. its starting point, i.e., $X^{m+\eta,t,\epsilon} = X^{m,t,\epsilon} + \eta$. Let $\mu^* \in \mathcal{A}_i^X$ be an optimal strategy for $v_i^\epsilon(t, m)$ so that

$$v_i^\epsilon(t, m) = \mathbb{E} \left[\int_t^T X_s^{m,t,\epsilon} \mathcal{I}_{\{\mu_s^*=1\}} ds - \sum_{t \leq \tau_n^* \leq T} c_{\mu_{\tau_{n-1}^*}^* \mu_{\tau_n}^*} \right].$$

We now perturb the initial condition m by η and consider $v_i^\eta(t, m + \eta)$. The strategy μ^* is sub-optimal for $v_i^\epsilon(t, m + \eta)$ and hence

$$\begin{aligned} v_i^\epsilon(t, m + \eta) &\geq \mathbb{E} \left[\int_t^T X_s^{m+\eta,t,\epsilon} \mathcal{I}_{\{\mu_s^*=1\}} ds - \sum_{t \leq \tau_n^* \leq T} c_{\mu_{\tau_{n-1}^*}^* \mu_{\tau_n}^*} \right] \\ &= \mathbb{E} \left[\int_t^T X_s^{m,t,\epsilon} \mathcal{I}_{\{\mu_s^*=1\}} ds - \sum_{t \leq \tau_n^* \leq T} c_{\mu_{\tau_{n-1}^*}^* \mu_{\tau_n}^*} + \eta \int_t^T \mathcal{I}_{\{\mu_s^*=1\}} ds \right] \\ &= v_i^\epsilon(t, m) + \eta f(t, m, i), \end{aligned} \tag{4.7}$$

where $f(t, m, i) = \int_t^T \mathcal{I}_{\{\mu_s^*=1\}} ds$ is the expected time spent in state 1, using the optimal strategy for the starting point (t, m, i) . Repeating the above for the optimal strategy associated to the initial value $(t, m + \eta)$ yields

$$v_i^\epsilon(t, m) \geq v_i^\epsilon(t, m + \eta) - \eta f(t, m + \eta, i).$$

Since $\int_t^T \mathcal{I}_{\{\mu_s^*=1\}} ds \geq 0$ it follows from (4.7) that $v_i^\epsilon(t, m)$ is non-decreasing in m . Furthermore, combining the inequalities above we find

$$v_i^\epsilon(t, m) + \eta f(t, m + \eta, i) \geq v_i^\epsilon(t, m + \eta) \geq v_i^\epsilon(t, m) + \eta f(t, m, i),$$

i.e., the time spent online increases with the initial starting point. Dividing by η and letting $\eta \rightarrow 0$ gives

$$f(t, m, i) \leq \lim_{\eta \searrow 0} \frac{v_i^\epsilon(t, m + \eta) - v_i^\epsilon(t, m)}{\eta} \leq \lim_{\eta \searrow 0} f(t, m + \eta, i).$$

Since $f(t, m, i)$ is non-decreasing in m , we can conclude that for any $\nu > 0$

$$D_x^+ v_i^\epsilon(t, m) \leq \lim_{\eta \rightarrow 0} f(t, m + \eta) \leq f(t, m + \nu) \leq D_x^+ v_i^\epsilon(t, m + \nu),$$

where $D_x^+ v_i^\epsilon(t, m)$ denotes the right spatial derivative of v_i^ϵ at (t, m) . We conclude that the right derivative of v_i^ϵ is increasing at m . Convexity of $v_i^\epsilon(t, m)$ now follows since m was arbitrary. \square

4.3 Proof of Theorem 3.2

By combining Proposition 4.3 and Theorem 2.1 we conclude that the vectors $(v_0^{\epsilon_k}, v_1^{\epsilon_k})$, $k \in \{1, 2\}$, solve, respectively, the systems of variational inequalities

$$\begin{aligned} \min \{ \varphi_0 - (\varphi_1 - c_{01}), -\partial_t \varphi_0 - \mathcal{L}^{\epsilon_k} \varphi_0 - \psi_0 \} &= 0 \\ \min \{ \varphi_1 - (\varphi_0 - c_{10}), -\partial_t \varphi_1 - \mathcal{L}^{\epsilon_k} \varphi_1 - \psi_1 \} &= 0, \end{aligned} \tag{4.8}$$

where

$$\mathcal{L}^{\epsilon_k} = \frac{1}{2} \tanh^2\left(\frac{t}{\epsilon_k}\right) \partial_{mm}$$

is the generator of the process X^{m,t,ϵ_k} . We now intend to prove that $(v_0^{\epsilon_2}, v_1^{\epsilon_2})$ is a subsolution to the system above with $k = 1$, i.e., that

$$\begin{aligned} (i) \quad & \min \{v_0^{\epsilon_2} - (v_1^{\epsilon_2} - c_{01}), -\partial_t v_0^{\epsilon_2} - \mathcal{L}^{\epsilon_1} v_0^{\epsilon_2} - \psi_0\} \leq 0 \\ (ii) \quad & \min \{v_1^{\epsilon_2} - (v_0^{\epsilon_2} - c_{10}), -\partial_t v_1^{\epsilon_2} - \mathcal{L}^{\epsilon_1} v_1^{\epsilon_2} - \psi_1\} \leq 0, \end{aligned} \tag{4.9}$$

and then apply the comparison principle for (4.8). We focus on (4.9) (i), the inequality in (4.9) (ii) being treated similarly. Firstly, on the region $S_0^{\epsilon_2}$ (4.9) (i) is trivially satisfied since $v_0^{\epsilon_2} = v_1^{\epsilon_2} - c_{01}$ by definition. Hence, we only need to show that on the region $C_0^{\epsilon_2}$, where $v_0^{\epsilon_2}$ is above its obstacle by construction, we have

$$-\partial_t v_0^{\epsilon_2} - \mathcal{L}^{\epsilon_1} v_0^{\epsilon_2} - \psi_0 \leq 0, \tag{4.10}$$

in the viscosity sense. Assume therefore the existence of a smooth function $\varphi(t, m)$ such that $\varphi - v_0^{\epsilon_2}$ has a local minimum equal to 0 at $(\hat{t}, \hat{m}) \in C_0^{\epsilon_2}$. Since $v_0^{\epsilon_2}$ is a viscosity solution to (4.8) with $k = 2$ we have

$$-\partial_t \varphi(\hat{t}, \hat{m}) - \mathcal{L}^{\epsilon_2} \varphi(\hat{t}, \hat{m}) - \psi_0(\hat{t}, \hat{m}) \leq 0.$$

In particular, we see that $-\partial_t \varphi(\hat{t}, \hat{m}) - \psi_0(\hat{t}, \hat{m}) \leq \mathcal{L}^{\epsilon_2} \varphi(\hat{t}, \hat{m})$. Hence, at (\hat{t}, \hat{m}) we have

$$\begin{aligned} & -\partial_t \varphi(\hat{t}, \hat{m}) - \mathcal{L}^{\epsilon_1} \varphi(\hat{t}, \hat{m}) - \psi_0(\hat{t}, \hat{m}) \\ & \leq \mathcal{L}^{\epsilon_2} \varphi(\hat{t}, \hat{m}) - \mathcal{L}^{\epsilon_1} \varphi(\hat{t}, \hat{m}) \\ & = \frac{1}{2} \left(\tanh^2\left(\frac{\hat{t}}{\epsilon_2}\right) - \tanh^2\left(\frac{\hat{t}}{\epsilon_1}\right) \right) \partial_{mm} \varphi(\hat{t}, \hat{m}). \end{aligned} \tag{4.11}$$

The monotonicity of $\tanh^2(t)$ for $t \geq 0$ and the assumption $\epsilon_1 \leq \epsilon_2$ gives

$$\frac{1}{2} \left(\tanh^2\left(\frac{\hat{t}}{\epsilon_2}\right) - \tanh^2\left(\frac{\hat{t}}{\epsilon_1}\right) \right) \leq 0.$$

Moreover, $\partial_{mm} \varphi(\hat{t}, \hat{m}) \geq 0$. Indeed, by Lemma 4.5 there exists a supporting line $\ell(m)$ for $v_0^{\epsilon_2}(\hat{t}, m)$ such that $\varphi(\hat{t}, \hat{m}) = v_0^{\epsilon_2}(\hat{t}, \hat{m}) = \ell(\hat{m})$ and, furthermore, we have $\varphi(\hat{t}, m) \geq v_0^{\epsilon_2}(\hat{t}, m) \geq \ell(m)$ in an m -neighbourhood of \hat{m} since $\varphi - v_0^{\epsilon_2}$ has a local minimum at (\hat{t}, \hat{m}) . It follows that the smooth function $\varphi(\hat{t}, m) - \ell(m)$ has a local minimum at \hat{m} . Differentiating with respect to m thus yields $\partial_{mm} \varphi(\hat{t}, \hat{m}) \geq \partial_{mm} \ell(\hat{m}) = 0$ which proves the claim.

We can thus conclude from (4.11) that (4.10) holds. This proves (4.9) (i) since $S_0^{\epsilon_2} \cup C_0^{\epsilon_2} = \mathbb{R}$. Repeating the above arguments for $v_1^{\epsilon_2}$ proves (4.9) (ii) and thus the subsolution property is proven. Since $v_i^{\epsilon_1}(T, x) = v_i^{\epsilon_2}(T, x)$, $i \in \mathcal{Q}$, the theorem now follows from the comparison principle for (4.8).

4.4 Proof of Theorem 3.3

That $v_i^0(t, m) \geq v_i^\epsilon(t, m)$ for any $\epsilon > 0$ is proven as Theorem 3.2 using $\tanh^2(t) \leq 1$ for any $t \geq 0$. We omit the details. By Theorem 2.2 there exists a finite optimal strategy $\mu^* \in \mathcal{A}_i^X$ for the full information problem (2.1). This strategy is sub-optimal in (4.6) and hence

$$\begin{aligned} v_i^0(t, m) - v_i^\epsilon(t, m) &= \mathbb{E} \left[\int_t^T W_s^{m,t} \mathcal{I}_{\{\mu_s^* = 1\}} ds \right] - \mathbb{E} \left[\sum_{t \leq \tau_n^* \leq T} c_{\mu_{\tau_n^* - 1}^* \mu_{\tau_n^*}^*} \right] \\ &\quad - \sup_{\mu \in \mathcal{A}_{i,t}^\epsilon} \mathbb{E} \left[\int_t^T X_s^{m,t,\epsilon} \mathcal{I}_{\{\mu_s = 1\}} ds - \sum_{t \leq \tau_n \leq T} c_{\mu_{\tau_n - 1} \mu_{\tau_n}} \right] \\ &\leq \mathbb{E} \left[\int_t^T (W_s^{m,t} - X_s^{m,t,\epsilon}) \mathcal{I}_{\{\mu_s^* = 1\}} ds \right], \end{aligned} \tag{4.12}$$

where $W_s^{m,t}$, $s \geq t$ is the Brownian motion W^x conditional on $W_t^x = m$. Following [F78] we introduce the notation

$$\begin{aligned} Q_{t,s}^\epsilon &= \int_t^s \left(1 - \tanh\left(\frac{u}{\epsilon}\right)\right) dW_u \\ N_{t,r}^\epsilon &= \sup_{s \in [t,r]} Q_{t,s}^\epsilon, \\ N_{t,\infty}^\epsilon &= \sup_{s \geq t} Q_{t,s}^\epsilon. \end{aligned}$$

By definition

$$W_s^{m,t} - X_s^{m,t,\epsilon} = \int_t^s \left(1 - \tanh\left(\frac{r}{\epsilon}\right)\right) dW_r$$

and hence (4.12) yields

$$\begin{aligned} v_i^0(t, m) - v_i^\epsilon(t, m) &\leq \mathbb{E} \left[\int_t^T Q_{t,s}^\epsilon \mathcal{I}_{\{\mu_s^* = 1\}} ds \right] \\ &\leq \mathbb{E} \left[\sup_{s \in [t,T]} Q_{t,s}^\epsilon \int_t^T \mathcal{I}_{\{\mu_s^* = 1\}} ds \right] \leq \mathbb{E} \left[\sup_{s \geq t} Q_{t,s}^\epsilon \int_t^T \mathcal{I}_{\{\mu_s^* = 1\}} ds \right] \\ &\leq \mathbb{E} [N_{t,\infty}^\epsilon] (T - t). \end{aligned}$$

Since $\mathbb{E} [N_{t,\infty}^\epsilon] = \mathbb{E} [\lim_{r \rightarrow \infty} N_{t,r}^\epsilon]$ and $N_{t,r}^\epsilon$ is monotone in r we may apply the monotone convergence theorem to find

$$\mathbb{E}[N_{t,\infty}^\epsilon] = \mathbb{E} \left[\lim_{r \rightarrow \infty} N_{t,r}^\epsilon \right] = \lim_{r \rightarrow \infty} \mathbb{E} [N_{t,r}^\epsilon].$$

By the reflection principle it follows that

$$\mathbb{E}[N_{t,r}^\epsilon] = \left(\frac{2}{\pi} \int_t^r \left(1 - \tanh\left(\frac{r}{\epsilon}\right)\right)^2 dr \right)^{1/2}.$$

To be more explicit, applying the reflection principle to the stochastic process $Q_{t,s}^\epsilon$ yields $\mathbb{P}(N_{t,r}^\epsilon \geq b) = 2\mathbb{P}(Q_{t,r}^\epsilon \geq b) = \mathbb{P}(|Q_{t,r}^\epsilon| \geq b)$, i.e., the distribution of $N_{t,r}^\epsilon$ coincides with that of $|Q_{t,r}^\epsilon|$. Furthermore, since $Q_{t,s}^\epsilon = \int_t^s (1 - \tanh(\frac{u}{\epsilon})) dW_u$ it is normally distributed,

$$Q_{t,s}^\epsilon \sim N\left(0, \int_t^s (1 - \tanh(\frac{u}{\epsilon}))^2 du\right),$$

and $|Q_{t,s}^\epsilon|$ follows the corresponding half-normal distribution. Hence

$$\mathbb{E}[N_{t,r}^\epsilon] = \mathbb{E}[|Q_{t,r}^\epsilon|] = \left(\frac{2}{\pi} \int_t^r (1 - \tanh(\frac{u}{\epsilon}))^2 du \right)^{1/2}$$

and

$$\begin{aligned} \mathbb{E}[N_{t,\infty}^\epsilon] &= \lim_{r \rightarrow \infty} \mathbb{E}[N_{t,r}^\epsilon] = \lim_{r \rightarrow \infty} \left(\frac{2}{\pi} \int_t^r (1 - \tanh(\frac{u}{\epsilon}))^2 du \right)^{1/2} \\ &\leq \lim_{r \rightarrow \infty} \left(\frac{2}{\pi} \int_0^r (1 - \tanh(\frac{u}{\epsilon}))^2 du \right)^{1/2} = \sqrt{\epsilon} \left(\frac{4}{\pi} \log(2) - \frac{2}{\pi} \right)^{1/2}. \end{aligned} \quad (4.13)$$

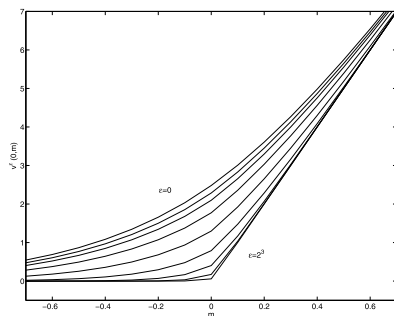
The result now follows by combining (4.12) and (4.13). □

Table 1: Parameter values.

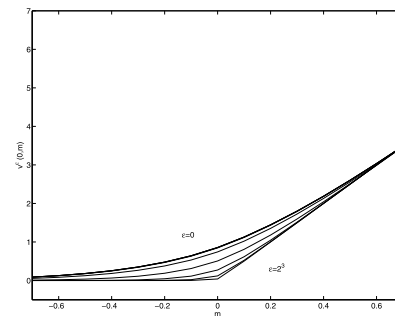
Description	Symbol	Value
Cost of opening	c_{01}	0.01
Cost of closing	c_{10}	0.001
Running payoff state 1	$\psi_1(x)$	$10x$
Running payoff state 0	$\psi_0(x)$	0
Terminal time	T	1

Table 2: Numerical results for $\epsilon = 2^{-4}/1/2^3$.

$v_1^\epsilon(t, m)$	$m = -0.5$	$m = 0$	$m = 0.5$
$t = 0$	0.7680/0.0631/-0.001	2.2860/0.7898/0.0575	5.6481/5.0367/5.0000
$t = 0.5$	0.1814/0.0349/-0.001	0.8567/0.5069/0.0396	2.6015/2.5097/2.5000



(a) $t = 0$



(b) $t = 0.5$

Figure 1: $v_1^\epsilon(t, m)$ for $\epsilon \in \{0, 2^{-4}, \dots, 2^3\}$.

5 Numerical example

We conclude with a numerical calculation showing some features stemming from the lack of information. In particular, we solve (4.8) using the Crank-Nicolson finite difference scheme with linear interpolation at the boundaries. The value function v_i^ϵ is found using the parameters in Table 1². Recall that the value function $v_i^\epsilon(t, m)$ is given w.r.t. $W_0^x = x$. For ease of exposition we only present numerical results for $v_1^\epsilon(t, m)$ subject to the initial condition $W_0^x = 0$.

The monotonicity proved in Theorem 3.2 is clearly seen in Figure 1 and Table 2. When the noise in the observation grows bigger it becomes less and less valuable and the value function tends towards the case of no information, i.e., $\lim_{\epsilon \rightarrow \infty} v_1^\epsilon = v_1^\infty(t, m)$ where $v_1^\infty(t, m) = 0$ for $x \leq 0$ and $v_1^\infty(t, m) = (T - t)\psi_1(m) = 10(T - t)m$ for $m > 0$.

6 Summary and conclusion

In this paper we studied a Brownian optimal switching problem under incomplete information. We showed that the value of information is positive and that the value function converges to the corresponding full information value function when the noise in the observation tends to 0. Although the problem studied here is simplistic, the results indicate that the method of reducing to a full information problem, a technique successfully used in the study of incomplete information optimal stopping, provides a

²The payoff function ψ_1 is chosen as $\psi_1(x) = 10x$ to get a convenient order of magnitude of the value function.

feasible way of tackling IIOSPs. An interesting and natural continuation of this paper is to study the IIOSP for more general stochastic processes, payoff functions, and switching costs, firstly in the general setting of linear Kalman-Bucy filters and ultimately for fully non-linear stochastic filters.

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