Brazilian Journal of Probability and Statistics
2019, Vol. 33, No. 3, 480–497
https://doi.org/10.1214/18-BJPS398

Fractional backward stochastic variational inequalities with non-Lipschitz coefficient

Katarzyna Jańczak-Borkowska
University of Technology and Life Sciences

Abstract. We prove the existence and uniqueness of the solution of backward stochastic variational inequalities with respect to fractional Brownian motion and with non-Lipschitz coefficient. We assume that $H > 1/2$.

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs for short) were first introduced by Pardoux and Peng (1990). They assumed that its generator—function f is a Lipschitz continuous function on space variables. Since it was found that BSDEs play an important role in many fields such as financial mathematics, stochastic games, optimal control and signal processing, many papers were devoted to their study (see example Hamadéne and Lepeltier (1995), El Karoui, Peng and Quenez (1997), Borkowski (2010)). Later, the theory of BSDEs has been extended on equations with stochastic integral with respect to fractional Brownian motion, called fractional BSDEs.

Let us now recall that a centered fractional Brownian motion (fBm for short) with Hurst parameter $H \in (0, 1)$ is a process $B^H = \{B_t^H, t \geq 0\}$ that satisfies

$$E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

The property of its self-similarity (i.e., B_{at}^H has the same law as $a^H B_t^H$ for any $a > 0$), makes this process a useful tool in models related to network traffic analysis, mathematical finance, physics, signal processing and many other fields. Note that for $H = 1/2$ we obtain a standard Wiener process, but for $H \neq 1/2$, the process B^H is not a semimartingale. Therefore, we cannot use the classical theory of stochastic calculus to define the fractional stochastic integral. Nevertheless an efficient stochastic calculus of B^H has been developed. To our best knowledge, Dai and Heyde and Lin were the first authors who defined the integral of Stratonovich type with respect to fBm (see Dai and Heyde (1996), Lin (1995)), but it did not satisfy the natural property $E \int_o^t f_s \, dB_s^H = 0$. Next, a new type of stochastic integral with respect to fBm was defined to satisfy the mentioned property. The definition

Key words and phrases. Backward stochastic differential equation, fractional Brownian motion, backward stochastic variational inequalities, subdifferential operator.
Received September 2017; accepted March 2018.

introduced in Decreusefond and Üstünel (1999) is the divergence operator (Skorokhod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus and the equivalent for $H > 1/2$ definition introduced in Duncan, Hu and Pasik-Duncan (2000) is based on the Wick product as the limit of Riemann sums.

The existence and uniqueness of nonlinear BSDEs and backward stochastic variational inequalities (BSVI for short) with respect to fBm, $H > 1/2$ was shown in Maticiuc and Nie (2015). They assumed in both cases the Lipschitz condition on the generator f on space variables. Also the Lipschitz function as a generator was considered in the generalized BSDEs and the generalized BSVI with respect to fBm in Jańczak-Borkowska (2013) and Borkowski and Jańczak-Borkowska (2016), respectively. By generalized, we mean the equation with additional component being an integral with respect to some increasing process. The authors of Wang and Huang (2009) omitted the Lipschitz condition on variable y in a generator f and assumed that

$$\left|f(t,x,y,z)-f\left(t,x,y',z\right)\right|^2 \le \rho\left(t,\left|y-y'\right|^2\right), \tag{1.1}$$

where ρ is a continuous, concave and nondecreasing function satisfying some technical conditions (see assumption (H_3)) and proved the existence and uniqueness of the solution of BSDE with respect to Wiener process. In Aïdara and Sow (2016), the non-Lipschitz assumption (1.1) was considered to show the existence and uniqueness of the solution to fractional generalized BSDE. In this paper, we treat a particular type of nonlinear drivers as described in (1.1) and with this assumption we prove the existence and the uniqueness of the solution to fractional BSVI.

The paper is organized as follows. In Section 2, we recall some definitions and result about a fractional stochastic integral which will be needed throughout the paper. In Section 3, we formulate the definition of fractional BSVI and introduce assumptions on generators. Section 4 contains some a priori estimates and finally Section 5 is devoted to the proof of the main theorem of the paper based on approximation of the solution to the fractional BSVI by Picard method.

2 Fractional calculus

Denote $\phi(x) = H(2H-1)|x|^{2H-2}$, $x \in \mathbb{R}$ and by $|\mathcal{H}|$ denote the Banach space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T \phi(u-v)\left|f(u)\right|\left|f(v)\right| du\, dv < \infty.$$

For $\xi, \eta \in |\mathcal{H}|$ we put

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)\, du\, dv$$

and $\|\xi\|_t^2 = \langle \xi, \xi \rangle_t$. Note that, for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the space of step function in $|\mathcal{H}|$ under this scalar product. The elements of $\mathcal{H}$ may be distributions (see Pipiras and Taqqu (2000)). Moreover, it is known that $\mathbb{L}^2([0, T]) \subset \mathbb{L}^{1/H}(0, T) \subset |\mathcal{H}| \subset \mathcal{H}$ (see, e.g., Nualart (2006)).

Let $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$ and by $\mathcal{P}_T$ denote the set of elementary random variables of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)\, dB_t^H, \dots, \int_0^T \xi_k(t)\, dB_t^H\right),$$

where f is a polynomial function of k variables. The Malliavin derivative operator D^H of an element $F \in \mathcal{P}_T$ is defined as follows:

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)\, dB_t^H, \dots, \int_0^T \xi_k(t)\, dB_t^H\right)\xi_i(s), \qquad s \in [0, T].$$

The divergence operator $D^H = (D_s^H)_{s\in[0,T]}$ is closable from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{F}, P; \mathcal{H})$. By $\mathbb{D}_{1,2}$ denote the Banach space being a completion of $\mathcal{P}_T$ with the following norm: $\|F\|_{1,2}^2 = E|F|^2 + E\|D^H F\|_T^2$. Now we introduce another derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-s) D_s^H F\, ds.$$

For a deeper discussion about the stochastic integral, we refer the reader to Duncan, Hu and Pasik-Duncan (2000), Hu (2005) and Nualart (2006). Here we will formulate only some theorems needed throughout the paper.

Theorem 2.1. *We denote by* $\mathbb{L}_H^{1,2}$ *the space of all stochastic processes* $F : (\Omega, \mathcal{F}, \mathcal{P}) \to \mathcal{H}$ *such that*

$$E\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2\, ds\, dt\right) < \infty.$$

If $F \in \mathbb{L}_H^{1,2}$, *then the stochastic integral denoted by* $\int_0^T F_s\, dB_s^H$ *exists in* $L^2(\Omega, \mathcal{F})$. *Moreover,* $E(\int_0^T F_s\, dB_s^H) = 0$ *and*

$$E\left(\int_0^T F_s\, dB_s^H\right)^2 = E\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s\, ds\, dt\right).$$

The above theorem can be found in Duncan, Hu and Pasik-Duncan (2000), Theorem 3.7 for an integral $\int_0^T F_s\, dB_s^H$ defined by the limit of Riemann sums involving Wick products, and in Hu (2005), Proposition 6.25 for an integral defined by Definition 6.11 in Hu (2005). Note that these two integrals coincide (see Hu (2005), Proposition 6.12).

Theorem 2.2 (Theorem 4.5 in Duncan, Hu and Pasik-Duncan (2000)). *Let $F \in \mathbb{L}_H^{1,2}$. Assume that there is $\alpha > 1 - H$ such that $E|F_u - F_v|^2 \le C|u - v|^{2\alpha}$, where $|u - v| \le \delta$ for some $\delta > 0$ and*

$$\lim_{0 \le u, v \le t, |u-v| \to 0} E\big|\mathbb{D}_u^H (F_u - F_v)\big|^2 = 0.$$

Set

$$X_t = X_0 + \int_0^t G_s\, ds + \int_0^t F_s\, dB_s^H, \qquad t \in [0, T],$$

where X_0 is a constant and $E\int_0^T |G_s|\, ds < \infty$. Let f be continuously differentiable with respect to t and twice continuously differentiable with respect to x and that these derivatives are bounded. Moreover, assume that $E\int_0^T |\mathbb{D}_s^H X_s F_s|\, ds < \infty$ and $(\partial f/\partial x(s, X_s) F_s)_{s \in [0,T]} \in \mathbb{L}_H^{1,2}$. Then

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) G_s\, ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, X_s) F_s\, dB_s^H + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) \mathbb{D}_s^H X_s F_s\, ds, \qquad t \in [0, T]. \end{aligned}$$

Theorem 2.3 (Theorem 10.3 in Hu (2005)). *Let $T \in (0, \infty)$ and let $f_1(s)$, $f_2(s)$, $g_1(s)$, $g_2(s)$ be in $\mathbb{D}_{1,2}$ and $E(\int_0^T (|f_i(s)| + |g_i(s)|)\, ds) < \infty$. Assume that $\mathbb{D}_t^H f_2(s)$ and $\mathbb{D}_t^H g_2(s)$ are continuously differentiable with respect to $(s, t) \in [0, T] \times [0, T]$ for almost all $\omega \in \Omega$. Suppose that*

$$E \int_0^T \int_0^T \big|\mathbb{D}_t^H f_2(s)\big|^2\, ds\, dt < \infty, \qquad E \int_0^T \int_0^T \big|\mathbb{D}_t^H g_2(s)\big|^2\, ds\, dt < \infty.$$

Denote

$$F(t) = \int_0^t f_1(s)\, ds + \int_0^t f_2(s)\, dB_s^H, \qquad t \in [0, T]$$

and

$$G(t) = \int_0^t g_1(s)\, ds + \int_0^t g_2(s)\, dB_s^H, \qquad t \in [0, T].$$

Then

$$\begin{aligned} F(t)G(t) = &\int_0^t F(s) g_1(s)\, ds + \int_0^t F(s) g_2(s)\, dB_s^H \\ &+ \int_0^t G(s) f_1(s)\, ds + \int_0^t G(s) f_2(s)\, dB_s^H \\ &+ \int_0^t \mathbb{D}_s^H F(s) g_2(s)\, ds + \int_0^t \mathbb{D}_s^H G(s) f_2(s)\, ds. \end{aligned}$$

3 BSVI with respect to fBm

Assume that

(H_1) $\sigma : [0,T] \to \mathbb{R}$ is a deterministic continuous differentiable function such that $\sigma(t) \neq 0$, for all $t \in [0,T]$ and $\eta_t = \eta_0 + \int_0^t \sigma(s)\,dB_s^H$, $t \in [0,T]$, where η_0 is a given constant.

Note that, since $\|\sigma\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma(u)\sigma(v)\,du\,dv$, we have

$$\frac{d}{dt}\big(\|\sigma\|_t^2\big) = 2H(2H-1)\int_0^t |t-u|^{2H-2}\sigma(u)\sigma(t)\,du = 2\sigma(t)\hat{\sigma}(t) > 0,$$

where $\hat{\sigma}(t) = \int_0^t \phi(t-u)\sigma(u)\,du$.

We will consider the following fractional backward stochastic variational inequality with non-Lipschitz coefficient:

$$\begin{cases} dY_t + f(t,\eta_t,Y_t,Z_t)\,dt - Z_t\,dB_t^H \in \partial\varphi(Y_t)\,dt, \\ Y_T = \xi. \end{cases} \tag{3.1}$$

We suppose that

(H_2) $\xi = h(\eta_T)$ for some function h with bounded derivative and such that $E|\xi|^2 < \infty$.

(H_3) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists positive constant $L > 1$ and for all $t \in [0,T]$, $x, x', y, y', z, z' \in \mathbb{R}$,

$$\big|f(t,x,y,z) - f\big(t,x',y',z'\big)\big|^2 \le \rho\big(t,|y-y'|^2\big) + L^2\big(|x-x'|^2 + |z-z'|^2\big)$$

and

$$\int_0^T \big|f(t,0,0,0)\big|^2\,dt < \infty,$$

where $\rho : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(a) for fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing function such that $\rho(t,0) = 0$

(b) the ordinary differential equation

$$v'(t) = -\rho\big(t,v(t)\big), \qquad v(T) = 0 \tag{3.2}$$

has a unique solution $v(t) = 0$, $t \in [0,T]$

(c) there exists two continuous functions $a, b : [0,T] \to \mathbb{R}^+$ such that

$$\rho(t,v) \le a(t) + b(t)\cdot v, \qquad \int_0^T \big(a(t)+b(t)\big)\,dt < \infty.$$

(H_4) $\varphi : \mathbb{R} \to (-\infty,\infty]$ is a proper, convex and lower semi-continuous function and satisfies $\varphi(y) \ge \varphi(0) = 0$.

We will denote

$$\partial\varphi(y) = \{\hat{y} \in \mathbb{R};\, \hat{y}\cdot(v-y) + \varphi(y) \le \varphi(v),\, \forall v \in \mathbb{R}\},$$

$$\mathrm{Dom}\,\varphi = \{y \in \mathbb{R};\, \varphi(y) < \infty\}, \qquad \mathrm{Dom}(\partial\varphi) = \{y \in \mathbb{R};\, \partial\varphi(y) \ne \varnothing\},$$

$$\langle y, \hat{y}\rangle \in \partial\varphi \quad \Leftrightarrow \quad y \in \mathrm{Dom}(\partial\varphi), \qquad \hat{y} \in \partial\varphi(y).$$

Remark 3.1. $\partial\varphi$ is maximal in this sense that

$$(\hat{y}-\hat{u})(y-u) \ge 0, \qquad (y,\hat{y}), (u,\hat{u}) \in \partial\varphi.$$

Let us mention here, that Mao (1995) considered a generator f satisfying the following non-Lipschitz assumption: there exists $L > 0$ and for all $t \in [0,T]$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times m}$,

$$|f(t,y,z) - f(t,y',z')|^2 \le \rho(|y-y'|^2) + L|z-z'|^2,$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave and nondecreasing function such that $\rho(0) = 0$, $\rho(u) > 0$, $u > 0$ and $\int_{0^+} \frac{du}{\rho(u)} = \infty$. With the help of Bihari's inequality (see Bihari (1956)) he proved the existence and uniqueness of the solution to BSDE (with respect to a Wiener process) under this assumption. Moreover, he introduced some examples of non-Lipschitz functions satisfying the above condition and noted that his condition includes also Lipschitz continuity. Wang and Wang (2003) showed the existence and uniqueness of the solution to BSDE under the assumption (H_3) with additional condition on ρ—continuity in both variables t and u. With the help of Bihari's inequality their also proved that their result includes that of Mao. Obviously our assumption, (H_3) includes both results and we have the following.

Remark 3.2.

1. If $f(t,x,y,z) = y\cdot t^{-1/4} + L(x+z)$, then assumption (H_3) is satisfied with $\rho(t,u) = 2u\cdot t^{-1/2}$.
2. If $f(t,x,y,z) = t^{-1/2}\kappa(|y|) + L(x+z)$, then assumption (H_3) is satisfied with $\rho(t,u) = \frac{2\kappa(u)}{t}$ where

$$\kappa(u) = \begin{cases} -u\ln u, & 0 \le u \le \delta, \\ -\delta\ln\delta + \kappa'(\delta-)(u-\delta), & u > \delta, \end{cases}$$

for some $\delta \in (0,1)$ small enough.

Now consider the set

$$\mathcal{V}_{[0,T]} = \left\{Y = \psi(\cdot,\eta) : \psi \in C^{1,2}_{\mathrm{pol}}([0,T]\times\mathbb{R}) \text{ and } \frac{\partial\psi}{\partial t} \text{ is bounded}\right\}.$$

By $\tilde{\mathcal{V}}^H_{[0,T]}$ denote the completion of the set of processes from $\mathcal{V}_{[0,T]}$ with the following norm

$$\|Y\|^2_H = E\int_0^T t^{2H-1}|Y_t|^2\,dt = E\int_0^T t^{2H-1}|\psi(t,\eta_t)|^2\,dt.$$

Definition 3.3. A solution to a fractional backward stochastic variational inequality (3.1) associated with data (ξ, f) is a triple of processes $(Y, Z, U) = (Y_t, Z_t, U_t)_{t\in[0,T]}$ satisfying

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s^H - \int_t^T U_s\, ds, \qquad t \in [0, T] \quad (3.3)$$

and such that

$$(Y_t, U_t) \in \partial\varphi, \qquad t \in [0, T] \quad \textit{and} \quad Y \in \tilde{\mathcal{V}}^{1/2}_{[0,T]}, \qquad Z, U \in \tilde{\mathcal{V}}^{H}_{[0,T]}.$$

4 A priori estimates

Proposition 4.1. *Let* (Y_t, Z_t, U_t) *be a solution of* (3.3). *Then*

(i) $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t, 0 \le t \le T$.
(ii) *There exists a constant* $M > 0$ *such that*

$$\frac{t^{2H-1}}{M} \le \frac{\hat{\sigma}(t)}{\sigma(t)} \le M t^{2H-1}, \qquad 0 \le t \le T.$$

Proof. (i) Since $Y \in \tilde{\mathcal{V}}^H_{[0,T]}$, $Y = \psi(\cdot, \eta)$ where $\psi \in C^{1,2}_{\text{pol}}([0, T] \times \mathbb{R})$. Applying Itô's formula and putting $\psi(T, \eta_T) = \xi$ we have

$$\begin{aligned}\psi(t, \eta_t) - \xi &= -\int_t^T \psi'_s(s, \eta_s)\, ds - \int_t^T \psi'_x(s, \eta_s)\sigma(s)\, dB_s^H \\ &\quad - \frac{1}{2}\int_t^T \psi''_{xx}(s, \eta_s)\sigma(s)\hat{\sigma}(s)\, ds \\ &= -\int_t^T \Big(\psi'_s(s, \eta_s) + \frac{1}{2}\psi''_{xx}(s, \eta_s)\sigma(s)\hat{\sigma}(s)\Big)\, ds \\ &\quad - \int_t^T \psi'_x(s, \eta_s)\sigma(s)\, dB_s^H.\end{aligned}$$

Comparing the above equation with (3.3), we deduce that $Z_t = \psi'_x(t, \eta_t)\sigma(t)$ and therefore

$$\begin{aligned}\mathbb{D}_t^H Y_t &= \int_0^T \phi(t - v) D_v^H \psi(t, \eta_t)\, dv = \psi'_x(t, \eta_t) \int_0^T \phi(t - v)\sigma(v)\, dv \\ &= \hat{\sigma}(t)\psi'_x(t, \eta_t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t.\end{aligned}$$

(ii) The proof is analogous to the proof of Proposition 25 in Maticiuc and Nie (2015). □

Proposition 4.2. *Let* (Y_t, Z_t, U_t) *be a solution of* (3.3). *Then there exists a constant* C *depending on* L, H, T, M *such that*

$$E|Y_t|^2 + E\int_t^T s^{2H-1}|Z_s|^2\,ds \le CE\Big(|\xi|^2 + \int_t^T \big(|\eta_s|^2 + \big|f(s,0,0,0)\big|^2\big)\,ds + \int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big).$$

Proof. By the Itô formula,

$$|Y_t|^2 = |\xi|^2 + 2\int_t^T Y_s f(s,\eta_s,Y_s,Z_s)\,ds - 2\int_t^T Y_s Z_s\,dB_s^H - 2\int_t^T Y_s U_s\,ds - 2\int_t^T \mathbb{D}_s^H Y_s Z_s\,ds.$$

Since $(Y,U)\in\partial\varphi$, $Y_tU_t \ge 0$ and by Proposition 4.1(i), we can write

$$|Y_t|^2 + 2\int_t^T \frac{\hat\sigma(s)}{\sigma(s)}|Z_s|^2\,ds \le |\xi|^2 + 2\int_t^T Y_s f(s,\eta_s,Y_s,Z_s)\,ds - 2\int_t^T Y_s Z_s\,dB_s^H. \tag{4.1}$$

By assumption (H_3) and using the inequality $2ab \le a^2/\varepsilon + \varepsilon b^2$ we have

$$\begin{aligned}
2yf(t,\eta,y,z) &\le 2|y|\sqrt{\rho\big(t,|y|^2\big) + L^2|\eta|^2 + L^2|z|^2} + 2|y|\big|f(t,0,0,0)\big| \\
&\le 2|y|\sqrt{\rho\big(t,|y|^2\big)} + 2L|y||\eta| + 2L|y||z| + 2|y|\big|f(t,0,0,0)\big| \\
&\le L^2|y|^2 + \frac{1}{L^2}\rho\big(t,|y|^2\big) + L^2|y|^2 + |\eta|^2 \\
&\quad + \frac{L^2M}{s^{2H-1}}|y|^2 + \frac{s^{2H-1}}{M}|z|^2 + |y|^2 + \big|f(t,0,0,0)\big|^2.
\end{aligned}$$

Therefore, using also Proposition 4.1(ii), from (4.1), we have

$$\begin{aligned}
E|Y_t|^2 + \frac{1}{M}E\int_t^T s^{2H-1}|Z_s|^2\,ds &\le E|\xi|^2 + E\int_t^T \Big(2L^2 + \frac{L^2M}{s^{2H-1}} + 1\Big)|Y_s|^2\,ds \\
&\quad + E\int_t^T \Big(\frac{1}{L^2}\rho\big(s,|Y_s|^2\big) + |\eta_s|^2 + \big|f(s,0,0,0)\big|^2\Big)\,ds.
\end{aligned}$$

Denote

$$\mu_t = L^2 E\Big(|\xi|^2 + \int_t^T (|\eta_s|^2 + |f(s,0,0,0)|^2)\,ds\Big),$$

then by the above

$$E|Y_t|^2 + \frac{1}{M} E\int_t^T s^{2H-1}|Z_s|^2\,ds \le \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) + E\int_t^T \Big(2L^2 + \frac{L^2 M}{s^{2H-1}} + 1\Big)|Y_s|^2\,ds. \tag{4.2}$$

Now by Gronwall's lemma, we get

$$\begin{aligned} E|Y_t|^2 &\le \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) \times \exp\Big(\int_t^T \Big(2L^2 + \frac{L^2 M}{s^{2H-1}} + 1\Big)\,ds\Big) \\ &= \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) \\ &\quad \times \exp\Big(\big(2L^2+1\big)(T-t) + \frac{L^2 M}{2-2H}\big(T^{2-2H} - t^{2-2H}\big)\Big). \end{aligned}$$

Choose α such that $(T^{2-2H} - t^{2-2H}) \le \alpha^{2-2H}(T-t)$ and let β satisfy:

$$\big(2L^2+1\big)(T-t) + \frac{L^2 M}{2-2H}\alpha^{2-2H}(T-t) = \beta(T-t).$$

Then by the above, we have

$$E|Y_t|^2 \le \frac{1}{L^2} e^{\beta(T-t)}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big). \tag{4.3}$$

Moreover putting (4.3) to (4.2), we get

$$\frac{1}{M} E\int_t^T s^{2H-1}|Z_s|^2\,ds \tag{4.4}$$

$$\begin{aligned} &\le \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) \\ &\quad + E\int_t^T \Big(2L^2 + \frac{L^2 M}{s^{2H-1}} + 1\Big)\frac{1}{L^2} e^{\beta(T-s)}\Big(\mu_s + E\int_s^T \rho\big(u,|Y_u|^2\big)\,du\Big)\,ds \\ &\le \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) \\ &\quad + \frac{1}{L^2} e^{\beta(T-t)}\Big(\mu_t + E\int_t^T \rho\big(u,|Y_u|^2\big)\,du\Big)\int_t^T \Big(2L^2 + \frac{L^2 M}{s^{2H-1}} + 1\Big)\,ds \\ &\le \frac{1}{L^2}\Big(\mu_t + E\int_t^T \rho\big(s,|Y_s|^2\big)\,ds\Big) \end{aligned} \tag{4.5}$$

$$+\frac{1}{L^2}e^{\beta(T-t)}\beta(T-t)\Big(\mu_t+E\int_t^T\rho\big(u,|Y_u|^2\big)\,du\Big)$$
$$=\frac{1}{L^2}\big(1+e^{\beta(T-t)}\beta(T-t)\big)\Big(\mu_t+E\int_t^T\rho\big(s,|Y_s|^2\big)\,ds\Big).$$

Now (4.5) together with (4.3) completes the proof. □

5 Picard method

We will consider Picard method for (3.3). Without loss of generality assume that $Y^0=0$ and define a sequence (Y^p,Z^p,U^p), $p\in\mathbb{N}$ as

$$\begin{aligned}Y_t^p=\xi+\int_t^T f\big(s,\eta_s,Y_s^{p-1},Z_s^p\big)\,ds\\ -\int_t^T Z_s^p\,dB_s^H-\int_t^T U_s^p\,ds,\qquad t\in[0,T].\end{aligned}\tag{5.1}$$

It is known that there exists a unique solution for (5.1). Indeed, since f is Lipschitz with respect to η and z^p and constant with respect to y^p it follows by Theorem 3.3 in Borkowski and Jańczak-Borkowska (2016) with $\Lambda=0$. Moreover, $(Y^p,U^p)\in\partial\varphi$.

Proposition 5.1. *There exists $\Gamma>0$ and $0\le T_1<T$ not depending on ξ such that for $p\ge 1$,*

$$E\big|Y_t^p\big|^2\le\Gamma,\qquad T_1\le t\le T.$$

Proof. Arguing similarly as in the proof of Proposition 4.2 we have

$$\begin{aligned}E\big|Y_t^p\big|^2+\frac{1}{M}E\int_t^T s^{2H-1}\big|Z_s^p\big|^2\,ds\le\frac{1}{L^2}\Big(\mu_t+E\int_t^T\rho\big(s,\big|Y_s^{p-1}\big|^2\big)\,ds\Big)\\ +E\int_t^T\Big(2L^2+\frac{L^2M}{s^{2H-1}}+1\Big)\big|Y_s^p\big|^2\,ds,\end{aligned}$$

where μ_t is as in the proof of Proposition 4.2. By Gronwall's lemma, we get

$$\begin{aligned}E\big|Y_t^p\big|^2&\le\frac{1}{L^2}\Big(\mu_t+E\int_t^T\rho\big(s,\big|Y_s^{p-1}\big|^2\big)\,ds\Big)\\ &\quad\times\exp\Big((2L^2+1)(T-t)+\frac{L^2M}{2-2H}\big(T^{2-2H}-t^{2-2H}\big)\Big)\\ &\le\frac{1}{L^2}\exp\big(\beta(T-t)\big)\Big(\mu_t+E\int_t^T\rho\big(s,\big|Y_s^{p-1}\big|^2\big)\,ds\Big),\end{aligned}\tag{5.2}$$

where β is the same as in the proof of Proposition 4.2. Put $\overline{T_1} = \max\{T - \frac{\ln L^2}{\beta}, 0\}$. Then for $t \in [\overline{T_1}, T]$ we have $\frac{1}{L^2}\exp(\beta(T-t)) \le 1$. Indeed, if $\beta T < \ln L^2$ then $\overline{T_1} = 0$ and

$$\frac{1}{L^2}\exp\bigl(\beta(T-t)\bigr) \le \frac{1}{L^2}\exp(\beta T) < \frac{1}{L^2}\exp\bigl(\ln L^2\bigr) = 1$$

and if $\beta T > \ln L^2$ then $\overline{T_1} = T - \beta^{-1}\ln L^2$ and for $t \in [\overline{T_1}, T]$,

$$\frac{1}{L^2}\exp\bigl(\beta(T-t)\bigr) \le \frac{1}{L^2}\exp\bigl(\beta(T-\overline{T_1})\bigr) = \frac{1}{L^2}\exp\bigl(\ln L^2\bigr) = 1.$$

Therefore by (5.2) and from the concavity of ρ for $t \in [\overline{T_1}, T]$, we can write

$$E|Y_t^p|^2 \le \mu_t + \int_t^T \rho\bigl(s, E|Y_s^{p-1}|^2\bigr)\,ds, \qquad t \in [\overline{T_1}, T]. \tag{5.3}$$

Let

$$\begin{aligned}\frac{1}{2}\Gamma &= \mu_0 + \int_0^T a(s)\,ds \\ &= L^2 E\biggl(|\xi|^2 + \int_0^T \bigl(|\eta_s|^2 + \bigl|f(s,0,0,0)\bigr|\bigr)\,ds\biggr) + \int_0^T a(s)\,ds < \infty.\end{aligned}$$

Choose $\hat{T}_1$ such that

$$\mu_0 + \int_t^T \rho(s,\Gamma)\,ds \le \Gamma, \qquad t \in [\hat{T}_1, T].$$

One can do that since by assumption $(H_3)(c)$,

$$\begin{aligned}\mu_0 + \int_t^T \rho(s,\Gamma)\,ds &\le \mu_0 + \int_t^T \bigl(a(s) + b(s)\Gamma\bigr)\,ds \\ &\le \frac{1}{2}\Gamma + \Gamma\int_t^T b(s)\,ds < \infty.\end{aligned}$$

And it is enough to take $\hat{T}_1$ satisfying $\int_{\hat{T}_1}^T b(s)\,ds = 1/2$.

Now let $T_1 = \max\{\overline{T_1}, \hat{T}_1\}$. Then for $t \in [T_1, T]$ the equation (5.3) is satisfied and in particular,

$$\begin{aligned}E|Y_t^1|^2 &\le \mu_t + \int_t^T \rho\bigl(s, E|Y_s^0|^2\bigr)\,ds \le \mu_0 + \int_t^T \rho(s,\Gamma)\,ds \le \Gamma, \\ E|Y_t^2|^2 &\le \mu_t + \int_t^T \rho\bigl(s, E|Y_s^1|^2\bigr)\,ds \le \mu_0 + \int_t^T \rho(s,\Gamma)\,ds \le \Gamma, \\ E|Y_t^3|^2 &\le \mu_t + \int_t^T \rho\bigl(s, E|Y_s^2|^2\bigr)\,ds \le \mu_0 + \int_t^T \rho(s,\Gamma)\,ds \le \Gamma.\end{aligned}$$

Therefore by induction,

$$E|Y_t^p|^2 \le \mu_t + \int_t^T \rho\big(s, E|Y_s^{p-1}|^2\big)\,ds \le \mu_0 + \int_t^T \rho(s,\Gamma)\,ds \le \Gamma, \qquad t \in [T_1, T]$$

which finishes the proof. □

Proposition 5.2. *Let* (Y^p, Z^p, U^p) *satisfies* (5.1). *Then*

$$\text{(i)}\qquad E|Y_t^p - Y_t^q|^2 + E\int_t^T s^{2H-1}|Z_s^p - Z_s^q|^2\,ds \le \frac{C}{L^2}e^{\beta(T-t)}\int_t^T \rho\big(s, E|Y_s^{p-1} - Y_s^{q-1}|^2\big)\,ds,$$

$$\text{(ii)}\qquad E\int_t^T s^{2H-1}|U_s^p|^2\,ds \le CE\Big(T^{2H-1}\varphi(\xi) + \int_t^T s^{2H-1}\big(\rho\big(s, |Y_s^{p-1}|^2\big) + |Z_s^p|^2 + |\eta_s|^2 + |f(s,0,0,0)|^2\big)\,ds\Big).$$

Proof. (i) Using theorems 2.2 and 2.3, we have

$$\begin{aligned}|Y_t^p - Y_t^q|^2 = 2\int_t^T &(Y_s^p - Y_s^q)\big(f\big(s,\eta_s,Y_s^{p-1},Z_s^p\big) - f\big(s,\eta_s,Y_s^{q-1},Z_s^q\big)\big)\,ds\\ &- 2\int_t^T (Y_s^p - Y_s^q)(Z_s^p - Z_s^q)\,dB_s^H\\ &- 2\int_t^T (Y_s^p - Y_s^q)(U_s^p - U_s^q)\,ds\\ &- 2\int_t^T \mathbb{D}_s^H(Y_s^p - Y_s^q)(Z_s^p - Z_s^q)\,ds.\end{aligned} \tag{5.4}$$

Analogously as in the proof of Proposition 4.1 one can show

$$\mathbb{D}_t^H(Y_t^p - Y_t^q) = \frac{\hat\sigma(t)}{\sigma(t)}(Z_t^p - Z_t^q), \qquad 0 \le t \le T.$$

Moreover,

$$\begin{aligned}2(y^p - y^q)&\big(f\big(s,\eta,y^{p-1},z^p\big) - f\big(s,\eta,y^{q-1},z^q\big)\big)\\ &\le 2|y^p - y^q|\big(\sqrt{\rho\big(s,|y^{p-1}-y^{q-1}|^2\big)} + L|z^p - z^q|\big)\\ &\le L^2|y^p - y^q|^2 + \frac{1}{L^2}\rho\big(s,|y^{p-1}-y^{q-1}|^2\big)\\ &\quad + \frac{L^2M}{s^{2H-1}}|y^p - y^q|^2 + \frac{s^{2H-1}}{M}|z^p - z^q|^2.\end{aligned}$$

Therefore integrating (5.4)

$$\begin{aligned} E|Y_t^p - Y_t^q|^2 &+ \frac{1}{M} E \int_t^T s^{2H-1} |Z_s^p - Z_s^q|^2 \, ds \\ &+ 2E \int_t^T s^{2H-1} (Y_s^p - Y_s^q)(U_s^p - U_s^q) \, ds \\ &\le \frac{1}{L^2} E \int_t^T \rho(s, |Y_s^{p-1} - Y_s^{q-1}|^2) \, ds \\ &+ E \int_t^T \left(L^2 + \frac{L^2 M}{s^{2H-1}} \right) |Y_s^p - Y_s^q|^2 \, ds. \end{aligned} \tag{5.5}$$

Note that since $(Y^p, U^p) \in \partial\varphi$ and $(Y^q, U^q) \in \partial\varphi$, we have $E \int_t^T (Y_s^p - Y_s^q)(U_s^p - U_s^q) \, ds \ge 0$ and using Gronwall's lemma

$$\begin{aligned} E|Y_t^p - Y_t^q|^2 &\le \frac{1}{L^2} \exp\left\{ L^2(T-t) + \frac{L^2 M}{2-2H} (T^{2-2H} - t^{2-2H}) \right\} \\ &\quad \times E \int_t^T \rho(s, |Y_s^{p-1} - Y_s^{q-1}|^2) \, ds \\ &\le \frac{1}{L^2} e^{\beta(T-t)} \int_t^T \rho(s, E|Y_s^{p-1} - Y_s^{q-1}|^2) \, ds, \end{aligned} \tag{5.6}$$

where β is as before. Putting (5.6) into (5.5) we get the result.

(ii) Since $(Y_t^p, U_t^p) \in \partial\varphi$ for any $t \in [0, T]$ by definition of $\partial\varphi(y)$, we have

$$U_r^p (Y_s^p - Y_r^p) + \varphi(Y_r^p) \le \varphi(Y_s^p).$$

Assume that $T \ge s > r \ge T_1$ and multiply the above inequality by s^{2H-1}.

$$\begin{aligned} s^{2H-1} \varphi(Y_s^p) &\ge s^{2H-1} \varphi(Y_r^p) + s^{2H-1} U_r^p (Y_s^p - Y_r^p) \\ &\ge r^{2H-1} \varphi(Y_r^p) + s^{2H-1} U_r^p (Y_s^p - Y_r^p). \end{aligned}$$

Take $s = t_{i+1} \vee T$, $r = t_i \vee T$, where $T_1 = t_0 < t_1 < t_2 < \cdots < t_n = T$ and $t_{i+1} - t_i = 1/n$. Summing up over i and passing to the limit as $n \to \infty$, we deduce

$$T^{2H-1} \varphi(Y_T^p) \ge t^{2H-1} \varphi(Y_t^p) + \int_t^T s^{2H-1} U_s^p \, dY_s^p, \qquad t \in [T_1, T]$$

and

$$\begin{aligned} t^{2H-1} \varphi(Y_t^p) &\le T^{2H-1} \varphi(Y_T^p) - \int_t^T s^{2H-1} U_s^p \, dY_s^p \\ &= T^{2H-1} \varphi(\xi) + \int_t^T s^{2H-1} U_s^p f(s, \eta_s, Y_s^{p-1}, Z_s^p) \, ds \\ &\quad - \int_t^T s^{2H-1} U_s^p Z_s^p \, dB_s^H - \int_t^T s^{2H-1} U_s^p U_s^p \, ds. \end{aligned} \tag{5.7}$$

Since

$$u \cdot f(s,\eta,y,z) \leq |u|\Big(\sqrt{\rho\big(s,|y|^2\big)+L^2|z|^2+L^2|\eta|^2}+\big|f(s,0,0,0)\big|\Big)$$
$$\leq \frac{1}{2}|u|^2+\rho\big(s,|y|^2\big)+L^2|z|^2+L^2|\eta|^2+\big|f(s,0,0,0)\big|^2$$

from (5.7) we get

$$Et^{2H-1}\varphi(Y_t^p)+E\int_t^T s^{2H-1}|U_s^p|^2\,ds$$
$$\leq ET^{2H-1}\varphi(\xi)+\frac{1}{2}E\int_t^T s^{2H-1}|U_s^p|^2\,ds$$
$$+E\int_t^T s^{2H-1}\big(\rho\big(s,|Y_s^{p-1}|^2\big)+L^2|Z_s^p|^2+L^2|\eta_s|^2+\big|f(s,0,0,0)\big|^2\big)\,ds$$

and therefore

$$E\int_t^T s^{2H-1}|U_s^p|^2\,ds \leq 2ET^{2H-1}\varphi(\xi)+2E\int_t^T s^{2H-1}\big(\rho\big(s,|Y_s^{p-1}|^2\big)$$
$$+L^2|Z_s^p|^2+L^2|\eta_s|^2+\big|f(s,0,0,0)\big|^2\big)\,ds,$$

which implies the result. □

Theorem 5.3. *There exists a unique solution of* (3.3).

Proof. We repeat here the arguments from the proof of Theorem 3.9 in Aïdara and Sow (2016) to show that (Y^p, Z^p) is a Cauchy sequence in the Banach space $\tilde{\mathcal{V}}^{1/2}_{[T_1,T]} \times \tilde{\mathcal{V}}^{H}_{[T_1,T]}$. Indeed, define the sequence $\{\tau_n(t)\}_{n\in\mathbb{N}}$ as follows:

$$\tau_0(t)=\int_t^T \rho(s,\Gamma)\,ds, \qquad \tau_{n+1}(t)=\int_t^T \rho\big(s,\tau_n(s)\big)\,ds.$$

For $t \in [T_1, T]$ we have

$$\tau_0(t)=\int_t^T \rho(s,\Gamma)\,ds \leq \Gamma,$$
$$\tau_1(t)=\int_t^T \rho\big(s,\tau_0(s)\big)\,ds \leq \int_t^T \rho(s,\Gamma)\,ds=\tau_0(t),$$
$$\tau_2(t)=\int_t^T \rho\big(s,\tau_1(s)\big)\,ds \leq \int_t^T \rho\big(s,\tau_0(s)\big)\,ds=\tau_1(t)$$

and by induction, for all $n \in \mathbb{N}$, $\tau_n(t)$ satisfies

$$0 \leq \tau_{n+1}(t) \leq \tau_n(t) \leq \cdots \leq \tau_1(t) \leq \tau_0(t) \leq \Gamma.$$

Therefore the sequence $\tau_n(t)$ is uniformly bounded. Moreover, for all $n \in \mathbb{N}$ and $t_1, t_2 \in [T_1, T]$, $t_1 < t_2$,

$$\left|\tau_n(t_1) - \tau_n(t_2)\right| = \left|\int_{t_1}^{t_2} \rho\big(s, \tau_{n-1}(s)\big)\, ds\right| \le \left|\int_{t_1}^{t_2} \rho(s, \Gamma)\, ds\right| < \infty$$

and we deduce that $\sup_n |\tau_n(t_1) - \tau_n(t_2)| \to 0$ as $t_1 - t_2 \to 0$. That means that the sequence $\tau_n(t)$ is equicontinuous family of functions and by the Arzelà–Ascoli theorem we can choose a subsequence of $\tau_n(t)$ which is convergent. Its limit denote by $\tau(t)$ and it satisfies

$$\tau(t) = \int_t^T \rho\big(s, \tau(s)\big)\, ds.$$

By assumption (H_3) (3.2), we have $\tau(t) = 0$ for $t \in [0, T]$.

Now, by (5.6) in the proof of Proposition 5.2 and by Proposition 5.1 for $t \in [T_1, T]$,

$$\begin{aligned}
E\big|Y_t^{p+1} - Y_t^1\big|^2 &\le \frac{1}{L^2} e^{\beta(T-t)} \int_t^T \rho\big(s, E\big|Y_s^p - Y_s^0\big|^2\big)\, ds \\
&\le \int_t^T \rho(s, \Gamma)\, ds = \tau_0(t), \\
E\big|Y_t^{p+2} - Y_t^2\big|^2 &\le \frac{1}{L^2} e^{\beta(T-t)} \int_t^T \rho\big(s, E\big|Y_s^{p+1} - Y_s^1\big|^2\big)\, ds \\
&\le \int_t^T \rho\big(s, \tau_0(s)\big)\, ds = \tau_1(t), \\
E\big|Y_t^{p+3} - Y_t^3\big|^2 &\le \frac{1}{L^2} e^{\beta(T-t)} \int_t^T \rho\big(s, E\big|Y_s^{p+2} - Y_s^2\big|^2\big)\, ds \\
&\le \int_t^T \rho\big(s, \tau_1(s)\big)\, ds = \tau_2(t)
\end{aligned}$$

and by induction,

$$\begin{aligned}
E\big|Y_t^{p+m} - Y_t^m\big|^2 &\le \frac{1}{L^2} e^{\beta(T-t)} \int_t^T \rho\big(s, E\big|Y_s^{p+m-1} - Y_s^{m-1}\big|^2\big)\, ds \\
&\le \int_t^T \rho\big(s, \tau_{m-2}(s)\big)\, ds = \tau_{m-1}(t).
\end{aligned}$$

In particular,

$$\sup_{T_1 \le t \le T} E\big|Y_t^{p+m} - Y_t^m\big|^2 \le \sup_{T_1 \le t \le T} \tau_{m-1}(t) = \tau_{m-1}(T_1) \to 0 \qquad \text{as } m \to \infty,$$

which together with Proposition 5.2(i) implies that (Y^p, Z^p) is a Cauchy sequence. and therefore there exists a pair of processes $(Y, Z) \in \tilde{\mathcal{V}}^{1/2}_{[T_1,T]} \times \tilde{\mathcal{V}}^{H}_{[T_1,T]}$ being a limit of (Y^p, Z^p), that is

$$\lim_{p\to\infty} E\Big(\int_{T_1}^T |Y_t^p - Y_t|^2\,dt + \int_{T_1}^T t^{2H-1}|Z_t^p - Z_t|^2\,dt\Big) = 0.$$

Now, note that from Proposition 5.2(ii) it follows that there exist a subsequence $p_n \to \infty$ and process U such that $U^{p_n} \to U$ weakly in $\tilde{\mathcal{V}}_{[0,T]}^H$ and from the Fatou lemma

$$E\int_0^T s^{2H-1}|U_s|^2\,ds \le C.$$

Since for any $t \in [T_1, T]$,

$$\begin{aligned}
&\lim_{p\to\infty}\Big(-Y_t^p + \xi + \int_t^T f\big(s, \eta_s, Y_s^{p-1}, Z_s^p\big)\,ds - \int_t^T U_s^p\,ds\Big)\\
&\quad = -Y_t + \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)\,ds - \int_t^T U_s\,ds\\
&\quad = \Phi(t) \qquad \text{in } \mathbb{L}^2(\Omega, \mathcal{F}, P)
\end{aligned}$$

and $Z^p\mathbf{1}_{[T_1,T]}$ converges to $Z\mathbf{1}_{[T_1,T]}$ in $\mathbb{L}^2(\Omega, \mathcal{F}, P, \mathcal{H})$, we have

$$\lim_{p\to\infty}\int_{T_1}^T Z_s^p\,dB_s^H = \int_{T_1}^T Z_s\,dB_s^H \qquad \text{in } \mathbb{L}^2(\Omega, \mathcal{F}, P).$$

Moreover since $U_t^p \in \partial\varphi(Y_t^p)$, for all $u \in \tilde{\mathcal{V}}_{[0,T]}^H$ we have

$$U_t^p \cdot \big(u_t - Y_t^p\big) + \varphi\big(Y_t^p\big) \le \varphi(u_t).$$

Therefore, we can deduce that

$$U_t \cdot (u_t - Y_t) + \varphi(Y_t) \le \varphi(u_t),$$

which means that $(Y_t, U_t) \in \partial\varphi$, $t \in [T_1, T]$.

Uniqueness. Assume that (Y, Z, U) and (Y', Z', U') are two solutions of (3.3). Then computing similarly as in the proof of Propositions 4.2 and 5.2 for $t \in [T_1, T]$ we have

$$E\big|Y_t - Y_t'\big|^2 \le \int_t^T \rho\big(s, E\big|Y_s - Y_s'\big|^2\big)\,ds.$$

From the comparison theorem of ODE, we know that $E|Y_t - Y_t'|^2 \le r(t)$, where $r(t)$ is the maximum left shift solution of

$$v'(t) = -\rho(t, v), \qquad v(T) = 0.$$

From (3.2), it follows that $r(t) = 0$ for $t \in [T_1, T]$. Hence, $E|Y_t - Y_t'|^2 = 0$. Moreover, since

$$E\int_t^T s^{2H-1}\big|Z_s - Z_s'\big|^2\,ds \le \frac{C}{L^2}e^{\beta(T-t)}\int_t^T \rho\big(s, E\big|Y_s - Y_s'\big|^2\big)\,ds,$$

then also $Z_t = Z_t'$ and in a consequence also $U_t = U_t'$ in $t \in [T_1, T]$.

Note that T_1 does not depend on ξ. Hence, one can deduce by iteration the existence and uniqueness on $[T_2, T_1]$ replacing T by T_1 and T_1 by T_2 and therefore the existence and uniqueness on the whole interval $[0, T]$. That completes the proof. □

References

Aïdara, S. and Sow, A. B. (2016). Generalized fractional BSDE with non Lipschitz coefficients. *Afrika Mathematica* **27**, 443–455. MR3500464

Bihari, I. (1956). A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations. *Acta Mathematica Academiae Scientiarum Hungaricae* **7**, 71–94. MR0079154

Borkowski, D. (2010). Smoothing, enhancing filters in terms of backward stochastic differential equations. In *Computer Vision and Graphics*. *Lect. Notes Comput. Sci.* **6374**, 233–240.

Borkowski, D. and Jańczak-Borkowska, K. (2016). Generalized backward stochastic variational inequalities driven by a fractional Brownian motion. *Brazilian Journal of Probability and Statistics* **30**, 502–519. MR3531696

Dai, W. and Heyde, C. C. (1996). Itô formula with respect to fractional Brownian motion and its application. *Journal of Applied Mathematics and Stochastic Analysis* **9**, 439–448. MR1429266

Decreusefond, L. and Üstünel, A. S. (1999). Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10**, 177–214. MR1677455

Duncan, T. E., Hu, Y. and Pasik-Duncan, B. (2000). Stochastic calculus for fractional Brownian motions. I. Theory. *SIAM Journal on Control and Optimization* **38**, 582–612. MR1741154

El Karoui, N., Peng, S. and Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Mathematical Finance* **7**, 1–71. MR1434407

Hamadéne, S. and Lepeltier, J. P. (1995). Zero-sum stochastic differential games and backward equations. *Systems & Control Letters* **24**, 259–263. MR1321134

Hu, Y. (2005). Integral transformations and anticipative calculus for fractional Brownian motions. *Memoirs of the American Mathematical Society* **175**(825). MR2130224

Jańczak-Borkowska, K. (2013). Generalized BSDEs driven by fractional Brownian motion. *Statistics & Probability Letters* **83**, 805–811. MR3040307

Lin, S. J. (1995). Stochastic analysis of fractional Brownian motions. *Stochastics and Stochastics Reports* **55**, 121–140. MR1382288

Mao, X. (1995). Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients. *Stochastic Processes and Their Applications* **58**, 281–292. MR1348379

Maticiuc, L. and Nie, T. (2015). Fractional backward stochastic differential equations and fractional backward variational inequalities. *Journal of Theoretical Probability* **28**, 337–395. MR3320972

Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. Berlin: Springer. MR2200233

Pardoux, É. and Peng, S. (1990). Adapted solutions of a backward stochastic differential equation. *Systems & Control Letters* **14**, 55–61. MR1037747

Pipiras, V. and Taqqu, M. (2000). Integration questions related to fractional Brownian motion. *Probability Theory and Related Fields* **118**, 251–291. MR1790083

Wang, Y. and Huang, Z. (2009). Backward stochastic differential equations with non-Lipschitz coefficients. *Statistics & Probability Letters* **79**, 1438–1443. MR2536503

Wang, Y. and Wang, Z. (2003). Adapted solutions of backward SDE with non-Lipschitz coefficients. *Chinese Journal of Applied Probability and Statistics* **19**, 245–251 (in Chinese). MR2080860

Institute of Mathematics and Physics
University of Technology
and Life Sciences
Bydgoszcz 85-796
Poland
E-mail: kaja@utp.edu.pl