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CRITICAL RADIUS AND SUPREMUM OF RANDOM SPHERICAL HARMONICS

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We first consider *deterministic* immersions of the d-dimensional sphere into high dimensional Euclidean spaces, where the immersion is via spherical harmonics of level n. The main result of the article is the, a priori unexpected, fact that there is a uniform lower bound to the critical radius of the immersions as $n \to \infty$. This fact has immediate implications for *random* spherical harmonics with fixed L^2-norm. In particular, it leads to an exact and explicit formulae for the tail probability of their (large deviation) suprema by the tube formula, and also relates this to the expected Euler characteristic of their upper level sets.

1. Introduction. The spherical harmonics, of level $n \geq 1$, on the d-dimensional unit sphere S^d, are the collection of the

$$k_n^d = \frac{2n+d-1}{n+d-1}\binom{n+d-1}{d-1} \tag{1.1}$$

eigenfunctions $\{\phi_j^{n,d}\}_{j=1}^{k_n^d}$ of the Laplacian Δ_d on S^d, satisfying,

$$\Delta_d \phi_j^{n,d}(x) = -n(n+d-1)\phi_j^{n,d}(x). \tag{1.2}$$

It is then immediate that for any vector $a = (a_1, \dots, a_{k_n^d})$ of reals, the functions

$$\Phi_n^d \triangleq \sum_{j=1}^{k_n^d} a_j \phi_j^{n,d} \tag{1.3}$$

solve the wave equation

$$\Delta_d \Phi_n^d = \alpha \Phi_n^d, \tag{1.4}$$

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where $\alpha = -n(n+d-1)$. Thus, with some ambiguity, both the Φ_n^d and their linear combinations are often also referred to as spherical harmonics, or wave functions on the sphere.

Instead of taking the a_j in (1.3) constant, they could also be taken to be random. Two classical choices are either to take the vector a to be uniformly distributed on $S^{k_n^d-1}$, or to take the a_j as independent, standard Gaussians. In the former case, we refer to random spherical harmonics under the spherical ensemble, while in the latter we refer to the Gaussian ensemble. The two are clearly related, due to the fact that, if the a_j are Gaussian, then normalizing $a \to a/\|a\|$ gives a uniform variable on $S^{k_n^d-1}$. Thus, the spherical harmonics under the spherical ensemble are a conditioned version of those under a Gaussian ensemble, with a corresponding statement going in the opposite direction.

The relationship between the two ensembles has been a recurrent theme in the general theory of Gaussian processes with a finite Karhunen–Loéve expansion, that is, processes which have a finite expansion similar to (1.3), although both the ϕ_j and the space over which they, and the process, are defined might be quite general (e.g., [2, 17, 20]). We shall give some more details below, but for the moment note that proofs based on this relationship typically only work when the expansion is finite. If the processes in question have an infinite expansion, then approximating them with a finite expansion and taking a passage to a limit has, to the best of our knowledge, only worked in situations in which the limit process is very smooth, typically at least C^2.

Smoothness of random spherical harmonics as $n \to \infty$ is most definitely not one of their properties, since the $n = \infty$ limit is not only not C^2, but rather is a generalized function (cf. [8]). Consequently, one would not expect the passage to the limit mentioned in the previous paragraph to be at all relevant for them. The result of this paper is that this is not exactly the case, and, with the right normalizations, connections between the spherical ensemble and integral geometry which hold for the finite n case still make sense as $n \to \infty$. In particular, we shall obtain explicit formulae for the tail probability of the supremum of random spherical harmonics above high levels, and for the expected Euler characteristic of the excursion sets (cf. (1.11)). The derivations will rely on the result about a certain immersion of S^d into the sphere $S^{k_n^d-1}$, which has its independent interest, and is really the main result of the paper. Thus we describe it first, then describe its implications for random spherical harmonics, and then close the Introduction with a roadmap to the remainder of the paper.

1.1. *Spherical harmonics and the immersion.* The main result of the paper is actually a deterministic one, and rather simple to state.

Consider the map $i_n^d : S^d \to \mathbb{R}^{k_n^d}$, defined by

$$i_n^d(x) = \sqrt{\frac{s_d}{k_n^d}}\,(\phi_1^{n,d}, \dots, \phi_{k_n^d}^{n,d}), \tag{1.5}$$

where

$$s_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)},$$

is the Euclidean surface area of S^d.

It is an easy calculation, that we shall carry out in Section 5, that $\|i_n^d(x)\| = 1$ for all $x \in S^d$, so that i_n^d is actually a mapping of spheres into spheres, namely

$$i_n^d : S^d \to S^{k_n^d - 1}. \tag{1.6}$$

As proved in [15, 24], this map is actually an immersion for sufficiently large n. Indeed, if n is odd it is an embedding, while if n is even then $i_n^d(S^d) \cong \mathbb{R}P^d$, the real projective space of dimension d. Furthermore, the pullback of the Euclidean metric to S^d has the leading order expansion

$$(i_n^d)^*(g_E) \cong c_d n^2 g_{S^d}, \tag{1.7}$$

where c_d is a constant depending on d and g_{S^d} is the standard round metric on S^d.

Hence, roughly speaking, a geodesic of unit length on the unit sphere S^d will be stretched by a factor of order n under the map i_n^d, and so it is reasonable to expects that its image, as with that of the entire sphere, becomes highly "twisted" as n grows. An informative measure of twistedness is provided by the notion of *reach* or *critical radius*, which we shall define and describe in Section 3, and which is a measure of both the local and global smoothness of a set. In general, the smaller the reach of a set, the less well behaved it is. In view of the last three sentences, the following result, which shows that there exists a uniform lower bound for the critical radii of the immersions as $n \to \infty$, is, a priori, somewhat surprising.

THEOREM 1.1. *For sufficiently large n, the reach of the immersion $i_n(S^d)$ in $\mathbb{R}^{k_n^d}$ has a strictly positive, uniform in n, lower bound, which depends only on d.*

An explicit lower bound for the two-dimensional case is given in (4.10), and for the general case in (5.5). From Theorem 1.1, it follows that there is a lower bound for the critical radius of $i_n^d(S^d)$ also when it is considered as a subset of $S^{k_n^d-1}$. We use ρ_d denote this new lower bound throughout the article.

Note that a model closely related to ours, but in the setting of complex geometry, was studied earlier in [18]. There the map (1.5) was replaced by the classical Kodaira embedding, defined via a basis of holomorphic sections of positive holomorphic line bundles over complex manifolds, and a result similar to our Theorem 1.1 was established. In particular, Sun derived a formula for the critical radius of the Kodaira embeddings in complex projective spaces in terms of the Bergman kernels. This allowed the exploitation of the fact that these kernels decay exponentially to carry out the detailed calculations behind his corresponding result. Our argument to prove Theorem 1.1 is similar to this, with the critical radius being expressed in terms of the spectral projection kernels of the sphere, which are of polynomial decay.

We now turn to the implications of the deterministic Theorem 1.1 in a random setting.

1.2. *Random spherical harmonics*. As we have already mentioned, the deterministic results of the previous subsection have implications for random spherical harmonics, under both the spherical and Gaussian ensembles. Both of these are objects of active research, much of the motivation coming from Berry's conjectures in the 1970s (e.g., [5]) linking them to the eigenstates of semiclassical, quantum, Hamiltonian systems, but more recently motivated by intrinsic mathematical interest. Thus, for example, there is a large and growing mathematical literature on the nodal domains of these systems (e.g., [14, 16]), although its roots too are in the quantum mechanical applications. There is also a rich literature on exceedence probabilities (e.g., [9, 12, 13]), which, while part of the general exceedence theory for Gaussian random fields (for which [2] will be our basic reference) is actually motivated by the statistical analysis of the cosmic microwave background radiation data.

Throughout this paper, we shall concentrate primarily on the spherical rather than the Gaussian ensemble. The reason is three-fold. First, the calculations on reach in Sections 4 and 5 are independent of the ensemble. Second, when applying these results one typically first treats the spherical ensemble, and then moves to the Gaussian ensemble via the conditioning argument described above. This is standard, and so we shall not treat it further. Finally, under the spherical ensemble, random spherical harmonics also have a property that makes them of intrinsic mathematical interest. It follows from the properties of (deterministic) spherical harmonics that, in the spherical case,

$$\|\Phi_n^d\|_{L^2} = \int_{S^d} |\Phi_n^d(x)|^2 \, dV_{g_{S^d}} = 1, \tag{1.8}$$

where we write $V_{g_{S^d}}$ for volume measure with respect to g_{S^d}. Put more simply, $V_{g_{S^d}}$ measures surface area on S^d, so that, for example, $s_d = V_{g_{S^d}}(S^d)$.

From this it follows, if we now write $\mathcal{H}_n^d$ to denote the nth eigenspace of Δ_d generated by the solutions of the wave equation (1.2), and $S\mathcal{H}_n^d$ to denote L^2-sphere in this space, that Φ_n^d, under the spherical ensemble, is a random element of $S\mathcal{H}_n^d$. Thus it provides a mathematical model for studying this space.

Two results that at first seem somewhat at odds with (1.8) are due to Burq and Lebeau [6]. To state them we need some notation. In particular, we shall denote probabilities and expectations under the spherical ensemble by $\mathbb{P}_{\mu_n^d}$ and $\mathbb{E}_{\mu_n^d}$, respectively. Then Burq and Lebeau showed that, for $u \geq 1$, and all $\alpha < s_d$,

$$\mathbb{P}_{\mu_n^d}\Big\{\sup_{S^d}|\Phi_n^d(x)| > u\Big\} \leq C n^{-d(1+d/2)} e^{-\alpha u^2}. \tag{1.9}$$

The result (1.9) is typical of what we referred to above as an exceedence probability. The second, related, result established the logarithmic growth of the expectation of suprema; namely, for some $0 < c < C < \infty$,

$$c\sqrt{\log n} \le \mathbb{E}_{\mu_n^d}\Big\{\sup_{S^d}\big|\Phi_n^d(x)\big|\Big\} \le C\sqrt{\log n}. \tag{1.10}$$

Combining (1.8)–(1.10), we obtain a picture of sample paths for Φ_n^d which, while almost surely L^2-integrable, have local behavior which grows increasingly erratic as $n \to \infty$, with the the supremum having an exponential concentration of measure around $\sqrt{\log n}$.

There are also analogues of (1.9) and (1.10) under the Gaussian ensemble. The close connection between the above results for the two ensembles is not coincidental but rather is related to the fact that the spherical ensemble is a conditional version of the Gaussian ensemble as we mentioned above.

However, it turns out that, despite the irregular behavior of random spherical harmonics for large n, the uniform lower bound that Theorem 1.1 provides for the critical radii of the immersions i_n^d actually allows one to exploit this general approach to prove a number of interesting results.

1.3. *Consequences for random spherical harmonics.* We need some notation. For $u > 0$, denote the excursion sets of Φ_n^d by

$$A_n^d(u) = \{x \in S^d : \Phi_n^d(x) > u\}. \tag{1.11}$$

THEOREM 1.2. *Let Φ_n^d be spherical harmonics under the spherical ensemble. Then there exist constants $\rho_d > 0$ such that, for sufficiently large n, and for all $u > \sqrt{k_n^d/s_d}\cos\rho_d$,*

$$\mathbb{P}_{\mu_n^d}\Big\{\sup_{S^d}\Phi_n^d(x) > u\Big\} = \kappa\mathbb{E}_{\mu_n^d}\{\chi(A_n^d(u))\}, \tag{1.12}$$

where $\kappa = 1/2$ if n is even and 1 if n is odd, and $\chi(A)$ denotes the Euler characteristic of the set A.

The factor of κ here is due to the fact that $\Phi_n^d(S^d)$ is isomorphic to S^d for n odd, and is isomorphic to $\mathbb{R}P^d$ for n even. This affects the corresponding tube formulae, which are the key to the probability calculation leading to (1.12), but not the Euler characteristic.

Note that (1.12) is an exact result (for quantifiably large u) and not an asymptotic equivalence as is more common, for example, in the Gaussian literature.

Precise expressions for the probability and expectation in Theorem 1.2 are basically already available in the literature, and lead to the following set of results, in which $P_{n,d}$ denotes the nth Legendre polynomial of order d.

PROPOSITION 1.3. *Under the conditions of Theorem 1.2,*

$$\begin{aligned}
&\mathbb{P}_{\mu_n^d}\Big\{\sup_{x\in S^d}\Phi_n^d(x)>u\Big\} \\
(1.13)\qquad &= \frac{\kappa}{s_{k_n^d-1}}\sum_{j=0}^{d} f_{k_n^d,j}\Big(\cos^{-1}\Big(u/\sqrt{k_n^d/s_d}\Big)\Big)\big[P'_{n,d}(1)\big]^{j/2}\mathcal{L}_j(S^d),
\end{aligned}$$

where $\mathcal{L}_j(S^d)$ are the jth Lipschitz–Killing curvatures of the unit sphere S^d, given explicitly by (6.10), and the $f_{k_n^d,j}$ are functions defined by (6.6) below.

As a direct corollary of Theorem 1.1 and Proposition 1.3, we have the following result for S^2:

COROLLARY 1.4. *For $u>\sqrt{(2n+1)/4\pi}\cos(\rho_2)$,*

$$\begin{aligned}
&\mathbb{P}_{\mu_n^2}\Big\{\sup_{S^2}\Phi_n^2(x)>u\Big\} \\
&\quad = \frac{\kappa\Gamma(n+\frac{1}{2})}{\pi^{1/2}\Gamma(n-1)}\int_0^{\cos^{-1}(u/\sqrt{(2n+1)/4\pi})}\sin^{2n-3}(r) \\
&\qquad \times\Big\{2(n^2+n)\Big(1-\frac{2n-1}{2n-2}\sin^2(r)\Big)+\frac{2\sin^2(r)}{n-1}\Big\}\,dr.
\end{aligned}$$

The simple structure of the two-dimensional result in Corollary 1.4 makes it easy to understand the large deviation nature of the result. In particular, since $\Phi_n^2(x)=\sum a_j\phi_j^{n,2}(x)$, it follows that

$$|\Phi_n^2(x)|^2\le\Bigg(\sum_{j=1}^{k_n^2}a_j^2\Bigg)\Bigg(\sum_{j=1}^{k_n^2}|\phi^{n,2}(x)|^2\Bigg)=\sum_{j=1}^{k_n^2}|\phi^{n,2}(x)|^2=\frac{2n+1}{4\pi},$$

the last equality coming from (2.1) and (2.4) below. Thus Corollary 1.4 relates only to the range $u\in[\sqrt{(2n+1)/4\pi}\cos(\rho_2),\sqrt{(2n+1)/4\pi}]$, which makes it a large deviation result. As opposed to most large deviation results, however, this one is quite unique in the fact that the exceedence probability is precise, and not just an approximation.

Obviously, a similar comment holds for Theorem 1.2 and Proposition 1.3 for general d and n.

1.4. *A roadmap.* We now turn to proving these results. In the following section, we collect some results on spherical harmonics, and in Section 3 we do the same for critical radii. Section 4 then proves Theorem 1.1 for the case $d=2$, while Section 5 treats the case of general d. Section 6 proves the remaining results, and in the final Section 7 we collect some comments relating our results to others in the literature and mention some interesting open problems.

2. Spherical harmonics on S^2. In this section, we shall collect a number of results specific to spherical harmonics on S^2, which we shall use in our proof of Theorem 1.1. Similar results hold in higher dimensions, but for the moment, we stay in dimension 2. We then look at immersions.

Since, for this and most of the following two sections, we shall be dealing with the case of S^2, we shall drop the the superscript 2 whenever it does not lead to ambiguities. Thus, i_n^2 becomes i_n, Φ_n^2 becomes Φ_n, $P_{n,2}$ becomes P_n, and so forth.

2.1. *Some basic facts.* Consider the unit sphere S^2 equipped with the round metric g_{S^2} and with associated Laplacian Δ. The spherical harmonics ϕ_j^n are then the eigenfunctions of

$$\Delta\phi_j^n(x) = -n(n+1)\phi_j^n(x).$$

We normalize the eigenfunctions so that the L^2 norm of ϕ_j^n is 1, and denote by $\mathcal{H}_n$ their span. The dimension of $\mathcal{H}_n$ is $2n+1$. Since the Laplacian is invariant under rotation, $\mathcal{H}_n$ is invariant under the action $\phi(x) \to \phi(Qx)$ for $Q \in SO(3)$. Moreover, if $\{\phi_j^n(x)\}$ is an orthonormal basis of $\mathcal{H}_n$, so is $\{\phi_j^n(Qx)\}$.

Let $\mathcal{H}_n$ be spanned by $\{\phi_{-n}^n, \dots, \phi_0^n, \dots, \phi_n^n\}$. We denote K_n as the spectral projection from the L^2-integrable functions to the spherical harmonics of level n, so that

$$K_n : L^2(S^2) \to \mathcal{H}_n(S^2).$$

Then the kernel of K_n is given by

$$K_n(x, y) = \sum_{j=-n}^{n} \phi_j^n(x)\phi_j^n(y). \tag{2.1}$$

In fact, the spectral projection kernel has the following explicit formula [4, 19]:

$$K_n(x, y) = \frac{2n+1}{4\pi} P_n\big(\cos\Theta(x, y)\big),$$

where $\Theta(x, y)$ is the angle between the vectors $x, y \in S^2$. The Legendre polynomials (of order 2) are defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}.$$

Some basic facts that we shall require are [3, 4]

(2.2) $P_n(1) = 1;\qquad P_n'(1) = \dfrac{n^2+n}{2};\qquad -1 \le P_n(x) \le 1,$ for $x \in [-1, 1]$,

and

$$P_n(-x) = (-1)^n P_n(x). \tag{2.3}$$

Thus, on the diagonal, the kernel satisfies

$$K_n(x, x) = \frac{2n+1}{4\pi} P_n(1) = \frac{2n+1}{4\pi}. \tag{2.4}$$

2.2. *Immersions.* Consider the map

$$i_n : S^2 \to \mathbb{R}^{2n+1}, \qquad x \to \sqrt{\frac{4\pi}{2n+1}}\big(\phi^n_{-n}(x), \dots, \phi^n_0(x), \dots, \phi^n_n(x)\big). \tag{2.5}$$

For large enough n, this map is an immersion [24].

Defining the normalized kernel

$$\Pi_n(x, y) = \frac{4\pi}{2n+1} K_n(x, y) = P_n\big(\cos\Theta(x, y)\big)$$

we have that the norm of $i_n(x)$ is given by

$$\|i_n(x)\|^2 = \frac{4\pi}{2n+1} \sum_{j=-n}^{n} |\phi_j(x)|^2 = \Pi_n(x, x) = 1.$$

Thus i_n is actually a map from S^2 to S^{2n-1}, and the pullback of the Euclidean metric is

$$g_n = i_n^*(g_E) = \frac{n^2+n}{2} g_{S^2}, \tag{2.6}$$

where we use g_E to denote the standard Euclidean metric. While this fact is well known (cf. [15, 24]) it will follow, en passant, from calculations below (cf. the argument surrounding (3.8)).

The distance between two points of the immersion is given by

$$\begin{aligned} \|i_n(x) - i_n(y)\|^2 &= \Pi_n(x, x) + \Pi_n(y, y) - 2\Pi_n(x, y) \\ &= 2\big(1 - P_n\big(\cos\Theta(x, y)\big)\big), \end{aligned} \tag{2.7}$$

and so it follows from (2.3) that i_n is an embedding for n odd but identifies antipodal points for n even. Thus, in the case of even n, it follows that $i_n(S^2) \cong \mathbb{R}P^2$.

3. The critical radius of $i_n(S^2)$. The modern notion of reach, or critical radius (terms which we shall use interchangeably) seems to have appeared first in the classic paper [10] of Federer, in which he introduced the notion of sets with positive reach and their associated curvatures and curvature measures. In doing so, Federer was able to include, in a single framework, Steiner's tube formula for convex sets and Weyl's tube formula for C^2 smooth submanifolds of $\mathbb{R}^n$. The importance of this framework extended, however, far beyond tube formulae, as it became clear that much of the theory surrounding convex sets could be extended to sets that were, in some sense, locally convex, and that the reach of a set was precisely the way to quantify this property.

To be just a little more precise, suppose N is a smooth manifold embedded in an ambient manifold $\widehat{N}$. Then the local reach at a point $x \in N$ is the furthest distance one can travel, along any geodesic in $\widehat{N}$ based at x but normal to N in $\widehat{N}$, without meeting a similar vector originating at another point in N. The (global) reach of N is then the infimum of all local reaches. As such it is related to local properties of N through its second fundamental form, but also to global structure, since points on N that are far apart in a geodesic sense, in the metric of N, might be quite close in the metric of the ambient space $\widehat{N}$.

There are many, equivalent, formal, definitions of reach, but we shall take as our definition a result which is actually a theorem of Takemura and Kuriki [20], that states that for a compact Riemannian manifold $N \subset \mathbb{R}^k$, the critical radius is given by

$$r_c(N) = \inf_{x,y \in N} \frac{\|x-y\|^2}{2\|P_y^{\perp}(x-y)\|}, \tag{3.1}$$

where $P_y^{\perp}(x-y)$ is the projection of $x-y$ to the normal bundle of N at y.

This is actually all we need for the remainder of the paper, and so for the reader interested to know more about critical radii we refer you to the review [21] for an excellent coverage of the history and uses of this notion in Mathematics as a whole, and to the expository sections of [1] to see why it is an important property in the theory of random processes.

Our interest now, however, is in the critical radii of the immersions $i_n(S^2)$ in $\mathbb{R}^{2n+1}$, and so we now concentrate solely on this.

Rewriting (3.1) for this setting, we have that the critical radius of $i_n(S^2)$ is given by

$$r_{c,n} := \inf_{x,y \in S^2} \frac{\|i_n(x) - i_n(y)\|^2}{2\|P_{i_n(y)}^{\perp}(i_n(x) - i_n(y))\|}. \tag{3.2}$$

The numerator here is given by (2.7), and the first step regarding the denominator is to compute the projection of the vector $i_n(x) - i_n(y)$ to the normal space, that is, the orthogonal complement, in $\mathbb{R}^{2n+1}$, of the tangent space $T_{i_n(y)} i_n(S^2)$.

To this end, we move to polar coordinates

$$x = (\sin\theta_x \sin\phi_x, \sin\theta_x \cos\phi_x, \cos\theta_x),$$

with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$, with a corresponding definition for y.

Note that the normalized projection kernel Π_n is constant on the diagonal, and so

$$\partial_\theta \Pi_n(y, y) = \partial_\phi \Pi_n(y, y) = 0. \tag{3.3}$$

We rewrite the normalized kernel in polar coordinates as

$$\begin{aligned} \Pi_n(x, y) = P_n(&\sin\theta_x \sin\phi_x \sin\theta_y \sin\phi_y \\ &+ \sin\theta_x \cos\phi_x \sin\theta_y \cos\phi_y + \cos\theta_x \cos\theta_y). \end{aligned} \tag{3.4}$$

This yields

$$\partial_{\theta_y}\Pi_n(x,y) = P_n'\big(\cos\Theta(x,y)\big)[\sin\theta_x\sin\phi_x\cos\theta_y\sin\phi_y + \sin\theta_x\cos\phi_x\cos\theta_y\cos\phi_y - \cos\theta_x\sin\theta_y]$$

and

$$\partial_{\phi_y}\Pi_n(x,y) = P_n'\big(\cos\Theta(x,y)\big)[\sin\theta_x\sin\phi_x\sin\theta_y\cos\phi_y - \sin\theta_x\cos\phi_x\sin\theta_y\sin\phi_y].$$

Further differentiation yields

$$\partial_{\theta_x}\partial_{\theta_y}\Pi_n(x,y)|_{x=y} = P_n'(1), \tag{3.5}$$

$$\partial_{\phi_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y} = P_n'(1)\sin^2\theta, \tag{3.6}$$

$$\partial_{\theta_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y} = 0. \tag{3.7}$$

An easy consequence of these three identities is the fact, given in (2.6), that the pullback, under i_n, of the Euclidean metric on $\mathbb{R}^{2n+1}$ is a scaled version of the standard metric on S^2. To see this, note that the pullback is just

$$\sum d\phi_j^n(x)\otimes d\phi_j^n(x), \tag{3.8}$$

which we can write as $d_x d_y \Pi_n(x,y)|_{x=y}$. Since the differential operator d is global, it is unchanged if we take derivatives with respect to the angle variables θ and ϕ. Applying now (3.5)–(3.7) and (2.2) immediately establishes (2.6).

With polar notation, it is easy to see that the tangent subspace at $i_n(y)$ is spanned by the vector $\{\frac{\partial i_n}{\partial\theta}(y), \frac{\partial i_n}{\partial\phi}(y)\}$. (3.7) implies that these two vectors are orthogonal, that is,

$$\left\langle \frac{\partial i_n}{\partial\theta}(y), \frac{\partial i_n}{\partial\phi}(y)\right\rangle = 0.$$

Thus the projection of $i_n(x)-i_n(y)$ to the tangent space is

$$p_x(y) := \frac{\langle i_n(x)-i_n(y), \frac{\partial i_n}{\partial\theta}(y)\rangle}{|\frac{\partial i_n}{\partial\theta}(y)|^2}\frac{\partial i_n}{\partial\theta}(y) + \frac{\langle i_n(x)-i_n(y), \frac{\partial i_n}{\partial\phi}(y)\rangle}{|\frac{i_n(y)}{\partial\phi}|^2}\frac{\partial i_n}{\partial\phi}(y),$$

which can be rewritten as

$$\frac{\partial_{\theta_y}\Pi_n(x,y)-\partial_{\theta_y}\Pi_n(y,y)}{\partial_{\theta_x}\partial_{\theta_y}\Pi_n(x,y)|_{x=y}}\frac{\partial i_n}{\partial\theta}(y) + \frac{\partial_{\phi_y}\Pi_n(x,y)-\partial_{\phi_y}\Pi_n(y,y)}{\partial_{\phi_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y}}\frac{\partial i_n}{\partial\phi}(y).$$

Applying (3.3) to the above gives

$$p_x(y) = \frac{\partial_{\theta_y}\Pi_n(x,y)}{\partial_{\theta_x}\partial_{\theta_y}\Pi_n(x,y)|_{x=y}}\frac{\partial i_n}{\partial\theta}(y) + \frac{\partial_{\phi_y}\Pi_n(x,y)}{\partial_{\phi_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y}}\frac{\partial i_n}{\partial\phi}(y). \tag{3.9}$$

It follows that the squared norm of the projection $p_x(y)$ in (3.9) can be written as

$$\|p_x(y)\|^2 = \frac{|\partial_{\theta_y}\Pi_n(x,y)|^2}{\partial_{\theta_x}\partial_{\theta_y}\Pi_n(x,y)|_{x=y}} + \frac{|\partial_{\phi_y}\Pi_n(x,y)|^2}{\partial_{\phi_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y}}. \tag{3.10}$$

Thus we can express the the critical radius (3.2) as

$$r_{c,n} = \inf_{x,y\in S^2} \frac{\|i_n(x)-i_n(y)\|^2}{2\sqrt{\|i_n(x)-i_n(y)\|^2 - \|p_x(y)\|^2}}.$$

By rotation invariance, it is clear that each of the terms within the infimum here is dependent only on the relative positions of x and y, and so the local radius is actually the same at the image of each point on the sphere. Thus, it suffices to consider the local critical radius at any point. Choosing $x=(0,0,1)$ for this point, the critical radius $r_{c,n}$ can be written as

$$r_{c,n} = \inf_{y\in S^2} \frac{\|i_n((0,0,1))-i_n(y)\|^2}{2\sqrt{\|i_n((0,0,1))-i_n(y)\|^2 - \|p_{(0,0,1)}(y)\|^2}}. \tag{3.11}$$

For $x=(0,0,1)$, we can write the coordinates of y as (θ,ϕ), and so (3.4)–(3.7) become

$$\begin{aligned}
\Pi_n\big((0,0,1),y\big) &= P_n(\cos\theta),\\
\partial_{\theta_x}\partial_{\theta_y}\Pi_n(x,y)|_{x=y} &= P_n'(1),\\
\partial_{\phi_x}\partial_{\phi_y}\Pi_n(x,y)|_{x=y} &= P_n'(1)\sin^2\theta,\\
\partial_{\theta_y}\Pi_n(x,y)|_{x=(0,0,1)} &= -P_n'(\cos\theta)\sin\theta,\\
\partial_{\phi_y}\Pi_n(x,y)|_{x=(0,0,1)} &= 0.
\end{aligned}$$

Similarly, (2.7) becomes

$$\big\|i_n\big((0,0,1)\big)-i_n(y)\big\|^2 = 2\big(1-P_n(\cos\theta)\big),$$

and (3.10) reads

$$\big\|p_{(0,0,1)}(y)\big\|^2 = \frac{[P_n'(\cos\theta)\sin\theta]^2}{P_n'(1)}.$$

Hence, we can now finally rewrite the critical radius of $i_n(S^2)$ in $\mathbb{R}^{2n+1}$ as

$$r_{c,n} = \inf_{\theta\in[0,\pi]} \frac{1-P_n(\cos\theta)}{\sqrt{2-2P_n(\cos\theta) - \frac{[P_n'(\cos\theta)\sin\theta]^2}{P_n'(1)}}}, \tag{3.12}$$

and we are now in a position to begin the more serious steps in the proof of Theorem 1.1, at least for the case $d=2$.

4. Proof of Theorem 1.1 for S^2. In view of the preceding section, in order to prove Theorem 1.1 for the case $d = 2$ we need to provide a lower bound for the expression given in (3.12) that is independent of n, at least for n large enough.

Note first that $P_n(\cos\theta)$ is symmetric (anti-symmetric) about $\theta = \pi/2$ for n even (odd). Thus, for n even, it suffices to consider $\theta \in [0, \pi/2]$ in (3.12). For the moment, we shall assume that n is even, and then discuss the case of odd n at the end of the section.

So, with n even, fix a positive constant c and divide $[0, \pi/2]$ into the three subintervals

$$[0, c/n], \qquad [c/n, n^{-3/4}], \qquad [n^{-3/4}, \pi/2].$$

For the first two, short range, subintervals, our strategy will be to study the rescaling limit of the projection kernel and its derivatives. The infimum for the third subinterval will follow directly from the rapid decay of the projection kernel and its derivatives. The entire proof is based on Hilb's asymptotics for Legendre polynomials [3]. Specifically, there exists a (uniform in n) constant c, for which

$$P_n(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\left(\left(n + \frac{1}{2}\right)\theta\right) + R_n(\theta), \tag{4.1}$$

where

$$R_n(\theta) = \begin{cases} \theta^2 O(1), & 0 \le \theta \le c/n, \\ \theta^{1/2} O(n^{-3/2}), & c/n \le \theta \le \pi/2, \end{cases} \tag{4.2}$$

and $J_0(\theta)$ is the Bessel function of order 0.

The global infimum is then

$$\inf_{\theta\in[0,\pi/2]} = \min\left\{\inf_{[0,c/n]}, \inf_{[c/n,n^{-3/4}]}, \inf_{[n^{-3/4},\pi/2]}\right\} =: \min\{I_n, II_n, III_n\}. \tag{4.3}$$

Consider the first infimum here:

$$I_n = \inf_{\theta\in[0,c/n]} \frac{1 - P_n(\cos\theta)}{\sqrt{2 - 2P_n(\cos\theta) - \frac{[P_n'(\cos\theta)\sin\theta]^2}{P_n'(1)}}}.$$

In order to investigate the error terms here, and to make the notation easier, we study a rescaling limit via a new parameter y, where $y = n\theta$, so that $y \in [0, c]$. By applying Hilb's asymptotic on $[0, c/n]$, we have

$$P_n(\cos(y/n)) = J_0(y) + O(n^{-1}).$$

Next, for, the rescaling of $P_n'(\cos\theta)$, we note the relation [4, 7]

$$P_n'(\cos\theta) = \frac{n+1}{\sin^2\theta}\left[\cos\theta P_n(\cos\theta) - P_{n+1}(\cos\theta)\right].$$

Again applying Hilb's asymptotic, we rescale $[P_n'(\cos\theta)\sin\theta]^2/P_n'(1)$ to obtain

$$\frac{[\frac{n+1}{y/n+O(n^{-3})}]^2[(1+O(n^{-2}))(J_0(y+\frac{y}{2n})+O(n^{-2}))-(J_0(y+\frac{3y}{2n})+O(n^{-2}))]^2}{P_n'(1)}.$$

We apply the Taylor expansion

$$J_0\left(y+\frac{y}{2n}\right)=J_0(y)+\frac{y}{2n}J_0'(y)+O(n^{-2})$$

to further get the rescaling

$$2[J_0'(y)]^2+O(n^{-1}).$$

Hence, as $n\to\infty$, I_n is asymptotic to

$$I_\infty=\inf_{y\in[0,c]}\frac{1-J_0(y)}{\sqrt{2-2J_0(y)-2[J'(y)]^2}}. \tag{4.4}$$

For II_n, we also apply the rescaling technique, the only difference between this and the previous case being in the estimates of the error terms, where we need to show that the leading terms in the rescaling limits will dominate the error terms. The details are as follows.

Again, take $y=n\theta$, so that now $y\in[c,n^{1/4}]$. Hilb's asymptotic gives

$$P_n(\cos(y/n))=(1+O(n^{-3/2}))^{1/2}J_0(y+y/(2n))+O(n^{-15/8}),$$

where the uniform bound $O(n^{-15/8})$ is achieved when $R_n(\theta)$ is evaluated at $\theta=n^{-3/4}$.

A Taylor expansion yields

$$J_0\left(y+\frac{y}{2n}\right)=J_0(y)+\frac{y}{2n}J_0'(y)+\cdots.$$

We now need two basic properties from [3] for Bessel functions. The first is that

$$J_n(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\qquad\text{as } x\to\infty. \tag{4.5}$$

The second is that

$$J_0'(x)=-J_1(x). \tag{4.6}$$

Combining these two properties, we have, for n large enough, the following uniform estimate for $y\in[c,n^{1/4}]$:

$$J_0\left(y+\frac{y}{2n}\right)=J_0(y)+O(n^{-3/4}).$$

Note that the leading term $J_0(y)$ will always dominate the error term, since, by (4.5), the growth of $J_0(y)$ is at least of order $O(n^{-1/8})$.

Hence, we have the rescaling limit

$$P_n\left(\cos\frac{y}{n}\right) = J_0(y) + O(n^{-3/4})$$

on the interval $y \in [c, n^{1/4}]$.

A similar argument shows that the rescaling of $[P_n'(\cos\theta)\sin\theta]^2/P_n'(1)$, for n large enough, will be dominated by the leading term $2[J_0'(y)]^2$. Hence, II_n will converge, as $n \to \infty$ to

$$II_\infty = \inf_{y\in[c,\infty]} \frac{1 - J_0(y)}{\sqrt{2 - 2J_0(y) - 2[J_0'(y)]^2}}. \tag{4.7}$$

We now turn to III_n, which is the last of the three terms to estimate. From the asymptotic expansion (4.5), we see that $J_0((n+\frac{1}{2})\theta)$ decays rapidly on $\theta \in [n^{-1/4}, \pi/2]$, and has, in fact, a uniform bound of $O(n^{-3/8})$. Thus by the Hilb asymptotic, the same is true of $P_n(\cos\theta)$. As for the derivative, Lemma 9.3 of [7] proves that, for $\theta \in [c/n, \pi/2]$,

$$P_n'(\cos\theta) = \sqrt{\frac{2}{\pi}}\frac{n^{1/2}}{\sin^{\frac{3}{2}}\theta}\left[\sin\phi^- - \frac{1}{8n\theta}\sin\phi^+\right] + O(n^{-1/2}\theta^{-5/2}),$$

where $\phi^\pm = (n+\frac{1}{2})\theta \pm \pi/4$. This implies the rapid decay of $[P_n'(\cos\theta)\sin\theta]^2/P_n'(1)$ if we apply the expression of $P_n'(1)$, and so the $n \to \infty$ limit of III_n is

$$III_\infty = \frac{1}{\sqrt{2}}. \tag{4.8}$$

Now fix (small) $\varepsilon > 0$. Combining (4.4), (4.7) and (4.8) with (4.3) and the definition (3.12) of $r_{c,n}$, it follows that there exists a finite n_ε such that, for all even $n > n_\varepsilon$, we have

$$r_{c,n} \geq \min\left\{\inf_{y\in[0,\infty]} \frac{1 - J_0(y)}{\sqrt{2 - 2J_0(y) - 2[J_0'(y)]^2}}, \frac{1}{\sqrt{2}}\right\} - \varepsilon. \tag{4.9}$$

As an aside, note that if we write the expansion of the Bessel function $J_0(y) = 1 - y^2/4 + y^4/64 + O(y^5)$ around $y = 0$, then the expression

$$\frac{1 - J_0(y)}{\sqrt{2 - 2J_0(y) - 2[J_0'(y)]^2}}$$

has the limit, as $y \to 0$, of $\sqrt{2/3}$. Since this is trivially positive, and ε was arbitrary, Theorem 1.1 is now proven for $d = 2$ and for even n, large enough.

However, we still need to treat the cases when n is odd. On the interval $\theta \in [0, \pi/2]$, exactly the same argument as above for the even case applies, and the same infimum is achieved. But when we consider on $\theta \in [\pi/2, \pi]$, there is a sign

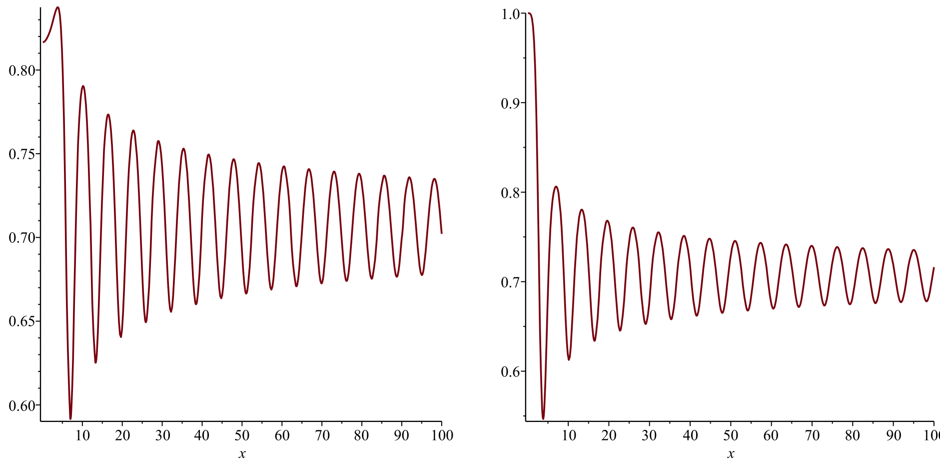

FIG. 1. *Behavior of the first and third terms in the lower bound for $r_{c,n}$ in the 2-dimensional case.*

change in the expression of $P_n(\cos\theta)$, since $P_n(-x) = -P_n(x)$ for n odd. Taking this into account, we obtain the global lower bound, for arbitrary ε and for all n large enough, of

$$
\begin{aligned}
r_{c,n} \geq \min\biggl\{ &\inf_{y\in[0,\infty]} \frac{1-J_0(y)}{\sqrt{2-2J_0(y)-2[J_0'(y)]^2}}, \frac{1}{\sqrt{2}}, \\
&\inf_{y\in[0,\infty]} \frac{1+J_0(y)}{\sqrt{2+2J_0(y)-2[J_0'(y)]^2}} \biggr\} - \varepsilon,
\end{aligned}
\tag{4.10}
$$

and the proof of Theorem 1.1 for the case $d = 2$ is done.

Figure 1 shows the behavior of the first and third terms in the lower bound for $r_{c,n}$ in the above inequality.

5. Proof of Theorem 1.1 for the general case. The proof of Theorem 1.1 for two dimensions can be generalized to higher dimensions without much difficulty. It relies on properties of spherical harmonics in high dimensions that parallel those of the two-dimensional case, and, not surprisingly, some heavier notation. (The additional notation was the main reason for handling the two-dimensional case first.) We shall sketch the main arguments in the proof now.

Retaining the earlier notation, we need to define the normalized spectral projection kernel

$$
\begin{aligned}
\Pi_n^d(x, y) &= \frac{s_d}{k_n^d} \sum_{j=1}^{k_n^d} \phi_j^{n,d}(x)\phi_j^{n,d}(y) \\
&= P_{n,d}\bigl(\Theta(x, y)\bigr),
\end{aligned}
$$

the second line following from [4].

Following the same arguments as those that led to and follow from (3.5)–(3.7), the pullback of the Euclidean metric is

$$(i_n^d)^*(g_E) = P'_{n,d}(1) g_{S^d} = \frac{n(n+d-1)}{d} g_{S^d} \tag{5.1}$$

and the critical radius of $i_n^d(S^d)$, as a subset of $\mathbb{R}^{k_n^d}$, is exactly the same as before, namely as given by (3.12). Once again, relying on rotation invariance, it suffices to study the local critical radius at the image of the point $(0, 0, \dots, 1)$.

As before, moving to polar coordinates on S^d, we have

$$\Pi_n^d\big((0, \dots, 1), y\big) = P_{n,d}(\cos\theta), \qquad \theta \in [0, \pi].$$

Taking derivatives of the normalized kernel and evaluating them at $(0, \dots, 1)$, the d-dimensional analogue of (3.10) now reads

$$\|p_{(0,\dots,1)}(y)\|^2 = \frac{[\partial_{\theta_y} \Pi_n((0, \dots, 1), y)]^2}{\partial_{\theta_x}\partial_{\theta_y} \Pi_n(x, y)|_{x=y}} = \frac{[P'_{n,d}(\cos\theta)\sin\theta]^2}{P'_{n,d}(1)}. \tag{5.2}$$

Hence, we can rewrite (3.11), now for the critical radius of the higher dimensional immersion as

$$r_{c,n}^d\big(S^d\big) = \inf_{\theta\in[0,\pi]} \frac{1 - P_{n,d}(\cos\theta)}{\sqrt{2 - 2P_{n,d}(\cos\theta) - \frac{[P'_{n,d}(\cos\theta)\sin\theta]^2}{P'_{n,d}(1)}}}. \tag{5.3}$$

We still have the following Hilb's asymptotic [19],

$$\begin{aligned} P_{n,d}(\cos\theta) = \Gamma\left(\frac{d}{2}\right)&\left[\frac{1}{2}\left(n + \frac{d-1}{2}\right)\sin\theta\right]^{-\frac{d}{2}+1}\left(\frac{\theta}{\sin\theta}\right)^{1/2} \\ &\times J_{\frac{d}{2}-1}\left(\left(n + \frac{d-1}{2}\right)\theta\right) + R_n(\theta), \end{aligned}$$

where

$$R_n(\theta) = \begin{cases} \theta^{d/2} O\big(n^{d/2-2}\big), & 0 \le \theta \le c/n, \\ \theta^{1/2} O\big(n^{-3/2}\big), & c/n \le \theta \le \pi/2, \end{cases} \tag{5.4}$$

with c a large, d-dependent, constant, and where $J_{\frac{d}{2}-1}(\theta)$ is the Bessel function

$$J_{\frac{d}{2}-1}(\theta) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\frac{d}{2})}\left(\frac{\theta}{2}\right)^{2j+\frac{d}{2}-1}.$$

Again, following the arguments of the preceding section, the global infimum is derived by considering each of the subintervals $[0, c/n]$, $[c/n, n^{-3/4}]$ and

$[n^{-3/4}, \pi/2]$. The infimum on the first two subintervals is expressed by the rescaling limit of the Hilb's asymptotic of the Legendre polynomials $P_{n,d}(\cos\theta)$. When we rescale $\theta \to y/n$ in the Hilb's asymptotic, we obtain the $n \to \infty$ limit

$$J_\infty^d(y) = \Gamma\left(\frac{d}{2}\right)\left(\frac{1}{2}y\right)^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(y) = \sum_{j=0}^\infty \frac{(-1)^j \Gamma(\frac{d}{2})}{j!\Gamma(j+\frac{d}{2})}\left(\frac{y}{2}\right)^{2j}.$$

On the remaining subinterval, $[n^{-3/4}, \pi/2]$, the rapid decay of $P_{n,d}(\cos\theta)$ and its derivative follow from standard properties of Bessel functions [see (4.5)], thus the infimum on this subinterval will tend to $1/\sqrt{2}$, as $n \to \infty$.

As before, combining arguments for the two cases of n add and n even, we find the following lower bound for the critical radii $r_{c,n}^d$ as $n \to \infty$:

$$\begin{aligned}\min\Bigg\{ &\inf_{y\in[0,\infty]} \frac{1 - J_\infty^d(y)}{\sqrt{2 - 2J_\infty^d(y) - d[(J_\infty^d)'(y)]^2}}, \frac{1}{\sqrt{2}}, \\ &\inf_{y\in[0,\infty]} \frac{1 + J_\infty^d(y)}{\sqrt{2 + 2J_\infty^d(y) - d[(J_\infty^d)'(y)]^2}}\Bigg\},\end{aligned} \tag{5.5}$$

which completes the proof.

6. Proof of Theorem 1.2 and Proposition 1.3. We break the proofs into three parts, starting with the proof of Theorem 1.2.

6.1. *The equivalence of mean Euler characteristics and exceedence probabilities.* The following lemma implies Theorem 1.2. It also sets up the relationship between exceedence probabilities and mean Euler characteristics, which we then evaluate in the following two subsections. To state it we need to define the tube

$$\mathrm{Tube}(i_n(S^d), \rho) \triangleq \Big\{x \in S^{k_n^d - 1} : \min_{y \in i_n(S^d)} d(x,y) \le \rho\Big\}, \tag{6.1}$$

where $d(x,y)$ is the geodesic distance on the sphere.

In addition, with a slight—but space saving—change of notation, we write V_{S^N} for volumetric measure with respect to the round metric on S^N.

LEMMA 6.1. *Under the conditions of Theorem 1.2, and for all $0 \le \rho \le \rho_d$,*

$$\begin{aligned}\mathbb{E}_{\mu_n^d}\{\chi(A_n^d(\cos\rho))\} &= \frac{1}{\kappa}\mathbb{P}_{\mu_n^d}\left\{\sup_{S^d} \frac{\Phi_n^d(x)}{\sqrt{k_n^d/s_d}} > \cos\rho\right\} \\ &= \frac{V_{S^{k_n^d-1}}(\mathrm{Tube}(i_n(S^d), \rho))}{\kappa s_{k_n^d-1}},\end{aligned} \tag{6.2}$$

where κ is $1/2$ if n is even and 1 if n is odd.

PROOF. We start by noting that, by (1.8), we can write

$$\frac{\Phi_n^d(x)}{\sqrt{k_n^d/s_d}} = \langle a, i_n^d(x)\rangle = \cos\Theta(a, i_n^d(x)),$$

with, as before $a = (a_1, \dots, a_{k_n^d}) \in S^{k_n^d-1}$, and where $\Theta(x, y)$ is the angle between vectors $x, y \in S^{k_n^d-1}$.

We now note the fact (e.g., [18], Lemma 3.1) that if M is a compact submanifold of a smooth manifold N, and $p \in N$, then the intersection between M and a ball of radius ρ around p will either be empty or contractible, as long as ρ is less than the reach of M.

Further, we know from Theorem 1.1 that there is a uniform lower bound for the critical radius of the immersion $i_n^d(S^d)$ in $\mathbb{R}^{k_n^d}$. From this and a little spherical geometry, it follows that the same is true, albeit with a different lower bound, for the critical radius of $i_n^d(S^d)$ considered as a subset of $S^{k_n^d-1}$. Let ρ_d denote this new lower bound.

Putting the last three paragraphs together, with $M = i_n^d(S^d)$ and $p = a \in N = S^{k_n^d-1}$, we have that the set

$$\{i_n^d(z) \in S^{k_n^d-1} : \langle a, i_n^d(z)\rangle > \cos\rho\}$$

is either empty or contractible for $0 \le \rho \le \rho_d$. Hence,

$$\begin{aligned}\kappa\mathbb{E}_{\mu_n^d}\{\chi(A_n^d(\cos\rho))\} &= \mathbb{E}_{\mu_n^d}\{\chi\{i_n^d(z) \in S^{k_n^d-1} : \langle a, i_n^d(z)\rangle > \cos\rho\}\} \\ &= \mathbb{P}_{\mu_n^d}\Big\{\sup_z \langle a, i_n^d(z)\rangle > \cos\rho\Big\},\end{aligned} \tag{6.3}$$

the factor of κ on the right-hand side coming from the fact that while i_n^d is an embedding if n is odd, it identifies antipodal points if n is even. Consequently, the Euler characteristic of the preimage on S^d will be double that of the image when n is even. This obviously completes the proof of the lemma. □

As an aside, we note that (6.3) is also proven in [20], although there the approach is to obtain expressions for the the expected Euler characteristic and the probability separately, and then note that they are identical.

6.2. *On tube formulae.* Returning to (6.3), and noting that, under the spherical ensemble, a is chosen uniformly on $S^{k_n^d-1}$, we have that we can write the final probability there as

$$\mathbb{P}_{\mu_n^d}\Big\{\sup_z \langle a, i_n^d(z)\rangle > \cos\rho\Big\} = \frac{V_{S^{k_n^d-1}}(\mathrm{Tube}(i_n(S^d), \rho))}{s_{k_n^d-1}}. \tag{6.4}$$

We now want to express the volume of the tube in (6.4) via Weyl's tube formula [2, 11, 22], and so spend the remainder of this section setting up some notation and facts.

Given an m-dimensional Riemannian submanifold (M, g) of S^{N-1}, the volume of a tube around M of radius ρ less than its critical radius, is given by (Theorem 10.5.7 in [2]),

$$V_{S^{N-1}}\big(\mathrm{Tube}(M,\rho)\big) = \sum_{j=0}^{m} f_{N,j}(\rho)\mathcal{L}_j(M), \tag{6.5}$$

where

$$f_{N,j}(\rho) = \sum_{k=0}^{[\frac{j}{2}]} (-4\pi)^{-k}\frac{1}{k!}\frac{j!}{(j-2k)!}G_{j-2k,N-1+2k-j}(\rho) \tag{6.6}$$

and

$$G_{a,b}(\rho) = \frac{b\pi^{b/2}}{\Gamma(\frac{b}{2}+1)}\int_0^{\rho}\cos^a(r)\sin^{b-1}(r)\,dr. \tag{6.7}$$

The Lipshitz–Killing curvatures $\mathcal{L}_j(M)$ are given by

$$\mathcal{L}_j = \begin{cases} \dfrac{(-2\pi)^{-(m-j)/2}}{(\frac{m-j}{2})!}\displaystyle\int_M \mathrm{Tr}\big(R_g^{(m-j)/2}\big)\,dV_g, & m-j \text{ even},\\ 0, & m-j \text{ odd}, \end{cases} \tag{6.8}$$

where R_g is the curvature tensor. In general, $\mathcal{L}_m(M) = V_g(M)$ is the volume of M and $\mathcal{L}_0(M) = \chi(M)$ is its Euler characteristic.

For two-dimensional surfaces of volume $V_g(M)$ and Euler characteristic $\chi(M)$, embedded in S^{N-1}, the tube formula simplifies to

$$\begin{aligned} &V_{S^{N-1}}\big(\mathrm{Tube}(M,\rho)\big)\\ &\quad = \frac{2\pi^{(N-3)/2}}{\Gamma(\frac{N-3}{2})}\\ &\qquad \times \int_0^{\rho}\sin^{N-4}(r)\left\{V_g(M)\left(1-\frac{N-2}{N-3}\sin^2(r)\right)+\frac{2\pi\chi(M)\sin^2(r)}{N-3}\right\}dr. \end{aligned} \tag{6.9}$$

One final fact that we shall need for later is the value of the Lipshitz–Killing curvatures for spheres. These are

$$\mathcal{L}_j(S^{N-1}) = \begin{cases} 2\dbinom{N-1}{j}\dfrac{s_N}{s_{N-j}}, & N-1-j \text{ even},\\ 0, & N-1-j \text{ odd}. \end{cases} \tag{6.10}$$

6.3. *Proof of Proposition* 1.3. The proof works by applying the tube formula (6.5) to the equivalence (6.4).

We tackle the notionally easier case for S^2 first, thus proving Corollary 1.4 directly. Then, by (6.9), for the surface $i_n(S^2)$ in the ambient space S^{2n+1}, we have

$$\begin{aligned}
&V_{S^{2n}}\big(\mathrm{Tube}\big(i_n^d(S^d),\rho\big)\big)/s_{2n}\\
&\quad = \left(\frac{2\pi^{n-1}/\Gamma(n-1)}{2\pi^{n+\frac{1}{2}}/\Gamma(n+\frac{1}{2})}\right)\int_0^\rho \sin^{2n-3}(r)\left\{V\big(i_n(S^2)\big)\left(1-\frac{2n-1}{2n-2}\sin^2 r\right)\right.\\
&\qquad \left. + \frac{2\pi\,\chi(i_n(S^2))\sin^2(r)}{2n-2}\right\}dr.
\end{aligned}\tag{6.11}$$

Recall (cf. (2.6)) that the pullback of the Euclidean metric is $((n^2+n)/2)g_{S^2}$. If we combine this with the fact that $i_n(S^2)\cong S^2$ for n odd and $i_n(S^2)\cong \mathbb{R}P^2$ for n even, we have

$$\begin{aligned}
&V\big(i_n(S^2)\big)=2(n^2+n)\pi, &&\chi\big(i_n(S^2)\big)=2, &&\text{for odd } n,\\
&V\big(i_n(S^2)\big)=(n^2+n)\pi, &&\chi\big(i_n(S^2)\big)=1, &&\text{for even } n.
\end{aligned}$$

Substituting this into (6.2) and noting (6.11) suffices to prove Corollary 1.4.

For the general, higher dimensional cases, (5.1) gives us that

$$\big(i_n^d\big)^*(g_E)=P'_{n,d}(1)g_{S^d},$$

implying that the curvature tensor of the pullback $(i_n^d)^*(g_E)$ is $[P'_{n,d}(1)]^{-1}R_{g_{S^d}}$, where $R_{g_{S^d}}$ is the curvature tensor of the round metric g_{S^d} so that

$$R^{(d-j)/2}_{(i_n^d)^*(g_E)}=\big[P'_{n,d}(1)\big]^{(j-d)/2}R^{(d-j)/2}_{g_{S^d}}.\tag{6.12}$$

Similarly, volume form is rescaled to give

$$dV_{(i_n^d)^*(g_E)}=\kappa\big[P'_{n,d}(1)\big]^{d/2}dV_{g_{S^d}},$$

where a factor of κ appears since the measure on $\mathbb{R}P^d$ induced from S^d is half of that on S^d. Hence, by definition of the Lipschitz–Killing curvatures in (6.10), the jth Lipschitz–Killing curvature of the pullback metric which involves the integration on $i_n^d(S^d)$ will be rescaled to be $\kappa[P'_{n,d}(1)]^{j/2}\mathcal{L}_j(S^d)$. Consequently,

$$\frac{V_{S^{k_n^d-1}}(\mathrm{Tube}(i_n^d(S^d),\rho))}{s_{k_n^d-1}}=\frac{\kappa}{s_{k_n^d-1}}\sum_{j=0}^{d} f_{k_n^d,j}(\rho)\big[P'_{n,d}(1)\big]^{j/2}\mathcal{L}_j(S^d),\tag{6.13}$$

which, on combining (6.2) and (6.13), completes the proof of Theorem 1.2.

7. Some closing comments. To conclude, we want to connect our results to some other recent ones, as well as pointing out some interesting open questions.

Given a Riemannian manifold M, [1] studied the random map

$$i_k : M \to \mathbb{R}^k, \qquad x \to k^{-1/2}(f_1, f_2, \dots, f_k), \tag{7.1}$$

where the f_j were independent and identically distributed copies of a smooth, mean zero, unit variance, Gaussian process f. For k large enough, the i_k become embeddings. It was shown that, as $k \to \infty$, the critical radius of the embedded manifold $i_k(M)$ converged, almost surely, to a constant known from Gaussian excursion theory, and which depended on a Riemannian metric on M induced by the Gaussian process f.

Consider an analogue of (7.1) in which we replace f by Gaussian spherical harmonics on S^d of level n. That is, we take for f the Φ_n^d in the form of (1.3), but with the a_j standard normal variables. Note that, as $n \to \infty$, we lose smoothness, and so leave the setting of [1].

Consider the random map

$$i_{k,d}^{(n)} : S^d \to \mathbb{R}^k, \qquad x \to \frac{1}{\sqrt{k}}\big(f_1^{(n)}, \dots, f_k^{(n)}\big),$$

where the $f_j^{(n)}$ are independent and identically distributed copies of Φ_n^d. When k is large enough, $i_{k,d}^{(n)}$ is still an embedding. However, as opposed to the setting (7.1), the interesting problem now is the decay rate of the critical radius of the embedded sphere as $n \to \infty$, but with fixed k, large enough. The method used in [1] highly depends on a central limit theorem as $k \to \infty$, and so their method is not applicable in this problem. The generic behavior of the critical radius of $i_{k,d}^{(n)}(S^d)$ as $n \to \infty$ is unclear.

Another problem, more closely related to what we have studied here, is to understand the critical radius for more general Riemannian manifolds. That is, given a d-dimensional Riemannian manifold (M, g), consider the eigenspace

$$\mathcal{H}_{[\lambda,\lambda+1]}^d := \{\phi : \Delta_g \phi = -\tilde{\lambda}\phi, \tilde{\lambda} \in [\lambda, \lambda+1]\},$$

for large λ. Then choose $\{\phi_1, \dots, \phi_{k_\lambda^d}\}$ as the orthogonal basis of $\mathcal{H}_{[\lambda,\lambda+1]}^d$ and define the immersion,

$$i_\lambda^d : M \to \mathbb{R}^{k_\lambda^d}, \qquad x \to \big(k_\lambda^d\big)^{-1/2}(\phi_1, \dots, \phi_{k_\lambda^d}), \tag{7.2}$$

where k_λ^d is the dimension of $\mathcal{H}_{[\lambda,\lambda+1]}^d$.

This map is not new, and was considered by Zelditch in [24], for Zoll and aperiodic manifolds. He obtained the leading order terms of the spectral projection kernel and its derivatives, from which he was able to derive asymptotics for the distribution of zeros of Gaussian random waves by the classical Kac–Rice formula.

In the results of the current paper, our computations regarding the critical radius for the immersion $i_n^d(S^d)$ relied on the fact that all the information of the immersion i_n^d (1.5) is contained in the spectral projection kernels. To be more precise, we needed the leading expansion and the rescaling limit of the spectral projection kernel and its derivatives up to order two. It seems that our method can be generalized to the case of Zoll and aperiodic manifolds. It is well known that the behavior of eigenfunctions highly depends on the dynamical system of the manifolds [23], and it should be very interesting to study the relation between the critical radius of $i_\lambda^d(M)$ and the dynamical system. We postpone these questions for further investigation.

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