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Existence of solution to scalar BSDEs with $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$-integrable terminal values*

Ying Hu[†] Shanjian Tang[‡]

Abstract

In this paper, we study a scalar linearly growing backward stochastic differential equation (BSDE) with an $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$-integrable terminal value. We prove that a BSDE admits a solution if the terminal value satisfies the preceding integrability condition with the positive parameter λ being less than a critical value λ_0, which is weaker than the usual L^p $(p>1)$ integrability and stronger than $L\log L$ integrability. We show by a counterexample that the conventionally expected $L\log L$ integrability and even the preceding integrability for $\lambda>\lambda_0$ are not sufficient for the existence of solution to a BSDE with a linearly growing generator.

Keywords: backward stochastic differential equation; $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$ integrability; terminal condition; dual representation.
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1 Introduction

Let $\{W_t, t\geq 0\}$ be a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, and $\{\mathcal{F}_t, t\geq 0\}$ its natural filtration augmented by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Let us fix a nonnegative real number $T>0$. The σ-field of predictable subsets of $\Omega\times[0,T]$ is denoted by $\mathcal{P}$.

For any real $p\geq 1$, denote by L^p the set of all $\mathcal{F}_T$-measurable random variables η such that $E|\eta|^p<\infty$, by $\mathcal{S}^p$ the set of all real-valued, adapted and càdlàg processes $\{Y_t, 0\leq t\leq T\}$ such that

$$||Y||_{\mathcal{S}^p} := \mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^p\right]^{1/p} < +\infty,$$

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[†]IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.
E-mail: ying.hu@univ-rennes1.fr

[‡]Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: sjtang@fudan.edu.cn http://math.fudan.edu.cn/

by $\mathcal{L}^p$ the set of all real-valued adapted processes $\{Y_t, 0 \le t \le T\}$ such that

$$||Y||_{\mathcal{L}^p} := \mathbb{E}\left[\int_0^T |Y_t|^p\, dt\right]^{1/p} < +\infty,$$

and by $\mathcal{M}^p$ the set of (equivalent class of) predictable processes $\{Z_t, 0 \le t \le T\}$ with values in $\mathbb{R}^{1\times d}$ such that

$$||Z||_{\mathcal{M}^p} := \mathbb{E}\left[\left(\int_0^T |Z_t|^2\, dt\right)^{p/2}\right]^{1/p} < +\infty.$$

Consider the following Backward Stochastic Differential Equation (BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{1.1}$$

Here, f (hereafter called the generator) is a real valued random function defined on the set $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$, measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1\times d})$, and continuous in the last two variables with the following linear growth:

$$|f(s, y, z) - f(s, 0, 0)| \le \beta|y| + \gamma|z|, \quad (s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$$

with $f_0 := f(\cdot, 0, 0) \in \mathcal{L}^1, \beta \ge 0$ and $\gamma > 0$. ξ is a real $\mathcal{F}_T$-measurable random variable, and hereafter called the terminal condition or terminal value.

Definition 1.1. *By a solution to BSDE (1.1), we mean a pair $\{(Y_t, Z_t), 0 \le t \le T\}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$ such that $\mathbb{P}$-a.s., $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2(0, T)$ and $t \mapsto f(t, Y_t, Z_t)$ is integrable, and $\mathbb{P}$-a.s. (Y, Z) verifies (1.1).*

By BSDE (ξ,f), we mean the BSDE of generator f and terminal condition ξ.

It is well known that for $(\xi, f_0) \in L^p \times \mathcal{L}^p$ (with $p > 1$), BSDE (1.1) admits an adapted solution (y, z) in the space $\mathcal{S}^p \times \mathcal{M}^p$. See e.g. [6, 4, 1] for more details. For $(\xi, f_0) \in L^1 \times \mathcal{L}^1$, one needs to restrict the generator f to grow sub-linearly with respect to z, i.e., with some $q \in [0, 1)$,

$$|f(t, y, z) - f_0(t)| \le \beta|y| + \gamma|z|^q, \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$$

for BSDE (1.1) to have an adapted solution (see [1]).

The objective of the paper is to search for a reasonably weakest possible integrability condition for the data (ξ, f_0) to guarantee the existence of an adapted solution for a linearly growing BSDE (1.1). It has been expected up till now that the $L \log L$ integrability is a sufficient one to guarantee the existence of an adapted solution to BSDE (1.1). In this paper, we show by a counterexample that such an expected condition is not sufficient, and further, we shall provide a novel integrability one.

Our sufficient condition is stated as follows: there exists $\lambda \in (0, \frac{1}{\gamma^2 T})$ such that

$$\mathbb{E}\left[|\xi| \exp\left(\sqrt{\frac{2}{\lambda}\log(1 + |\xi|)}\right)\right] + \mathbb{E}\left[\int_0^T |f_0(t)|\, dt \exp\left(\sqrt{\frac{2}{\lambda}\log\left(1 + \int_0^T |f_0(t)|\, dt\right)}\right)\right] < +\infty. \tag{1.2}$$

Define for $\lambda > 0$,

$$\psi(x) = xe^{\sqrt{\frac{2}{\lambda}\log(1+x)}}, \quad x \ge 0.$$

We have for $x > 0$,

$$\psi'(x) = e^{\sqrt{\frac{2}{\lambda}\log(x+1)}}\left[1 + \frac{x}{(1+x)\sqrt{2\lambda\log(1+x)}}\right] > 0,$$

and

$$\psi''(x) = \frac{\sqrt{2\lambda}(4+2x)\log(1+x) + 2x\log^{1/2}(1+x) - \sqrt{2\lambda}x}{4\lambda(1+x)^2\log^{3/2}(1+x)} e^{\sqrt{\frac{2}{\lambda}\log(1+x)}} > 0.$$

Obviously,

$$\lim_{x\to 0+}\psi'(x) = 1, \quad \lim_{x\to 0+}\psi''(x) = +\infty,$$

and ψ is strictly increasing and strictly convex on $(0, +\infty)$. Therefore, our sufficient condition (1.2) in fact requires that both the terminal value ξ and the integral $\int_0^T |f_0|\, ds$ lies in the Orlicz space L^ψ of random variables associated to the convex function ψ.

Remark 1.2. Note that the $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$-integrability is stronger than L^1, weaker than L^p for any $p > 1$, because for any $\varepsilon > 0$, we have,

$$x \le xe^{\sqrt{\frac{2}{\lambda}\log(1+x)}} \le xe^{\varepsilon\log(1+x)+\frac{1}{2\varepsilon\lambda}} \le e^{\frac{1}{2\varepsilon\lambda}}x(1+x)^\varepsilon, \quad x \ge 0.$$

Moreover, for any $p \ge 1$, there exists a constant $C_p > 0$ such that

$$xe^{\sqrt{\frac{2}{\lambda}\log(1+x)}} \ge C_p\, x\log^p(1+x).$$

We will show by giving a simple example in Example 2.3 that even the condition

$$\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right)\right] + \mathbb{E}\left[\int_0^T |f_0(t)|\, dt\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(t)|\, dt)}\right)\right] < +\infty$$

for some $\lambda > \frac{1}{\gamma^2 T}$ (which implies that $\int_0^T |f_0(t)|\, dt\log^p(1+\int_0^T |f_0(t)|\, dt) \in L^1$ and $|\xi|\log^p(1+|\xi|) \in L^1$) is still too weak to ensure the existence of solution.

Our method applies the dual representation of solution to BSDE with convex generator (see, e.g. [4, 7, 3]) in order to establish some a priori estimate and then the localization procedure of real-valued BSDE [2].

The rest of the paper is organized as follows. Section 2 provides a necessary and sufficient condition for the existence of solution to BSDE (1.1) for the typical form of generator $f(t, y, z) = f_0(t) + \beta y + \gamma|z|$, and establishes that the $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$ integrability for some λ small enough is a sufficient condition for the existence of solution to BSDE (1.1) for the typical form of the generator $f(t, y, z) = f_0(t) + \beta y + \gamma|z|$. Section 3 is devoted to the sufficiency of the $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$ integrability condition for the existence of solution to BSDE (1.1) of the general linearly growing generator.

2 Typical case

Let us first consider the following BSDE:

$$Y_t = \xi + \int_t^T (f_0(s) + \beta Y_s + \gamma|Z_s|)\, ds - \int_t^T Z_s\, dW_s, \tag{2.1}$$

where $f_0 \in \mathcal{L}^1$, and $\beta \ge 0$ and $\gamma > 0$ are some real constants. We suppose further that both the terminal condition ξ and f_0 are nonnegative. Let us denote Σ_T the set of all stopping time τ such that $\tau \le T$; we recall that, for a process $Y = \{Y_t\}_{0\le t\le T}$, Y belongs to the class D if the family of random variables $\{Y_\tau, \tau \in \Sigma_T\}$ are uniformly integrable. Note that if Y is a solution belonging to the class D, then as $\{e^{\beta t}Y_t, 0 \le t \le T\}$ is a local supermartingale, it is a supermatingale, from which we deduce that $Y \ge 0$. In this section, we restrict ourselves to nonnegative solution.

For $\xi + \int_0^T |f_0|\, ds \in L^p$ (with $p > 1$), BSDE (2.1) has a unique solution. It has a dual representation as follows (see, e.g. [4, 3])

$$Y_t = \operatorname*{ess\,sup}_{q\in\mathcal{A}} \mathbb{E}_q \left[e^{\beta(T-t)}\xi + \int_t^T e^{\beta(s-t)} f_0(s)\, ds \,\middle|\, \mathcal{F}_t \right], \quad t \in [0,T] \tag{2.2}$$

where $\mathcal{A}$ is the set of progressively measurable processes q such that $|q| \le \gamma$, and $\mathbb{E}_q$ is the expectation with respect to the equivalent probability $\mathbb{Q}^q$ which is defined as follows:

$$d\mathbb{Q}^q := M_T^q \, d\mathbb{P},$$

with

$$M_t^q := \exp\left(\int_0^t q_s\, dW_s - \frac{1}{2}\int_0^t |q_s|^2\, ds \right), \quad t \in [0,T].$$

2.1 An equivalent condition

Theorem 2.1. *Assume that $\xi \ge 0$ and $f_0 \ge 0$. Then BSDE (2.1) admits a solution (Y,Z) such that $Y \ge 0$ if and only if the following process $\widehat{Y}$ defined by*

$$\widehat{Y}_t := \operatorname*{ess\,sup}_{q\in\mathcal{A}} \mathbb{E}_q \left[e^{\beta(T-t)}\xi + \int_t^T e^{\beta(s-t)} f_0(s)\, ds \,\middle|\, \mathcal{F}_t \right], \quad t \in [0,T]$$

is locally bounded.

Proof. If BSDE (2.1) admits a solution (Y,Z) such that $Y \ge 0$, then we define a sequence of stopping times

$$\sigma_n = T \wedge \inf\{t \ge 0 : |Y_t| > n\},$$

with the convention that $\inf \emptyset = +\infty$. Since Y is continuous, it is locally bounded, which implies that $\sigma_n \to T$ as $n \to +\infty$.

As $W_s^q = W_s - \int_0^s q_r dr$ is a Brownian motion under $\mathbb{Q}^q$, we have

$$Y_{t\wedge\sigma_n} = Y_{\sigma_n} + \int_{t\wedge\sigma_n}^{\sigma_n} (f_0(s) + \beta Y_s + \gamma|Z_s| - Z_s q_s) ds - \int_{t\wedge\sigma_n}^{\sigma_n} Z_s dW_s^q.$$

Applying Itô's formula to $e^{\beta s}Y_s$, we deduce

$$e^{\beta(t\wedge\sigma_n)} Y_{t\wedge\sigma_n} = e^{\beta\sigma_n} Y_{\sigma_n} + \int_{t\wedge\sigma_n}^{\sigma_n} e^{\beta s}(f_0(s) + \gamma|Z_s| - Z_s q_s) ds - \int_{t\wedge\sigma_n}^{\sigma_n} e^{\beta s} Z_s dW_s^q.$$

Taking the conditional $\mathbb{Q}^q$-expectation with respect to $\mathcal{F}_t$, using the fact that $\gamma|Z_s| - Z_s q_s \ge 0$, and the fact that $\mathcal{F}_{t\wedge\sigma_n} \subset \mathcal{F}_t$, we obtain

$$\mathbb{E}_q \left[e^{\beta(\sigma_n - t\wedge\sigma_n)} Y_{\sigma_n} + \int_{t\wedge\sigma_n}^{\sigma_n} e^{\beta(s - t\wedge\sigma_n)} f_0(s) ds \,\middle|\, \mathcal{F}_t \right] \le Y_{t\wedge\sigma_n}.$$

As $\sigma_n \to T$, Fatou's lemma yields that

$$\mathbb{E}_q \left[e^{\beta(T-t)}\xi + \int_t^T e^{\beta(s-t)} f_0(s)\, ds \,\middle|\, \mathcal{F}_t \right] \le Y_t,$$

which implies that $\widehat{Y}$ is locally bounded.

On the other hand, if the process $\widehat{Y}$ is locally bounded, then we construct the solution by use of a localization method (see e.g. [2]). We describe this method here for completeness. Consider the following BSDE:

$$Y_t^n = n \wedge \xi + \int_t^T \left[n \wedge f_0(s) + \beta Y_s^n + \gamma |Z_s^n|\right] ds - \int_t^T Z_s^n \, dW_s.$$

Since the terminal value $n \wedge \xi$ and $n \wedge f_0$ are bounded (hence square-integrable) and the generator is uniformly Lipschitz with respect to (y, z), in view of the well-known existence and uniqueness theorem of Pardoux and Peng [6], the last BSDE has a unique solution (Y^n, Z^n) in $\mathcal{S}^2 \times \mathcal{M}^2$. By comparison theorem, Y^n is nonnegative and nondecreasing with respect to n. Moreover, setting $q_s^n = \gamma \; \mathrm{s}gn(Z_s^n)$, we obtain

$$\begin{aligned} Y_t^n &= \mathbb{E}_{q^n}\left[e^{\beta(T-t)} n \wedge \xi + \int_t^T e^{\beta(s-t)} n \wedge f_0(s)\, ds \,\middle|\, \mathcal{F}_t\right] \\ &\le \mathbb{E}_{q^n}\left[e^{\beta(T-t)} \xi + \int_t^T e^{\beta(s-t)} f_0(s)\, ds \,\middle|\, \mathcal{F}_t\right] \\ &\le \widehat{Y}_t. \end{aligned}$$

Set

$$\tau_k := T \wedge \inf\left\{t \ge 0 : \widehat{Y}_t > k\right\},$$

and

$$Y_k^n(t) := Y_{t\wedge\tau_k}^n, \quad Z_k^n(t) := Z_t^n \mathbf{1}_{t \le \tau_k}.$$

Then (Y_k^n, Z_k^n) satisfies

$$Y_k^n(t) = Y_k^n(T) + \int_t^T \mathbf{1}_{s \le \tau_k}\left[n \wedge f_0(s) + \beta Y_k^n(s) + \gamma |Z_k^n(s)|\right] ds - \int_t^T Z_k^n(s)\, dW_s. \tag{2.3}$$

For fixed k, Y_k^n is nondecreasing with respect to n and remains bounded by k. We can now apply the stability property of BSDE with bounded terminal data (see e.g. [2, Lemma 3, page 611]). Setting $Y_k(t) := \sup_n Y_k^n(t) \ge 0$, there exists Z_k such that $\lim_n Z_k^n = Z_k$ in $\mathcal{M}^2$ and

$$Y_k(t) = \sup_n Y_{\tau_k}^n + \int_t^{\tau_k} \left(f_0(s) + \beta Y_k(s) + \gamma |Z_k(s)|\right) ds - \int_t^{\tau_k} Z_k(s)\, dW_s. \tag{2.4}$$

Finally, noting that

$$Y_{k+1}(t \wedge \tau_k) = Y_k(t \wedge \tau_k) \ge 0, \quad Z_{k+1}\mathbf{1}_{t\le\tau_k} = Z_k \mathbf{1}_{t \le \tau_k},$$

we conclude the existence of solution (Y, Z) with $Y \ge 0$. □

Remark 2.2. Consider the case $d = 1$ and $f_0 \equiv 0$. If BSDE (2.1) admits a solution (Y, Z) such that $Y \ge 0$, by taking $q = \gamma$ and $q = -\gamma$, we deduce from Theorem 2.1 that both $\xi e^{\gamma W_T}$ and $\xi e^{-\gamma W_T}$ are in $L^1(\Omega)$, which implies that $\xi e^{\gamma |W_T|} \in L^1(\Omega)$, as

$$\xi e^{\gamma |W_T|} \le \xi e^{\gamma W_T} + \xi e^{-\gamma W_T}.$$

Example 2.3. Let us set $d = 1$, $T = 1$, $f_0 \equiv 0$, $\beta = 0$, $\gamma = 1$, $\mu \in (0, 1)$, and

$$\xi := e^{\frac{1}{2}W_1^2 - \mu |W_1| + \frac{1}{2}\mu^2} - 1.$$

Write ξ_μ for ξ to indicate the dependence on the parameter μ whenever it is necessary.

Since $\xi e^{|W_1|}$ does not belong to $L^1(\Omega)$ by the following direct calculus:

$$\mathbb{E}\left[\xi_\mu e^{|W_1|}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(e^{\frac{1}{2}|x|^2-\mu|x|+\frac{1}{2}\mu^2}-1\right)e^{|x|}e^{-\frac{1}{2}|x|^2}dx = +\infty,$$

we see that BSDE (2.1) with the terminal value ξ_μ does not admit a solution (Y, Z) such that $Y \geq 0$. We then arrive at the following two assertions.

(i) The $L\log L$ integrability of the terminal value is not a sufficient condition for the existence of solution to BSDE (2.1), for it is straightforward to see that $\xi_\mu \log^p(\xi_\mu+1) \in L^1(\Omega)$ for any $p \geq 1$.

(ii) Since

$$\xi_\mu = e^{\frac{1}{2}(|W_1|-\mu)^2}-1,$$

we have

$$|\xi_\mu|\exp\left(\sqrt{\frac{2}{\lambda}\log\left(1+|\xi_\mu|\right)}\right) \in L^1 \quad \text{for } \lambda > \tfrac{1}{\mu^2}$$

via the following direct calculus:

$$\begin{aligned}&\mathbb{E}\left[|\xi_\mu|\exp\left(\sqrt{\frac{2}{\lambda}\log\left(1+|\xi_\mu|\right)}\right)\right]\\ = \;& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(e^{\frac{1}{2}|x|^2-\mu|x|+\frac{1}{2}\mu^2}-1\right)e^{\frac{1}{\sqrt{\lambda}}\left||x|-\mu\right|}e^{-\frac{1}{2}|x|^2}dx < +\infty.\end{aligned}$$

Therefore, the $L\exp\left(\sqrt{\frac{2}{\lambda}\log(1+L)}\right)$-integrability for $\lambda > \frac{1}{\mu^2}$ of the terminal value is not a sufficient condition for the existence of solution to BSDE (2.1). The upcoming Theorem 2.7 will provide a critical value λ_0 such that this integrability for $\lambda \in (0, \lambda_0)$ of the terminal value is sufficient for the existence of solution to BSDE (2.1).

2.2 Sufficient condition

Let us now look for a sufficient condition for the local boundedness of the process $\widehat{Y}$ defined by

$$\widehat{Y}_t := \operatorname*{ess\,sup}_{q\in\mathcal{A}} \mathbb{E}_q\left[e^{\beta(T-t)}\xi + \int_t^T e^{\beta(s-t)}f_0(s)\,ds \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,T].$$

We have the following elementary inequality.

Lemma 2.4. *For any $x \in \mathbb{R}$ and $y \geq 0$, we have*

$$e^x y \leq e^{\frac{\lambda}{2}x^2} + e^{\frac{2}{\lambda}} y e^{\left[\frac{2}{\lambda}\log(y+1)\right]^{1/2}}. \tag{2.5}$$

Proof. Set

$$z = \left(\frac{2}{\lambda}\log(y+1)\right)^{1/2} \geq 0.$$

Then

$$y = e^{\frac{\lambda}{2}z^2}-1.$$

It is equivalent to prove that for any $x\in\mathbb{R}$ and $z \geq 0$,

$$e^{\frac{\lambda}{2}x^2-x} + \left(e^{\frac{\lambda}{2}z^2}-1\right)\left(e^{z+\frac{2}{\lambda}-x}-1\right) \geq 0.$$

It is evident to see that the above inequality holds when $z+\frac{2}{\lambda}-x \geq 0$.

Consider the case $z + \frac{2}{\lambda} - x < 0$. Then $x - \frac{1}{\lambda} > z + \frac{1}{\lambda} > 0$. Hence

$$\begin{aligned} & e^{\frac{\lambda}{2}x^2 - x} + \left(e^{\frac{\lambda}{2}z^2} - 1\right)\left(e^{z+\frac{2}{\lambda}-x} - 1\right) \\ = \; & e^{\frac{\lambda}{2}(x-\frac{1}{\lambda})^2 - \frac{1}{2\lambda}} + \left(e^{\frac{\lambda}{2}z^2} - 1\right)e^{z+\frac{2}{\lambda}-x} - e^{\frac{\lambda}{2}z^2} + 1 \\ \geq \; & e^{\frac{\lambda}{2}(z+\frac{1}{\lambda})^2 - \frac{1}{2\lambda}} - e^{\frac{\lambda}{2}z^2} + 1 \\ \geq \; & e^{\frac{\lambda}{2}z^2 + z} - e^{\frac{\lambda}{2}z^2} \\ \geq \; & e^{\frac{\lambda}{2}z^2}(e^z - 1) \;\geq\; 0. \quad \square \end{aligned}$$

Remark 2.5. For $\lambda > 0$, define the following function:

$$\varphi(x) := e^{\frac{\lambda}{2}\log^2 x}, \quad x > 0.$$

Recalling that

$$\psi(x) = xe^{\sqrt{\frac{2}{\lambda}\log(1+x)}}, \quad x \geq 0,$$

the inequality (2.5) has the following form

$$xy \leq \varphi(x) + e^{\frac{2}{\lambda}}\psi(y), \quad x > 0, y \geq 0. \tag{2.6}$$

It has the flavor of a Young inequality. Is it exactly a Young inequality? Recall that a Young inequality is the following one

$$xy \leq \int_0^x g(s)\,ds + \int_0^y h(s)\,ds, \quad x \geq 0, y \geq 0$$

for some strictly increasing function g with $g(0) = 0$ and h being the inverse function of g. We have for $x > 0$,

$$\varphi'(x) = \frac{\lambda}{x}\varphi(x)\log x, \quad \varphi''(x) = \frac{\lambda}{x^2}\varphi(x)[\lambda\log^2 x - \log x + 1].$$

Therefore, φ is convex only when the parameter $\lambda \geq \frac{1}{4}$. Since the derivative $\varphi'(x) < 0$ for $x \in (0, 1)$, the inequality (2.6) is very far from a real Young inequality.

Lemma 2.6. *Let $0 < \lambda < \frac{1}{\gamma^2 T}$. For any $q \in \mathcal{A}$,*

$$\mathbb{E}\left[e^{\frac{\lambda}{2}|\int_t^T q_s dW_s|^2} \,\middle|\, \mathcal{F}_t\right] \leq \frac{1}{\sqrt{1 - \lambda\gamma^2(T-t)}}.$$

Proof. Firstly, by use of Girsanov's lemma, for $\theta \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}\left[e^{\theta\int_t^T q_s dW_s} \,\middle|\, \mathcal{F}_t\right] \\ = \; & \mathbb{E}\left[e^{\theta\int_t^T q_s dW_s - \frac{\theta^2}{2}\int_t^T |q_s|^2 ds} e^{\frac{\theta^2}{2}\int_t^T |q_s|^2 ds} \,\middle|\, \mathcal{F}_t\right] \\ \leq \; & e^{\frac{\theta^2\gamma^2}{2}(T-t)}. \end{aligned}$$

Then we apply the equality

$$e^{\frac{\lambda}{2}x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\sqrt{\lambda}yx - \frac{y^2}{2}}\,dy,$$

together with Fubini's theorem and a change of variable to deduce that

$$\begin{aligned}
&\mathbb{E}\left[e^{\frac{\lambda}{2}(\int_t^T q_s dW_s)^2} \,\Big|\, \mathcal{F}_t\right] \\
=\;& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathbb{E}\left[e^{\sqrt{\lambda}y\int_t^T q_s dW_s - \frac{y^2}{2}} \,\Big|\, \mathcal{F}_t\right] dy \\
\le\;& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{(\sqrt{\lambda}y\gamma)^2}{2}(T-t)-\frac{y^2}{2}} dy \\
=\;& \frac{1}{\sqrt{1-\lambda\gamma^2(T-t)}}. \quad \square
\end{aligned}$$

Applying the last two lemmas, we deduce the following sufficient condition.

Theorem 2.7. *Assume that there exists* $\lambda \in (0, \frac{1}{\gamma^2 T})$ *such that*

$$\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right)\right] + \mathbb{E}\left[\int_0^T |f_0(t)|\,dt\,\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(t)|\,dt)}\right)\right] < +\infty.$$

Then

$$\operatorname*{ess\,sup}_{q\in\mathcal{A}} \left\{\mathbb{E}_q\left[e^{\beta(T-t)}\xi + \int_t^T e^{\beta(s-t)} f_0(s)ds \,\Big|\, \mathcal{F}_t\right]\right\} \le \bar{Y}_t, \tag{2.7}$$

with the process

$$\begin{aligned}
\bar{Y}_t \;:=\;& \frac{2}{\sqrt{1-\lambda\gamma^2(T-t)}} e^{\beta(T-t)} + e^{\frac{2}{\lambda}+\beta(T-t)}\,\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \,\Big|\, \mathcal{F}_t\right] \\
&+ e^{\frac{2}{\lambda}+\beta(T-t)}\,\mathbb{E}\left[\int_0^T |f_0(s)|\,ds\,\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(s)|\,ds)}\right) \,\Big|\, \mathcal{F}_t\right], \quad t\in[0,T]
\end{aligned}$$

being locally bounded. Furthermore, if $\xi \ge 0$ *and* $f_0 \ge 0$*, BSDE (2.1) admits a solution* (Y, Z) *such that*

$$Y_t \le \bar{Y}_t.$$

Proof. Since the two random variables

$$|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \quad \text{and} \quad \int_0^T |f_0(s)|\,ds\,\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(s)|\,ds)}\right)$$

are integrable and the filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ is generated by the Brownian motion, both processes

$$\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \,\Big|\, \mathcal{F}_t\right], \quad 0\le t\le T$$

and

$$\mathbb{E}\left[\int_0^T |f_0(s)|\,ds\,\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(s)|\,ds)}\right) \,\Big|\, \mathcal{F}_t\right], \quad 0\le t\le T$$

are continuous. Therefore, the process $\bar{Y}$ is continuous and then locally bounded.

Applying the last two lemmas, we deduce

$$\begin{aligned}\mathbb{E}_q\left[\xi \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[M_T^q (M_t^q)^{-1}\xi \,\middle|\, \mathcal{F}_t\right] \le \mathbb{E}\left[e^{\int_t^T q_s dW_s}|\xi|\middle|\mathcal{F}_t\right] \\ &\le \mathbb{E}\left[e^{\frac{\lambda}{2}|\int_t^T q_s dW_s|^2} \,\middle|\, \mathcal{F}_t\right] + \mathbb{E}\left[e^{\frac{2}{\lambda}}|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \,\middle|\, \mathcal{F}_t\right] \\ &\le \frac{1}{\sqrt{1-\lambda\gamma^2(T-t)}} + e^{\frac{2}{\lambda}}\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \,\middle|\, \mathcal{F}_t\right]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_q\left[\int_t^T f_0(s)\,ds \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[M_T^q (M_t^q)^{-1}\int_t^T f_0(s)\,ds \,\middle|\, \mathcal{F}_t\right] \\ \le\ & \mathbb{E}\left[e^{\int_t^T q_s dW_s}\left|\int_t^T f_0(s)\,ds\right| \,\middle|\, \mathcal{F}_t\right] \\ \le\ & \mathbb{E}\left[e^{\frac{\lambda}{2}|\int_t^T q_s dW_s|^2} \,\middle|\, \mathcal{F}_t\right] + \mathbb{E}\left[e^{\frac{2}{\lambda}}\left|\int_t^T f_0(s)\,ds\right|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+\left|\int_t^T f_0(s)\,ds\right|)}\right) \,\middle|\, \mathcal{F}_t\right] \\ \le\ & \frac{1}{\sqrt{1-\lambda\gamma^2(T-t)}} + e^{\frac{2}{\lambda}}\mathbb{E}\left[\int_0^T |f_0(s)|\,ds\ \exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(s)|\,ds)}\right) \,\middle|\, \mathcal{F}_t\right].\end{aligned}$$

Then we get (2.7) and the rest follows from Theorem 2.1. □

3 An existence result for the general generator

Consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s, \tag{3.1}$$

where f satisfies

$$|f(s,y,z) - f_0(s,0,0)| \le \beta|y| + \gamma|z|, \tag{3.2}$$

with $f_0 := f(\cdot,0,0) \in \mathcal{L}^1, \beta \ge 0$ and $\gamma > 0$.

Theorem 3.1. *Let f be a generator which is continuous with respect to (y,z) and verifies inequality (3.2), and ξ be a terminal condition. Let us suppose that there exists $\lambda \in (0, \frac{1}{\gamma^2 T})$ such that*

$$\mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right)\right] + \mathbb{E}\left[\int_0^T |f_0(t)|\,dt\ \exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(t)|\,dt)}\right)\right] < +\infty.$$

Then BSDE (3.1) admits a solution (Y,Z) such that

$$\begin{aligned}|Y_t| \le\ & \frac{2}{\sqrt{1-\lambda\gamma^2(T-t)}}e^{\beta(T-t)} + e^{\frac{2}{\lambda}+\beta(T-t)}\ \mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log(1+|\xi|)}\right) \,\middle|\, \mathcal{F}_t\right] \\ & + e^{\frac{2}{\lambda}+\beta(T-t)}\ \mathbb{E}\left[\int_t^T |f_0(s)|\,ds\ \exp\left(\sqrt{\frac{2}{\lambda}\log(1+\int_0^T |f_0(s)|\,ds)}\right) \,\middle|\, \mathcal{F}_t\right].\end{aligned}$$

Proof. Let us fix $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$. Set

$$\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p, \quad f_0^{n,p} := f_0^+ \wedge n - f_0^- \wedge p, \quad f^{n,p} := f - f_0 + f_0^{n,p}.$$

As the terminal value $\xi^{n,p}$ and $f^{n,p}(\cdot,0,0)$ are bounded (hence square-integrable) and $f^{n,p}$ is a continuous generator with a linear growth, in view of the existence result of Lepeltier and San Martin [5], the BSDE $(\xi^{n,p}, f^{n,p})$ has a minimal solution $(Y^{n,p}, Z^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$. Set

$$\bar{f}^{n,p}(s,y,z) = |f_0^{n,p}(s)| + \beta y + \gamma|z|, \quad (s,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}.$$

Again in view of Pardoux and Peng [6], the BSDE $(|\xi^{n,p}|, \bar{f}^{n,p})$ has a unique solution $(\bar{Y}^{n,p}, \bar{Z}^{n,p})$ in $\mathcal{S}^2 \times \mathcal{M}^2$.

By comparison theorem,

$$|Y_t^{n,p}| \le \bar{Y}_t^{n,p}.$$

Setting $q_s^{n,p} = \gamma\ sgn(Z_s^{n,p})$, we obtain,

$$\begin{aligned} |Y_t^{n,p}| &\le \bar{Y}_t^{n,p} \\ &= \mathbb{E}_{q^{n,p}}\left[e^{\beta(T-t)}|\xi^{n,p}|\Big|\mathcal{F}_t\right] + \int_t^T e^{\beta(s-t)}|f_0^{n,p}(s)|\,ds. \end{aligned}$$

From inequality (2.7), we have

$$|Y_t^{n,p}| \le \bar{Y}_t$$

with

$$\begin{aligned} \bar{Y}_t &= \frac{2}{\sqrt{1-\lambda\gamma^2(T-t)}}e^{\beta(T-t)} + e^{\frac{2}{\lambda}+\beta(T-t)}\ \mathbb{E}\left[|\xi|\exp\left(\sqrt{\frac{2}{\lambda}\log\left(1+|\xi|\right)}\right)\ \Big|\ \mathcal{F}_t\right] \\ &\quad + e^{\frac{2}{\lambda}+\beta(T-t)}\ \mathbb{E}\left[\int_t^T|f_0(s)|\,ds\ \exp\left(\sqrt{\frac{2}{\lambda}\log\left(1+\int_0^T|f_0(s)|\,ds\right)}\right)\ \Big|\ \mathcal{F}_t\right]. \end{aligned}$$

Moreover, $Y^{n,p}$ is nondecreasing with respect to n, and nonincreasing with respect to p. Once again, we apply the localization method as follows to conclude the existence of solution.

Set

$$\tau_k = T \wedge \inf\left\{t \ge 0 : \bar{Y}_t > k\right\}$$

and

$$Y_k^{n,p}(t) = Y_{t\wedge\tau_k}^{n,p}, \quad Z_k^{n,p}(t) = Z_t^{n,p}\mathbf{1}_{t\le\tau_k}.$$

Then $(Y_k^{n,p}, Z_k^{n,p})$ satisfies

$$Y_k^{n,p}(t) = Y_k^{n,p}(T) + \int_t^T \mathbf{1}_{s\le\tau_k} f^{n,p}(s, Y_k^{n,p}(s), Z_k^{n,p}(s))ds - \int_t^T Z_k^{n,p}(s)dW_s. \tag{3.3}$$

For fixed k, $Y_k^{n,p}$ is nondecreasing with respect to n and nonincreasing with respect to p, and remains bounded by k. We can now apply the stability property of BSDEs with bounded terminal data. Setting $Y_k(t) = \inf_p \sup_n Y_k^{n,p}$, there exists Z_k in $\mathcal{M}^2$ such that $\lim_p \lim_n Z_k^{n,p} = Z_k$ in $\mathcal{M}^2$ and

$$Y_k(t) = \inf_p \sup_n Y_{\tau_k}^{n,p} + \int_t^{\tau_k} f(s, Y_k(s), Z_k(s))ds - \int_t^{\tau_k} Z_k(s)dW_s. \tag{3.4}$$

Finally, noting that

$$Y_{k+1}(t\wedge\tau_k) = Y_k(t\wedge\tau_k), \quad Z_{k+1}\mathbf{1}_{t\le\tau_k} = Z_k\mathbf{1}_{t\le\tau_k},$$

we conclude the existence of solution (Y, Z). □

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