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Stationary distributions of the Atlas model*

Li-Cheng Tsai†

Abstract

In this article we study the Atlas model, which consists of Brownian particles on $\mathbb{R}$, independent except that the Atlas (i.e., lowest ranked) particle $X_{(1)}(t)$ receives drift γdt, $\gamma \in \mathbb{R}$. For any fixed shape parameter $a > 2\gamma_-$, we show that, up to a shift $\frac{a}{2}t$, the *entire* particle system has an invariant distribution ν_a, written in terms an explicit Radon-Nikodym derivative with respect to the Poisson point process of density $ae^{a\xi}d\xi$. We further show that ν_a indeed has the product-of-exponential gap distribution π_a derived in [ST17]. As a simple application, we establish a bound on the fluctuation of the Atlas particle $X_{(1)}(t)$ uniformly in t, with the gaps initiated from π_a and $X_{(1)}(0) = 0$.

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1 Introduction

In this article we study the (infinite) Atlas model. Such a model consists of a semi-infinite collection of particles $X_i(t)$, $i = 1, 2, \ldots$, performing independent Brownian motions on $\mathbb{R}$, except that the Atlas (i.e., lowest ranked) particle receives a drift of strength $\gamma \in \mathbb{R}$. To rigorously define the model, we recall that $x = (x_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ is **rankable** if there exists a ranking permutation $p : \mathbb{N} \to \mathbb{N}$ such that $x_{p(i)} \leq x_{p(j)}$, for all $i < j \in \mathbb{N}$. To ensure that such a ranking permutation is unique, we resolve ties in lexicographic order. That is, if $x_{p(i)} = x_{p(j)}$ for $i < j$, then $p(i) < p(j)$. We then let $p_x(\cdot) : \mathbb{N} \to \mathbb{N}$ denote the unique ranking permutation for a given, rankable x. Fix independent standard Brownian motions $W_1, W_2, \ldots$. For suitable initial conditions, the infinite Atlas model $X(t) = (X_i(t))_{i=1}^{\infty}$ is given by the unique weak solution of the following system of Stochastic Differential Equations (SDEs)

$$dX_i(t) = \gamma \mathbf{1}\{p_{X(t)}(i) = 1\}dt + dW_i(t), \quad i \in \mathbb{N}. \tag{1.1}$$

To state the well-posedness results of (1.1), consider the following configuration space

$$\mathcal{U} = \Big\{ x = (x_i)_{i=1}^{\infty} : \lim_{i\to\infty} x_i = \infty, \text{ and } \sum_{i=1}^{\infty} e^{-ax_i^2} < \infty, \forall a > 0 \Big\}, \tag{1.2}$$

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†Columbia University, United States of America. E-mail: lcstai.math@gmail.com

and note that $\lim_{i\to\infty} x_i = \infty$ necessarily implies that x is rankable. It is shown in [Sar17a, Theorem 3.2], for any fixed $\gamma \in \mathbb{R}$ and any given $x \in \mathcal{U}$, the system (1.1) admits a unique weak solution $X(t)$ starting from the initial condition x, such that $\mathbf{P}(X(t) \in \mathcal{U},\ \forall t \geq 0) = 1$. See also [Shk11, IKS13].

The interest of the Atlas model originates from the study of diffusions with rank-based drifts [Fer02, FK09]. In particular, the Atlas model was first introduced, in finite dimensions, as a simple special case of rank-based diffusions [Fer02]. Due to their intriguing properties, rank-based diffusions have been intensively studied in various generality. See [BFK05, BFI$^+$11, IKS13, Sar17b] and the references therein. The infinite-dimensional system (1.1) considered here was introduced by Pal and Pitman [PP08]. Parts of the motivation was to understand the effect of a drift exerted on a large (but finite) collection of Brownian particles [Ald02, TT15]. In particular, it was shown in [PP08] that, for $\gamma > 0$, the system (1.1) admits a stationary gap distribution of i.i.d. $\mathrm{Exp}(2\gamma)$, which indicates that the drift γdt is balanced by the push-back of a crowd of particles of density 2γ. To state the previous result more precisely, given a rankable $x = (x_i)_{i=1}^\infty$, we let $(x_{(1)} \leq x_{(2)} \leq \ldots)$ denote the corresponding ranked points, i.e., $x_{(i)} = x_{(p_x)^{-1}(i)}$, and consider the corresponding gaps $z_i := x_{(i+1)} - x_{(i)}$. It was shown in [PP08] that $\pi := \bigotimes_{i=1}^\infty \mathrm{Exp}(2\gamma)$ is a stationary distribution of the gap process $Z(t) := (X_{(i+1)}(t) - X_{(i)}(t))_{i=1}^\infty$ of the Atlas model (1.1).

It addition to the i.i.d. $\mathrm{Exp}(2\gamma)$ distribution, it was recently shown in [ST17] that the Atlas model has a different type of stationary gap distributions. That is, for each $a > 2\gamma_-$, $\pi_a := \bigotimes_{i=1}^\infty \mathrm{Exp}(2\gamma + ia)$ is also a stationary gap distribution of the Atlas model. Unlike π, the distribution π_a has exponentially growing particle density away from the Atlas particle. In this article, we go one step further and show that, in fact, up to a deterministic shift $\frac{at}{2}$ of each particle, the *entire* particle system $\{X_i(t) + \frac{at}{2}\}_{i=1}^\infty$ has a stationary distribution. This extends the result of [ST17] on stationary *gap* distributions. In the following we use $\{x_i\}_{i=1}^\infty \subset \mathbb{R}$ to denote a configuration of *indistinguishable* particles, in contrast with $(x_i)_{i=1}^\infty$, which denotes labeled (named) particles. Let

$$\mathcal{V} = \Big\{ \{x_i\}_{i=1}^\infty \ : \ (x_i)_{i=1}^\infty \in \mathcal{U} \Big\}$$

denote the corresponding configuration space, and let μ_a denote the Poisson point process on $\mathbb{R}$ with density $ae^{a\xi}d\xi$. It is standard to show (e.g., using techniques from [Pan13, Section 2.2]) that μ_a is supported on $\mathcal{V}$. Let $\Gamma(\alpha) := \int_0^\infty \xi^{-1-\alpha} e^{-\xi} d\xi$ denote the Gamma function, and let $\mathrm{Gamma}(\alpha, \beta) \sim \frac{1}{\Gamma(\alpha)} \beta^\alpha \xi^{-1-\alpha} e^{-\beta\xi} \mathbf{1}_{\{\xi>0\}} d\xi$ denote the Gamma distribution. The following is the main result.

Theorem 1.1.

(a) For any fixed $\gamma \in \mathbb{R}$ *and* $a > 2\gamma_-$, $\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}}) = \Gamma(\frac{2\gamma}{a} + 1) \in (0, \infty)$, *so that*

$$\nu_a(\bullet) := \frac{1}{\Gamma(\frac{2\gamma}{a} + 1)} \mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}} \bullet) \tag{1.3}$$

defines a probability distribution supported on $\mathcal{V}$. *Furthermore, under* ν_a, *we have that* $e^{aX_{(1)}} \sim \mathrm{Gamma}(\frac{2\gamma}{a} + 1, 1)$, *and that*

$$Z := (X_{(i+1)} - X_{(i)})_{i=1}^\infty \sim \pi_a = \bigotimes_{i=1}^\infty \mathrm{Exp}(2\gamma + ia). \tag{1.4}$$

(b) The distribution ν_a *is a stationary distribution of* $\{X_i(t) + \frac{at}{2}\}_{i=1}^\infty$, *where* $(X_i(t))_{i=1}^\infty$ *evolves under* (1.1).

Remark 1.2. Under ν_a, the Atlas particle $X_{(1)}$ and the gap process $Z = (Z_i)_{i=1}^{\infty}$ are *not* independent.

For the special case $\gamma = 0$, the Atlas model (1.1) reduces to independent Brownian motions. In this case, it is well known that the Poisson point process μ_a is quasi-stationary [Lig78], and the shift $-\frac{a}{2}t$ can be easily calculated from the motion of independent Brownian particles. Here we show that, with a drift γdt exerted on the Atlas particle $X_{(1)}(t)$, a stationary distribution is obtained by taking $V(x) := 2\gamma x_{(1)}$ to be the potential. Indeed, under such a choice of $V(x)$, we have that $\gamma \mathbf{1}\{p_x(i) = 1\} = \frac{1}{2}\partial_{x_i} e^{V(x)}$. This explains why we should expect the stationary distribution ν_a as in (1.3). The proof of Theorem 1.1 amounts to justifying the aforementioned heuristic in the setting of infinite dimensional diffusions with discontinuous drift coefficients. We achieve this through finite-dimensional, smooth approximations, and using the explicit expressions of semigroups from Girsanov's theorem to take limits.

Due to their simplicity, product-of-exponential stationary gap distributions have been intensively searched within competing Brownian particle systems, in both finite and infinite dimensions. See [Sar17a] and the references therein. To date, derivations of product-of-exponential stationary gap distributions have been relying on the theory of Semimartingale Reflecting Brownian Motions (SRBM), e.g., [Wil95]. On the other hand, given the expression (1.3) of ν_a, the gap distribution (1.4) follows straightforwardly from Rényi's representation [Rén53]. Theorem 1.1 hence provides an alternative derivation of the product-of-exponential distribution π_a without going through SRBM.

Our methods should generalize to the case of competing Brownian particle systems with finitely many non-zero drift coefficients, i.e.,

$$dX_i(t) = \sum_{j=1}^{m} \gamma_j \mathbf{1}\{p_{X(t)}(i) = j\} dt + dW_i(t), \quad i \in \mathbb{N},$$

yielding the stationary distribution $\nu_a(\,\cdot\,) := \frac{1}{J}\mathbf{E}_{\mu_a}(e^{2\sum_{j=1}^{m}\gamma_j X_{(j)}}\,\cdot\,)$, for some normalizing constant $J < \infty$. Here we consider only the Atlas model for simplicity of notations.

A natural question, following the discovery a stationary gap distribution, is the longtime behavior of the Atlas particle $X_{(1)}(t)$ under such a gap distribution. For the i.i.d. $\mathrm{Exp}(2)$ gap distribution π, this question was raised in [PP08] and answered in [DT17]. It was shown in [DT17] that $X_{(1)}(t)$ fluctuates at order $t^{\frac{1}{4}}$ around its starting location, and scales to a $\frac{1}{4}$-fractional Brownian motion, as $t \to \infty$. As a simple application of Theorem 1.1, under the stationary gap distribution π_a and $X_{(1)}(0) = 0$, we establish an exponential tail bound, uniformly in t, of the fluctuation Atlas particle around its expected location $-\frac{at}{2}$. This shows that the fluctuation of $X_{(1)}(t)$ stays bounded under π_a, in sharp contrast with the $t^{\frac{1}{4}}$ fluctuation obtained in [DT17].

Corollary 1.3. *Fix $\gamma \in \mathbb{R}$ and $a > 2\gamma_-$. Starting the Atlas model* (1.1) *from the initial distribution $X_{(i)}(0) = 0$, $(X_{(i+1)}(0) - X_{(i)}(0))_{i=1}^{\infty} \sim \pi_a$, we have that*

$$\mathbf{P}(|X_{(1)}(t) + \tfrac{at}{2}| \geq \xi) \leq c e^{-\frac{1}{2}(2\gamma+a)\xi}, \quad \forall t, \xi \in \mathbb{R}_+,$$

for some constant $c = c(a, \gamma) < \infty$ depending only on a, γ.

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2 Proof

2.1 Theorem 1.1(a)

Fix $\gamma \in \mathbb{R}$, $a > 2\gamma_-$, and let $\{X_i\}_{i=1}^\infty$ denote a sample from the Poisson point process μ_a. Let $N(\xi) := \#\{X_i \in (-\infty, \xi]\}$ denote the number of particles in $(-\infty, \xi]$, whereby $N(\xi) \sim \mathrm{Pois}(e^{a\xi})$. Indeed, $\mathbf{P}_{\mu_a}(X_{(1)} > \xi) = \mathbf{P}(N(\xi) = 0) = e^{-e^{a\xi}}$. From this we calculate

$$\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}}\mathbf{1}\{X_{(1)} \le \xi\}) = \int_{-\infty}^{\xi} e^{2\gamma\zeta}\frac{d}{d\zeta}(1 - e^{-e^{a\zeta}})d\zeta.$$

Performing the change of variable $\zeta' := e^{a\zeta}$, we see that $\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}}\mathbf{1}\{e^{aX_{(1)}} \le e^{\xi}\}) = \int_0^{e^\xi} \zeta'^{\frac{2\gamma}{a}} e^{-\zeta'}d\zeta'$. From this it follows that $\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}}) = \Gamma(\frac{2\gamma}{a}+1)$ and that $e^{aX_{(1)}} \sim \mathrm{Gamma}(\frac{2\gamma}{a}+1, 1)$ under ν_a.

Turning to showing (1.4), we let $\{X_i\}_{i=1}^\infty$ be sampled from μ_a and let $(Z_k)_{k=1}^\infty = (X_{(i+1)} - X_{(i)})_{i=1}^\infty$ denote the gap process. Fix arbitrary $m < \infty$. Our goal is to show that

$$\frac{\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i \ge \xi_i\})}{\mathbf{E}_{\mu_a}(e^{2\gamma X_{(1)}})} = \prod_{i=1}^{m-1} e^{-(2\gamma+ia)\xi_i} =: \eta. \tag{2.1}$$

For any given threshold $\xi \in \mathbb{R}$, we let $\mu_{a,\xi}$ denote the restriction of the Poisson point process μ_a onto $(-\infty, \xi]$. For the restricted process, we have $\mu_{a,\xi} \sim \{\xi - Y_1, \ldots, \xi - Y_{N(\zeta)}\}$, where $Y_1, Y_2, \ldots$ are i.i.d. $\mathrm{Exp}(a)$ variables, independent of $N(\xi)$. Let $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ denote the ranking of $(Y_1, \ldots, Y_n)$. We then have that, conditionally on $N(\xi) \ge m$, $(X_{(1)}, \ldots, X_{(m)}) = (\xi - Y_{N(\xi)}, \ldots, \xi - Y_{N(\xi)-m+1})$. Further, by Rényi's representation [Rén53],

$$(Y_{(k)})_{k=1}^n \stackrel{\mathrm{d}}{=} \Big(\sum_{i=k}^n G_i\Big)_{k=1}^n, \quad \text{where } \big(G_i\big)_{i=1}^n \sim \prod_{i=1}^n \mathrm{Exp}(ia).$$

Using this we calculate

$$\begin{aligned}
&\mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i \ge \xi_i\}\Big| N(\xi) \ge m\Big) \\
&\quad = \mathbf{E}\Big(e^{2\gamma(N(\xi)\xi - \sum_{i=m}^{N(\xi)} G_i)}\prod_{i=1}^{m-1} e^{-2\gamma G_i}\mathbf{1}\{G_i \ge \xi_i\}\Big| N(\xi) \ge m\Big) \\
&\quad = \mathbf{E}\Big(e^{2\gamma(N(\xi)\xi - \sum_{i=m}^{N(\xi)} G_i)}\Big| N(\xi) \ge m\Big)\prod_{i=1}^{m-1}\frac{iae^{-(ai+2\gamma)\xi_i}}{ai+2\gamma}.
\end{aligned}$$

Further use $\frac{ia}{ai+2\gamma} = \mathbf{E}e^{-2\gamma G_i}$ to write $\prod_{i=1}^{m-1}\frac{iae^{-(ai+2\gamma)\xi_i}}{ai+2\gamma} = \mathbf{E}(e^{-2\gamma(G_1+\ldots+G_{m-1})})\eta$, We then obtain

$$\mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i \ge \xi_i\}\Big| N(\xi) \ge m\Big) = \mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\Big| N(\xi) \ge m\Big)\eta.$$

Taking into account the case $N(\xi) < m$, we write

$$\begin{aligned}
&\mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i \ge \xi_i\}\Big) \\
&= \mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\mathbf{1}\{N(\xi) \ge m\}\Big)\eta + \mathbf{E}_{\mu_a}\Big(e^{2\gamma X_{(1)}}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i \ge \xi_i\})\mathbf{1}\{N(\xi) < m\}\Big).
\end{aligned} \tag{2.2}$$

Since $a > -2\gamma$, fixing $q > 1$ with $|q-1|$ small enough, we have

$$\mathbf{E}_{\mu_a}(e^{2q\gamma X_{(1)}}) = \mathbf{E}_{\mu_a}(|e^{2\gamma X_{(1)}}|^q) = \Gamma(\tfrac{2q\gamma}{a}+1) < \infty. \tag{2.3}$$

That is, $e^{2q\gamma X_{(1)}}$ has bounded q-th moment with $q>1$, so in particular $\{e^{2\gamma X_{(1)}}\mathbf{1}\{N(\xi) \geq m\}\}_{\xi>0}$ is uniformly integrable. For fixed $m<\infty$, $\mathbf{1}\{N(\xi)<m\} \to_{\mathrm{P}} 0$, as $\xi\to\infty$. Using this to take the limit $\xi\to\infty$ in (2.2), we thus obtain $\mathbf{E}_{\mu_a}(e^{2\gamma X_1}\prod_{i=1}^{m-1}\mathbf{1}\{Z_i\geq\xi_i\}) = \mathbf{E}_{\mu_a}(e^{2\gamma X_1})\eta$. This concludes (2.1).

2.2 Theorem 1.1(b)

Samples from μ_a have, almost surely, no repeated points, i.e., $X_{(1)} < X_{(2)} < X_{(3)} < \ldots$. Fix arbitrary $m<\infty$ and $\phi\in C_c^\infty(\mathcal{W})$, where $\mathcal{W} := \{(x_1 < x_2 < \ldots < x_m)\}$ denote the Weyl chamber. Let $\overline{X}_i(t) := X_i(t) + \frac{a}{2}t$, and $\overline{X}_{(i)}(t) := X_{(i)}(t)+\frac{a}{2}t$ denote the compensated particle locations. It then suffices to show that

$$\mathbf{E}_{\mu_a}\big(e^{2\gamma\overline{X}_{(1)}(0)}\phi(\overline{X}_{(1)}(t),\ldots,\overline{X}_{(m)}(t))\big) = \mathbf{E}_{\mu_a}\big(e^{2\gamma\overline{X}_{(1)}(0)}\phi(\overline{X}_{(1)}(0),\ldots,\overline{X}_{(m)}(0))\big). \tag{2.4}$$

As will be convenient for notations, for $n\geq m$, we consider the symmetric extension ϕ^{s} of ϕ, defined for $n\geq m$ as

$$\phi^{\mathrm{s}}:\mathbb{R}^n\to\mathbb{R}, \quad \phi^{\mathrm{s}}(x) := \phi(x_{(1)},\ldots,x_{(m)}). \tag{2.5}$$

We have slightly abused notations by using the same symbol ϕ^{s} to denote the function for *all* $n\in\mathbb{N}_{\geq m}\cup\{\infty\}$. Note that, by definition, the function ϕ vanishes near the boundary $\{(x_1\leq\ldots x_i = x_{i+1}\ldots\leq x_m) : i=1,\ldots,n-1\}$ of $\mathcal{W}$, so, for $n<\infty$, $\phi^{\mathrm{s}}\in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$.

The strategy of proving (2.4) is to approximate the infinite system $\overline{X}(t)$ by finite systems. Fixing $m\leq n<\infty$, we consider the following n-dimensional analog of $\overline{X}(t)$:

$$\overline{X}^n_i(t) = x_i + \int_0^t \big(\gamma\mathbf{1}\{p_{\overline{X}^n(s)}(i)=1\} + \tfrac{a}{2}\big)ds + W_i(t), \quad i=1,\ldots,n, \tag{2.6}$$

where the ranking permutation $p_x(\bullet):\{1,\ldots,n\}\to\{1,\ldots,n\}$ is defined similarly to the case of infinite particles. As the discontinuity of $x\mapsto\mathbf{1}\{p_x(i)=1\}$ imposes unwanted complication in the subsequence analysis, we consider further the mollified system as follows. Fix a mollifier $r\in C^\infty(\mathbb{R}^n)$, i.e., $r\geq 0$, $r|_{\|x\|\geq 1}=0$ and $\int_{\mathbb{R}^n} r(y)dy = 1$. Let $V(x) := 2\gamma x_{(1)} = 2\gamma\min(x_1,\ldots,x_n)$. For $\varepsilon\in(0,1)$, we define the mollified potential as $V^\varepsilon(x) := \int_{\mathbb{R}^n} V(y)r(\varepsilon^{-1}(x-y))\varepsilon^{-n}dy$. Under these notations, we have that

$$\tfrac{1}{2}\partial_i V^\varepsilon(x) = \gamma\mathbf{1}\{p_{\overline{X}^n(s)}(i)=1\}, \quad \text{on } \Omega_\varepsilon := \{x\in\mathbb{R}^n : |x_i-x_j|>\varepsilon,\ \forall i<j\}. \tag{2.7}$$

We then consider the following mollified system

$$\overline{X}^{n,\varepsilon}_i(t) = x_i + \int_0^t\big(\tfrac{1}{2}\partial_i V^\varepsilon(\overline{X}^{n,\varepsilon}(s)) + \tfrac{a}{2}\big)ds + W_i(t), \quad i=1,\ldots,n. \tag{2.8}$$

With $\partial_i V^\varepsilon$ being smooth and bounded, the well-posedness of (2.8) follows from standard theory, e.g., [SV07]. Furthermore, letting $u(t,x) := \mathbf{E}_x(\phi^{\mathrm{s}}(\overline{X}^{n,\varepsilon}(t)))$, we have that $u\in C^\infty(\mathbb{R}_+\times\mathbb{R}^n)$, and that u solves the following PDE:

$$\partial_t u = \sum_{i=1}^n\big(\tfrac{1}{2}\partial_{ii} + \tfrac{a}{2}\partial_i + \tfrac{1}{2}\partial_i V^\varepsilon\big)u, \quad u(0,x) = \phi^{\mathrm{s}}(x). \tag{2.9}$$

With $\partial_i V^\varepsilon$ being bounded and ϕ^{s} being compactly supported, applying the Feynman-Kac formula to the solution u of (2.9), we see that u decays exponentially as $|x|\to\infty$, i.e.,

$$\sup_{s\leq t, x\in\mathbb{R}^n}\big\{|u(t,x)|e^{\xi(|x_1|+\ldots+|x_n|)}\big\} < \infty, \quad \forall\xi,t<\infty. \tag{2.10}$$

Such an exponential estimate (2.10) progresses to higher order derivatives of u. More precisely, with $\partial_i V^\varepsilon \in C^\infty(\mathbb{R}^n)$ and $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$, taking derivative ∂_i in (2.9), we see that $\partial_i u$ solves the following equation:

$$\partial_t(\partial_i u) = \sum_{j=1}^n \big(\big(\tfrac{1}{2}\partial_{jj} + \tfrac{a}{2}\partial_j + \partial_j V^\varepsilon\big)(\partial_i u) + (\partial_{ij} V^\varepsilon) u \big), \tag{2.11}$$

$$\partial_i u(0,x) = \partial_i \phi^{\mathrm{s}}(x) \in C_c^\infty(\mathbb{R}^n),$$

A similarly procedure applied to the solution $\partial_i u$ of (2.11) yields

$$\sup_{s\le t, x\in\mathbb{R}^n, i=1,\ldots,n} \big\{ |\partial_i u(t,x)| e^{\xi(|x_1|+\ldots+|x_n|)} \big\} < \infty, \quad \forall \xi, t < \infty.$$

Iterating this argument to higher order derivatives, we obtain

$$\sup_{s\le t, x\in\mathbb{R}^n, |\beta|\le k} \big\{ |\partial_\beta u(t,x)| e^{\xi(|x_1|+\ldots+|x_n|)} \big\} < \infty, \quad \forall \xi, t, k < \infty. \tag{2.12}$$

The PDE (2.9) has stationary distribution $e^{V^\varepsilon(x)} \prod_{i=1}^n e^{a x_i} dx_i$ (*not* a probability distribution, since the total mass is infinite). More precisely, integrate $u(t,x)$ against the aforementioned distribution to get

$$v(t) := \int_{\mathbb{R}^n} u(t,x) e^{V^\varepsilon(x)} \prod_{i=1}^n e^{a x_i} dx_i.$$

Taking time derivative using (2.9) and (2.12), followed by integrations by parts

$$\int_{\mathbb{R}^n} \tfrac{1}{2}(\partial_{ii} u(t,x)) e^{V^\varepsilon(x)} \prod_{j=1}^n e^{a x_j} dx_j = -\int_{\mathbb{R}^n} (\partial_i u(t,x))(\tfrac{1}{2}\partial_i V^\varepsilon(x) + \tfrac{a}{2}) e^{V^\varepsilon(x)} \prod_{j=1}^n e^{a x_j} dx_j,$$

we obtain that $\frac{d}{dt} v(t) = 0$. Consequently,

$$\int_{\mathbb{R}^n} \mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n,\varepsilon}(t))\big) e^{V^\varepsilon(x)} \prod_{i=1}^n e^{a x_i} dx_i = \int_{\mathbb{R}^n} \phi^{\mathrm{s}}(x) e^{V^\varepsilon(x)} \prod_{i=1}^n e^{a x_i} dx_i. \tag{2.13}$$

The next step is to take the limit $\varepsilon \to 0$ in (2.13), for *fixed* n. This amounts to establishing the convergence of the term $\mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n,\varepsilon}(t))\big)$. To this end, we use Girsanov's theorem to write

$$\mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n}(t))\big) = \mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t)) F(t)\big), \tag{2.14}$$

$$\mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n,\varepsilon}(t))\big) = \mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t)) F^\varepsilon(t)\big), \tag{2.15}$$

where $H(t) := (W_i(t) + \frac{at}{2} + x_i)_{i=1}^n$ consists of independent, drifted Brownian motions starting from $x = (x_i)_{i=1}^n$, and the terms $F(t)$ and $F^\varepsilon(t)$ are stochastic exponentials given by

$$F(t) := \exp\big(M(t) - \tfrac{1}{2}\langle M\rangle(t)\big), \qquad M(t) := \int_0^t \sum_{i=1}^n \gamma \mathbf{1}\{p_{H(t)}(i) = 1\} dW_i(s), \tag{2.16}$$

$$F^\varepsilon(t) := \exp\big(M^\varepsilon(t) - \tfrac{1}{2}\langle M^\varepsilon\rangle(t)\big), \qquad M^\varepsilon(t) := \int_0^t \sum_{i=1}^n \frac{1}{2} \partial_i V^\varepsilon(H(s)) dW_i(s). \tag{2.17}$$

Taking the difference of (2.14)–(2.15), followed by using the Cauchy–Schwarz inequality, we obtain

$$\big|\mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n}(t))\big) - \mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^{n,\varepsilon}(t))\big)\big| = \big|\mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t)) F(t)(1 - \tfrac{F^\varepsilon(t)}{F(t)})\big)\big|$$

$$\leq \Big(\mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t))^2F(t)^2\big)\Big)^{\frac{1}{2}}\Big(\mathbf{E}_x(1-\tfrac{F^\varepsilon(t)}{F(t)})^2\Big)^{\frac{1}{2}}. \tag{2.18}$$

For the two terms in (2.18), we next show that: *i*) the first term is bounded; and *ii*) the second term vanishes as $\varepsilon \to 0$. Hereafter, we use $c(a_1, a_2, \ldots)$ to denote a finite, deterministic constant, that may change from line to line, but depends only on the designated variables $a_1, a_2, \ldots$.

i) Recall that ϕ^{s} is defined in terms of ϕ through (2.5). We fix $\lambda < \infty$, *independently* of n, such that $\mathrm{supp}(\phi^{\mathrm{s}}) \subset [-\lambda, \lambda]^n$. Under these notations,

$$\begin{aligned}\mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t))^2F(t)^2\big) &\leq \|\phi\|^2_{L^\infty}\mathbf{E}_x\big(\mathbf{1}_{\{H(t)\in[-\lambda,\lambda]^n\}}F(t)^2\big)\\ &\leq \|\phi\|^2_{L^\infty}\big(\mathbf{E}F(t)^4\big)^{\frac{1}{2}}\mathbf{P}_x\big(H(t)\in[-\lambda,\lambda]^n\big)^{\frac{1}{2}}.\end{aligned} \tag{2.19}$$

With $F(t)$ defined in (2.16), and $\langle M\rangle(t) = \gamma^2 t$, it follows that

$$\mathbf{E}_x(F(t)^4) = \mathbf{E}_x\big(e^{4M(t)}e^{-2\langle M\rangle(t)}\big) = e^{\frac{1}{2}(16-4)\gamma^2 t} = c(\gamma, t).$$

Let $\Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$ denote the Gaussian distribution function. With $H_i(t) = x_i + \frac{a}{2}t + W_i(t)$, we have

$$\mathbf{P}_x\big(H(t)\in[-\lambda,\lambda]^n\big) \leq \prod_{i=1}^n \Phi\Big(\frac{\lambda - \frac{a}{2}t - x_i}{\sqrt{t}}\Big).$$

Inserting these bounds into (2.19), we obtain

$$\mathbf{E}_x\big(\phi^{\mathrm{s}}(H(t))^2F(t)^2\big) \leq c(a,\gamma,\lambda,t)\prod_{i=1}^n \Phi\Big(\frac{\lambda - \frac{a}{2}t - x_i}{\sqrt{t}}\Big) \tag{2.20}$$

$$\leq c(a,\gamma,\lambda,t,n)\exp\Big(-\frac{x_1^2+\ldots+x_n^2}{4(t+1)}\Big). \tag{2.21}$$

ii) Expand the expression $\mathbf{E}_x(1-\frac{F^\varepsilon(t)}{F(t)})^2$ into

$$\mathbf{E}_x(1-\tfrac{F^\varepsilon(t)}{F(t)})^2 = 1 + \mathbf{E}_x(\tfrac{F^\varepsilon(t)}{F(t)})^2 - 2\mathbf{E}_x\tfrac{F^\varepsilon(t)}{F(t)}. \tag{2.22}$$

From (2.16)–(2.17), we have

$$\tfrac{F^\varepsilon(t)}{F(t)} = \exp(M(t) - M^\varepsilon(t))\exp(\tfrac{1}{2}\langle M\rangle(t) - \tfrac{1}{2}\langle M^\varepsilon\rangle(t)). \tag{2.23}$$

Set $b_i^\varepsilon(s) := \frac{1}{2}\partial_i V^\varepsilon(H(s))$ to simplify notations. As $V(x)$ is Lipschitz with Lipschitz seminorm $2|\gamma|$, (i.e., $|V(x)-V(y)| \leq 2\gamma|x-y|$, $\forall x, y \in \mathbb{R}^n$), we have $|b_i^\varepsilon(s)| \leq |\gamma|$. Consequently,

$$\langle M\rangle(t) = \gamma^2 t, \quad \langle M^\varepsilon\rangle(t) \leq n\gamma^2 t. \tag{2.24}$$

To estimate the expression (2.23), we use (2.7) and $|b_i^\varepsilon(s)| \leq |\gamma|$ to write

$$|\langle M - M^\varepsilon\rangle(t)| = \int_0^t \sum_{i=1}^n \big(b_i^\varepsilon(s) - \gamma\mathbf{1}\{p_{H(s)}(i)=1\}\big)^2 ds \leq 4n\gamma^2\int_0^t \mathbf{1}\{H(s)\notin\Omega_\varepsilon\}ds. \tag{2.25}$$

Let $L_{i,j}(s,\xi)$ denote the localtime process of $H_i(s) - H_j(s) = W_i(s) - W_j(s) + (x_j - x_i)$ at a given level ξ. We further bound the r.h.s. of (2.25) as

$$\int_0^t \mathbf{1}\{H(s)\notin\Omega_\varepsilon\}ds \leq \sum_{i<j}\int_0^t\int_{|\xi|\leq\varepsilon} L_{i,j}(s,\xi)d\xi ds \longrightarrow_{\mathrm{P}} 0, \quad \text{as } \varepsilon\to 0.$$

Consequently, $|\langle M - M^\varepsilon\rangle(t)| \to_{\mathrm{P}} 0$. Since, by (2.24), $\langle M\rangle(t)$ and $\langle M^\varepsilon\rangle(t)$ are bounded (for fixed t), it also follows that $\mathbf{E}_x|\langle M - M^\varepsilon\rangle(t)| \to 0$ and hence $M(t) - M^\varepsilon(t) \to_{\mathrm{P}} 0$. Referring back to the expression (2.23), we see that $\frac{F^\varepsilon(t)}{F(t)} \to_{\mathrm{P}} 1$. Using again the fact that $\langle M\rangle(t)$ and $\langle M^\varepsilon\rangle(t)$ are bounded, (which implies the uniform integrability of $(\frac{F^\varepsilon(t)}{F(t)})^k$, $k = 1, 2$), we obtain $\mathbf{E}_x(\frac{F^\varepsilon(t)}{F}), \mathbf{E}_x(\frac{F^\varepsilon(t)}{F})^2 \to 1$. Inserting these into (2.22) yields

$$\mathbf{E}_x(1 - \tfrac{F^\varepsilon(t)}{F(t)})^2 \to 0, \text{ as } \varepsilon \to 0, \quad \text{for any fixed } x \in \mathbb{R}^n. \tag{2.26}$$

Now, combine (2.21), (2.26) with (2.18), and insert the result into the l.h.s. of (2.13). After taking the $\varepsilon \to 0$ limit with $n < \infty$ being fixed, we obtain

$$\int_{\mathbb{R}^n} \mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^n(t))\big)e^{2\gamma x_{(1)}} \prod_{i=1}^n e^{a x_i} dx_i = \int_{\mathbb{R}^n} \phi^{\mathrm{s}}(x) e^{2\gamma x_{(1)}} \prod_{i=1}^n e^{a x_i} dx_i. \tag{2.27}$$

Recall that $\mu_{a,\zeta}$ denote the restriction of the Poisson point process μ_a on $(-\infty, \zeta]$ and that $N(\zeta)$ denote the number of particles on $(-\infty, \zeta]$. As mentioned previously, $\mu_{a,\zeta} \sim \{\zeta - Y_1, \ldots, \zeta - Y_{N(\zeta)}\}$, where $Y_1, Y_2, \ldots$ are i.i.d. $\mathrm{Exp}(a)$ variables, independent of $N(\zeta)$. Conditionally on $N(\zeta) = n$, the process $\{\zeta - Y_1, \ldots, \zeta - Y_{N(\zeta)}\}$ have joint distribution $\prod_{i=1}^n a e^{a(x_i - \zeta)} dx_i \mathbf{1}_{\{x_i \le \zeta\}}$. With this, multiplying both sides of (2.27) by $a^n e^{-an\zeta}$, and averaging over $\{N(\zeta) \ge m\}$, we obtain that

$$\begin{aligned}
&\mathbf{E}_{\mu_{a,\zeta}}\Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(t)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \mathbf{1}\{N(\zeta) \ge m\}\Big) + \mathbf{E}(R_{N(\zeta)}(\zeta)\mathbf{1}\{N(\zeta) \ge m\}) \\
=&\mathbf{E}_{\mu_{a,\zeta}}\Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(0)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \mathbf{1}\{N(\zeta) \ge m\}\Big) + \mathbf{E}(S_{N(\zeta)}(\zeta)\mathbf{1}\{N(\zeta) \ge m\}),
\end{aligned} \tag{2.28}$$

where the terms $R_n(\zeta)$ and $S_n(\zeta)$ are given by

$$R_n(\zeta) := \int_{\cup_{i=1}^n \{x_i > \zeta\}} \mathbf{E}_x\big(\phi^{\mathrm{s}}(\overline{X}^n(t))\big) e^{2\gamma x_{(1)}} \prod_{i=1}^n a e^{a(x_i - \zeta)} dx_i, \tag{2.29}$$

$$S_n(\zeta) := \int_{\cup_{i=1}^n \{x_i > \zeta\}} \mathbf{E}_x\big(\phi^{\mathrm{s}}(x)\big) e^{2\gamma x_{(1)}} \prod_{i=1}^n a e^{a(x_i - \zeta)} dx_i.$$

Recall that $\mathrm{supp}(\phi^{\mathrm{s}}) \subset [-\lambda, \lambda]^n$. Hence

$$S_n(\zeta) = 0, \quad \text{for all } \zeta > \lambda. \tag{2.30}$$

As for $R_n(\zeta)$, inserting the bound (2.20) into (2.29) gives

$$|R_n(\zeta)| \le c(a, \gamma, \lambda, t) \int_{\cup_{i=1}^n \{x_i > \zeta\}} e^{2\gamma x_{(1)}} \prod_{i=1}^n \Phi\Big(\frac{\lambda - \frac{a}{2}t - x_i}{\sqrt{t}}\Big) a e^{a(x_i - \zeta)} dx_i.$$

Indeed, $x_{(1)} \le \zeta + \sum_{i=1}^n (x_i - \zeta)_+$, so, after a change of variable $x_i - \zeta \mapsto x_i$, we obtain

$$|R_n(\zeta)| \le c(a, \gamma, \lambda, t) e^{\zeta} \int_{\cup_{i=1}^n \{x_i > 0\}} \prod_{i=1}^n \Phi\Big(\frac{\lambda - \frac{a}{2}t - x_i - \zeta}{\sqrt{t}}\Big) a e^{a x_i + a(x_i)_+} dx_i.$$

To bound the last integral, we split the integration over x_i into $\{x_i > 0\}$ and $\{x_i \le 0\}$ for *each* x_i, and thereby express the integral as

$$\sum_{k=1}^n \sum_{\{i_1, \ldots, i_k\}} \Big(\prod_{j \in \{i_1, \ldots, i_k\}} \int_{\{x_j > 0\}} (\cdots) dx_j\Big)\Big(\prod_{j \notin \{i_1, \ldots, i_k\}} \int_{\{x_j \le 0\}} (\cdots) dx_j\Big),$$

where $\{i_1, \ldots, i_k\}$ ranges over all distinct k-indices from $\{1, \ldots, n\}$. Further, for each integral over $\{x > 0\}$ and over $\{x \le 0\}$, we have that

$$\int_{\{x>0\}} \Phi\Big(\frac{\lambda - \frac{a}{2}t - x - \zeta}{\sqrt{t}}\Big) a e^{ax + a(x)_+} dx \le c(a, \lambda, \gamma, t) e^{-\frac{\zeta^2}{4(t+1)}},$$
$$\int_{\{x\le 0\}} \Phi\Big(\frac{\lambda - \frac{a}{2}t - x - \zeta}{\sqrt{t}}\Big) a e^{ax + a(x)_+} dx < \int_{\{x\le 0\}} a e^{ax} dx = 1.$$

Consequently,

$$|R_n(\zeta)| \le c(a, \gamma, \lambda, t) e^{\zeta} \sum_{k=1}^{n} \binom{n}{k} c(a, \gamma, \lambda, t)^k e^{-\frac{k\zeta^2}{4(t+1)}}.$$

Now, with $N(\zeta) \sim \mathrm{Pois}(e^{a\zeta})$, we have $\mathbf{E}(\binom{N(\zeta)}{k}) = \frac{1}{k!}\mathbf{E}(N(\zeta)\cdots(N(\zeta)-k+1)) = \frac{1}{k!}e^{ka\zeta}$. Given this identity, setting $n = N(\zeta)$ and taking expected value, we obtain

$$\mathbf{E}|R_{N(\zeta)}(\zeta)| \le c(a, \gamma, \lambda, t) e^{\zeta} \sum_{k=1}^{\infty} \frac{1}{k!} c(a, \lambda, t)^k e^{ka\zeta - \frac{k\zeta^2}{4(t+1)}}, \tag{2.31}$$

which converges to zero as $\zeta \to \infty$.

Using (2.30)–(2.31) in (2.28), and taking the limit $\zeta \to \infty$, we arrive at

$$\lim_{\zeta\to\infty} \Big(\mathbf{E}_{\mu_{a,\zeta}} \Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(t)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \mathbf{1}\{N(\zeta) \ge m\} \Big) - \mathbf{E}_{\mu_{a,\zeta}} \Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(0)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \mathbf{1}\{N(\zeta) \ge m\} \Big) \Big) = 0. \tag{2.32}$$

It remains to show that, under the limit $\zeta \to \infty$, we can exchange the finite system $\overline{X}^{N(\zeta)}$ for the infinite system $\overline{X}$ within the expressions in (2.32). As $\zeta \to \infty$, we have that

$$\overline{X}^{N(\zeta)}_{(i)}(t) \Rightarrow \overline{X}_{(i)}(t), \text{ as } \zeta \to \infty, \quad i = 1, \ldots, m, \tag{2.33}$$

where $\overline{X}^{N(\zeta)}(0) \sim \mu_{a,\zeta}$ and $\overline{X}(0) \sim \mu_a$. Such a statement (2.33) can be proven by techniques from [Sar17a] and [ST17, Section 3(a)]. We omit repeating the standard arguments here. Combining (2.33) and (2.3), we obtain that

$$\lim_{\zeta\to\infty} \mathbf{E}_{\mu_{a,\zeta}} \Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(t)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \Big) = \mathbf{E}_{\mu_a} \Big(\phi^{\mathrm{s}}(\overline{X}(t)) e^{2\gamma \overline{X}_{(1)}(0)} \Big), \tag{2.34}$$

$$\lim_{\zeta\to\infty} \mathbf{E}_{\mu_{a,\zeta}} \Big(\phi^{\mathrm{s}}(\overline{X}^{N(\zeta)}(0)) e^{2\gamma \overline{X}^{N(\zeta)}_{(1)}(0)} \Big) = \mathbf{E}_{\mu_a} \Big(\phi^{\mathrm{s}}(\overline{X}(0)) e^{2\gamma \overline{X}_{(1)}(0)} \Big). \tag{2.35}$$

Combining (2.34)–(2.35) with (2.32), we thus obtain (2.4), and hence complete the proof.

2.3 Corollary 1.3

Fixing $\gamma \in \mathbb{R}$ and $a > 2\gamma_-$, we let $c = c(a, \gamma) < \infty$ denote a generic finite constant that depends only on these two variables. Let $Y(t) = (Y_i(t))_{i=1}^{\infty}$ be a solution to (1.1) starting from the distribution $\{Y_i(0)\}_{i=1}^{\infty} \sim \nu_a$, so that $\{Y_i(t) + \frac{at}{2}\}_{i=1}^{\infty} \sim \nu_a$, for all $t \in \mathbb{R}_+$. Since, by (1.4), the gap process $(Y_{(i+1)}(0) - Y_{(i)}(0))_{i=1}^{\infty}$ is distributed as π_a, setting $X_i(t) = Y_i(t) - Y_{(1)}(0)$, we have that $X(t)$ is a solution to (1.1) with the designated initial distribution as in Corollary 1.3. Under these notations, for any given $\xi \ge 0$,

$$\mathbf{P}(|X_{(1)}(t)| \ge \xi) = \mathbf{P}(|Y_{(1)}(t) - Y_{(1)}(0)| \ge \xi) \le \mathbf{P}(|Y_{(1)}(0)| \ge \tfrac{\xi}{2}) + \mathbf{P}(|Y_{(1)}(t)| \ge \tfrac{\xi}{2}) = 2\mathbf{P}(|Y_{(1)}(0)| \ge \tfrac{\xi}{2}). \tag{2.36}$$

With $e^{aY_{(1)}} \sim \mathrm{Gamma}(\frac{2\gamma}{a}, 1)$, we have that

$$\mathbf{P}(Y_{(1)}(0) \le -\tfrac{\xi}{2}) = \frac{1}{\Gamma(\frac{2\gamma}{a})} \int_0^{e^{-\frac{1}{2}a\xi}} \zeta^{\frac{2\gamma}{a}} e^{-\zeta} d\zeta \le c \int_0^{e^{-\frac{1}{2}a\xi}} \zeta^{\frac{2\gamma}{a}} d\zeta = c e^{-\frac{1}{2}(2\gamma+a)\xi},$$
$$\mathbf{P}(Y_{(1)}(0) \ge \tfrac{\xi}{2}) = \frac{1}{\Gamma(\frac{2\gamma}{a})} \int_{e^{\frac{1}{2}a\xi}}^{\infty} \zeta^{\frac{2\gamma}{a}} e^{-\zeta} d\zeta \le c \int_{e^{\frac{1}{2}a\xi}}^{\infty} e^{-\frac{1}{2}\zeta} d\zeta = c e^{-\frac{1}{2}e^{\frac{1}{2}a\xi}} \le c e^{-\frac{1}{2}(2\gamma+a)\xi}.$$

Combining these bounds with (2.36) yields the desired result.

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