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On the ladder heights of random walks attracted to stable laws of exponent 1

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Abstract

Let Z be the first ladder height of a one dimensional random walk $S_n = X_1 + \cdots + X_n$ with i.i.d. increments X_j which are in the domain of attraction of a stable law of exponent α, $0 < \alpha \leq 1$. We show that $P[Z > x]$ is slowly varying at infinity if and only if $\lim_{n\to\infty} n^{-1} \sum_1^n P[S_k > 0] = 0$. By a known result this provides a criterion for $S_{T(R)}/R \xrightarrow{\mathrm{P}} \infty$ as $R \to \infty$, where $T(R)$ is the time when S_n crosses over the level R for the first time. The proof mostly concerns the case $\alpha = 1$.

Keywords: random walk; stable law of exponent 1; domain of attraction; slowly varying; ladder height; large deviation.
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1 Introduction

This paper concerns the one dimensional random walk S_n with i.i.d. increments which are in the domain of attraction of a stable law. When the limiting law is strictly stable, quite fine results are established for fundamental questions [17],[13],[3], [5], [6], [4] etc. (and references in them), while some of them have remained unsatisfactorily answered in case it is not strictly stable (which may occur only if the exponent of the stable law is one). In the very recent work [1] Q. Berger has addressed some of such questions and obtained natural results on the large deviations, the first ladder epoch, recurrence-transience criteria and the Green function (in transient case). In this paper we address a problem concerning the first ladder height of S_n, which we denote by Z. It is known or readily derived by known results that under the existence of $r = \lim n^{-1} \sum_1^n P[S_k > 0]$, if $r > 0$ then the integrated tail $\int_0^x P[Z > y]dy$ varies regularly and vice versa, whereas in case $r = 0$ the corresponding result, which one might well expect to be likely, has been missing. We prove that $P[Z > x]$ varies slowly if and only if $n^{-1} \sum_{k=1}^n P[S_n > 0] \to 0$, and thus fill the gap in the literature.

Let $S_n = X_1 + \cdots + X_n$ be a random walk on the real line $\mathbb{R}$ with i.i.d. increments $X_k, k = 1, 2, \ldots$ and denote by F the common distribution function of X_k. Let P be the probability law for X_k and E the expectation by P. We need the condition

$$(\mathrm{H}) \begin{cases} (1) & EX = 0 \text{ if } E|X| < \infty, \text{ and} \\ (2) & F \text{ belongs to the domain of attraction of a stable law.} \end{cases}$$

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Here X is a random variable having the distribution F. Under (H1), for the condition (H2) to hold, namely for the law of the normalized sum

$$(S_n - \tau_n)/a_n \tag{1.1}$$

to converge to a stable law of exponent α for some $0 < \alpha \le 2$ and non-stochastic sequences $(a_n), (\tau_n)$, it is necessary and sufficient that as $x \to \infty$

$$\begin{cases} \int_{-x}^{x} y^2 dF(y) \sim L(x) & \text{if} \quad \alpha = 2 \\ F(-x) \sim (1-p)x^{-\alpha}L(x) \quad \text{and} \quad 1 - F(x) \sim px^{-\alpha}L(x) & \text{if} \quad 0 < \alpha < 2 \end{cases} \tag{1.2}$$

where $L(x)$ is positive and slowly varying at infinity (in the sense of Karamata) (cf. [10], [3]), $\sim$ means that the ratio of two sides of it approaches one and $0 \le p \le 1$. In the sequel we suppose (H) as well as (1.2) to hold unless the contrary is stated.

Let Z be the first (strictly ascending) ladder height of the walk: $Z = S_T$ where $T = \inf\{n \ge 1 : S_n > 0\}$. For $P[S_n > 0 \ \text{ i.o.}] = 1$ to hold so that Z is defined as a proper random variable, it is necessary and sufficient that

$$\sum n^{-1} P[S_n > 0] = \infty \tag{1.3}$$

(cf. [10, Section XII,7]). This is valid if $EX = 0$, while under the condition $E|X| = \infty$ (1.3) holds if and only if

$$\int_0^\infty \frac{xdF(x)}{1 + \int_0^x F(-y)dy} = \infty$$

—in general, i.e., without assuming (H)— according to [8]; it is then readily seen that (1.3) possibly fails to hold only if $\alpha \le 1$, $p = 0$ and $E|X| = \infty$ (under (H)).

Suppose that if $\alpha \ne 1$ then $\tau_n = 0$ in (1.1) (as is standard), let Y be a random variable with the limiting stable law and put $\rho = P[Y > 0]$. The following results are mostly obtained by Rogozin [17].

(i) If $\alpha = 2$, then $\int_0^x P[Z > u]du$ is slowly varying.

(ii) Let $1 < \alpha < 2$. Then $0 < \rho\alpha < 1$ if $p > 0$ and $\rho\alpha = 1$ if $p = 0$, and $\int_0^x P[Z > u]du$ is regularly varying with index $-\rho\alpha + 1$ in either case.

(iii) If $\alpha < 1$ and $p > 0$, then $0 < \rho\alpha < 1$ and $P[Z > x]$ is regularly varying with index $-\rho\alpha$ (or, equivalently, $\int_0^x P[Z > u]du$ is regularly varying with index $-\rho\alpha + 1$ in view of Karamata's theorem).

(iv) Let $\alpha = 1$. In case $p = 1/2$ suppose that Spitzer's condition holds, namely, there exists

$$\lim \frac{1}{n} \sum_{k=1}^{n} P[S_k > 0] = r. \tag{1.4}$$

Then $\int_0^x P[Z > y]dy$ is slowly varying in each of the following cases

(a) $EX = 0$ and $p < 1/2$; (b) $E|X| = \infty$ and $p > 1/2$; (c) $p = 1/2$ with $r = 1$.

If $p = 1/2$ and $0 < r < 1$, then $P[Z > x]$ is regularly varying with index $-r\alpha$.

(Cf. [17] except for the case $r = 1$ of (iv) where a result from [15] is employed;for more detais see [19] in which $EX = 0$ is assumed but the arguments apply also to the case $E|X| = \infty$.)

From the above list the following case is excluded:

$$(\sharp) \begin{cases} \text{either} \quad \alpha = 1 \ \text{ and } \begin{cases} p > 1/2 \text{ or } p = 1/2 \text{ with } r = 0, \text{ if } EX = 0, \\ p < 1/2 \text{ or } p = 1/2 \text{ with } r = 0, \text{ if } E|X| = \infty; \end{cases} \\ \text{or} \qquad \alpha < 1 \ \text{ and } \ p = 0. \end{cases}$$

In [17] it is shown that if X itself is subject to the stable law of exponent 1 with skewness parameter in $(-1,0)$ (that corresponds to $0<p<1/2$), then $P[Z>x]$ is slowly varying. The following theorem, the main result of the present work, asserts that the same consequence holds in and only in the above exceptional case, provided Z is proper. Note that (1.4) follows from (H) with $r=\rho$ unless $\alpha=1$, and hence the condition ($\sharp$) may be rephrased as (1.4) being valid with $r=0$ (without assuming the existence of the limit in (1.4) in advance), and also that (1.4) implies $P[S_n>0]\to r$ according to [2].

Theorem *Let $\alpha=1$ and the condition (1.3) be satisfied. Then $P[Z>x]$ is slowly varying at infinity if and only if*

$$\lim P[S_n>0]=0,$$

and if this is the case,

$$\log P[Z>x]=-\sum_{1\le n\le x/L(x)}\frac{1}{n}P[S_n>0]+o(1). \tag{1.5}$$

In [1] Q. Berger obtains an asymptotic form of $P[S_n>0]$ expressed in terms of L and τ_n (see (2.11)). The sum on the RHS of (1.5) accordingly has the corresponding expression; if $p\neq 1/2$ in (1.5) in particular, this leads to a simple explicit expression of $P[Z>x]$, which we state as a corollary below. (Its proof is given in Section 3.4 in which we also give a brief discussion for the case $p=1/2$ with $r=0$.)

Let L^* be the slowly varying function defined by

$$L^*(x)=\begin{cases}\int_x^\infty y^{-1}L(y)dy & \text{if}\quad EX=0,\\ \int_1^x y^{-1}L(y)dy & \text{if}\quad EX=\infty.\end{cases} \tag{1.6}$$

Corollary *Under the assumption of Theorem, if $P[S_n>0]\to 0$ and $p\neq 1/2$, then*

$$P[Z>x]=[L^*(x)]^{p/(2p-1)+o(1)}. \tag{1.7}$$

The behaviour of the tail of the distribution of Z is related to that of the overshoots in crossing high levels. Let $T(R)$ be the first time when S_n enters a half line (R,∞) after time 0: $T(R)=\inf\{n\ge 1: S_n>R\}$ and put

$$Z(R)=S_{T(R)}-R,$$

the overshoot of the walk in crossing over the level R for the first time. (Note that $T(0)=T$ and $Z(0)=Z$.) It is then known that $\int_0^x P[Z>u]du$ is regularly varying with index $-s+1$, $0\le s\le 1$ (for $0\le s<1$, this implies the regular variation of $P[Z>x]$) if and only if as $R\to\infty$

$$\frac{Z(R)}{R}\begin{cases}\xrightarrow{\text{P}}\begin{cases}0 & \text{if}\quad s=1,\\ \infty & \text{if}\quad s=0,\end{cases}\\ \Longrightarrow \zeta^{(s)} \qquad \text{if}\quad 0<s<1,\end{cases}$$

where $\Rightarrow$ denotes the convergence in law and $\zeta^{(s)}$ a random variable having the probability law with density $(\pi^{-1}\sin\pi s)/x^s(1+x)$, $x>0$ (cf. [17]; see also [7], [10, Theorem XIV.3] for the convergence to $\zeta^{(s)}$). Under the conditions (1.3) and (1.4) the following table summarizes the asymptotic behaviour of $Z(R)$ as well as the results that are mentioned in (i) through (iv) above or given by Theorem (our Theorem deals with the second column of the table). In the table $\theta_Z(x)$ stands for $P[Z>x]$ and $f\in R_\nu$ means that f varies regularly at infinity with index ν.

	$Z(R)/R \xrightarrow{\mathrm{P}} \infty$	$Z(R)/R \Longrightarrow \zeta^{(r\alpha)}$	$Z(R)/R \xrightarrow{\mathrm{P}} 0$
	$\theta_Z \in R_0$	$\theta_Z \in R_{-r\alpha}$	$\int_0^{\cdot} \theta_Z(u)du \in R_0$
	$r\alpha = 0$	$0 < r\alpha < 1$	$r\alpha = 1$
$\alpha = 2$	$*$	$*$	$0 \le p \le 1$, $\rho = \frac{1}{2}$
$1 < \alpha < 2$	$*$	$p > 0$, $0 < \rho\alpha < 1$	$p = 0$, $\rho\alpha = 1$
$\alpha = 1$ $EX = 0$	$p > \frac{1}{2}$ or $p = \frac{1}{2}$ with $r = 0$	$0 < r < 1$	$p < \frac{1}{2}$ or $p = \frac{1}{2}$ with $r = 1$
$\alpha = 1$ $E\|X\| = \infty$	$p < \frac{1}{2}$ or $p = \frac{1}{2}$ with $r = 0$	(necessarily $p = \frac{1}{2}$)	$p > \frac{1}{2}$ or $p = \frac{1}{2}$ with $r = 1$
$0 < \alpha < 1$	$p = 0$, $\rho = 0$	$p > 0$, $0 < \rho \le 1$	$*$

※ In case $\alpha = 1$ the second (fourth) column is simply represented by $r = 0$ ($r = 1$).
※ The value of ρ depends on the choice of τ_n. If $\alpha \ne 1$, we can take $\tau_n \equiv 0$ and then r agrees with ρ. In case $\alpha = 1$, r is related to ρ quite differently (cf. Section 2.1).

Recalling the well-known result that if $\theta_T(t) := P[T > t]$ then $\theta_T \in R_{-r}$ for $0 \le r < 1$ and $\int_0^{\cdot} \theta_T(s)ds \in R_0$ for $r = 1$ under (1.3) and (1.4) (without assuming (H); cf. [17], [3]), one reads off the (expected) parallel between the distributions of Z and T from the table.

2 Preliminaries

Throughout this section, consisting of three subsections, we suppose that $\alpha = 1$ and (1.2) holds, namely

$$F(-x) \sim (1-p)L(x)/x \quad \text{and} \quad 1 - F(x) \sim pL(x)/x \tag{2.1}$$

($0 \le p \le 1$). In the first subsection we review fundamental properties of norming and centering constants a_n and τ_n, and give elementary facts concerning them. We show that nb_n is essentially slowly varying in the second one. In the third we briefly discuss on large deviation estimates for S_n. We shall apply fundamental results concerning regularly varying functions as given in Section VIII, 8 of [10] or Sections 1.5-7 of [3] usually without specifying them.

2.1. Relations of a_n and b_n to $L(x)$

Let a_n be any sequence satisfying

$$\frac{nE[X^2; |X| \le a_n]}{a_n^2} \to 1 \tag{2.2}$$

and put

$$b_n = E[\sin(X/a_n)].$$

Then the normalized walk $S_n/a_n - nb_n$ converges in law to a stable variable Y whose characteristic function $Ee^{itY} = e^{-\Psi(t)}$ is given by

$$\Psi(t) = |t|\{\tfrac{1}{2}\pi + i(\operatorname{sgn} t)\beta \log|t|\}$$

where $\beta = 2p-1$ and $\operatorname{sgn} t = t/|t|$ (cf. [10, (XVII.3.18-19)]). Spitzer's condition (1.4) holds if and only if there exists $m := \lim nb_n \in \mathbb{R} \cup \{-\infty, +\infty\}$, and in this case $r = P[Y + m > 0]$; in particular $r = 0$ or 1 according as $m = -\infty$ or $+\infty$.

Let L^* be the slowly varying function given in (1.6). By integrating by parts

$$E[\sin Xt] = t\int_{-\infty}^{\infty}[1-F(y)-F(-y)]\cos ty\,dy, \tag{2.3}$$

and if $\beta \neq 0$,

$$\frac{E[\sin Xt]}{t} = \begin{cases} -\beta L^*(1/t)\{1+o(1)\} & \text{if} \quad EX=0, \\ \beta L^*(1/t)\{1+o(1)\} & \text{if} \quad E|X|=\infty: \end{cases}$$

see [16, Theorem7], [3, Theorems 4.3.1-2] (the direct verification is not hard (cf. (2.7)); in [16] a precise expression of the error term is given). Substitution of $1/a_n$ for t in the relation above yields

$$a_nb_n = \pm\beta L^*(a_n)\{1+o(1)\} \quad \text{or equivalently} \quad nb_n = \pm\beta\frac{L^*(a_n)}{L(a_n)}\{1+o(1)\}, \tag{2.4}$$

where $+$ or $-$ prevails according as $E|X|=\infty$ or $EX=0$. This entails that if $\beta\neq 0$, $|nb_n|$ is slowly varying and tends to infinity (since $L(x)/L^*(x)\to 0$ for the latter). In the next subsection we shall see that nb_n is slowly varying also in case $\beta = 0$ as far as it is bounded away from zero.

It is easy to see $E[X^2;|X|\le a]\sim aL(a)$, so that a_n is determined by

$$a_n/L(a_n)\sim n,$$

which entails that if $a(t)$ is an asymptotic inverse of $x/L(x)\sim 1/[1-F(x)+F(-x)]$, then $a_n\sim a(n)$. Since $a(st)/a(t)\to s$, it follows that $a_{n+k}/a_n\to 1$ if $k/n\to 0$. The function $a(t)$ is uniquely determined by F within asymptotic equivalence. (Cf. [3, Section 1.5.7].)

Sometimes it is convenient to choose $L(x)$ so that it is given in the form $e^{\int_1^x \varepsilon(u)du/u}$ with a measurable function $\varepsilon(u)$ tending to zero as $u\to\infty$ so that L is continuous and $a(t)$, $t>0$ can be the exact inverse of $x/L(x)$, i.e.,

$$a(t) = tL(a(t)) \tag{2.5}$$

for all sufficiently large x, $x/L(x)$ being ultimately increasing. According to the uniqueness of $a(t)$, nb_n is determined uniquely apart from an additive error of magnitude $o(nb_n)$.

2.2. Slow variation of nb_n

Put

$$\tau_n = na_nb_n \quad \text{and} \quad \tilde S_n = S_n - \tau_n$$

so that $\tilde S_n/a_n$ converges in law to Y.

For the proof of the converse half of Theorem we need the following

Lemma 2.1. $mb_m = nb_n\{1+o(1)\}+o(1)$ *$(n\to\infty)$ uniformly for* $n<m\le 2n$

The proof is given after showing two preparatory lemmas. In the first one L may be any positive function that is right-continuous and slowly varying at infinity.

Lemma 2.2. *Let $\eta(x)$ be a positive function such that $\eta(x)/x$ is ultimately non-increasing and tends to zero as $x\to 0$. Put for $a>0$*

$$\phi(a) = \int_0^\infty \frac{\eta(x)}{x}\cos\frac{x}{a}\,dx.$$

If $\eta(x)/L(x)\to 0$, then uniformly for $0<h\le a$,

$$\phi(a+h)-\phi(a) = o(L(a)) \quad \textit{as} \quad a\to\infty, \tag{2.6}$$

and if $\eta(x)\le CL(x)$ for some constant C, then (2.6) holds whenever $h/a\to 0$.

Proof. Put
$$I_a(u,v;\eta) = \int_u^v \frac{\eta(x)}{x}\cos\frac{x}{a}\,dx$$
$(0<u<v)$. Then by the assumed monotonicity of $\eta(x)/x$ we have for any $a, a', M > 0$
$$|\phi(a') - I_{a'}(0, Ma;\eta)| \le \int_{Ma}^{Ma+\pi a'} \frac{\eta(y)dy}{y}. \tag{2.7}$$
Observing that for $0 \le h \le a$,
$$|I_{a+h}(0,Ma;\eta) - I_a(0,Ma;\eta)| \le \frac{h}{a^2}\int_0^{Ma}\eta(x)dx, \text{ and}$$
$$\int_{Ma}^{Ma+\pi(a+h)} \frac{\eta(y)dy}{y} \le \Big[\sup_{Ma\le x\le Ma+2\pi a}\eta(x)\Big]\frac{2\pi}{M}$$
and then applying the inequality (2.7) with $a'=a$ and $a'=a+h$, we see that
$$|\phi(a+h)-\phi(a)| \le \frac{h}{a^2}\int_0^{Ma}\eta(x)dx + \Big[\sup_{Ma\le x\le Ma+2\pi a}\eta(x)\Big]\frac{4\pi}{M}.$$
Since $\int_0^{Ma} L(x)dx \sim MaL(Ma)$, we readily deduce the assertions of the lemma. □

Lemma 2.3. *Put for $a>0$,*
$$L^\dagger(a) = aE[\sin(X/a)].$$
Then for $p=1/2$, $L^\dagger(a+h) - L^\dagger(a) = o(L(a))$ $(a\to\infty)$ uniformly for $0<h\le a$.

Proof. Put $\eta_+(x) = x(1-F(x))$, $\eta_-(x) = xF(-x)$ and
$$\phi_\pm(a) = \int_0^\infty \frac{\eta_\pm(x)}{x}\cos\frac{x}{a}\,dx,$$
so that $L^\dagger(a) = \phi_+(a) - \phi_-(a)$ in view of (2.3). Let $I_a(u,v;\eta_\pm)$ be defined as in the proof of the preceding lemma and put $\eta = \eta_+ - \eta_-$. Although $\eta(x)/x$ is not necessarily decreasing, the inequalities of (2.7) with η_+ or η_- in place of η are valid. Since $I_a(u,v;\eta)$ is linear in η, namely $I_a(u,v;\eta_+) - I_a(u,v;\eta_-) = I_a(u,v;\eta)$, we have
$$|L^\dagger(a+h) - L^\dagger(a)| \le |I_{a+h}(0,Ma;\eta) - I_a(0,Ma;\eta)| + \sum_{+,-}\Big[\sup_{Ma\le x\le Ma+2\pi a}\eta_\pm(x)\Big]\frac{2\pi}{M}.$$
If $p=1/2$, then $|\eta(x)| = o(L(x))$, so that the first term on the RHS is dominated by
$$ha^{-2}\int_0^{Ma}|\eta(x)|dx = a^{-1}\int_0^{Ma}L(x)dx \times o(1) = o(L(a)).$$
We also have $\eta_+(x)+\eta_-(x) \sim L(x)$, hence the second term is at most a constant multiple of $L(Ma)/M$. Since M may be arbitrarily large we conclude the asserted relation of the lemma. □

Proof of Lemma 2.1. We have seen in (2.4) that nb_n is slowly varying if $p \ne 1/2$. So let $p=1/2$. We first note that by Lemma 2.3 this implies that uniformly for for $n < m \le 2n$,
$$L^\dagger(a_m) - L^\dagger(a_n) = o(L(a_n)) \quad \text{as} \quad n\to\infty.$$

We may suppose that $a_n = nL(a(n))$ (cf. (2.5) and a remark given right after it) so that

$$nb_n = nL^\dagger(a_n)/a_n = L^\dagger(a_n)/L(a_n), \quad a_n = a(n). \tag{2.8}$$

One then deduces that

$$\begin{aligned} mb_m - nb_n = \frac{L^\dagger(a_m)}{L(a_m)} - \frac{L^\dagger(a_n)}{L(a_n)} &= \frac{L^\dagger(a_m) - L^\dagger(a_n)}{L(a_m)} + \left(\frac{1}{L(a_m)} - \frac{1}{L(a_n)}\right)L^\dagger(a_n) \\ &= o(1) + \frac{L(a_n) - L(a_m)}{L(a_m)} nb_n = o(nb_n) + o(1) \end{aligned}$$

(for the third equality use the relations noted above). This concludes the proof. □

2.3. Large deviations

For the proof of Theorem we need an upper bound of $P[S_n \geq y]$ as $y \wedge n \to \infty$ in case $\alpha < 1$ with $p = 0$ as well as in case $\alpha = 1$ with $nb_n \to -\infty$. When $\alpha \neq 1$ and $p > 0$ the exact asymptotic order is known. Here we consider the case $\alpha = 1$; the case $\alpha < 1$ with $p = 0$ will be dealt with in Section 3.2. Let $\tau_n = na_nb_n$ and $\tilde{S}_n = S_n - \tau_n$ as in the preceding subsection.

Let $\alpha = 1$. According to [1, Theorem 2.1] as $x \wedge n \to \infty$

$$P[\tilde{S}_n \geq xa_n] \sim pnL(xa_n)/xa_n \tag{2.9}$$

with the obvious interpretation if $p = 0$. Since $P[S_n > y] = P[\tilde{S}_n > y - \tau_n]$ and $P[|X| > x] \sim x/L(x)$, it plainly follows that if $nb_n \to -\infty$ (or, equivalently, $P[S_n > 0] \to 0$), then for $y \geq 0$,

$$P[S_n \geq y] \leq C_1 nP[|X| > |\tau_n| + y]. \tag{2.10}$$

This is sufficient for our proof of Theorem, and could have been relatively easy to prove (the method used by Heyde [14] may apply). By (2.4) and (2.8) we get that $|\tau_n| \sim |\beta| nL^*(a_n)$ if $p \neq 1/2$ and $|\tau_n| \sim nL^\dagger(a_n)$ if $p = 1/2$. It therefore follows from (2.9) that if $nb_n \to -\infty$,

$$P[S_n \geq 0] \sim \frac{pnL(|\tau_n|)}{|\tau_n|} \sim \begin{cases} \frac{p}{|2p-1|} L(nL^*(a_n))/L^*(a_n) & \text{if } p \neq \frac{1}{2}, \\ \frac{1}{2} L(nL^\dagger(a_n))/L^\dagger(a_n) & \text{if } \ p = \frac{1}{2}. \end{cases} \tag{2.11}$$

We will use (2.11) to obtain the expression of $P[Z > x]$ given in (1.7) (see Section 3.4).

3 Proof of Theorem and Corollary

Put

$$f(\lambda) = 1 - E[e^{-\lambda Z}].$$

Let (1.3) holds. Then $f(\lambda) = \lambda \int_0^\infty e^{-\lambda x} P[Z > x]dx$, $\lambda > 0$ and by Tauberian theorem (cf. [10], [3]) the slow variation of $P[Z > x]$ at infinity is equivalent to that of $f(\lambda)$ at zero. We are going to prove the latter. The proof is based on the representation

$$f(\lambda) = \exp\left\{ -\sum_{n=1}^\infty \frac{1}{n} E[e^{-\lambda S_n}; S_n > 0] \right\} \tag{3.1}$$

due to Spitzer [18, Proposition 17.5] (cf. [10, Section XVIII,3]). Put

$$U(x) = \sum_{n=1}^\infty \frac{1}{n} P[0 < S_n \leq x] \quad \text{and} \quad \hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} dU(x); \ (\lambda > 0),$$

so that $f(\lambda) = e^{-\hat{U}(\lambda)}$. It is shown by Greenwood et al [12] that for $0 \le \nu < 1$, $P[Z > x]$ is regularly varying with index $-\nu$ if and only if for any $s > 1$, $U(sx) - U(x) \to \nu \log s$ as $x \to \infty$ (called Sinai's condition) [3, Theorem 8.9.17]. Here we are concerned only with the case $\nu = 0$ for which this is a consequence of the equivalence

$$\forall s > 1,\ \lim_{\lambda\to 0}[\hat{U}(s\lambda) - \hat{U}(\lambda)] = 0 \iff \forall s > 1,\ \lim_{x\to\infty}[U(sx) - U(x)] = 0;$$

moreover if this is the case,

$$\hat{U}(\lambda) = U(1/\lambda) + o(1) \tag{3.2}$$

(cf. [3, Theorem 3.9.1]). Although we apply this polished result below (to bypass the standard arguments for subsidiary results), we could avoid their use and derive the result directly from (3.1) with the essential part of the proof unaltered.

The rest of this section is divided into four subsections. The sufficiency of the condition $\lim P[S_n > 0] = 0$ for the slow variation of $f(\lambda)$ is shown in the first subsection in case $\alpha = 1$. The necessity part and the case $\alpha < 1$ are dealt with in the second and third ones, respectively. In the last one Corollary is proved. The conditions (1.2) and (1.3) are supposed to hold throughout.

3.1. Sufficiency in Case $\alpha = 1$

It suffices to show that if $\lim P[S_n > 0] = 0$, then

$$U(x) = \sum_{1\le n<x/L(x)} \frac{1}{n} P[S_n > 0] + o(1). \tag{3.3}$$

Indeed (3.3) together with $P[S_n > 0] \to 0$ implies that for each $s > 1$,

$$|U(sx) - U(x)| \le \sup_{n>x/L(x)} P[S_n > 0] \log[sL(x)/L(sx)] + o(1) \to 0,$$

and hence by (3.2)

$$f(\lambda) = e^{-\hat{U}(\lambda)} = \exp\left\{ - \sum_{1\le n<1/\lambda L(1/\lambda)} \frac{1}{n} P[S_n > 0] + o(1)\right\},$$

which verifies the slow variation of $f(\lambda)$ and, by virtue of a Tauberian theorem, $P[Z > x] \sim f(1/x)$, hence the relation (1.5).

For the derivation of (3.3) we bring in the function $M = M(x)$ defined by

$$M = x/L(x)$$

(we shall usually drop x from the notation). For the present purpose we may suppose that $a(t)$ is given as in (2.5) and $a_n = a(n)$ so that $a_M = ML(a_M)$. By uniqueness of the inverse function it follows that

$$a_M = x \tag{3.4}$$

for x large enough. (3.3) is derived by showing the following two lemmas, where we assume $P[S_n > 0] \to 0$ or, what amounts to the same, $nb_n \to -\infty$ (hence $-\tau_n = |\tau_n|$).

Lemma 3.1.

$$\sum_{n\ge M(x)} \frac{1}{n} P[0 < S_n \le x] \to 0 \quad (x \to \infty). \tag{3.5}$$

Proof. The proof rests on the local limit theorem, of which here we need only the following one-sided estimate

$$a_n P[z \le S_n < z + 1] \le Kg(z/a_n - nb_n) + o(1) \tag{3.6}$$

($n \to \infty$) valid uniformly for $z \in \mathbb{R}$, where $g(y) = P[Y \in dy]/dy$ and K is a constant depending only on F. Since $nb_n \to -\infty$ so that $g(-nb_n + y) \to 0$ uniformly for $y \geq 0$, this implies that

$$P[0 < S_n < x] = o(x/a_n).$$

We accordingly deduce that

$$\sum_{n \geq M} \frac{1}{n} P[0 < S_n \leq x] = \sum_{n \geq M} \frac{x \times o(1)}{na_n} = \frac{x \times o(1)}{a_M} = o(1), \tag{3.7}$$

where the second equality is due to the regular variation of a_n with index 1. □

Lemma 3.2.

$$\sum_{n \leq M(x)} \frac{1}{n} P[S_n \geq x] \to 0.$$

Proof. For each $\varepsilon > 0$, for x large enough

$$\sum_{n \leq \varepsilon M} \frac{1}{n} P[S_n \geq x] \leq C_1 \sum_{n < \varepsilon M} P[|X| > |\tau_n| + x] \leq C\varepsilon M P[|X| > x] \leq 2C\varepsilon, \tag{3.8}$$

where the first inequality follows from (2.10) and the third from $P[|X| > x] \sim L(x)/x$. By $P[S_n > 0] \to 0$ it follows that for each $\varepsilon > 0$, $\sum_{\varepsilon M < n \leq M} \frac{1}{n} P[S_n > 0] \to 0$, which combined with (3.8) shows the lemma, for ε can be chosen arbitrarily small in (3.8). □

Obviously Lemmas 3.1 and 3.2 together show (3.3) as required.

3.2. Necessity in Case $\alpha = 1$

It suffices to show that if $\quad r^* := \limsup_{n \to \infty} P[S_n > 0] > 0, \quad$ then

$$\limsup_{x \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} P[x < S_n < 2x] > 0. \tag{3.9}$$

If $r_* := \liminf P[S_n > 0] = 1$, we know that $\int_0^x P[Z > y]dy$ varies slowly. Hence our present task reduces to showing that if $r^* > 0$ and $r_* < 1$, then (3.9) holds.

Suppose $r^* > 0$ and $r_* < 1$. Put $B_n = nb_n$. Since then $Y_n := \bar{S}_n/a_n = S_n/a_n - B_n$ converges in law to Y and

$$P[S_n > 0] = P[Y_n > -B_n], \tag{3.10}$$

and since $B_{n+1} = B_n\{1 + o(1)\} + o(1)$ in view of Lemma 2.1, the increment of $P[S_n > 0]$ tends to zero. This shows that there exists an increasing sequence $(n(k))$ of integers such that $n(k) \to \infty$ ($k \to \infty$) and

$$0 < (r_* + r^*)/4 \leq P[S_{n(k)} > 0] \leq (r_* + r^* + 2)/4 < 1,$$

which combined with (3.10) implies that $B_{n(k)}$ is bounded. Put

$$x_k = a_{n(k)}.$$

Then, for $n(k) \leq n \leq 2n(k)$, we have on the one hand

$$\begin{aligned} P[x_k \leq S_n < 2x_k] &= P[x_k \leq \tilde{S}_n + a_n B_n < 2x_k] \\ &= P[a_{n(k)}/a_n \leq Y_n + B_n < 2a_{n(k)}/a_n], \end{aligned} \tag{3.11}$$

and on the other hand, by virtue of Lemma 2.1 again and by the regularity of a_n,

$$B_n = B_{n(k)}\{1 + o(1)\} + o(1) \quad \text{and} \quad a_{n(k)}/a_n = [n(k)/n]\{1 + o(1)\},$$

and hence the convergence of Y_n to Y implies that the last probability in (3.11) is bounded away from zero so that there exists a positive constant q such that $P[x_k \le S_n < 2x_k] \ge q$, and consequently

$$\sum \frac{1}{n} P[x_k < S_n < 2x_k] \ge q \sum_{n(k) \le n \le 2n(k)} \frac{1}{n} \ge q \log 2$$

showing (3.9). The proof of necessity part is complete.

3.3. In case $\alpha < 1$ and $p = 0$

The following lemma is a consequence of the inequality

$$P[S_n \ge y] \le nP[X > z] + \exp\left\{\frac{y}{z} - \frac{y}{z}\log\left(1 + \frac{y}{nE[X; 0 < X \le z]}\right)\right\} \tag{3.12}$$

($y, z > 0$) due to Fuk and Nagaev [11](Theorem 1) (valid for any real random variable X).

Lemma 3.3. *For $y > 0$, $P[S_n > y] \le e \cdot ny^{-1} \int_0^y (1 - F(u))du$.*

Proof. Take $z = y$ in (3.12) and omit 1 in the round parentheses. Then substitution of the expression $\int_0^y P[X > u]du - yP[X > y]$ for $E[X; 0 < X \le y]$ yields the inequality of the lemma. □

Now we turn to the proof of Theorem. First note that choosing a_n in (1.1) to be the sequence satisfying

$$\frac{nE[X^2; |X| \le a_n]}{a_n^2} \longrightarrow \frac{\alpha}{2 - \alpha},$$

which is the same as imposing that

$$a_n^\alpha / L(a_n) \sim n.$$

Then the normalized walk S_n/a_n converges in law to a stable variable Y.

Since $y^{-1} \int_0^y (1 - F(u))du = o(F(-y))$, we have in view of Lemma 3.3

$$\frac{1}{n} P[S_n > x] \le \frac{e}{x} \int_0^x [1 - F(u)]du = o(x^{-\alpha} L(x)).$$

Define $M = M(x)$ analogously to the one for $\alpha = 1$ so that for x large,

$$x = a_M = (ML(a_M))^{1/\alpha} \quad \text{or equivalently} \quad M(x) = x^\alpha / L(x)$$

Then

$$\sum_{n<M} \frac{1}{n} P[S_n > x] \le M \times o(x^{-\alpha} L(x)) \to 0 \quad (x \to \infty).$$

On the other hand since $p = 0$ entails $P[Y > 0] = 0$, the local limit theorem yields

$$P[0 < S_n \le x] = o(x/a_n)$$

and we see

$$\sum_{n>M} \frac{1}{n} P[0 < S_n < x] \le x \sum_{n>M} o(1/na_n) = x \times o(1/a_M) = o(1).$$

Thus $U(x) = -\sum_{n \le M(x)} n^{-1} P[S_n > 0] + o(1)$ and we obtain the result in the same way as discussed in the beginning of the sufficiency part.

3.4. Proof of Corollary

Let $p \neq 1/2$ and $a(t)$ be as given at the end of Section 2.1 so that $a(t) = tL(a(t))$. Recall $(d/dx)L^*(x) = \pm L(x)/x$. Here and below $\pm$ is as in (2.4). Then, by applying Theorems 2.3.2 and 2.3.1 of [3] in turn (with $f(x) = L^*(x)/L(x)$ and $\ell = L^*$) or by Lemma 4.3 of [1] we obtain

$$L^*(xL^*(x)/L(x)) \sim L^*(x). \tag{3.13}$$

Put $R(t) = tL^*(a(t))$. Substitution of $a_n = nL(a_n)$ for x then shows $L^*(R(n)) \sim L^*(a_n)$, hence (2.11) gives $P[S_n > 0] \sim (p/|1-2p|)L(R(n))/L^*(R(n))$. On using $R'(t) \sim L^*(a(t))$ observe

$$\frac{d}{dt}[\log L^*(R(t))] = \pm\frac{L(R(t))R'(t)}{R(t)L^*(R(t))} \sim \pm\frac{L(R(t))}{tL^*(R(t))} \quad (t \to \infty)$$

and we accordingly obtain that if $P[S_n > 0] \to 0$ (entailing $\pm(1-2p) > 0$),

$$-\sum_{k=1}^{n}\frac{1}{k}P[S_k > 0] \sim \frac{-p}{|1-2p|}\int_1^n \frac{L(R(t))}{tL^*(R(t))}dt \sim \frac{p}{2p-1}\log L^*(R(n)).$$

Now the identity $a(x/L(x)) = x$ (valid for x large enough) together with (3.13) shows that $L^*(R(x/L(x))) = L^*(xL^*(x)/L(x)) \sim L^*(x)$, hence (1.7) by virtue of (1.5).

In case $p = 1/2$ with $r = 0$ one may write down an expression of $P[Z > x]$ by substituting into (1.7) that of $P[S_n \geq 0]$ given by (2.11). If $L(nL^\dagger(a_n)) \sim L(a_n)$, which relation seems to hold quite generally, using $a'(t) \sim L(a(t))$ we readily obtain

$$P[Z > x] \sim e^{\{\frac12 + o(1)\}\int_1^x \varepsilon(u)du/u}, \quad \varepsilon(x) = -L(x)/L^\dagger(x). \tag{3.14}$$

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References

[1] Q. Berger, Notes on random walks in the Cauchy domain of attraction, arXiv:1706.07924v2

[2] J. Bertoin and R.A. Doney, Spitzer's condition for random walks and Lévy Processes, Ann. Inst. Henri Poincaré, **33** (1997), 167-178. MR-1443955

[3] N.H. Bingham, G.M. Goldie and J.L. Teugels, Regular variation, *Cambridge Univ. Press*, Cambridge, 1989. MR-0898871

[4] F. Caravenna and R.A. Doney, Local large deviations and the strong renewal theorem. arXiv:1612.07635

[5] R.A. Doney, One-sided local large deviation and renewal theorems in the case of infinite mean, *Probab. Theory Rel. Fields* **107** (1997), 451-465. MR-1440141

[6] R.A. Doney and R.A. Maller, The overshoot for Lévy processes, *Ann. Probab.* **30** (2002), 188-212. MR-1894105

[7] E.B. Dynkin, Some limit theorems for sums of independent random variables with infinite mathematical expectations, *Selected Trans. in Math. Statist. Probab.* vol. 1 IMS-AMS, (1961), 171-189. MR-0116376

[8] K.B. Erickson, The strong law of large numbers when the mean is undefined, *Trans. Amer. Math. Soc.* **185** (1973), 371-381 MR-0336806

[9] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, 2nd ed. *John Wiley and Sons, Inc.* NY. (1957) MR-0088081

[10] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, 2nd ed. *John Wiley and Sons, Inc.* NY. (1971) MR-0088081

[11] D.X. Fuk and S.V. Nagaev, Probability inequalities for sums of independent random variables, *Theor. Probab. Appl.* **16** (1971), 643-660 MR-0293695

[12] P.E. Greenwood, E. Omey and J.I. Teugels, Harmonic renewal measures, *Z. Wahrsch. verw. Geb.* **59** (1982), 391-409. MR-0721635

[13] P.E. Greenwood, E. Omey and J.I. Teugels, Harmonic renewal measures and bivariate domains of attraction in fluctuation theory, *Z. Wahrsch. verw. Geb.* **61** (1982), 527-539. MR-0682578

[14] C.C. Heyde, A contribution to the theory of large deviations for sums of independent random variables, *Z. Wahrsch. verw. Geb.* **7** (1967), 303-308. MR-0216549

[15] H. Kesten and R.A. Maller, Infinite limits and infinite limit points of random walks and trimmed sums, *Ann. Probab.*, **22** (1994), 1473-1513 MR-1303651

[16] E.J.G. Pitman, On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin, *J. Australian Math. Soc. Series A* **8** (1968), 422-443. MR-0231423

[17] B.A. Rogozin, On the distribution of the first ladder moment and height and fluctuations of a random walk, *Theory Probab. Appl.* **16** (1971), 575-595. MR-0290473

[18] F. Spitzer, Principles of Random Walks, *Van Nostrand*, Princeton, 1964. MR-0171290

[19] K. Uchiyama, Estimates of Potential functions of random walks on $\mathbb{Z}$ with zero mean and infinite variance and their applications I. arXiv:1802.09832