

Application of weighted and unordered majorization orders in comparisons of parallel systems with exponentiated generalized gamma components

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Abstract. Consider two parallel systems, say A and B , with respective lifetimes T_1 and T_2 wherein independent component lifetimes of each system follow exponentiated generalized gamma distribution with possibly different exponential shape and scale parameters. We show here that T_2 is smaller than T_1 with respect to the usual stochastic order (reversed hazard rate order) if the vector of logarithm (the main vector) of scale parameters of System B is weakly weighted majorized by that of System A , and if the vector of exponential shape parameters of System A is unordered majorized by that of System B . By means of some examples, we show that the above results can not be extended to the hazard rate and likelihood ratio orders. However, when the scale parameters of each system divide into two homogeneous groups, we verify that the usual stochastic and reversed hazard rate orders can be extended, respectively, to the hazard rate and likelihood ratio orders. The established results complete and strengthen some of the known results in the literature.

1 Introduction

Assume that a manufacturer produces a particular system whose components are ordered from different vendors. Sometimes, these components are expensive or they are not available in a specific time. In such cases, the manufacturer can replace the components of the system by new components that are cheaper or are available. But, this replacement may affect some lifetime characteristics of the system such as mean time to failure, survival function, hazard rate function, etc. So, to determine the influence of this replacement on the life characteristics of the system, a tool is needed and the theory of stochastic ordering is found to be useful for this purpose. Stochastic orderings also play an important role in other fields such as management science, financial economics, insurance, actuarial science, operations research, queuing theory and survival analysis. Interested readers may refer to the books by Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Belzunce, Martínez-Riquelme and Mulero (2016) for elaborate discussions on various stochastic orderings and their applications.

Recently, various ordering results have been established for comparisons of lifetime characteristics of two parallel systems consisting of heterogeneous component. The existing results are commonly based on comparisons of a parameter vector of the systems in some mathematical sense; see Balakrishnan and Zhao (2013a) for more details on this topic. However, there exists some situations in which the heterogeneity in the component lifetimes is due to difference in more than one parameter. For example, consider two parallel systems with their component lifetimes following the generalized exponential (GE) distribution with possibly different exponential shape and scale parameters. In this case, the including parameters

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can be represented in the matrix format and the ordering results can be obtained by comparison of these matrices in some mathematical sense, as discussed by [Balakrishnan, Haidari and Masoumifard \(2015\)](#). Next, assume that the component lifetimes of two parallel systems follow gamma distribution with possibly different shape and scale parameters. Now, we have two parameter vectors for each system. [Zhao, Hu and Zhang \(2015\)](#) considered this case and established some ordering results based on comparisons between the vectors of shape parameters and the vectors of scale parameters separately. The heterogeneity of two parameters of the lifetimes allow us to compare more systems in contrast with the case in which the lifetimes differ in one parameter.

In this paper, we follow the above framework and compare the lifetimes of parallel systems with respect to some known stochastic orderings. It is assumed here that the component lifetimes of each system follow the exponentiated generalized gamma distribution differing in two parameters. The ordering results established here are based on the concepts of weighted and unordered majorization orders, which to the best of the author's knowledge, have not utilized so far in comparison of reliability systems. Furthermore, these results reinforce and extend some previous ones in the literature.

The rest of this paper is organized as follows. In Section 2, we present definitions and notions of some concepts including stochastic orders, exponentiated generalized gamma distribution, and weighted and unordered majorization orders. Some useful results, which are fundamental for the main results of the paper, are also stated there. Section 3 contains main results concerning the comparison of two parallel systems with independent heterogeneous exponentiated generalized gamma components. Finally, some discussions on the ordering results established in Section 3 are presented in Section 4.

2 Preliminaries

Here, we present some definitions along with some useful results which play important roles in the sequel.

2.1 Definitions and notions

The first definition briefly reviews some stochastic orders. Interested readers may refer to [Shaked and Shanthikumar \(2007\)](#) and [Belzunce, Martínez-Riquelme and Mulero \(2016\)](#) for elaborate discussions on stochastic orders and their applications.

Definition 1. Suppose X_1 and X_2 are two positive absolutely continuous random variables with corresponding distribution functions F_{X_1} and F_{X_2} , survival functions \bar{F}_{X_1} and \bar{F}_{X_2} , density functions f_{X_1} and f_{X_2} , hazard rate functions r_{X_1} and r_{X_2} , and reversed hazard rate functions \tilde{r}_{X_1} and \tilde{r}_{X_2} , respectively. Then,

- (i) X_1 is said to be smaller than X_2 in the likelihood ratio order, denoted by $X_1 \leq_{lr} X_2$, if $f_{X_2}(t)/f_{X_1}(t)$ is increasing in $t \in \mathbb{R}^+$;
- (ii) X_1 is said to be smaller than X_2 in the reversed hazard rate order, denoted by $X_1 \leq_{rh} X_2$, if $F_{X_2}(t)/F_{X_1}(t)$ is increasing in $t \in \mathbb{R}^+$, or equivalently, $\tilde{r}_{X_1}(t) \leq \tilde{r}_{X_2}(t)$ for all $t \in \mathbb{R}^+$;
- (iii) X_1 is said to be smaller than X_2 in the hazard rate order, denoted by $X_1 \leq_{hr} X_2$, if $\bar{F}_{X_2}(t)/\bar{F}_{X_1}(t)$ is increasing in $t \in \mathbb{R}^+$, or equivalently, $r_{X_2}(t) \leq r_{X_1}(t)$ for all $t \in \mathbb{R}^+$;
- (iv) X_1 is said to be smaller than X_2 in the usual stochastic order, denoted by $X_1 \leq_{st} X_2$, if $\bar{F}_{X_1}(t) \leq \bar{F}_{X_2}(t)$ for all $t \in \mathbb{R}^+$.

It is well known that the likelihood ratio order implies the hazard rate and reversed hazard rate orders, and both of these orders result in the usual stochastic order.

Let \mathcal{P}_n denotes the set of all permutations of $(1, \dots, n)$. For any $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n) \in \mathcal{P}_n$, set

$$\begin{aligned} \mathcal{D}_n^\boldsymbol{\pi} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{\pi_1} \geq \dots \geq x_{\pi_n}\}, \\ \mathcal{G}_n^\boldsymbol{\pi} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{\pi_1} \geq \dots \geq x_{\pi_n} > 0\}. \end{aligned}$$

For the special case of $\boldsymbol{\pi} = (1, \dots, n)$, the spaces $\mathcal{D}_n^\boldsymbol{\pi}$ and $\mathcal{G}_n^\boldsymbol{\pi}$ are denoted by \mathcal{D}_n and \mathcal{G}_n , respectively. In the next definition, we state the weighted majorization orders.

Definition 2. Consider two vectors $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{v} = (v_1, \dots, v_n)$ in \mathbb{R}^n and suppose that $\boldsymbol{p} = (p_1, \dots, p_n)$ is a vector with positive components. Then,

- (i) \boldsymbol{u} is said to be p -majorized by \boldsymbol{v} on $\mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$, denoted by $\boldsymbol{u} \prec_p \boldsymbol{v}$ on $\mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$, if $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$, $\sum_{j=1}^i p_{\pi_j} u_{\pi_j} \leq \sum_{j=1}^i p_{\pi_j} v_{\pi_j}$ for $i = 1, \dots, n - 1$, and $\sum_{j=1}^n p_{\pi_j} u_{\pi_j} = \sum_{j=1}^n p_{\pi_j} v_{\pi_j}$;
- (ii) \boldsymbol{u} is said to be weakly p -majorized by \boldsymbol{v} on $\mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$, denoted by $\boldsymbol{u} \prec_p^w \boldsymbol{v}$ on $\mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$, if $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{D}_n^\boldsymbol{\pi}(\mathcal{G}_n^\boldsymbol{\pi})$ and $\sum_{j=i}^n p_{\pi_j} u_{\pi_j} \geq \sum_{j=i}^n p_{\pi_j} v_{\pi_j}$ for $i = 1, \dots, n$.

It is easy to observe that the p -majorization order implies the weak p -majorization order. Note that, we always have

$$(\boldsymbol{u}_{wa}, \dots, \boldsymbol{u}_{wa}) \prec_p (\boldsymbol{u}_1, \dots, \boldsymbol{u}_n), \quad \boldsymbol{u}_{wa} = \frac{\sum_{i=1}^n p_i u_i}{\sum_{i=1}^n p_i}.$$

The classic majorization and weak majorization orders, denoted respectively by \prec and \prec^w , are special cases of the above weighted versions of majorization when $p_1 = \dots = p_n$. Interested readers may refer to Cheng (1977) for a comprehensive discussion on the weighted versions of majorization order and their properties.

Below, the concept of unordered majoriation order is presented.

Definition 3. Consider two vectors $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{v} = (v_1, \dots, v_n)$ in \mathbb{R}^n . If $\sum_{i=1}^k u_i \leq \sum_{i=1}^k v_i$ for $k = 1, \dots, n - 1$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$, then \boldsymbol{u} is said to be unordered majorized by \boldsymbol{v} , denoted by $\boldsymbol{u} \prec^{uo} \boldsymbol{v}$.

Indeed, to compare vectors in the classic majorization order, it is necessary to arrange their components whereas this arrangement is unnecessary in unordered majorization order. For the case of monotone vectors, it can be easily seen that there exists a connection between the unordered majorization and classic majorization orders. Except this case, there does not exist a connection between the two vector orders. For example, setting $\boldsymbol{u} = (3, 0.1, 9, 7)$ and $\boldsymbol{v} = (11.1, 0.2, 2, 5.8)$, one can easily observe that $\boldsymbol{u} \prec^{uo} \boldsymbol{v}$ while the classic majorization order does not hold between the two vectors. On the other hand, taking $\boldsymbol{u} = (8, 2, 5.5, 4.6)$ and $\boldsymbol{v} = (9, 0.1, 5, 6)$, it then readily follows that $\boldsymbol{u} \prec \boldsymbol{v}$, however, we have $\boldsymbol{u} \not\prec^{uo} \boldsymbol{v}$ and $\boldsymbol{v} \not\prec^{uo} \boldsymbol{u}$. For more details on the unordered and classic majorization orders along with their applications, see Parker and Ram (1997) and Marshall, Olkin and Arnold (2011).

2.2 Exponentiated generalized gamma distribution

A random variable X is said to have the exponentiated generalized gamma (EGG) distribution with shape parameters α, ν and τ , and scale parameter λ (denote by $X \sim \text{EGG}(\alpha, \nu, \tau, \lambda)$) if its cumulative distribution function is given by

$$F(t; \alpha, \nu, \tau, \lambda) = \left[\int_0^t \frac{\nu(\lambda t)^\tau}{\Gamma(\frac{\tau}{\nu})} u^{\tau-1} e^{-(\lambda u)^\nu} \right]^\alpha,$$

$$t \in \mathbb{R}^+, \alpha \in \mathbb{R}^+, \nu \in \mathbb{R}^+, \tau \in \mathbb{R}^+, \lambda \in \mathbb{R}^+.$$

We call the parameter α as the exponential parameter. Many well-known distributions are sub-models of the EGG distribution. For $\tau = \nu$, it becomes the exponentiated Weibull distribution proposed by [Mudholkar and Srivastava \(1993\)](#). If $\nu = \tau = 1$, it reduces to the GE distribution introduced by [Gupta and Kundu \(1999\)](#). When $\nu = 1$, it reduces to exponentiated gamma distribution initiated by [Gupta, Gupta and Gupta \(1998\)](#). If $\tau = \nu$ and $\alpha = 1$, the two-parameter Weibull distribution is obtained, while for $\tau = \nu = \alpha = 1$ the exponential distribution is deduced. The EGG distribution has a flexible hazard rate function that admits increasing, decreasing, bathtub and upside-down bathtub shapes. For more details on some general properties of the EGG distribution and its applications, one may refer to [Cordeiro, Ortega and Silva \(2009\)](#).

2.3 Some useful results

The following lemma deals with the preservation of the unordered majorization order by weighted sum functions.

Lemma 1 (Marshall, Olkin and Arnold (2011, p. 639)). *Consider the real vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$, and assume that $(w_1, \dots, w_n) \in \mathcal{D}_n$. If $\mathbf{u} \stackrel{uo}{\prec} \mathbf{v}$, then we have $\sum_{i=1}^n w_i u_i \leq \sum_{i=1}^n w_i v_i$.*

It is interesting to find specific conditions under which those functions preserving the weighted majorization order can be determined. In the following, we discuss this problem by recalling a general statement. Consider the function $\phi : \mathbb{R}^n \times \mathbb{R}^{+n} \rightarrow \mathbb{R}$ satisfying the following properties:

$$\phi(\mathbf{u}; \mathbf{p}) = \phi(\mathbf{u}^\pi; \mathbf{p}^\pi) \quad \text{for all } \mathbf{u} \in \mathbb{R}^n, \mathbf{p} \in \mathbb{R}^{+n} \text{ and } \pi \in \mathcal{P}_n, \quad (2.1)$$

$$\mathbf{u} \prec_p \mathbf{v} \quad \text{on } \mathcal{D}_n \quad \Rightarrow \quad \phi(\mathbf{u}; \mathbf{p}) \leq \phi(\mathbf{v}; \mathbf{p}). \quad (2.2)$$

If $\mathbf{u} \prec_p \mathbf{v}$ on \mathcal{D}_n^π , it then easily follows that $\mathbf{u}^\pi \prec_{p^\pi} \mathbf{v}^\pi$ on \mathcal{D}_n , and hence by (2.1) and (2.2) we have

$$\phi(\mathbf{u}; \mathbf{p}) = \phi(\mathbf{u}^\pi; \mathbf{p}^\pi) \leq \phi(\mathbf{v}^\pi; \mathbf{p}^\pi) = \phi(\mathbf{v}; \mathbf{p}).$$

Thus, if the function ϕ is permutation invariant (the property given in (2.1)) and preserves the p -majorization order on \mathcal{D}_n , then it also preserves the p -majorization order on \mathcal{D}_n^π for all $\pi \in \mathcal{P}_n$. This statement allows us to consider the preserving property of the permutation invariant functions only on the space \mathcal{D}_n . Let us now consider a special case in which $\phi(\mathbf{u}; \mathbf{p}) = \sum_{i=1}^n p_i \varphi(u_i)$, where φ is a real-valued function on a sub-interval I of \mathbb{R} . It is clear that ϕ is permutation invariant. The next theorem gives sufficient conditions for preserving the p -majorization and weakly p -majorization orders by the mentioned function ϕ .

Theorem 1 (Pečarić, Proschan and Tong (1992, p. 323)). *Let $\mathbf{u} = (u_1, \dots, u_n)$ be a vector in \mathbb{R}^n , and that $\mathbf{p} = (p_1, \dots, p_n)$ be a vector with positive components. Set $\phi(\mathbf{u}; \mathbf{p}) = \sum_{i=1}^n p_i \varphi(u_i)$, where φ is a real-valued continuous function on a sub-interval I of \mathbb{R} .*

- (i) *If φ is a convex function on I , then ϕ preserves the p -majorization order on \mathcal{D}_n ;*
- (ii) *If φ is a decreasing convex function on I , then ϕ preserves the weakly p -majorization order on \mathcal{D}_n .*

Remark 1. In view of the proof of Theorem 1, one can easily find that its results remain true if the spaces \mathbb{R}^n and \mathcal{D}_n are replaced by \mathbb{R}^{+n} and \mathcal{G}_n , respectively.

It should be noted here that the weighted majorization order allows us to compare only similarly ordered vectors. Cheng (1977) presented an example to show that when two vectors (u_1, \dots, u_n) and (v_1, \dots, v_n) are ordered in different direction and $(u_1, \dots, u_n) \prec_{(p_1, \dots, p_n)} (v_1, \dots, v_n)$, then the inequality $\sum_{i=1}^n p_i m(u_i) \leq \sum_{i=1}^n p_i m(v_i)$ can either be true or false for every convex function m . However, according to Theorem 1, the mentioned inequality holds for every convex function m if the vector (u_1, \dots, u_n) and (v_1, \dots, v_n) are similarly ordered.

In the case of differentiable functions, we have the following theorem to check the preservation property of the weighted majorization order.

Theorem 2 (Cheng (1977, p. 25)). Consider the differentiable function $\phi : \mathbb{R}^n \times \mathbb{R}^{+n} \rightarrow \mathbb{R}$ satisfying (2.1). Then, (2.2) is satisfied iff for all $\mathbf{u} \in \mathbb{R}^n$ and all $i, j = 1, \dots, n$,

$$(u_i - u_j) \left(\frac{1}{p_i} \frac{\partial \phi(\mathbf{u}, \mathbf{p})}{\partial u_i} - \frac{1}{p_j} \frac{\partial \phi(\mathbf{u}, \mathbf{p})}{\partial u_j} \right) \geq 0. \tag{2.3}$$

Remark 2. The result of Theorem 2 remains true if the spaces \mathbb{R}^n and \mathcal{D}_n are replaced respectively, by the spaces \mathbb{R}^{+n} and \mathcal{G}_n . Now, assume that $\mathbb{B} \subset \mathbb{R}^+$ and consider the differentiable function $\phi : \mathbb{R}^n \times \mathbb{B}^n \rightarrow \mathbb{R}$ satisfying

$$\phi(\mathbf{u}; \mathbf{p}) = \phi(\mathbf{u}^\pi; \mathbf{p}^\pi) \quad \text{for all } \mathbf{u} \in \mathbb{R}^n, \mathbf{p} \in \mathbb{B}^n \text{ and } \pi \in \mathcal{P}_n.$$

In view of the proof of Theorem 2, one can observe that its sufficient part still holds, that is, the function ϕ preserves the p -majorization order if (2.3) holds.

We now present a series of lemmas pertinent to the hazard rate and likelihood ratio orders established in the next section.

Lemma 2. Suppose the functions $d, l_1, l_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are defined as

$$d(y) = \frac{e^{-y^\nu}}{\int_0^1 z^{\frac{\tau}{\nu}-1} e^{-y^\nu z} dz}, \quad l_1(y) = y^\nu + d(y), \quad l_2(y) = d(y)l_1'(y),$$

where $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$. Then,

- (i) $d(y)$ is decreasing in $y \in \mathbb{R}^+$ for all $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$;
- (ii) $l_1(y)$ is increasing in $y \in \mathbb{R}^+$ for all $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$;
- (iii) $l_2(y)$ is decreasing in $y \in \mathbb{R}^+$ for all $0 < \tau \leq \nu \leq 1$.

Proof.

- (i) Taking $f(y) = e^{-y}/(\int_0^1 z^{\frac{\tau}{\nu}-1} e^{-yz} dz)$, $y \in \mathbb{R}^+$, we can observe that $d(y) = f(y^\nu)$, $x \in \mathbb{R}^+$. Therefore, it immediately follows, for $y \in \mathbb{R}^+$, that $d'(y) = \nu y^{\nu-1} f'(y^\nu)$. Misra and Misra (2013) showed that $f(y)$ is decreasing in $y \in \mathbb{R}^+$ for any $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, thus completing the proof of Part (i);
- (ii) Assume that $g(y) = y + e^{-y}/(\int_0^1 z^{\frac{\tau}{\nu}-1} e^{-yz} dz)$, $y \in \mathbb{R}^+$. We then easily have $l_1(y) = g(y^\nu)$, $y \in \mathbb{R}^+$. Based on Lemma 3.3 of Zhao and Balakrishnan (2015), we know that $g(y)$ is increasing in $y \in \mathbb{R}^+$ for any $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, and the desired result then follows;
- (iii) Based on the proofs of Parts (i) and (ii), we can write $l_2(y) = \nu y^{\nu-1} \varepsilon(y^\nu)$, $y \in \mathbb{R}^+$, where $\varepsilon(y) = f(y)g'(y)$. We know from Lemma 3.5 of Zhao and Balakrishnan (2015) that $\varepsilon(y)$ is non-negative and decreasing in $y \in \mathbb{R}^+$ for $0 < \tau \leq \nu$. On the other hand, for $0 < \nu \leq 1$, $y^{\nu-1}$ is non-negative and decreasing in $y \in \mathbb{R}^+$. By combining the above observations, the required result is obtained. □

Lemma 3. Assume that $y_i \in \mathbb{R}^+$ and $\gamma_i \geq 1$ for $i = 1, 2$. Then, for $0 < \tau \leq \nu$, the following inequality holds:

$$\frac{\gamma_1 d(y_1) + \gamma_2 d(y_2)}{1 - [\rho(y_1)]^{\gamma_1} [\rho(y_2)]^{\gamma_2}} \geq l_1(y_{\min}),$$

where $y_{\min} = \min\{y_1, y_2\}$, $\rho(y) = \int_0^1 \frac{y^\tau}{\Gamma(\frac{\tau}{\nu})} u^{\frac{\tau}{\nu}-1} e^{-y^\nu u} du$ for $y \in \mathbb{R}^+$, and the functions d and l_1 are as defined in Lemma 2.

Proof. For $y_i \in \mathbb{R}^+$, $i = 1, 2$, we have

$$\begin{aligned} \frac{\delta_1 d(y_1) + \delta_2 d(y_2)}{1 - [\rho(y_1)]^{\delta_1} [\rho(y_2)]^{\delta_2}} &\geq \frac{\delta_1 d(y_1) + \delta_2 d(y_2)}{\delta_1 (1 - \rho(y_1)) + \delta_2 (1 - \rho(y_2))} \\ &\geq \frac{\delta_1 d(y_1) + \delta_2 d(y_2)}{\delta_1 \frac{d(y_1)}{l_1(y_1)} + \delta_2 \frac{d(y_2)}{l_1(y_2)}} \\ &\geq \min\{l_1(y_1), l_1(y_2)\} = l_1(y_{\min}), \end{aligned}$$

wherein the first inequality obtains from Weierstrass inequality (Mitrinović, Pečarić and Fink (1993, p. 71)), the second inequality follows by using an argument similar to that utilized in the proof of Proposition 3 of Balakrishnan and Zhao (2013b), the third inequality holds based on inequality (8.1) of Mitrinović, Pečarić and Fink (1993, p. 340), and finally the equality follows from Part (ii) of Lemma 2. \square

Lemma 4. Suppose the function $\Upsilon : \mathbb{R}^2 \times [1, \infty)^2 \rightarrow \mathbb{R}^+$ is defined as

$$\Upsilon(\mathbf{y}; \boldsymbol{\delta}) = \frac{[\rho(e^{y_1})]^{\delta_1} [\rho(e^{y_2})]^{\delta_2} (\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}))}{1 - [\rho(e^{y_1})]^{\delta_1} [\rho(e^{y_2})]^{\delta_2}},$$

where $0 < \tau \leq \nu$, $\mathbf{y} = (y_1, y_2)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2)$. If $\mathbf{y}^* \prec_{\boldsymbol{\delta}} \mathbf{y}$ on \mathcal{D}_2 , then we have $\Upsilon(\mathbf{y}; \boldsymbol{\delta}) \leq \Upsilon(\mathbf{y}^*; \boldsymbol{\delta})$.

Proof. It is easy to observe that Υ is permutation invariant. Further, after some computations, we have

$$\begin{aligned} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_1} &= (1 - A)A^2(\delta_1 \nu d(e^{y_1})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_1})A] + \delta_1 \tau d(e^{y_1})A) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_2} &= (1 - A)A^2(\delta_2 \nu d(e^{y_2})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_2})A] + \delta_2 \tau d(e^{y_2})A), \end{aligned}$$

where $A = 1 - [\rho(e^{y_1})]^{\delta_1} [\rho(e^{y_2})]^{\delta_2}$. Hence, we obtain

$$\begin{aligned} \frac{1}{\delta_1} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_1} - \frac{1}{\delta_2} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_2} &\stackrel{\text{sgn}}{=} \tau A(d(e^{y_1}) - d(e^{y_2})) \\ &\quad + \nu \{d(e^{y_1})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_1})A] \\ &\quad - d(e^{y_2})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_2})A]\} \\ &= \Theta_1 + \Theta_2, \quad \text{say,} \end{aligned}$$

where $a \stackrel{\text{sgn}}{=} b$ means that a and b have the same sign. Assume that $y_1 \leq y_2$. Based on Part (i) of Lemma 2, it readily follows that $\Theta_1 \geq 0$. On the other hand, from Parts (i) and (ii) of Lemma 2 and the result in Lemma 3, we find that

$$\begin{aligned} \Theta_2 &\geq \nu d(e^{y_2})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_1})A] \\ &\quad - d(e^{y_2})[\delta_1 d(e^{y_1}) + \delta_2 d(e^{y_2}) - l_1(e^{y_2})A] \\ &= \nu A d(e^{y_2})(l_1(e^{y_2}) - l_1(e^{y_1})) \\ &\geq 0. \end{aligned}$$

These findings result in

$$(y_1 - y_2) \left(\frac{1}{\delta_1} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_1} - \frac{1}{\delta_2} \frac{\partial \Upsilon(\mathbf{y}; \boldsymbol{\delta})}{\partial y_2} \right) \leq 0,$$

and the conclusion is now obtained from Remark 2. □

Lemma 5. Suppose the function $\psi : \mathbb{R}^{+2} \times \mathbb{R}^{+2} \rightarrow \mathbb{R}^+$ is defined as

$$\psi(\mathbf{y}; \boldsymbol{\delta}) = \frac{\delta_1 d(y_1) + \delta_2 d(y_2)}{\delta_1 d(y_1)l_1(y_1) + \delta_2 d(y_2)l_1(y_2)},$$

where $0 < \tau \leq \nu \leq 1$, $\mathbf{y} = (y_1, y_2)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2)$. If $\mathbf{y}^* \prec_{\boldsymbol{\delta}} \mathbf{y}$ on \mathcal{G}_2 , then we have $\psi(\mathbf{y}^*; \boldsymbol{\delta}) \leq \psi(\mathbf{y}; \boldsymbol{\delta})$.

Proof. It is clear that ψ is permutation invariant. Moreover, the derivatives of $\psi(y_1, y_2; \delta_1, \delta_2)$ with respect to y_1 and y_2 are, respectively,

$$\frac{\partial \psi(\mathbf{y}; \boldsymbol{\delta})}{\partial y_1} = \frac{\delta_1(\delta_2 d'(y_1)d(y_2)(l_1(y_2) - l_1(y_1)) - l_2(y_1)(\delta_1 d(y_1) + \delta_2 d(y_2)))}{(\delta_1 d(y_1)l_1(y_1) + \delta_2 d(y_2)l_1(y_2))^2}$$

and

$$\frac{\partial \psi(\mathbf{y}; \boldsymbol{\delta})}{\partial y_2} = \frac{\delta_2(\delta_1 d'(y_2)d(y_1)(l_1(y_1) - l_1(y_2)) - l_2(y_2)(\delta_1 d(y_1) + \delta_2 d(y_2)))}{(\delta_1 d(y_1)l_1(y_1) + \delta_2 d(y_2)l_1(y_2))^2}.$$

From these observations, we have

$$\begin{aligned} (y_1 - y_2) \left(\frac{1}{\delta_1} \frac{\partial \psi(\mathbf{y}; \boldsymbol{\delta})}{\partial y_1} - \frac{1}{\delta_2} \frac{\partial \psi(\mathbf{y}; \boldsymbol{\delta})}{\partial y_2} \right) \\ \stackrel{\text{sgn}}{=} (y_1 - y_2)(l_1(y_2) - l_1(y_1))(\delta_1 d'(y_2)d(y_1) + \delta_2 d'(y_1)d(y_2)) \\ + (y_1 - y_2)(l_2(y_2) - l_2(y_1))(\delta_1 d(y_1) + \delta_2 d(y_2)). \end{aligned} \tag{2.4}$$

Let us assume that $y_1 \geq y_2$. Based on Parts (i) and (ii) of Lemma 2, it immediately follows that $d'(y_i) \leq 0$ for $i = 1, 2$, and $l_1(y_1) \geq l_1(y_2)$ for any $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$. These observations show that the first term on the right-hand side of (2.4) is non-negative. On the other hand, using Part (iii) of Lemma 2, we find that $l_2(y_2) \geq l_2(y_1)$ for any $0 < \tau \leq \nu \leq 1$. Thus, the second term on the right-hand side of (2.4) is also non-negative. The conclusion now follows from Remark 2. □

Suppose $X_1, \dots, X_n, Z_1, \dots, Z_n$ and Y_1, \dots, Y_n are three sets of independent positive random variables such that

$$X_i \sim [F(\lambda_i x)]^{\alpha_i}, \quad Z_i \sim [F(\mu_i x)]^{\alpha_i}, \quad Y_i \sim [F(\mu_i x)]^{\beta_i}, \tag{2.5}$$

where for $i = 1, \dots, n$, $\alpha_i \in \mathbb{R}^+$, $\beta_i \in \mathbb{R}^+$, $\lambda_i \in \mathbb{R}^+$, $\mu_i \in \mathbb{R}^+$ and F (the baseline distribution) is an absolutely continuous distribution function centered on \mathbb{R}^+ with the corresponding reversed hazard rate function \tilde{r} . Set $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$. For every vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^{+n}$, set $\log \mathbf{x} = (\log x_1, \dots, \log x_n)$.

In the upcoming theorems, we discuss the usual stochastic and reversed hazard rate orders between the largest order statistics $X_{n:n}$ and $Y_{n:n}$ by using the concepts of weighted and unordered majorization orders.

Theorem 3. Assume that $t\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$. If $\log \boldsymbol{\mu} \prec_{\boldsymbol{\alpha}}^w \log \boldsymbol{\lambda}$ on \mathcal{D}_n and $\boldsymbol{\alpha} \prec^{uo} \boldsymbol{\beta}$, then we have $Y_{n:n} \leq_{st} X_{n:n}$.

Proof. Assume that $\log \boldsymbol{\mu} \prec_{\boldsymbol{\alpha}}^w \log \boldsymbol{\lambda}$ on \mathcal{D}_n and $\boldsymbol{\alpha} \prec^{uo} \boldsymbol{\beta}$. Setting $\lambda_i^* = \log \lambda_i$ and $\mu_i^* = \log \mu_i$, $i = 1, \dots, n$, it then follows that $\boldsymbol{\mu}^* \prec_{\boldsymbol{\alpha}}^w \boldsymbol{\lambda}^*$ on \mathcal{D}_n . Note that the function $\varphi(\lambda^*) = \log[F(xe^{\lambda^*})]$ is increasing in λ^* for each fixed $x \in \mathbb{R}^+$. On the other hand, because $t\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$, it follows that $\varphi(\lambda^*)$ is concave in λ^* for each fixed $x \in \mathbb{R}^+$. Now, based on the above observations, Par(ii) of Theorem 1 and Remark 1, we find that

$$\sum_{i=1}^n \alpha_i \log[F(xe^{\mu_i^*})] \geq \sum_{i=1}^n \alpha_i \log[F(xe^{\lambda_i^*})] \quad \text{for all } x \in \mathbb{R}^+,$$

and so $Z_{n:n} \leq_{st} X_{n:n}$. Further, taking $u_i = \alpha_i$, $v_i = \beta_i$ and $w_i = \log F(\mu_i x)$, $i = 1, \dots, n$, in Lemma 1, we find $Y_{n:n} \leq_{st} Z_{n:n}$, thus completing the proof of the theorem. \square

Theorem 4. Assume that $t\tilde{r}(t)$ is decreasing and convex in $t \in \mathbb{R}^+$. If $\boldsymbol{\mu} \prec_{\boldsymbol{\alpha}}^w \boldsymbol{\lambda}$ on \mathcal{G}_n and $\boldsymbol{\alpha} \prec^{uo} \boldsymbol{\beta}$, then we have $Y_{n:n} \leq_{rh} X_{n:n}$.

Proof. Assume that $\boldsymbol{\mu} \prec_{\boldsymbol{\alpha}}^w \boldsymbol{\lambda}$ on \mathcal{D}_n and $\boldsymbol{\alpha} \prec^{uo} \boldsymbol{\beta}$. Setting $u_i = \alpha_i$, $v_i = \beta_i$ and $w_i = -\mu_i \tilde{r}(\mu_i x)$, $i = 1, \dots, n$, in Lemma 1, it readily follows from the decreasing property of $t\tilde{r}(t)$ that $Y_{n:n} \leq_{rh} Z_{n:n}$. Therefore, the desired result obtains if we show that $Z_{n:n} \leq_{rh} X_{n:n}$. The reversed hazard rate function of $X_{n:n}$ and $Z_{n:n}$ can be rewritten respectively as

$$\tilde{r}_{X_{n:n}}(x) = \frac{1}{x} \sum_{i=1}^n \alpha_i \phi(\lambda_i x) \quad \text{and} \quad \tilde{r}_{Z_{n:n}}(x) = \frac{1}{x} \sum_{i=1}^n \alpha_i \phi(\mu_i x), \quad x \in \mathbb{R}^+,$$

where $\phi(t) = t\tilde{r}(t)$, $t \in \mathbb{R}^+$. Because $t\tilde{r}(t)$ is decreasing and convex in $t \in \mathbb{R}^+$, then we easily observe that $\phi(t)$ is also decreasing and convex in $t \in \mathbb{R}^+$. Now, from Part (ii) of Theorem 1 and Remark 1, we find that $\tilde{r}_{Z_{n:n}}(x) \leq \tilde{r}_{X_{n:n}}(x)$ for all $x \in \mathbb{R}^+$, as required. \square

Let us now assume that $\mu_1 = \dots = \mu_k = \mu$ and $\mu_{k+1} = \dots = \mu_n = \acute{\mu}$, where $k \in \{1, \dots, n-1\}$. Set

$$\delta_1 = \sum_{i=1}^k \alpha_i, \quad \delta_2 = \sum_{i=k+1}^n \alpha_i, \quad \xi_1 = \sum_{i=1}^k \beta_i, \quad \xi_2 = \sum_{i=k+1}^n \beta_i.$$

Below, the random variables $Z_{n:n}$ and $Y_{n:n}$ are compared in the sense of the likelihood ratio order by means of the unordered majorization order. For this purpose, the following lemma is needed.

Lemma 6. Consider the real functions h , h_1 and h_2 on \mathbb{R}^+ such that $h(x) > h_2(x) \geq h_1(x) > 0$ for all $x \in \mathbb{R}^+$. If $h_2(x)/h_1(x)$ and $h(x)/h_1(x)$ are, respectively, increasing and

decreasing in $x \in \mathbb{R}^+$, then the ratio

$$\frac{h(x) - h_1(x)}{h(x) - h_2(x)}$$

is increasing in $x \in \mathbb{R}^+$.

Proof. We can easily observe that

$$\frac{h(x) - h_1(x)}{h(x) - h_2(x)} = 1 + m_1(x)m_2(x),$$

where

$$m_1(x) = \frac{h_2(x)}{h_1(x)} - 1 \quad \text{and} \quad m_2(x) = \frac{1}{\frac{h(x)}{h_1(x)} - m_1(x) - 1}.$$

From the assumptions, it immediately follows that both functions m_1 and m_2 are non-negative and increasing, thus completing the proof of the lemma. \square

Theorem 5. Assume that both functions $t\tilde{r}(t)$ and $t\tilde{r}'(t)/\tilde{r}(t)$ are decreasing in $t \in \mathbb{R}^+$. If $(\mu, \acute{\mu}) \in \mathcal{G}_2$ and $(\delta_1, \delta_2) \stackrel{uo}{\prec} (\xi_1, \xi_2)$, then we have $Y_{n:n} \leq_{lr} Z_{n:n}$.

Proof. From the assumption $(\delta_1, \delta_2) \stackrel{uo}{\prec} (\xi_1, \xi_2)$, we have $\delta_1 \leq \xi_1$ and $\delta_1 + \delta_2 = \xi_1 + \xi_2 = c$. Under this setting, we can write the ratio of the density functions of $Z_{n:n}$ and $Y_{n:n}$ as

$$\frac{f_{Z_{n:n}}(x)}{f_{Y_{n:n}}(x)} = c(x) \frac{F_{Z_{n:n}}(x)}{F_{Y_{n:n}}(x)}, \quad x \in \mathbb{R}^+,$$

where

$$\begin{aligned} c(x) &= \frac{c\acute{\mu}\tilde{r}(\acute{\mu}x) - \delta_1(\acute{\mu}\tilde{r}(\acute{\mu}x) - \mu\tilde{r}(\mu x))}{c\acute{\mu}\tilde{r}(\acute{\mu}x) - \xi_1(\acute{\mu}\tilde{r}(\acute{\mu}x) - \mu\tilde{r}(\mu x))} \\ &= \frac{h(x) - h_1(x)}{h(x) - h_2(x)}, \quad x \in \mathbb{R}^+. \end{aligned}$$

Using the facts $t\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$ and $(\mu, \acute{\mu}) \in \mathcal{G}_2$, we find that $\acute{\mu}\tilde{r}(\acute{\mu}x) - \mu\tilde{r}(\mu x) \geq 0$ for all $x \in \mathbb{R}^+$. Now, based on this fact and the assumption $\delta_1 \leq \xi_1$, it follows that $F_{Z_{n:n}}(x)/F_{Y_{n:n}}(x)$ is increasing in $x \in \mathbb{R}^+$. Thus, the desired result follows if $c(x)$ is increasing in $x \in \mathbb{R}^+$. The assumptions $\delta_1 \leq \xi_1$ and $0 < \xi_1 < c$ immediately imply that $h(x) > h_2(x) \geq h_1(x) > 0$ for all $x \in \mathbb{R}^+$. Also, $h_1(x)/h(x) = (\delta_1/c)(1 - \mu\tilde{r}(\mu x)/(\acute{\mu}\tilde{r}(\acute{\mu}x)))$ is increasing in $x \in \mathbb{R}^+$ by the assumptions that $(\mu, \acute{\mu}) \in \mathcal{G}_2$ and $t\tilde{r}'(t)/\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$. Finally, it is evident that $h_2(x)/h_1(x) = \xi_1/\delta_1$ is increasing in $x \in \mathbb{R}^+$. Now, from the above observations and the result in Lemma 6, it follows that $c(x)$ is increasing in $x \in \mathbb{R}^+$, as required. \square

3 Main ordering results for the comparison of two parallel systems

Suppose we have two parallel systems, say A and B , with their component lifetimes being independent random variables X_1, \dots, X_n and Y_1, \dots, Y_n , respectively, satisfying $X_i \sim \text{EGG}(\alpha_i, \nu, \tau, \lambda_i)$ and $Y_i \sim \text{EGG}(\beta_i, \nu, \tau, \mu_i)$ for $i = 1, \dots, n$. Clearly, the lifetimes of Systems A and B correspond to $X_{n:n} = \max\{X_1, \dots, X_n\}$ and $Y_{n:n} = \max\{Y_1, \dots, Y_n\}$, respectively. In this section, we obtain some sufficient conditions to compare the lifetimes of Systems A and B with respect to the usual stochastic, reversed hazard rate, hazard rate and likelihood ratio orders. Set $\alpha^\pi = (\alpha_{\pi_1}, \dots, \alpha_{\pi_n})$ and $\beta^\pi = (\beta_{\pi_1}, \dots, \beta_{\pi_n})$, where $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{P}_n$. In what follows, we adopt the above setting.

To prove the main results, we need the following lemma.

Lemma 7. Assume that \tilde{r} is the reversed hazard rate function of the generalized gamma distribution with shape parameters ν and τ , and scale parameter 1. Then:

- (i) $t\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$ for all $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, and is convex in $t \in \mathbb{R}^+$ for all $0 < \nu \leq 1$ and $\tau \in \mathbb{R}^+$ (Khaledi, Farsinezhad and Kochar (2011));
- (ii) $t\tilde{r}'(t)/\tilde{r}(t)$ is decreasing in $t \in \mathbb{R}^+$ for all $0 < \tau \leq \nu$ (Kochar and Torrado (2015)).

Below, we present a theorem concerning the usual stochastic order between the lifetimes of Systems A and B.

Theorem 6. If $\log \mu \stackrel{w}{<}_{\alpha} \log \lambda$ on \mathcal{D}_n and $\alpha \stackrel{uo}{<} \beta$, then for all $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, we have $Y_{n:n} \leq_{st} X_{n:n}$.

Proof. By taking the baseline distribution in Theorem 3 as the generalized gamma, the required result follows immediately from Part (i) of Lemma 7. □

Khaledi, Farsinezhad and Kochar (2011) established the result in Theorem 6 for the special case when $\alpha_i = \beta_i = 1, i = 1, \dots, n$, that is, the case in which the component lifetimes have the generalized gamma distribution with possibly different scale parameters.

Remark 3. The result of Theorem 6 is restricted to the weak weighted majorization order only on the space \mathcal{D}_n . However, we can extend its result to the more general case as follows. Assume that $\log \mu \stackrel{w}{<}_{\alpha} \log \lambda$ on \mathcal{D}_n^{π} , where $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{P}_n$, and $\alpha^{\pi} \stackrel{uo}{<} \beta^{\pi}$. Then, for any $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, we have $Y_{n:n} \leq_{st} X_{n:n}$.

We now explain the result given in Theorem 6 by means of a numerical example. Assume that $n = 4, \nu = 0.4, \tau = 0.8, \alpha = (1, 3, 5, 0.6), \beta = (4.6, 4.4, 0.5, 0.1), \lambda = (9, 6, 1, 0.7)$ and $\mu = (8, 5, 0.8, 0.75)$. Then, we have $\log \mu \stackrel{w}{<}_{\alpha} \log \lambda$ on \mathcal{D}_4 and $\alpha \stackrel{uo}{<} \beta$, which according to Theorem 6, imply that $Y_{4:4} \leq_{st} X_{4:4}$. To see this result graphically, we plot the survival functions of Systems A and B in Figure 1 under the above setting.

Next theorem enables us to compare the lifetimes of Systems A and B with respect to the reversed hazard rate order.

Theorem 7. If $\mu \stackrel{w}{<}_{\alpha} \lambda$ on \mathcal{G}_n and $\alpha \stackrel{uo}{<} \beta$, then for any $0 < \nu \leq 1$ and $\tau \in \mathbb{R}^+$, we have $Y_{n:n} \leq_{rh} X_{n:n}$.

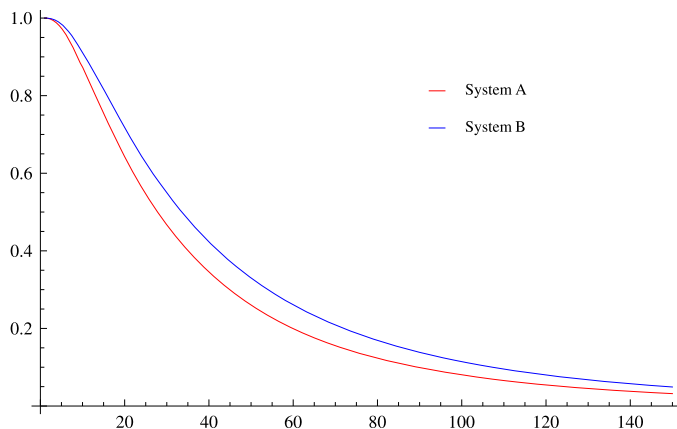


Figure 1 Plots of survival functions of Systems A and B for $n = 4, \nu = 0.4, \tau = 0.8, \alpha = (1, 3, 5, 0.6), \beta = (4.6, 4.4, 0.5, 0.1), \lambda = (9, 6, 1, 0.7)$ and $\mu = (8, 5, 0.8, 0.75)$.

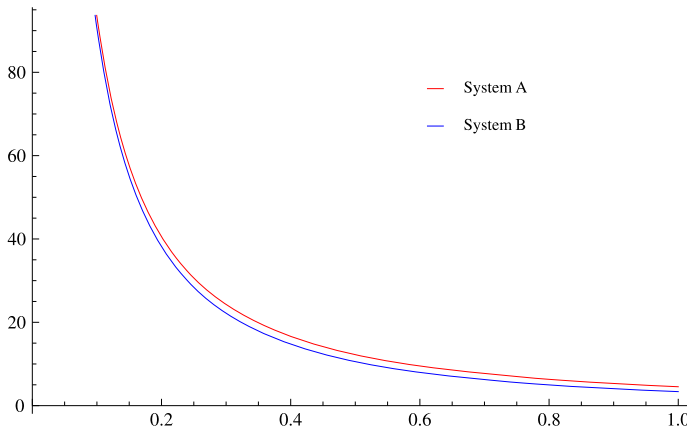


Figure 2 Plots of reversed hazard rate functions of Systems A and B for $n = 4$, $\nu = 0.5$, $\tau = 2$, $\alpha = (4, 0.8, 3.3, 5)$, $\beta = (1, 3, 2.1, 7)$, $\lambda = (2, 11, 12, 13)$ and $\mu = (5, 6, 10, 14)$.

Proof. By assuming the baseline distribution in Theorem 4 as the generalized gamma, the proof readily obtains and Part (i) of Lemma 7. \square

The result of Theorem 7 extends that of Misra and Misra (2013) which is established in the generalized gamma framework, that is, the case when $\alpha_i = \beta_i = 1$ for $i = 1, \dots, n$.

Remark 4. The result in Theorem 7 can be extended as follows. Assume that $\mu \stackrel{w}{\prec}_\alpha \lambda$ on \mathcal{G}_n^π , where $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{P}_n$, and $\alpha^\pi \stackrel{uo}{\prec} \beta^\pi$. Then, for any $0 < \nu \leq 1$ and $\tau \in \mathbb{R}^+$, we have $Y_{n:n} \leq_{rh} X_{n:n}$.

Next example illustrates the result of Remark 4. Set $n = 4$, $\nu = 0.5$, $\tau = 2$, $\alpha = (4, 0.8, 3.3, 5)$, $\beta = (1, 3, 2.1, 7)$, $\lambda = (2, 11, 12, 13)$ and $\mu = (5, 6, 10, 14)$. Setting $\pi = (4, 3, 2, 1)$, we can observe that $\mu \stackrel{w}{\prec}_\alpha \lambda$ on \mathcal{G}_4^π and $\alpha^\pi \stackrel{uo}{\prec} \beta^\pi$, which according to Remark 4, result in $Y_{4:4} \leq_{rh} X_{4:4}$. We plot the reversed hazard rate functions of Systems A and B in Figure 2 to observe the established result graphically.

A natural question which arises here is whether the results in Theorems 6 and 7 can be extended, respectively, to the hazard rate and likelihood ratio orders. Unfortunately, the answer is no as shown in the following examples. Set $n = 3$, $\nu = 0.5$, $\tau = 0.2$, $\alpha = \beta = (4.5, 2, 3)$, $\lambda = (9, 3.5, 0.1)$ and $\mu = (10, 2.9, 0.3)$. We then find

$$\frac{\bar{F}_{X_{3:3}}(4)}{\bar{F}_{Y_{3:3}}(4)} = 1.13809, \quad \frac{\bar{F}_{X_{3:3}}(5)}{\bar{F}_{Y_{3:3}}(5)} = 1.14207, \quad \frac{\bar{F}_{X_{3:3}}(15)}{\bar{F}_{Y_{3:3}}(15)} = 1.12199.$$

This observation means that the ratio of the survival functions of $X_{3:3}$ and $Y_{3:3}$ is not monotone, i.e., the lifetimes of Systems A and B can not be compared in the sense of the hazard rate order. However, $\log \mu \stackrel{w}{\prec}_\alpha \log \lambda$ on \mathcal{D}_3 and $\alpha \stackrel{uo}{\prec} \beta$. Let us now assume that $n = 3$, $\nu = 0.5$, $\tau = 0.1$, $\alpha = (0.1, 0.6, 0.4)$, $\beta = (0.7, 0.2, 0.2)$, $\lambda = (7, 6, 1)$ and $\mu = (100, 20, 1)$. Then, it can be checked that $\mu \stackrel{w}{\prec}_\alpha \lambda$ on \mathcal{G}_3 and $\alpha \stackrel{uo}{\prec} \beta$, while

$$\begin{aligned} \frac{f_{X_{3:3}}(0.3)}{f_{Y_{3:3}}(0.3)} &\approx 2.92344, \\ \frac{f_{X_{3:3}}(0.4)}{f_{Y_{3:3}}(0.4)} &\approx 2.95206, \\ \frac{f_{X_{3:3}}(0.6)}{f_{Y_{3:3}}(0.6)} &\approx 2.91266. \end{aligned}$$

Therefore, the ratio of the density functions of $X_{3:3}$ and $Y_{3:3}$ is not monotone. So, the likelihood ratio order does not hold between the lifetimes of Systems A and B .

A natural way to extend the result in Theorem 6 (Theorem 7) to the hazard rate order (the likelihood ratio order) is that we reduce the heterogeneity of the scale parameters. To this end, from now on, we assume that

$$\begin{aligned} \lambda_1 = \dots = \lambda_k = \lambda, & \quad \lambda_{k+1} = \dots = \lambda_n = \hat{\lambda}, \\ \mu_1 = \dots = \mu_k = \mu, & \quad \mu_{k+1} = \dots = \mu_n = \hat{\mu}, \end{aligned}$$

where $k \in \{1, \dots, n - 1\}$. Set $\delta_1 = \sum_{i=1}^k \alpha_i$, $\delta_2 = \sum_{i=k+1}^n \alpha_i$, $\xi_1 = \sum_{i=1}^k \beta_i$, and $\xi_2 = \sum_{i=k+1}^n \beta_i$. Further, suppose Z_1, \dots, Z_n is a set of independent random variables such that $Z_i \sim \text{EGG}(\alpha_i, \nu, \tau, \mu)$ and $Z_j \sim \text{EGG}(\alpha_j, \nu, \tau, \hat{\mu})$ for $i = 1, \dots, k$ and $j = k + 1, \dots, n$.

Before discussing the hazard rate and likelihood ratio orders between the lifetimes of System A and B , we prove a useful lemma.

Lemma 8.

- (i) Assume that $\delta_i \geq 1, i = 1, 2$. If $(\log \mu, \log \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\log \lambda, \log \hat{\lambda})$ on \mathcal{D}_2 , then for $0 < \tau \leq \nu$, we have $Z_{n:n} \leq_{hr} X_{n:n}$;
- (ii) If $(\mu, \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\lambda, \hat{\lambda})$ on \mathcal{G}_2 , then for $0 < \tau \leq \nu \leq 1$, we have $Z_{n:n} \leq_{lr} X_{n:n}$;
- (iii) If $\hat{\lambda} = \hat{\mu}$ and $\hat{\lambda} \leq \mu \leq \lambda$, then for any $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, we have $Z_{n:n} \leq_{lr} X_{n:n}$.

Proof.

- (i) The hazard rate functions of $X_{n:n}$ and $Z_{n:n}$ can be rewritten, respectively, as

$$\begin{aligned} r_{X_{n:n}}(x) &= \frac{1}{x} \Upsilon(e^{\log \lambda x}, e^{\log \hat{\lambda} x}; \delta_1, \delta_2) \quad \text{and} \\ r_{Z_{n:n}}(x) &= \frac{1}{x} \Upsilon(e^{\log \mu x}, e^{\log \hat{\mu} x}; \delta_1, \delta_2), \quad x \in \mathbb{R}^+, \end{aligned}$$

where the function Υ is defined in Lemma 4. Clearly, we have $(\log \mu x, \log \hat{\mu} x) \prec_{(\delta_1, \delta_2)} (\log \lambda x, \log \hat{\lambda} x)$ on \mathcal{D}_2 for all $x \in \mathbb{R}^+$, and so the required result follows from Lemma 4;

- (ii) Assume that $(\mu, \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\lambda, \hat{\lambda})$. The ratio of the density functions of $X_{n:n}$ and $Z_{n:n}$ can be written as

$$\frac{f_{X_{n:n}}(x)}{f_{Z_{n:n}}(x)} = s(x) \frac{F_{X_{n:n}}(x)}{F_{Z_{n:n}}(x)}, \quad x \in \mathbb{R}^+,$$

where

$$s(x) = \frac{\frac{\delta_1 e^{-(\lambda x)^\nu}}{\int_0^x z^{\tau-1} e^{-(\lambda z)^\nu} dz} + \frac{\delta_2 e^{-(\hat{\lambda} x)^\nu}}{\int_0^x z^{\tau-1} e^{-(\hat{\lambda} z)^\nu} dz}}{\frac{\delta_1 e^{-(\mu x)^\nu}}{\int_0^x z^{\tau-1} e^{-(\mu z)^\nu} dz} + \frac{\delta_2 e^{-(\hat{\mu} x)^\nu}}{\int_0^x z^{\tau-1} e^{-(\hat{\mu} z)^\nu} dz}}.$$

According to Theorem 4 and Part (i) of Lemma 7, it immediately follows that $F_{X_{n:n}}(x)/F_{Z_{n:n}}(x)$ is increasing in $x \in \mathbb{R}^+$. Further, taking derivative of $s(x)$ with respect to x , we find after some algebraic computations that $s'(x) \geq 0$ for all $x \in \mathbb{R}^+$ iff

$$\psi(\mu x, \hat{\mu} x; \delta_1, \delta_2) \leq \psi(\lambda x, \hat{\lambda} x; \delta_1, \delta_2), \quad \text{for all } x \in \mathbb{R}^+, \quad (3.1)$$

where the function ψ is as defined in Lemma 5. Now, the inequality in (3.1) follows immediately from Lemma 5 which results in $s(x)$ is increasing in $x \in \mathbb{R}^+$;

(iii) The proof is similar to that of Theorem 3.1 in Zhao and Balakrishnan (2015). □

Next Theorem gives some sufficient conditions to compare the lifetimes of Systems A and B in the sense of the hazard rate and likelihood ratio orders.

Theorem 8.

- (i) Assume that $\mu \leq \lambda$ and $\delta_i \geq 1$ for $i = 1, 2$. If $(\log \mu, \log \hat{\mu}) \prec_{(\delta_1, \delta_2)}^w (\log \lambda, \log \hat{\lambda})$ on \mathcal{D}_2 and $(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2)$, then for $0 < \tau \leq \nu$, we have $Y_{n:n} \leq_{hr} X_{n:n}$;
- (ii) Assume that $\mu \leq \lambda$, $(\mu, \hat{\mu}) \prec_{(\delta_1, \delta_2)}^w (\lambda, \hat{\lambda})$ on \mathcal{G}_2 and $(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2)$. Then for $0 < \tau \leq \nu \leq 1$, we have $Y_{n:n} \leq_{lr} X_{n:n}$.

Proof.

- (i) Suppose $(\log \mu, \log \hat{\mu}) \prec_{(\delta_1, \delta_2)}^w (\log \lambda, \log \hat{\lambda})$ on \mathcal{D}_2 and $(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2)$. Because the likelihood ratio order implies the hazard rate order, it follows from Theorem 5 and Part (ii) of Lemma 7 that

$$(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2) \Rightarrow Y_{n:n} \leq_{hr} Z_{n:n}, \quad \text{for } 0 < \tau \leq \nu.$$

The required result now follows if we could show that $Z_{n:n} \leq_{hr} X_{n:n}$. Based on the existing assumptions, we have $\hat{\lambda} \leq \hat{\mu} \leq \mu \leq \lambda$ and $\mu^{\delta_1} \hat{\mu}^{\delta_2} \geq \lambda^{\delta_1} \hat{\lambda}^{\delta_2}$. If $\mu^{\delta_1} \hat{\mu}^{\delta_2} = \lambda^{\delta_1} \hat{\lambda}^{\delta_2}$, we then have from Part (i) of Lemma 8 that $Z_{n:n} \leq_{hr} X_{n:n}$ for $0 < \tau \leq \nu$. Let us now assume that $\mu^{\delta_1} \hat{\mu}^{\delta_2} > \lambda^{\delta_1} \hat{\lambda}^{\delta_2}$. Set

$$\lambda_0 = \left(\frac{\mu}{\lambda}\right)^{\frac{\delta_1}{\delta_2}} \hat{\mu},$$

and suppose W_1, \dots, W_n are independent random variables with $W_i \sim \text{EGG}(\alpha_i, \nu, \tau, \lambda)$ and $W_j \sim \text{EGG}(\alpha_j, \nu, \tau, \lambda_0)$ for $i = 1, \dots, k$ and $j = k + 1, \dots, n$. It is easy to observe that $\hat{\lambda} \leq \lambda_0 \leq \hat{\mu} \leq \lambda$ and $(\log \mu, \log \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\log \lambda, \log \lambda_0)$ on \mathcal{D}_2 . Therefore, we can conclude from Part (i) of Lemma 8 that $Z_{n:n} \leq_{hr} W_{n:n}$. On the other hand, because the likelihood ratio order implies the hazard rate order, it follows from Part (iii) of Lemma 8 that $W_{n:n} \leq_{hr} X_{n:n}$ for $0 < \tau \leq \nu$, thus completing the proof of the theorem;

- (ii) The proof is obtained by virtue of Part (ii) of Lemma 8 and an argument similar to the one used in Part (i). □

In the following, we present some examples to illustrate the result of Theorem 8. At first, set $n = 4, k = 1, \alpha = (1.1, 2, 0.2, 4), \beta = (1.2, 1, 1.5, 3.6), (\lambda, \hat{\lambda}) = (8, 2)$ and $(\mu, \hat{\mu}) = (6, 3)$. We then observe that $(\delta_1, \delta_2) = (1.1, 6.2)$ and $(\xi_1, \xi_2) = (1.2, 6.1)$. It can be easily checked that $(\log 6, \log 3) \prec_{(1.1, 6.2)}^w (\log 8, \log 2)$ on $\mathcal{D}_2, (1.1, 6.2) \prec^{uo} (1.2, 6.1)$ and $\mu \leq \lambda$. Hence, from Part (i) of Theorem 8, it follows that $Y_{4:4} \leq_{hr} X_{4:4}$. The graphs of the hazard rate functions of Systems A and B are plotted in Figure 3. Let us now set $n = 5, k = 3, \alpha = (0.4, 0.4, 1, 5, 5.7), \beta = (3, 0.8, 3.6, 1.2, 3.9), (\lambda, \hat{\lambda}) = (20, 0.1)$ and $(\mu, \hat{\mu}) = (18, 2)$. It can be easily seen that $\delta_1 = 1.8, \delta_2 = 10.7, \xi_1 = 7.4$ and $\xi_2 = 5.1$. Since $(20, 0.1) \prec_{(1.8, 10.7)}^w (18, 2)$ on $\mathcal{G}_2, (1.8, 10.7) \prec^{uo} (7.4, 5.1)$ and $\mu \leq \lambda$, we can conclude from Part (ii) of Theorem 8 that $Y_{5:5} \leq_{lr} X_{5:5}$.

Remark 5. Note that, if $\mu \leq \lambda$, one can easily see that

$$\begin{aligned} & ((\log \mu) \mathbf{1}_k, (\log \hat{\mu}) \mathbf{1}_{n-k}) \prec_{\alpha}^w ((\log \lambda) \mathbf{1}_k, (\log \hat{\lambda}) \mathbf{1}_{n-k}) \\ & \text{on } \mathcal{D}_n \Leftrightarrow (\log \mu, \log \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\log \lambda, \log \hat{\lambda}) \quad \text{on } \mathcal{D}_2 \end{aligned}$$

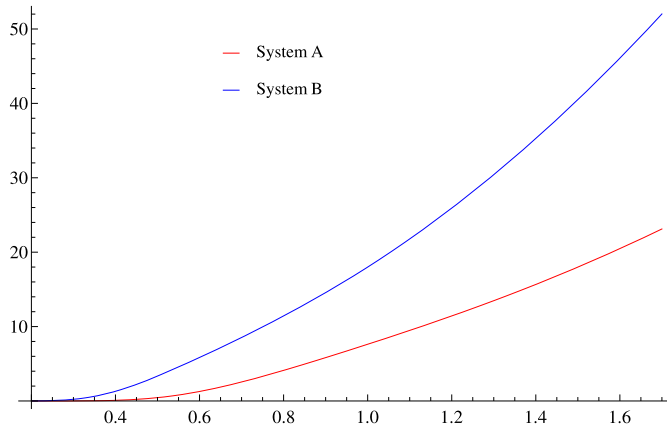


Figure 3 Plots of hazard rate functions of Systems A and B for $n = 4$, $k = 1$, $\alpha = (1.1, 2, 0.2, 4)$, $\beta = (1.2, 1, 1.5, 3.6)$, $(\lambda, \hat{\lambda}) = (8, 2)$ and $(\mu, \hat{\mu}) = (6, 3)$.

and

$$\begin{aligned}
 (\mu \mathbf{1}_k, \hat{\mu} \mathbf{1}_{n-k}) &\stackrel{w}{\prec}_{\alpha} (\lambda \mathbf{1}_k, \hat{\lambda} \mathbf{1}_{n-k}) \\
 \text{on } \mathcal{G}_n &\Leftrightarrow (\mu, \hat{\mu}) \prec_{(\delta_1, \delta_2)} (\lambda, \hat{\lambda}) \quad \text{on } \mathcal{G}_2,
 \end{aligned}$$

where $\mathbf{1}_r$ is a r -dimensional vector with all values being 1. According to these observations, we find that the results in Theorem 8 extend those of Zhao and Balakrishnan (2015) from the gamma framework to the exponentiated generalized gamma framework. Further, it is easy to verify that

$$(\alpha_1, \dots, \alpha_n) \stackrel{uo}{\prec} (\beta_1, \dots, \beta_n) \Rightarrow (\delta_1, \delta_2) \stackrel{uo}{\prec} (\xi_1, \xi_2).$$

Therefore, the result in Theorem 8 remains true under the unordered majorization order between the vectors of exponential parameters.

4 Discussion

In Theorem 7, we have observed that the restriction $0 < \nu \leq 1$ is appeared in the comparison of the lifetimes of Systems A and B, with respect to the reversed hazard rate order. We shall now present a numerical example to show that the mentioned restriction is necessary for the revered hazard order to hold. Set $n = 3$, $\nu = 2$, $\tau = 0.2$, $\alpha = (5, 4, 9)$, $\beta = (6, 3.5, 8.5)$, $\lambda = (10, 1, 1)$ and $\mu = (7, 5, 2)$. The reversed hazard rate functions of Systems A and B are plotted in Figure 4 to see that the lifetimes of Systems A and B can not be compared in the reversed hazard rate order. Although, one can easily check that $\mu \stackrel{w}{\prec}_{\alpha} \lambda$ on \mathcal{G}_3 and $\alpha \stackrel{uo}{\prec} \beta$.

In the following, we find a different condition to compare the lifetimes of the two Systems in the reversed hazard rate order, without any restriction on the shape parameter ν . Let X_1^*, \dots, X_n^* and Y_1^*, \dots, Y_n^* be two sets of independent random variables with $X_i^* \sim \text{EGG}(\alpha_i, 1, \frac{\tau}{\nu}, \lambda_i^{\nu})$ and $Y_i^* \sim \text{EGG}(\beta_i, 1, \frac{\tau}{\nu}, \mu_i^{\nu})$, $i = 1, \dots, n$. Set $\lambda^{\nu} = (\lambda_1^{\nu}, \dots, \lambda_n^{\nu})$ and $\mu^{\nu} = (\mu_1^{\nu}, \dots, \mu_n^{\nu})$. Now, if $\mu^{\nu} \stackrel{w}{\prec}_{\alpha} \lambda^{\nu}$ on \mathcal{G}_n and $\alpha \stackrel{uo}{\prec} \beta$, then from Theorem 7, it readily follows for $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$ that $Y_{n:n}^* \leq_{rh} X_{n:n}^*$. Now, from this observation and Theorem 1.B.43 of Shaked and Shanthikumar (2007, p. 38), we have $Y_{n:n}^{*\frac{1}{\nu}} \leq_{rh} X_{n:n}^{*\frac{1}{\nu}}$. But, it is easy to observe that $Y_{n:n}^{*\frac{1}{\nu}} \stackrel{st}{=} Y_{n:n}^*$ and $X_{n:n}^{*\frac{1}{\nu}} \stackrel{st}{=} X_{n:n}^*$, where the notation $\stackrel{st}{=}$ means equality in distribution. Thus, the following theorem is established.

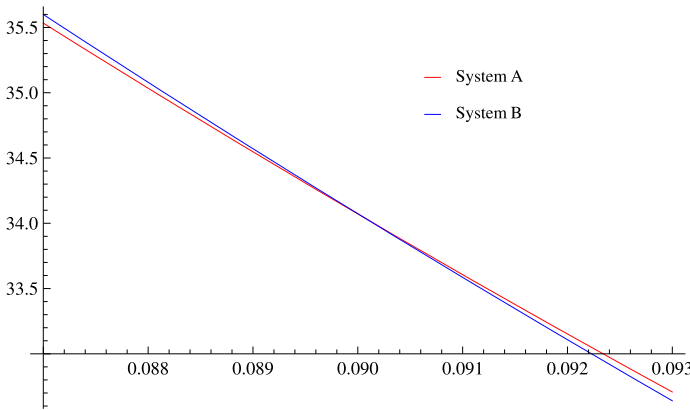


Figure 4 Plots of reversed hazard rate functions of Systems A and B for $n = 3, \nu = 2, \tau = 0.2, \alpha = (5, 4, 9), \beta = (6, 3.5, 8.5), \lambda = (10, 1, 1)$ and $\mu = (7, 5, 2)$.

Theorem 9. If $\mu^\nu \prec_\alpha^w \lambda^\nu$ on \mathcal{G}_n and $\alpha \prec^{uo} \beta$, then for $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}^+$, we have $Y_{n:n} \leq_{rh} X_{n:n}$.

Next, we discuss the connection between the weighted orderings $\mu \prec_\alpha^w \lambda$ and $\mu^\nu \prec_\alpha^w \lambda^\nu$. Set $\phi(x) = x^\nu, x \in \mathbb{R}^+, 0 < \nu \leq 1$. Clearly, $\phi(x)$ is increasing and concave in $x \in \mathbb{R}^+$. Assume that $\mu \prec_\alpha^w \lambda$ on \mathcal{G}_n . Then, we easily have $\mu_r \prec_{\alpha_r}^w \lambda_r$ on \mathcal{G}_{n-r+1} , where $\alpha_r = (\alpha_r, \dots, \alpha_n), \lambda_r = (\lambda_r, \dots, \lambda_n)$ and $\mu_r = (\mu_r, \dots, \mu_n)$ for $r = 1, \dots, n$. From the above observations and Remark 1, it follows for $0 < \nu \leq 1$ that $\sum_{i=r}^n \alpha_i \phi(\lambda_i) \leq \sum_{i=r}^n \alpha_i \phi(\mu_i)$ or equivalently $\sum_{i=r}^n \alpha_i \lambda_i^\nu \leq \sum_{i=r}^n \alpha_i \mu_i^\nu, r = 1, \dots, n$. Therefore, we have the following implication:

$$\mu \prec_\alpha^w \lambda \quad \text{on } \mathcal{G}_n \quad \Rightarrow \quad \mu^\nu \prec_\alpha^w \lambda^\nu \quad \text{on } \mathcal{G}_n, \quad \text{for } 0 < \nu \leq 1. \quad (4.1)$$

Consequently, based on (4.1), it readily follows that Theorem 9 contains less restricted condition than Theorem 7 and allows us to compare more parallel systems, with respect to the reversed hazard rate order for the case when $0 < \nu < 1$. We shall now illustrate this finding by a numerical example. Set $n = 4, \nu = 0.5, \tau = 0.8, \alpha = (4, 0.8, 3.3, 5), \beta = (1, 3, 2.1, 7), \lambda = (\sqrt{2}, \sqrt{11}, \sqrt{12}, \sqrt{13})$ and $\mu = (\sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{14})$. It is therefore easy to check that $\lambda \not\prec_\alpha^w \mu$, however, one can observe that $\mu^\nu \prec_\alpha^w \lambda^\nu$ on \mathcal{G}_4 . Thus, since $\alpha \prec^{uo} \beta$, we find from Theorem 9 that $X_{4:4} \leq_{rh} X_{4:4}$, while this ordering result can not be concluded from Theorem 7.

Now, for $k \in \{1, \dots, n - 1\}$, set

$$\begin{aligned} \lambda_1 = \dots = \lambda_{k+1} = \lambda, & \quad \lambda_{k+1} = \dots = \lambda_n = \hat{\lambda}, \\ \mu_1 = \dots = \mu_k = \mu, & \quad \mu_{k+1} = \dots = \mu_n = \hat{\mu} \end{aligned}$$

Based on Part (ii) of Theorem 8, we have $Y_{n:n}^* \leq_{lr} X_{n:n}^*$ when $(\mu^\nu, \hat{\mu}^\nu) \prec_{(\delta_1, \delta_2)}^w (\lambda^\nu, \hat{\lambda}^\nu)$ on \mathcal{G}_2 and $(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2)$ for $0 < \tau \leq \nu$. Using this observation and Theorem 1.C.8 of Shaked and Shanthikumar (2007, p. 46), it follows that $Y_{n:n}^{*\frac{1}{\nu}} \leq_{lr} X_{n:n}^{*\frac{1}{\nu}}$ or equivalently $Y_{n:n} \leq_{lr} X_{n:n}$. This result is stated in the following theorem.

Theorem 10. Assuming $\mu \leq \lambda$, if $(\mu^\nu, \hat{\mu}^\nu) \prec_{(\delta_1, \delta_2)}^w (\lambda^\nu, \hat{\lambda}^\nu)$ on \mathcal{G}_2 and $(\delta_1, \delta_2) \prec^{uo} (\xi_1, \xi_2)$, then for $0 < \tau \leq \nu$, we have $Y_{n:n} \leq_{lr} X_{n:n}$.

According to (4.1), for $0 < \nu \leq 1$, we have

$$\begin{aligned}
 (\mu, \hat{\mu}) \stackrel{w}{\prec}_{(\delta_1, \delta_2)} (\lambda, \hat{\lambda}) \\
 \text{on } \mathcal{G}_2 \quad \Rightarrow \quad (\mu^\nu, \hat{\mu}^\nu) \stackrel{w}{\prec}_{(\delta_1, \delta_2)} (\lambda^\nu, \hat{\lambda}^\nu) \quad \text{on } \mathcal{G}_2,
 \end{aligned}
 \tag{4.2}$$

which results in, for $0 < \tau \leq \nu \leq 1$, Theorem 10 contains less restricted condition than Part (ii) of Theorem 8 for the likelihood ratio order between the lifetimes of Systems A and B to hold.

5 Conclusions

Let X_1, \dots, X_n and Y_1, \dots, Y_n , representing the component lifetimes of two parallel systems, be two sets of independent random variables such that

$$X_i \sim \text{EGG}(\alpha_i, \nu, \tau, \lambda_i), \quad Y_i \sim \text{EGG}(\beta_i, \nu, \tau, \mu_i), \quad i = 1, \dots, n.$$

We have established here that

$$\begin{aligned}
 \alpha \stackrel{uo}{\prec} \beta \quad \text{and} \quad \log \mu \stackrel{w}{\prec}_\alpha \log \lambda \\
 \text{on } \mathcal{D}_n \quad \Rightarrow \quad Y_{n:n} \leq_{st} X_{n:n}, \quad \nu \in \mathbb{R}^+, \tau \in \mathbb{R}^+,
 \end{aligned}
 \tag{5.1}$$

$$\begin{aligned}
 \alpha \stackrel{uo}{\prec} \beta \quad \text{and} \quad \mu^\nu \stackrel{w}{\prec}_\alpha \lambda^\nu \\
 \text{on } \mathcal{G}_n \quad \Rightarrow \quad Y_{n:n} \leq_{rh} X_{n:n}, \quad \nu \in \mathbb{R}^+, \tau \in \mathbb{R}^+.
 \end{aligned}
 \tag{5.2}$$

By means of two counterexamples, we have showed that the results in (5.1) and (5.2) can not reinforce, respectively, to the hazard rate and likelihood ratio orders. So, to achieve the hazard rate and likelihood ratio orders, we have reduced the heterogeneity of the scale parameters λ_i 's and μ_i 's. For $k \in \{1, \dots, n - 1\}$, taking

$$\lambda_1 = \dots = \lambda_k, \quad \lambda_{k+1} = \dots = \lambda_n, \quad \mu_1 = \dots = \mu_k, \quad \mu_{k+1} = \dots = \mu_n$$

and

$$\delta_1 = \sum_{i=1}^k \alpha_i, \quad \delta_2 = \sum_{i=k+1}^n \alpha_i, \quad \xi_1 = \sum_{i=1}^k \beta_i, \quad \xi_2 = \sum_{i=k+1}^n \beta_i,$$

we have obtained the following result for the hazard rate order to hold:

$$\begin{aligned}
 (\delta_1, \delta_2) \stackrel{uo}{\prec} (\xi_1, \xi_2) \quad \text{and} \quad (\log \mu_1, \log \mu_n) \stackrel{w}{\prec}_{(\delta_1, \delta_2)} (\log \lambda_1, \log \lambda_n) \\
 \text{on } \mathcal{D}_2 \quad \Rightarrow \quad Y_{n:n} \leq_{hr} X_{n:n}, \quad 0 < \tau \leq \nu,
 \end{aligned}
 \tag{5.3}$$

wherein $\mu \leq \lambda$ and $\delta_i \geq 1$. Also, it is proved for $\mu \leq \lambda$ that

$$\begin{aligned}
 (\delta_1, \delta_2) \stackrel{uo}{\prec} (\xi_1, \xi_2) \quad \text{and} \quad (\mu_1^\nu, \mu_n^\nu) \stackrel{w}{\prec}_{(\delta_1, \delta_2)} (\lambda_1, \lambda_n) \\
 \text{on } \mathcal{G}_2 \quad \Rightarrow \quad Y_{n:n} \leq_{lr} X_{n:n}, \quad 0 < \tau \leq \nu.
 \end{aligned}
 \tag{5.4}$$

The ordering results in (5.1)–(5.4) complete and strengthen some of the known results in the literature.

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