The Annals of Probability
2019, Vol. 47, No. 2, 835–895
https://doi.org/10.1214/18-AOP1273

LIMIT THEORY FOR GEOMETRIC STATISTICS OF POINT PROCESSES HAVING FAST DECAY OF CORRELATIONS

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Let $\mathcal{P}$ be a simple, stationary point process on $\mathbb{R}^d$ having fast decay of correlations, that is, its correlation functions factorize up to an additive error decaying faster than any power of the separation distance. Let $\mathcal{P}_n := \mathcal{P} \cap W_n$ be its restriction to windows $W_n := [-\frac{1}{2}n^{1/d}, \frac{1}{2}n^{1/d}]^d \subset \mathbb{R}^d$. We consider the statistic $H_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)$ where $\xi(x, \mathcal{P}_n)$ denotes a score function representing the interaction of x with respect to $\mathcal{P}_n$. When ξ depends on local data in the sense that its radius of stabilization has an exponential tail, we establish expectation asymptotics, variance asymptotics and central limit theorems for H_n^ξ and, more generally, for statistics of the re-scaled, possibly signed, ξ-weighted point measures $\mu_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)\delta_{n^{-1/d}x}$, as $W_n \uparrow \mathbb{R}^d$. This gives the limit theory for nonlinear geometric statistics (such as clique counts, the number of Morse critical points, intrinsic volumes of the Boolean model and total edge length of the k-nearest neighbors graph) of α-determinantal point processes (for $-1/\alpha \in \mathbb{N}$) having fast decreasing kernels, including the β-Ginibre ensembles, extending the Gaussian fluctuation results of Soshnikov [*Ann. Probab.* **30** (2002) 171–187] to nonlinear statistics. It also gives the limit theory for geometric U-statistics of α-permanental point processes (for $1/\alpha \in \mathbb{N}$) as well as the zero set of Gaussian entire functions, extending the central limit theorems of Nazarov and Sodin [*Comm. Math. Phys.* **310** (2012) 75–98] and Shirai and Takahashi [*J. Funct. Anal.* **205** (2003) 414–463], which are also confined to linear statistics. The proof of the central limit theorem relies on a factorial moment expansion originating in [*Stochastic Process. Appl.* **56** (1995) 321–335; *Statist. Probab. Lett.* **36** (1997) 299–306] to show the fast decay of the correlations of ξ-weighted point measures. The latter property is shown to imply a condition equivalent to Brillinger mixing, and consequently yields the asymptotic normality of μ_n^ξ via an extension of the cumulant method.

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Received July 2016; revised January 2018.

[1]Supported by DST-INSPIRE faculty award, CPDA from the Indian Statistical Institute and TOPOSYS grant.

[2]Supported in part by NSF Grant DMS-1406410.

MSC2010 subject classifications. Primary 60F05, 60D05; secondary 60G55, 52A22, 05C80.

Key words and phrases. Point processes having fast decay of correlations, determinantal point process, permanental point process, Gaussian entire functions, Gibbs' point process, U-statistics, stabilization, difference operators, cumulants, Brillinger mixing, central limit theorem.

1. Introduction and main results. Functionals of geometric structures on finite point sets $\mathcal{X} \subset \mathbb{R}^d$ often consist of sums of spatially dependent terms admitting the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1.1}$$

where the $\mathbb{R}$-valued *score function* ξ, defined on pairs $(x, \mathcal{X})$, $x \in \mathcal{X}$, represents the *interaction* of x with respect to $\mathcal{X}$, called the *input*. The sums (1.1) typically describe a global geometric feature of a structure on $\mathcal{X}$ in terms of local contributions $\xi(x, \mathcal{X})$.

It is frequently the case in stochastic geometry, statistical physics and spatial statistics that one seeks the large n limit behavior of

$$H_n^\xi := H_n^\xi(\mathcal{P}) := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n), \tag{1.2}$$

where ξ is an appropriately chosen score function, $\mathcal{P}$ is a simple, stationary point process on $\mathbb{R}^d$, and $\mathcal{P}_n$ is the restriction of $\mathcal{P}$ to $W_n := [-\frac{1}{2}n^{1/d}, \frac{1}{2}n^{1/d}]^d$. For example, if $\mathcal{P}_n$ is either a Poisson or binomial point process and if ξ is either a local U-statistic or an exponentially stabilizing score function, then the limit theory for H_n^ξ is established in [5, 15, 27, 29, 40, 42, 45, 48]. If $\mathcal{P}_n$ is a rarified Gibbs' point process on W_n and ξ is exponentially stabilizing, then [50, 54] treat the limit theory for H_n^ξ.

It is natural to ask whether the limit theory of these papers extends to more general input satisfying a notion of "asymptotic independence" for point processes. Recall that if $\xi \equiv 1$ and if $\mathcal{P}$ is an α-determinantal point process with $\alpha = -1/m$ or an α-permanental point process with $\alpha = 2/m$ for some m in the set of positive integers $\mathbb{N}$ (resp., $\mathcal{P}$ is the zero set of a Gaussian entire function), then remarkable results of Soshnikov [53], Shirai and Takahashi [52] (resp., Nazarov and Sodin [37]), show that the counting statistic $\mathcal{P}_n(W_n) := \sum_{x \in \mathcal{P}_n} \mathbf{1}[x \in W_n]$ is asymptotically normal. One may ask whether asymptotic normality of H_n^ξ still holds when ξ is either a local U-statistic or an exponentially stabilizing score function. We answer these questions affirmatively. Loosely speaking, subject to a mild growth condition on $\operatorname{Var} H_n^\xi$, our approach shows that H_n^ξ is asymptotically normal whenever $\mathcal{P}$ is a point process having fast decay of correlations.

Heuristically, when the score functions depend on "local data" and when the input is "asymptotically independent," one might expect that the statistics H_n^ξ obey a strong law and a central limit theorem. The notion of dependency on "local data" for score functions is formalized via stabilization in [5, 15, 40, 42, 45]. Here we formalize the idea of asymptotically independent input $\mathcal{P}$ via the notion of "fast decay of correlation functions." We thereby extend the limit theory of the aforementioned papers to input having fast decay of correlation functions. A point process $\mathcal{P}$ on $\mathbb{R}^d$ has fast decay of correlations if for all $p, q \in \mathbb{N}$ and all $x_1, \dots, x_{p+q} \in \mathbb{R}^d$, its correlation functions $\rho^{(p+q)}(x_1, \dots, x_{p+q})$ factorize into $\rho^{(p)}(x_1, \dots, x_p)\rho^{(q)}(x_{p+1}, \dots, x_{p+q})$ up to an additive error decaying faster than any power of the separation distance

$$\begin{aligned} s &:= d\bigl(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\}\bigr) \\ &:= \inf_{i \in \{1,\dots,p\}, j \in \{p+1,\dots,p+q\}} |x_i - x_j| \end{aligned} \tag{1.3}$$

as at (1.10) below. Here $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. Roughly speaking, such point processes exhibit asymptotic independence at large distances. Examples of such point processes are given in Section 2.2. Point processes with fast decay of correlations are called "clustering point processes" in statistical physics [32, 34, 37]. We shall avoid this terminology since, at least from the point of view of spatial statistics, it suggests that the points of $\mathcal{P}$ clump or aggregate together, which need not be the case.

If either $\mathcal{P}$ has fast decay of correlations and ξ is a local U-statistic or if $\mathcal{P}$ has exponentially fast decay of correlations and ξ is an exponentially stabilizing score function, then with δ_x denoting the point mass at x, our main results establish expectation and variance asymptotics as $n \to \infty$, as well as central limit theorems for the re-scaled, possibly signed, ξ-weighted point measures

$$\mu_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)\delta_{n^{-1/d}x}, \tag{1.4}$$

thereby also establishing the limit theory for the total mass of μ_n^ξ given by the nonlinear statistics H_n^ξ.

As shown in Theorems 1.12–1.15, this yields the limit theory for general nonlinear statistics of α-determinantal and α-permanental point processes, the point process given by the zero set of a Gaussian entire function, as well as rarified Gibbsian input.

The benefit of the general approach taken here is four-fold: (i) we establish the asymptotic normality of the random measures μ_n^ξ, with $\mathcal{P}$ either an α-permanental point process (with $1/\alpha \in \mathbb{N}$), an α-determinantal point process (with $-1/\alpha \in \mathbb{N}$) or the zero set of a Gaussian entire function, thereby extending the work of Soshnikov [53], Shirai and Takahashi [52] and Nazarov and Sodin [37], who restrict to linear statistics, (ii) we extend the limit theory of [5, 29, 40–42, 45], which is confined to Poisson and binomial input, to point processes having fast decay of correlations, (iii) we apply our general results to deduce asymptotic normality and variance asymptotics for geometric statistics of input having fast decay of correlations, including statistics of simplicial complexes and germ-grain models, clique counts, Morse critical points, as well of statistics of random graphs (cf. Section 2.3 of [10]), (iv) our general proof of the asymptotic normality of μ_n^ξ relates the fast decay of correlations of the input process $\mathcal{P}$ to a similar fast correlation decay for the family of *ξ-weighted (point) measures*

$$\sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P}_n)\delta_x, \tag{1.5}$$

consequently implying Brillinger mixing of these measures, and thus directly relating the two concepts: Fast decay of correlations implies Brillinger mixing.

Given input $\mathcal{P}$ having fast decay of correlations, an interesting feature of the measures μ_n^ξ is that their variances are at most of order $\mathrm{Vol}_d(W_n)$, the volume of the window W_n (Theorem 1.12). This holds also for the statistic $\hat{H}_n^\xi := \sum_{x \in \mathcal{P}_n} \xi(x, \mathcal{P})$, which involves summands having no boundary effects. An interesting feature of this statistic is that if its variance is $o(\mathrm{Vol}_d(W_n))$ then it has to be $O(\mathrm{Vol}_{d-1}(\partial W_n))$, where ∂W_n denotes the boundary of W_n and $\mathrm{Vol}_{d-1}(\cdot)$ stands for the $(d-1)$th intrinsic volume (Theorem 1.15). In other words, if the fluctuations of $\hat{H}_n^\xi$ are not of volume order, then they are at most of surface order.

Coming back to our setup, when a functional $H_n^\xi(\mathcal{P})$ is expressible as a sum of local U-statistics or, more generally, as a sum of exponentially stabilizing score functions ξ, then a key step towards proving the central limit theorem is to show that the correlation functions of the ξ-weighted measures defined via Palm expectations $\mathbb{E}_{x_1,\dots,x_k}$ (cf. Section 1.1) and given by

$$\begin{aligned} & m^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n) \\ & \quad := \mathbb{E}_{x_1,\dots,x_{p+q}}\big(\xi(x_1,\mathcal{P}_n)^{k_1}\dots\xi(x_{p+q},\mathcal{P}_n)^{k_{p+q}}\big) \\ & \qquad \times \rho^{(p+q)}(x_1,\dots,x_{p+q}), \end{aligned} \tag{1.6}$$

similar to those of the input process $\mathcal{P}$, approximately factorize into

$$m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n)m^{(k_{p+1},\dots,k_{p+q})}(x_{p+1},\dots,x_{p+q};n),$$

uniformly in n $\leq \infty$, up to an additive error decaying faster than any power of the separation distance s, defined at (1.3). Here $x_1,\dots,x_{p+q}$ are distinct points in W_n and $k_1,\dots,k_{p+q} \in \mathbb{N}$. This result, spelled out in Theorem 1.11, is at the heart of our approach. We then give two proofs of the central limit theorem (Theorem 1.13) for the purely atomic random measures (1.4) via the cumulant method, and as a corollary, derive the asymptotic normality of $H_n^\xi(\mathcal{P})$ and $\int f\,\mathrm{d}\mu_n^\xi$, f a test function, as $n \to \infty$. The proof of expectation and variance asymptotics (Theorem 1.12) mainly relies upon the refined Campbell theorem.

In contrast to the aforementioned works, our proof of the fast decay of correlations of the ξ-weighted measures depends heavily on a factorial moment expansion for expected values of functionals of a general point process $\mathcal{P}$. This expansion, which originates in [8, 9], is expressed in terms of iterated difference operators of the considered functional on the null configuration of points and integrated against factorial moment measures of the point process. It is valid for general point processes, in contrast to the Fock space representation of Poisson functionals, which involves the same difference operators but is deeply related to chaos expansions [30]. Further connections with the literature are discussed in the remarks following Theorems 1.14 and 1.15.

Our interest in these issues was stimulated by similarities in the methods of [4, 5, 32, 50] and [37]. The articles [5, 50] prove central limit theorems for stabilizing functionals of Poisson and rarified Gibbsian point processes, respectively, while [37] proves central limit theorems for linear statistics $\sum_{x\in\mathcal{P}_n}\xi(x)$ of point processes having fast decay of correlations. These papers all establish the fast decay of correlations of the ξ-weighted measures as at (1.21) below, and then use the resulting volume order cumulant bounds to show asymptotic normality. This paper unifies and extends the results of [4, 5, 37, 50, 52] to input having fast decay of correlations. The idea of using correlation functions to show asymptotic normality via cumulants goes back to [32]. The earlier work of [34] has stimulated our investigation of variance asymptotics.

Having described the goals and context of this paper, we now describe more precisely the assumptions on allowable score and input pairs $(\xi, \mathcal{P})$ as well as our main results. The generality of allowable pairs $(\xi, \mathcal{P})$ considered here necessitates several definitions which go as follows.

1.1. *Admissible point processes having fast decay of correlations.* Throughout $\mathcal{P} \subset \mathbb{R}^d$ denotes a simple point process. By a simple point process, we mean a random element taking values in $\mathcal{N}$, the space of locally finite simple point sets in $\mathbb{R}^d$ [or equivalently Radon counting measures μ such that $\mu(\{x\}) \in \{0, 1\}$ for all $x \in \mathbb{R}^d$] and equipped with the canonical σ-algebra $\mathcal{B}$. Given a simple point process $\mathcal{P}$, we interchangeably use the following representations of $\mathcal{P}$:

$$\mathcal{P}(\cdot) := \sum_i \delta_{X_i}(\cdot) \qquad \text{(random measure);}$$

$$\mathcal{P} := \{X_i\}_{i \geq 1} \qquad \text{(random set),}$$

where $X_i, i \geq 1$, are $\mathbb{R}^d$-valued random variables (given a measurable numbering of points, which is irrelevant for the results presented in this paper). Points of $\mathbb{R}^d$ are denoted by x or y whereas points of $(\mathbb{R}^d)^{k-1}$ are denoted by $\mathbf{x}$ or $\mathbf{y}$. We let $\mathbf{0}$ denote a point at the origin of $\mathbb{R}^d$.

For a bounded function f on $\mathbb{R}^d$ and a simple counting measure μ, let $\mu(f) := \langle f, \mu \rangle$ denote the integral of f with respect to μ. For a bounded set $B \subset \mathbb{R}^d$, we let $\mu(B) = \mu(\mathbf{1}_B) = \operatorname{card}(\mu \cap B)$, with μ in the last expression interpreted as the set of its atoms.

For a simple Radon counting measure μ and $k \in \mathbb{N}$, its kth factorial power is

$$\mu^{(k)} := \begin{cases} \displaystyle\sum_{\text{distinct } x_1, \ldots, x_k \in \mu} \delta_{(x_1, \ldots, x_k)} & \text{when } \mu(\mathbb{R}^d) \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu^{(k)}$ is a Radon counting measure on $(\mathbb{R}^d)^k$. Consistently, for a set $\mathcal{X} \subset \mathbb{R}^d$, we denote $\mathcal{X}^{(k)} := \{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k : x_i \in \mathcal{X}, x_i \neq x_j \text{ for } i \neq j\}$. The *kth order factorial moment measure* of the (simple) point process $\mathcal{P}$ is defined as $\alpha^{(k)}(\cdot) := \mathbb{E}(\mathcal{P}^{(k)}(\cdot))$ on $(\mathbb{R}^d)^k$, that is, $\alpha^{(k)}(\cdot)$ is the intensity measure of the point process $\mathcal{P}^{(k)}(\cdot)$. Its Radon–Nikodyn density $\rho^{(k)}(x_1, \ldots, x_k)$ (provided it exists) is the *k-point correlation function* (or kth joint intensity) and is characterized by the relation

$$\begin{aligned} \alpha^{(k)}(B_1 \times \cdots \times B_k) &= \mathbb{E}\Big(\prod_{1 \leq i \leq k} \mathcal{P}(B_i)\Big) \\ &= \int_{B_1 \times \cdots \times B_k} \rho^{(k)}(x_1, \ldots, x_k)\, \mathrm{d}x_1 \ldots \mathrm{d}x_k, \end{aligned} \tag{1.7}$$

where $B_1, \ldots, B_k$ are mutually disjoint bounded Borel sets in $\mathbb{R}^d$. Since $\mathcal{P}$ is simple, we may put $\rho^{(k)}$ to be zero on the diagonals of $(\mathbb{R}^d)^k$, that is on the subsets of $(\mathbb{R}^d)^k$ where two or more coordinates coincide. The disjointness assumption is crucial as illustrated by the following useful relation: For any bounded Borel set $B \subset \mathbb{R}^d$ and $k \geq 1$, we have

$$\begin{aligned} \alpha^{(k)}(B^k) &= \mathbb{E}\big(\mathcal{P}(B)\big(\mathcal{P}(B)-1\big)\dots\big(\mathcal{P}(B)-k+1\big)\big) \\ &= \int_{B^k} \rho^{(k)}(x_1,\dots,x_k)\,\mathrm{d}x_1\dots\mathrm{d}x_k. \end{aligned} \tag{1.8}$$

Heuristically, the *kth Palm measure* $P_{x_1,\dots,x_k}$ of $\mathcal{P}$ is the probability distribution of $\mathcal{P}$ conditioned on $\{x_1,\dots,x_k\} \subset \mathcal{P}$. More formally, if $\alpha^{(k)}$ is locally finite, there exists a family of probability distributions $P_{x_1,\dots,x_k}$ on $(\mathcal{N},\mathcal{B})$, unique up to an $\alpha^{(k)}$-null set of $(\mathbb{R}^d)^k$, called the kth Palm measures of $\mathcal{P}$, and satisfying the disintegration formula

$$\begin{aligned} &\mathbb{E}\bigg(\sum_{(x_1,\dots,x_k)\in\mathcal{P}^{(k)}} f(x_1,\dots,x_k;\mathcal{P})\bigg) \\ &\qquad = \int_{(\mathbb{R}^d)^k}\int_{\mathcal{N}} f(x_1,\dots,x_k;\mu)P_{x_1,\dots,x_k}(\mathrm{d}\mu)\alpha^{(k)}(\mathrm{d}x_1\dots\mathrm{d}x_k) \end{aligned} \tag{1.9}$$

for any (say nonnegative) measurable function f on $(\mathbb{R}^d)^k \times \mathcal{N}$. Formula (1.9) is also known as the refined Campbell theorem.

To simplify notation, write $\int_{\mathcal{N}} f(x_1,\dots,x_k;\mu)P_{x_1,\dots,x_k}(\mathrm{d}\mu) = \mathbb{E}_{x_1,\dots,x_k}(f(x_1,\dots,x_k;\mathcal{P}))$, where $\mathbb{E}_{x_1,\dots,x_k}$ is the expectation corresponding to the Palm probability $\mathbb{P}_{x_1,\dots,x_k}$ on a canonical probability space on which $\mathcal{P}$ is also defined. To further simplify notation, denote by $\mathbb{P}^!_{x_1,\dots,x_k}$ the *reduced Palm probabilities* and their expectation by $\mathbb{E}^!_{x_1,\dots,x_k}$, which satisfies $\mathbb{E}^!_{x_1,\dots,x_k}(f(x_1,\dots,x_k;\mathcal{P})) = \mathbb{E}_{x_1,\dots,x_k}(f(x_1,\dots,x_k;\mathcal{P}\setminus\{x_1,\dots,x_k\}))$.[3]

All Palm probabilities (expectations) are meaningfully defined only for $\alpha^{(k)}$ almost all $x_1,\dots,x_k \in \mathbb{R}^d$. Consequently, all expressions involving these measures should be understood in the $\alpha^{(k)}$ a.e. sense and suprema should likewise be understood as *essential* suprema with respect to $\alpha^{(k)}$.

The following definition is reminiscent of the so-called weak exponential decrease of correlations introduced in [32] and subsequently used in [4, 34, 37].

DEFINITION 1.1 (ω-mixing correlation functions). The correlation functions of a point process $\mathcal{P}$ are *ω-mixing* if there exists a decreasing function $\omega : \mathbb{N}\times\mathbb{R}^+ \to \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, $\lim_{x\to\infty}\omega(n,x) = 0$ and for all $p,q \in \mathbb{N}$, $x_1,\dots,x_{p+q} \in \mathbb{R}^d$, we have

$$\big|\rho^{(p+q)}(x_1,\dots,x_{p+q}) - \rho^{(p)}(x_1,\dots,x_p)\rho^{(q)}(x_{p+1},\dots,x_{p+q})\big| \leq \omega(p+q,s),$$

where $s := d(\{x_1,\dots,x_p\},\{x_{p+1},\dots,x_{p+q}\})$ is as at (1.3).

[3]It can be shown that $\mathbb{P}_{x_1,\dots,x_k}(x_1,\dots,x_k \in \mathcal{P}) = 1$ for $\alpha^{(k)}$ a.e. $x_1,\dots,x_k \in \mathbb{R}^d$.

By an admissible point process $\mathcal{P}$ on $\mathbb{R}^d, d \geq 2$, we mean that $\mathcal{P}$ is simple, stationary (i.e., $\mathcal{P} + x \overset{d}{=} \mathcal{P}$ for all $x \in \mathbb{R}^d$, where $\mathcal{P} + x$ denotes the translation of $\mathcal{P}$ by the vector x), with nonnull and finite intensity $\rho^{(1)}(\mathbf{0}) = \mathbb{E}(\mathcal{P}(W_1))$, and has k-point correlation functions of all orders $k \in \mathbb{N}$. By a *fast decreasing function* $\phi : \mathbb{R}^+ \to [0, 1]$, we mean ϕ satisfies $\lim_{x\to\infty} x^m \phi(x) = 0$ for all $m \geq 1$.

DEFINITION 1.2 (Admissible point process having fast decay of correlations). Let $\mathcal{P}$ be an admissible point process. $\mathcal{P}$ is said to have *fast decay of correlations* if its correlation functions are ω-mixing as in Definition 1.1 with $\omega(n, x) = C_n \phi(c_n x)$ for some *correlation decay constants* $c_n, C_n \in (0, \infty)$ and a fast decreasing function $\phi : \mathbb{R}^+ \to [0, 1]$, called *a correlation decay function*.

More explicitly, an admissible point process has *fast decay of correlations*, if for all $p, q \in \mathbb{N}$ and all $(x_1, \ldots, x_{p+q}) \in (\mathbb{R}^d)^{p+q}$,

$$\begin{aligned} &\left|\rho^{(p+q)}(x_1, \ldots, x_{p+q}) - \rho^{(p)}(x_1, \ldots, x_p)\rho^{(q)}(x_{p+1}, \ldots, x_{p+q})\right| \\ &\quad \leq C_{p+q}\phi(c_{p+q}s), \end{aligned} \tag{1.10}$$

where $s := d(\{x_1, \ldots, x_p\}, \{x_{p+1}, \ldots, x_{p+q}\})$ is as at (1.3) and C_k, c_k, ϕ are as in Definition 1.2. Without loss of generality, we assume that c_k is nonincreasing in k, and that $C_k \in [1, \infty)$ is nondecreasing in k. As a by-product of our proof of the asymptotic normality of μ_n^ξ in (1.4), we establish that the fast decay of correlations of $\mathcal{P}$ implies that it is Brillinger mixing; cf. Remark (vi) in Section 1.4 and Remarks at the end of Section 4.4.2.

Admissible point processes having fast decay of correlations are ubiquitous and include certain determinantal, permanental and Gibbs point processes, as explained in Section 2.2. The k-point correlation functions of admissible point processes having fast decay of correlations are bounded, that is,

$$\sup_{(x_1,\ldots,x_k)\in(\mathbb{R}^d)^k} \rho^{(k)}(x_1, \ldots, x_k) \leq \kappa_k < \infty, \tag{1.11}$$

for some constants κ_k, which without loss of generality are assumed nondecreasing in k. Also without loss of generality, assume $\kappa_0 := \max\{\rho^{(1)}(\mathbf{0}), 1\}$. For stationary $\mathcal{P}$ with intensity $\rho^{(1)}(\mathbf{0}) \in (0, \infty)$ we have that (1.10) implies (1.11) with

$$\kappa_k \leq \left(\rho^{(1)}(\mathbf{0})\right)^k + \sum_{i=2}^{k} C_i \left(\rho^{(1)}(\mathbf{0})\right)^{k-i} \leq kC_k\kappa_0^k. \tag{1.12}$$

The bound (1.12) helps to determine when point processes having fast decay of correlations also have exponential moments, as in Section 2.1.

1.2. *Admissible score functions*. Throughout we restrict to translation-invariant score functions $\xi : \mathbb{R}^d \times \mathcal{N} \to \mathbb{R}$, that is, those which are measurable in each coordinate, $\xi(x, \mathcal{X}) = 0$ if $x \notin \mathcal{X} \in \mathcal{N}$, and for all $y \in \mathbb{R}^d$, satisfy $\xi(\cdot + y, \cdot + y) = \xi(\cdot, \cdot)$.

We introduce classes (A1) and (A2) of *admissible* score and input pairs $(\xi, \mathcal{P})$. Specific examples of admissible input pairs of both classes are provided in Sections 2.2 and 2.3. The first class allows for admissible input $\mathcal{P}$ as in Definition 1.2 whereas the second considers admissible input $\mathcal{P}$ having fast decay of correlations (1.10), subject to $c_k \equiv 1$ and growth conditions on the decay constants C_k and the decay function ϕ.

DEFINITION 1.3 (Class (A1) of admissible score and input pairs $(\xi, \mathcal{P})$). Admissible input $\mathcal{P}$ consists of admissible point processes having fast decay of correlations as in Definition 1.2. Admissible score functions are of the form

$$\xi(x, \mathcal{X}) := \frac{1}{k!} \sum_{\mathbf{x} \in \mathcal{X}^{(k-1)}} h(x, \mathbf{x}), \tag{1.13}$$

for some $k \in \mathbb{N}$ and a symmetric, translation-invariant function $h : \mathbb{R}^d \times (\mathbb{R}^d)^{k-1} \to \mathbb{R}$ such that $h(x_1, \dots, x_k) = 0$ whenever either $\max_{2 \le i \le k} |x_i - x_1| > r$ for some given $r > 0$ or when $x_i = x_j$ for some $i \neq j$. When $k = 1$, we set $\xi(x, \mathcal{X}) = h(x)$. Further, assume $\|h\|_\infty := \sup_{\mathbf{x} \in (\mathbb{R}^d)^{k-1}} |h(\mathbf{0}, \mathbf{x})| < \infty$.

The interaction range for h is at most r, showing that the functionals H_n^ξ defined at (1.2) generated via scores (1.13) are local U-statistics of order k as in [48]. Before introducing a more general class of score functions, we recall [5, 29, 40, 42, 45] a few definitions formalizing the notion of the local dependence of ξ on its input. Let $B_r(x) := \{y : |y - x| \le r\}$ denote the ball of radius r centered at x and $B_r^c(x)$ its complement.

DEFINITION 1.4 (Radius of stabilization). Given a score function ξ, input $\mathcal{X}$, and $x \in \mathcal{X}$, define the radius of stabilization $R^\xi(x, \mathcal{X})$ to be the smallest $r \in \mathbb{N}$ such that $\xi(x, \mathcal{X} \cap B_r(x)) = \xi(x, (\mathcal{X} \cap B_r(x)) \cup (\mathcal{A} \cap B_r^c(x)))$, for all $\mathcal{A} \subset \mathbb{R}^d$ locally finite. If no such finite r exists, we set $R^\xi(x, \mathcal{X}) = \infty$.

If ξ is a translation invariant function then so is $R^\xi(x, \mathcal{X})$. Score functions (1.13) of class (A1) have radius of stabilization upper-bounded by r.

DEFINITION 1.5 (Stabilizing score function). We say that ξ *is stabilizing on* $\mathcal{P}$ if for all $l \in \mathbb{N}$ there are constants $a_l > 0$, such that

$$\sup_{1 \le n \le \infty} \sup_{x_1, \dots, x_l \in W_n} \mathbb{P}_{x_1, \dots, x_l}\big(R^\xi(x_1, \mathcal{P}_n) > t\big) \le \varphi(a_l t) \tag{1.14}$$

with $\varphi(t) \downarrow 0$ as $t \to \infty$. Without loss of generality, the a_l are nonincreasing in l and $0 \le \varphi \le 1$. In (1.14) and elsewhere, we adopt the convention that $W_\infty := \mathbb{R}^d$ and $\mathcal{P}_\infty := \mathcal{P}$. The second sup in (1.14) is understood as ess sup with respect to the measure $\alpha^{(l)}$ at (1.7).

DEFINITION 1.6 (Exponentially stabilizing score function). We say that ξ is *exponentially stabilizing* on $\mathcal{P}$ if ξ is stabilizing on $\mathcal{P}$ as in Definition 1.5 with φ satisfying

$$\liminf_{t\to\infty} \frac{\log \varphi(t)}{t^c} \in (-\infty, 0) \tag{1.15}$$

for some $c \in (0, \infty)$.

We define a general class of score functions exponentially stabilizing on their input.

DEFINITION 1.7 [Class (A2) of admissible score and input pairs $(\xi, \mathcal{P})$]. Admissible input $\mathcal{P}$ consists of admissible point processes having fast decay of correlations as in Definition 1.2 with correlation decay constants satisfying $c_k \equiv 1$,

$$C_k = O(k^{ak}), \tag{1.16}$$

for some $a \in [0, 1)$ and correlation decay function ϕ satisfying the exponential decay condition

$$\liminf_{t\to\infty} \frac{\log \phi(t)}{t^b} \in (-\infty, 0) \tag{1.17}$$

for some constant $b \in (0, \infty)$. Admissible score functions ξ for this class are exponentially stabilizing on the input $\mathcal{P}$ and satisfy a *power growth condition*, namely there exists $\hat{c} \in [1, \infty)$ such that for all $r \in (0, \infty)$

$$\left|\xi\big(x, \mathcal{X} \cap B_r(x)\big)\right| \mathbf{1}\big[\operatorname{card}\big(\mathcal{X} \cap B_r(x)\big) = n\big] \le \big(\hat{c} \max(r, 1)\big)^n. \tag{1.18}$$

The condition $c_k \equiv 1$ is equivalent to $c_* := \inf c_k > 0$. This follows since we may replace the fast decreasing function $\phi(\cdot)$ by $\phi(c_* \times \cdot)$, with $c_k \equiv 1$ for this new fast decreasing function. Score functions of class (A1) also satisfy the power growth condition (1.18) since in this case the left-hand side of (1.18) is at most $\|h\|_\infty n^{(k-1)}/k$. Thus the generalization from (A1) to (A2) consists in replacing local U-statistics by exponentially stabilizing score functions satisfying the power growth condition. This is done at the price of imposing stronger conditions on the input process, requiring in particular that it has finite exponential moments, as explained in Section 2.1.

1.3. *Fast decay of correlations of the ξ-weighted measures.* The following p-moment condition involves the score function ξ and the input $\mathcal{P}$. We shall describe in Section 2.1 ways to control the p-moments of input pairs of class (A1) and (A2).

DEFINITION 1.8 (Moment condition). Given $p \in [1, \infty)$, say that the pair $(\xi, \mathcal{P})$ satisfies the p-moment condition if

$$\sup_{1 \le n \le \infty} \sup_{1 \le p' \le \lfloor p \rfloor} \sup_{x_1,\dots,x_{p'} \in W_n} \mathbb{E}_{x_1,\dots,x_{p'}} \left| \xi(x_1, \mathcal{P}_n) \right|^p \le \tilde{M}_p < \infty \tag{1.19}$$

for some constant $\tilde{M}_p := \tilde{M}_p^{\xi} \in [1, \infty)$, where sup signifies ess sup with respect to $\alpha^{(p)}$. Without loss of generality, we assume that $\tilde{M}_p$ is increasing in p for all p such that (1.19) holds.

We next consider the decay of the functions at (1.6), the so-called correlation functions of the ξ-weighted measures at (1.5). These functions indeed play the same role as the k-point correlation functions of the simple point process $\mathcal{P}$. When $\xi \equiv 1$ they obviously reduce to the correlation functions of $\mathcal{P}$. For general ξ and $k_i \equiv 1$, they are densities ("mixed moment densities" in the language of [5]) of the higher-order moment measures of the ξ-weighted measures with all distinct arguments. In the case of repeated arguments, the moment measures of a simple point process "collapse" to appropriate lower dimensional ones. This is neither the case for nonsimple point processes nor for our ξ-weighted measures, where general exponents k_i are required to properly take into account repeated arguments.

When $k_i \equiv 1$ for all $1 \le i \le p$, we write $m_{(p)}(x_1, \dots, x_p; n)$ instead of $m^{(1,\dots,1)}(x_1, \dots, x_p; n)$. Abbreviate $m^{(k_1,\dots,k_p)}(x_1, \dots, x_p; \infty)$ by $m^{(k_1,\dots,k_p)}(x_1, \dots, x_p)$. These functions exist whenever (1.19) is satisfied for p set to $k_1 + \cdots + k_p$ and provided the p-point correlation function $\rho^{(p)}$ exists. As for the input process $\mathcal{P}$, we consider mixing properties and fast decay of correlations for the ξ-weighted measures at (1.5).

DEFINITION 1.9 ($\tilde{\omega}$-mixing correlation functions of ξ-weighted measures). The correlation functions (1.6) are said to be *ω-mixing* if there exists a decreasing function $\tilde{\omega} : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $p \in \mathbb{N}$, $\lim_{x \to \infty} \tilde{\omega}(p, x) = 0$ and for all $p, q \in \mathbb{N}$, distinct $x_1, \dots, x_{p+q} \in \mathbb{R}^d$ and $n \in \mathbb{N} \cup \{\infty\}$:

$$\begin{aligned} &\big| m^{(k_1,\dots,k_{p+q})}(x_1, \dots, x_{p+q}; n) \\ &\quad - m^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n) m^{(k_{p+1},\dots,k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n) \big| \\ &\quad \le \tilde{\omega}(K, s), \end{aligned} \tag{1.20}$$

where $K := \sum_{i=1}^{p+q} k_i$ and $s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$ is as at (1.3).

DEFINITION 1.10 (Fast decay of correlations of the ξ-weighted measures). The ξ-weighted measures are said to have *fast decay of correlations* if their correlations functions are $\tilde{\omega}$-mixing as in Definition 1.9 with $\tilde{\omega}(n,x) = \tilde{C}_n \tilde{\phi}(\tilde{c}_n x)$ for some fast decreasing function $\tilde{\phi} : \mathbb{R}^+ \to [0,1]$ and some constants $\tilde{c}_n > 0$ and $\tilde{C}_n < \infty$.

More explicitly, the ξ-weighted measures (1.5) have fast decay of correlations if there exists a fast-decreasing function $\tilde{\phi}$ and constants $\tilde{C}_k < \infty$, $\tilde{c}_k > 0, k \in \mathbb{N}$ such that for all $n \in \mathbb{N} \cup \{\infty\}$, $p, q \in \mathbb{N}$ and any collection of positive integers $k_1, \dots, k_{p+q}$, we have

$$
\begin{aligned}
&\big|m^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n) \\
&\quad - m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n) m^{(k_{p+1},\dots,k_{p+q})}(x_{p+1},\dots,x_{p+q};n)\big| \\
&\quad \le \tilde{C}_K \tilde{\phi}(\tilde{c}_K s),
\end{aligned}
\tag{1.21}
$$

where $x_1, \dots, x_{p+q}$, K and s are as in Definition 1.9.

Our first theorem shows that the fast decay of correlations is inherited from the input process $\mathcal{P}$ by the ξ-weighted measures for a wide class of score functions and input. This key result forms the starting point of our approach.

THEOREM 1.11. *Let $(\xi, \mathcal{P})$ be an admissible score and input pair of class* (A1) *or* (A2) *such that the p-moment condition* (1.19) *holds for all $p \in (1,\infty)$. Then the correlations of the ξ-weighted measures decay fast as at* (1.21).

We prove this theorem in Section 3, where it is also shown that it subsumes more specialized results of [5, 50].

1.4. *Main results*. We give the limit theory for the measures μ_n^ξ, $n \ge 1$, defined at (1.4). Given a score function ξ on admissible input $\mathcal{P}$ we set [4]

$$
\sigma^2(\xi) := \mathbb{E}_{\mathbf{0}} \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) + \int_{\mathbb{R}^d} \big(m_{(2)}(\mathbf{0}, x) - m_{(1)}(\mathbf{0})^2\big)\,\mathrm{d}x.
\tag{1.22}
$$

The following result provides expectation and variance asymptotics for $\mu_n^\xi(f)$, with f belonging to the space $\mathcal{B}(W_1)$ of bounded measurable functions on W_1.

THEOREM 1.12. *Let $\mathcal{P}$ be an admissible point process on $\mathbb{R}^d$.*

(i) *If ξ satisfies exponential stabilization* (1.15) *and if $(\xi, \mathcal{P})$ satisfies the p-moment condition* (1.19) *for some $p \in (1,\infty)$ then for all $f \in \mathcal{B}(W_1)$*

$$
\left| n^{-1} \mathbb{E} \mu_n^\xi(f) - \mathbb{E}_{\mathbf{0}} \xi(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \int_{W_1} f(x)\,\mathrm{d}x \right| = O\big(n^{-1/d}\big).
\tag{1.23}
$$

[4]For a stationary point process $\mathcal{P}$, its Palm expectation $\mathbb{E}_{\mathbf{0}}$ [and consequently $m_{(1)}(\mathbf{0})$, $m_{(2)}(\mathbf{0}, x)\,\mathrm{d}x$] is meaningfully defined, for example, via the Palm–Matthes approach.

If ξ only satisfies stabilization (1.14) *and the p-moment condition* (1.19) *for some $p \in (1, \infty)$, then the right-hand side of* (1.23) *is $o(1)$.*

(ii) *Assume that the second correlation function $\rho^{(2)}$ of $\mathcal{P}$ exists and is bounded as in* (1.11), *that ξ satisfies* (1.14), *and that $(\xi, \mathcal{P})$ satisfies the p-moment condition* (1.19) *for some $p \in (2, \infty)$. If the second-order correlations of the ξ-weighted measures decay fast, that is, satisfy* (1.21) *with $p = q = k_1 = k_2 = 1$ and all $n \in \mathbb{N} \cup \{\infty\}$, then for all $f \in \mathcal{B}(W_1)$*

$$\lim_{n\to\infty} n^{-1} \operatorname{Var} \mu_n^{\xi}(f) = \sigma^2(\xi) \int_{W_1} f(x)^2 \,\mathrm{d}x \in [0, \infty), \tag{1.24}$$

whereas for all $f, g \in \mathcal{B}(W_1)$

$$\lim_{n\to\infty} n^{-1} \operatorname{Cov}\big(\mu_n^{\xi}(f), \mu_n^{\xi}(g)\big) = \sigma^2(\xi) \int_{W_1} f(x)g(x)\,\mathrm{d}x. \tag{1.25}$$

We remark that (1.23) and (1.24) together show convergence in probability

$$n^{-1} \mu_n^{\xi}(f) \xrightarrow{P} \mathbb{E}_{\mathbf{0}} \xi(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \int_{W_1} f(x)\,\mathrm{d}x \qquad \text{as } n \to \infty.$$

The proof of variance asymptotics (1.24) requires fast decay of the second-order correlations of the ξ-weighted measures. Fast decay of *all* higher-order correlations as in Definition 1.10 yields Gaussian fluctuations of μ_n^{ξ}, $n \geq 1$, under moment conditions on the atom sizes (i.e., under moment conditions on ξ) and a variance lower bound. Let N denote a mean zero normal random variable with variance 1. We write $f(n) = \Omega(g(n))$ when $g(n) = O(f(n))$, that is, when $\liminf_{n\to\infty} |f(n)/g(n)| > 0$.

THEOREM 1.13. *Let $\mathcal{P}$ be an admissible point process on $\mathbb{R}^d$ and let the pair $(\xi, \mathcal{P})$ satisfy the p-moment condition* (1.19) *for all $p \in (1, \infty)$. If the correlations of the ξ-weighted measures at* (1.5) *decay fast as in Definition* 1.10 *and if $f \in \mathcal{B}(W_1)$ satisfies*

$$\operatorname{Var} \mu_n^{\xi}(f) = \Omega\big(n^{\nu}\big) \tag{1.26}$$

for some $\nu \in (0, \infty)$, then as $n \to \infty$

$$\frac{\mu_n^{\xi}(f) - \mathbb{E}\mu_n^{\xi}(f)}{\sqrt{\operatorname{Var} \mu_n^{\xi}(f)}} \xrightarrow{\mathcal{D}} N. \tag{1.27}$$

Combining Theorem 1.11 and Theorem 1.13 yields the following theorem, which is well suited for off-the-shelf use in applications, as seen in Section 2.3.

THEOREM 1.14. *Let $(\xi, \mathcal{P})$ be an admissible pair of class* (A1) *or* (A2) *such that the p-moment condition* (1.19) *holds for all $p \in (1, \infty)$. If $f \in \mathcal{B}(W_1)$ satisfies condition* (1.26) *for some $\nu \in (0, \infty)$, then $\mu_n^{\xi}(f)$ is asymptotically normal as in* (1.27), *as $n \to \infty$.*

Theorems 1.12 and 1.13 are proved in Section 4. We next compare our results with those in the literature. Point processes mentioned below are defined in Section 2.2.

Remarks. (i) *Theorem* 1.12. In the case of Poisson and binomial input, the limits (1.23) and (1.24) are shown in [44] and [5, 40], respectively (the binomial point processes are not the restriction of an infinite point process to windows, but rather a re-scaled binomial point process on $[0,1]^d$). In the case of Gibbsian input, the limits (1.23) and (1.24) are established in [50]. Theorem 1.12 shows these limits hold for general stationary input. The paper [55] gives a weaker version of Theorem 1.12 for specific ξ and for $f = \mathbf{1}[x \in W_1]$. In full generality, the convergence rate (1.23) is new.

(ii) *Theorems* 1.13 *and* 1.14. Under condition (1.26), Theorems 1.13 and 1.14 provide a central limit theorem for nonlinear statistics of either α-determinantal and α-permanental input ($|\alpha|^{-1} \in \mathbb{N}$) with a fast-decaying kernel as at (2.7), the zero set $\mathcal{P}_{\mathrm{GEF}}$ of a Gaussian entire function, or rarified Gibbsian input. When $\xi \equiv 1$, then $\mu_n^\xi(f)$ reduces to the *linear statistic* $\sum_{x \in \mathcal{P}_n} f(x)$. These theorems extend the central limit theorem for linear statistics of $\mathcal{P}_{\mathrm{GEF}}$ as established in [37]. When the input is determinantal with a fast decaying kernel as at (2.7), then Theorems 1.13 and 1.14 also extend the main result of Soshnikov [53], whose pathbreaking paper gives a central limit theorem for linear statistics for any determinantal input, provided the variance grows as least as fast as a power of the expectation. Proposition 5.7 of [52] shows central limit theorems for linear statistics of α-determinantal point processes with $\alpha = -1/m$ or α-permanental point processes with $\alpha = 2/m$ for some $m \in \mathbb{N}$. During the revision of this article, we noticed the recent work [46]. This paper shows that when the kernel satisfies (2.7) with $\omega(s) = o(s^{-(d+\varepsilon)/2})$ and when $|\xi|$ is bounded with a deterministic radius of stabilization, then H_n^ξ at (1.2) is asymptotically normal. The generality of the score functionals and point processes considered in our article necessitates assumptions on the determinantal kernel which are more restrictive than those of [46, 53].

(iii) *Variance lower bounds.* To prove asymptotic normality, it is customary to require variance lower bounds as at (1.26); [37] and [53] both require assumptions of this kind. Showing condition (1.26) is a separate problem and it fails in general. For example, the variance of the point count of some determinantal point processes, including the GUE point process, grows at most logarithmically. This phenomena is especially pronounced in dimensions $d = 1, 2$. Additionally, given input $\mathcal{P}_{\mathrm{GEF}}$ and $\xi \equiv 1$, the bound (1.26) may fail even when f is a smooth cut-off that equals one in a neighborhood of the origin (cf. Proposition 5.2 of [36]). On the other hand, if $\xi \equiv 1$, and if the kernel K for a determinantal point process satisfies $\int_{\mathbb{R}^d} |K(\mathbf{0}, x)|^2 \,\mathrm{d}x < K(\mathbf{0}, \mathbf{0}) = \rho^{(1)}(\mathbf{0})$, then recalling the definition of $\sigma^2(\xi)$ at (1.22), we have $\sigma^2(\xi) = \sigma^2(1) = \rho^{(1)}(\mathbf{0}) - \int_{\mathbb{R}^d} |K(\mathbf{0}, x)|^2 \,\mathrm{d}x > 0$. In the case of rarified Gibbsian input, the bound (1.26) holds with $\nu = 1$, as shown in of [54], Theorem 1.1. Theorem 1.14 allows for surface-order variance growth, which arises for linear statistics $\sum_{x \in \mathcal{P}_n} \xi(x)$ of determinantal point processes; see [16], (4.15).

(iv) *Poisson, binomial and Gibbs input*. When $\mathcal{P}$ is Poisson or binomial input and when ξ is a functional which stabilizes exponentially fast as at (1.15), then μ_n^ξ is asymptotically normal (1.27) under moment conditions on ξ; see the survey [56]. When $\mathcal{P}$ is a rarified Gibbs' point process with "ancestor clans" decaying exponentially fast, and when ξ is an exponentially stabilizing functional, then μ_n^ξ satisfies normal convergence (1.27), as established in [50, 54].

(v) *Mixing conditions.* Central limit theorems for geometric functionals of mixing point processes (random fields) are established in [1, 12, 21–23, 25, 46]. The geometric functionals considered in these papers are different than the ones considered here; furthermore, the relation between the mixing conditions in these papers and ω-mixing correlation functions as in Definition 1.1 is unclear. Though correlation functions are simpler than mixing coefficients, which depend on σ-algebras generated by the point processes, our decay rates appear more restrictive than those needed in aforementioned papers. A careful investigation of the relations between the various notions of mixing and fast decay of correlations lies beyond the scope of our limit results and will be treated in a separate paper. In the case of point processes on discrete spaces, such a study is easier; cf. [47].

(vi) *Brillinger mixing and fast decay of correlations.* Brillinger mixing [25], Section 3.5, is defined via finiteness of integrals of the *reduced cumulant measures* (see Section 4.3.2). The very definition of Brillinger mixing implies volume-order growth of cumulants; the converse follows using the ideas in the proof of [7], Theorem 3.2. The key to proving our announced central limit theorems is to show that the fast decay of correlations of the ξ-weighted measures (1.5) implies volume-order growth of cumulants, and hence Brillinger mixing; see the remarks at the beginning of Section 4.3 and also those and at the end of Section 4.4.2.

(vii) *Multivariate central limit theorem.* We may use the Cramér–Wold device to extend Theorems 1.12 and 1.14 to the multivariate setting as follows. Let $(\xi, \mathcal{P})$ be a pair satisfying the hypotheses of Theorems 1.12 and 1.14. If $f_i \in \mathcal{B}(W_1)$, $1 \leq i \leq k$, satisfy the variance limit (1.24) with $\sigma^2(\xi) > 0$, then as $n \to \infty$ the fidis

$$\left(\frac{\mu_n^\xi(f_1) - \mathbb{E}\mu_n^\xi(f_1)}{\sqrt{n}}, \dots, \frac{\mu_n^\xi(f_k) - \mathbb{E}\mu_n^\xi(f_k)}{\sqrt{n}}\right)$$

converge to that of a centred Gaussian field having covariance kernel $f, g \mapsto \sigma^2(\xi) \int_{W_1} f(x)g(x)\,\mathrm{d}x$.

(viii) *Deterministic radius of stabilization*. It may be shown that our main results go through without the condition (1.17) if the radius of stabilization $R^\xi(x, \mathcal{P})$ is bounded by a nonrandom (deterministic) constant and if (1.16) and (1.18) are satisfied. However, we are unable to find any interesting examples of point processes satisfying (1.10) but not (1.17).

(ix) *Fast decay of the correlation of the ξ-weighted measures; Theorem* 1.11. Though the cumulant method is common to [5, 37, 50] and this article, a distinguishing and novel feature of our approach is the proof of fast decay of correlations of the ξ-weighted measures (1.21), and consequently their Brillinger mixing, for a wide class of functionals and point processes. As mentioned in the Introduction, the proof of this result is via factorial moment expansions, which differs from the approach of [5, 37, 50] (see the remarks at the beginning of Section 3). Fast decay of correlations of the ξ-weighted measures (1.21) appears to be of independent interest. It features in the proofs of moderate deviation principles and laws of the iterated logarithms for stabilizing functionals of Poisson point process [3, 14]. Fast decay of correlations (1.21) yields volume order cumulant bounds, useful in establishing concentration inequalities as well as moderate deviations, as explained in [18], Lemma 4.2.

(x) *Normal approximation.* Difference operators (which appear in our factorial moment expansions) are also a key tool in the Malliavin–Stein method [38, 39]. This method yields presumably optimal rates of normal convergence for various statistics (including many considered in Section 2.3) in stochastic geometric problems [27–29, 48]. However, these methods currently apply only to functionals defined on Poisson and binomial point processes. It is an open question whether a refined use of these methods would yield rates of convergence in our central limit theorems.

(xi) *Cumulant bounds.* As mentioned, we establish that the kth order cumulants for $\langle f, \mu_n^{\xi}\rangle$ grow *at most linearly* in n for $k \geq 1$. Thus, under assumption (1.26), the cumulant C_n^k for $\langle f, \mu_n^{\xi}\rangle/\sqrt{\operatorname{Var}\langle f, \mu_n^{\xi}\rangle}$ satisfies $C_n^k \leq D(k) n^{1-(\nu k/2)}$, with $D(k)$ depending only on k. For $k = 3, 4, \dots$ and $\nu > 2/3$, we have $C_n^k \leq D(k)/(\Delta(n))^{k-2}$, where $\Delta(n) := n^{(3\nu-2)/2}$. When $D(k)$ satisfies $D(k) \leq (k!)^{1+\gamma}$, γ a constant, we obtain the Berry–Esseen bound (cf. [18], Lemma 4.2):

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\mu_n^{\xi}(f) - \mathbb{E}\mu_n^{\xi}(f)}{\sqrt{\operatorname{Var} \mu_n^{\xi}(f)}} \leq t \right) - \mathbb{P}(N \leq t) \right| = O\big(\Delta(n)^{-1/(1+2\gamma)}\big).$$

Determining conditions on input pairs $(\xi, \mathcal{P})$ insuring the bounds $\nu > 2/3$ and $D(k) \leq (k!)^{1+\gamma}$, γ a constant, is beyond the scope of this paper. When $\mathcal{P}$ is Poisson input, this issue is addressed by [14].

We next consider the case when the fluctuations of $H_n^{\xi}(\mathcal{P})$ are not of volume-order, that is to say $\sigma^2(\xi) = 0$. Though this may appear to be a degenerate condition, interesting examples involving determinantal point processes or zeros of GEF in fact satisfy $\sigma^2(1) = 0$. Such point processes are termed "super-homogeneous point processes" [37], Remark 5.1. Put

$$\widehat{H}_n^{\xi}(\mathcal{P}) := \sum_{x\in\mathcal{P}_n} \xi(x, \mathcal{P}). \tag{1.28}$$

The summands in $\widehat{H}_n^{\xi}(\mathcal{P})$, in contrast to those of $H_n^{\xi}(\mathcal{P})$, are not sensitive to boundary effects. We shall show that under volume-order scaling the asymptotic variance of $\hat{H}_n^{\xi}(\mathcal{P})$ also equals $\sigma^2(\xi)$. However, when $\sigma^2(\xi) = 0$ we derive surface-order variance asymptotics for $\hat{H}_n^{\xi}(\mathcal{P})$. Though a similar result should plausibly hold for $H_n^\xi(\mathcal{P})$, a proof seems beyond the scope of the current paper. Letting Vol_d denote the d-dimensional Lebesgue volume, for $y \in \mathbb{R}^d$ and $W \subset \mathbb{R}^d$, put

$$\gamma_W(y) := \mathrm{Vol}_d\big(W \cap \big(\mathbb{R}^d \setminus W - y\big)\big). \tag{1.29}$$

By [34], Lemma 1(a), we are justified in writing $\gamma(y) := \lim_{n\to\infty} \gamma_{W_n}(y)/n^{(d-1)/d}$.

THEOREM 1.15. *Under the assumptions of Theorem* 1.12(ii), *suppose also that ξ is exponentially stabilizing on $\mathcal{P}$ as in* (1.15). *Then*

$$\lim_{n\to\infty} n^{-1} \operatorname{Var} H_n^\xi(\mathcal{P}) = \sigma^2(\xi). \tag{1.30}$$

If moreover $\sigma^2(\xi) = 0$ in (1.24), *then*

$$\begin{aligned}\lim_{n\to\infty} & n^{-(d-1)/d} \operatorname{Var} \widehat{H}_n^\xi(\mathcal{P}) \\ &= \sigma^2(\xi, \gamma) \\ &:= \int_{\mathbb{R}^d} \big(m_{(1)}(\mathbf{0})^2 - m_{(2)}(\mathbf{0}, x)\big)\gamma(x)\,\mathrm{d}x \in [0, \infty).\end{aligned} \tag{1.31}$$

Remarks. (i) Checking positivity of $\sigma^2(\xi, \gamma)$ is not always straightforward, though we note if ξ has the form (1.13), then the disintegration formula (1.9) yields

$$\sigma^2(\xi, \gamma) = \sum_{j=0}^{k} \int_{\mathbb{R}^d} \frac{\gamma(x)\zeta_j(x)}{j!(k-j-1)!(k-j-1)!}\,\mathrm{d}x,$$

where $\zeta_j(x) = \int_{A_j(x)} h(\mathbf{0}, \mathbf{y}, \mathbf{z})h(x, \mathbf{x}, \mathbf{z})[\rho^{(k)}(\mathbf{0}, \mathbf{y}, \mathbf{z})\rho^{(k)}(x, \mathbf{x}, \mathbf{z}) - \rho^{(2k-j)}(\mathbf{0}, \mathbf{y}, \mathbf{z}, x, \mathbf{x})]\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}$ and $A_j(x) = (B_r(\mathbf{0}) \cap B_r(x))^j \times B_r(\mathbf{0})^{k-j-1} \times B_r(x)^{k-j-1}$.

(ii) Theorem 1.12 and Theorem 1.15 extend [34], Propositions 1 and 2, which are valid only for $\xi \equiv 1$, to general functionals. If an admissible pair $(\xi, \mathcal{P})$ of type (A1) or (A2) is such that $\widehat{H}_n^\xi(\mathcal{P})$ does not have volume-order variance growth, then Theorems 1.12 and 1.15 show that $\widehat{H}_n^\xi(\mathcal{P})$ has at most surface-order variance growth.

2. Examples and applications. Before providing examples and applications of our general results, we briefly discuss the moment assumptions involved in our main theorems.

2.1. *Moments of point processes having fast decay of correlations*. We say that $\mathcal{P}$ has exponential moments if for all bounded Borel $B \subset \mathbb{R}^d$ and all $t \in \mathbb{R}^+$ we have

$$\mathbb{E}\big[t^{\mathcal{P}(B)}\big] < \infty. \tag{2.1}$$

Similarly, say that $\mathcal{P}$ has all moments if for all bounded Borel $B \subset \mathbb{R}^d$ and all $k \in \mathbb{N}$, we have

$$\mathbb{E}[\mathcal{P}(B)^k] < \infty. \tag{2.2}$$

Remarks. (i) The point process $\mathcal{P}$ has exponential moments whenever $\sum_{k=1}^{\infty} \kappa_k t^k / k! < \infty$ for all $t \in \mathbb{R}^+$ with κ_k as in (1.11) (cf. the expansion of the probability generating function of a random variable in terms of factorial moments [13], Proposition 5.2.III). By (1.12), an admissible point process having fast decay of correlations has exponential moments provided

$$\sum_{k=1}^{\infty} \frac{C_k t^k}{(k-1)!} < \infty, \qquad t \in \mathbb{R}^+. \tag{2.3}$$

Note that input of type (A2) has exponential moments since by (1.16), we have $C_k = O(k^{ak})$, $a \in [0, 1)$, making (2.3) summable. For pairs $(\xi, \mathcal{P})$ of type (A2) with radius of stabilization bounded by $r_0 \in [1, \infty)$, by (1.18) the p-moment in (1.19) is consequently controlled by a finite exponential moment, that is, for $x_1, \dots, x_{p'} \in W_n$,

$$\mathbb{E}_{x_1,\dots,x_{p'}} |\xi(x_1, \mathcal{P}_n)|^p \le \mathbb{E}_{x_1,\dots,x_{p'}} (\hat{c} r_0)^{p\mathcal{P}(B_{r_0}(x_1))}. \tag{2.4}$$

Finally, if $\mathcal{P}$ has exponential moments under its stationary probability $\mathbb{P}$, the same is true under $\mathbb{P}_{x_1,\dots,x_k}$ for $\alpha^{(k)}$ almost all $x_1, \dots, x_k$.[5]

(ii) For pairs $(\xi, \mathcal{P})$ of type (A1), the p-moment (1.19) satisfies for $x_1, \dots, x_{p'} \in W_n$

$$\mathbb{E}_{x_1,\dots,x_{p'}} |\xi(x_1, \mathcal{P}_n)|^p \le \left(\frac{\|h\|_\infty}{k}\right)^p \mathbb{E}_{x_1,\dots,x_{p'}} \big[\big(\mathcal{P}(B_r(x_1))\big)^{(k-1)p}\big]. \tag{2.5}$$

We next show that (2.5) may be controlled by moments of Poisson random variables. For any Borel set $B \subset (\mathbb{R}^d)^k$, the definition of factorial moment measures gives $\alpha^{(k)}(B) \le \kappa_k \operatorname{Vol}_{dk}(B)$. Since moments may be expressed as a linear combination of factorial moments, for $k \in \mathbb{N}$ and a bounded Borel subset $B \subset \mathbb{R}^d$, using (1.8) we have

$$\begin{aligned} \mathbb{E}\big[\big(\mathcal{P}(B)\big)^k\big] &= \sum_{j=0}^{k} \left\{ k \atop j \right\} \alpha^{(j)}(B^j) \le \kappa_k \sum_{j=0}^{k} \left\{ k \atop j \right\} \operatorname{Vol}_{jd}(B)^j \\ &= \kappa_k \mathbb{E}\big(\operatorname{Po}\big(\operatorname{Vol}_d(B)\big)^k\big), \end{aligned} \tag{2.6}$$

---

[5] Indeed, if $\mathbb{E}_{x_1,\dots,x_k}[\rho^{\mathcal{P}(B_r(x_1))}] = \infty$ for $x_1, \dots, x_k \in B'$ for some bounded $B' \in \mathbb{R}^d$ such that $\alpha^{(k)}(B'^k) > 0$ then $\mathbb{E}_{x_1,\dots,x_k}[\rho^{\mathcal{P}(B_r(x_1))}] \le \mathbb{E}_{x_1,\dots,x_k}[\rho^{\mathcal{P}(B'_r)}] = \infty$ with $B'_r = B' \oplus B_r(\mathbf{0}) = \{y' + y : y' \in B', y \in B_r(\mathbf{0})\}$ the r-parallel set of B'. Integrating with respect to $\alpha^{(k)}$ in B'^k, by the Campbell formula $\mathbb{E}[(\mathcal{P}(B'_r))^k \rho^{\mathcal{P}(B'_r)}] = \infty$, which contradicts the existence of exponential moments under $\mathbb{P}$.

where $\left\{ {k \atop j} \right\}$ stand for the *Stirling numbers of the second kind*, $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ and where κ_j's are nondecreasing in j. Thus by (1.12), an admissible point process having fast decay of correlations has all moments, as in (2.2). If $\mathcal{P}$ has all moments under its stationary probability $\mathbb{P}$, the same is true under $\mathbb{P}_{x_1,\dots,x_k}$ for $\alpha^{(k)}$ almost all $x_1,\dots,x_k$ (by the same arguments as in Footnote 5).

2.2. *Examples of point processes having fast decay of correlations*. The notion of a stabilizing functional is well established in the stochastic geometry literature but since the notion of fast decay of correlations for point processes (1.10) is less well studied, we first establish that some well-known point processes enjoy this property. For more details on the first five examples, we refer to [24].

2.2.1. *Class* $A1$ *input*.

Permanental input. The point process $\mathcal{P}$ is permanental if its correlation functions are defined by $\rho^{(k)}(x_1,\dots,x_k) := \mathrm{per}(K(x_i,x_j))_{1\le i,j\le k}$, where the permanent of an $n\times n$ matrix M is $\mathrm{per}(M) := \sum_{\pi\in S_n}\prod_{i=1}^n M_{i,\pi(i)}$, with S_n denoting the permutation group of the first n integers and $K(\cdot,\cdot)$ is the Hermitian kernel of a locally trace class integral operator $\mathcal{K}: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ [24], Assumption 4.2.3. A kernel K is *fast-decreasing* if

$$\big|K(x,y)\big| \le \omega\big(|x-y|\big), \qquad x,y\in\mathbb{R}^d, \tag{2.7}$$

for some fast-decreasing $\omega:\mathbb{R}^+\to\mathbb{R}^+$. [11], Lemma 1.5, in the Supplementary Material shows that if a stationary permanental point process has a fast-decreasing kernel as at (2.7), then it is an admissible point process having fast decay of correlations with decay function $\phi=\omega$ and with correlation decay constants satisfying

$$C_k := kk!\|K\|^{k-1}, \qquad c_k \equiv 1, \tag{2.8}$$

where $\|K\| := \sup_{x,y}|K(x,y)|$ and we can choose $\kappa_k = k!\|K\|^k$. However, a trace class permanental point process in general does not have exponential moments, that is, the right-hand side of (2.1) might be infinite for some bounded B and ρ large enough. [6]

The permanental point process with kernel K may be represented as a Cox point process (see Section 2.2.3) directed by the random measure $\eta(B) := \int_B (Z_1(x)^2 + Z_2(x)^2)\,\mathrm{d}x$, $B\subset\mathbb{R}^d$, where the intensity $Z_1(x)^2+Z_2(x)^2$ is a sum of i.i.d. Gaussian random fields with zero mean and covariance function $K/2$ [52], Theorem 6.13. Thus mean zero Gaussian random fields with a fast decaying covariance function $K/2$ yield a permanental (Cox) point process with kernel K and having fast decay of correlations.

[6]This is because, the number of points of a (trace-class) permanental p.p. in a compact set B is a sum of independent geometric random variables $\mathrm{Geo}(1/(1+\lambda))$ where λ runs over all eigenvalues of the integral operator defining the process truncated to B.

α-permanental point processes. See [24], Section 4.10, [35] and [52] for more details on this class of point processes which generalize permanental point processes. Given $\alpha \ge 0$ and a kernel K which is Hermitian, nonnegative definite and locally trace class, a point process $\mathcal{P}$ is said to be *α-permanental*[7] if its correlation functions satisfy

$$\rho^{(k)}(x_1, \dots, x_k) = \sum_{\pi \in S_k} \alpha^{k-\nu(\pi)} \prod_{i=1}^{k} K(x_i, x_{\pi(i)}), \tag{2.9}$$

where S_k stands for the usual symmetric group and $\nu(\cdot)$ denotes the number of cycles in a permutation. The right-hand side is the α-permanent of the matrix $((K(x_i, x_j))_{i,j \le k}$. The special cases $\alpha = 0$ and $\alpha = 1$, respectively, give the Poisson point process with intensity $K(\mathbf{0}, \mathbf{0}))$ and the permanental point process with kernel K. In what follows, we assume $\alpha = 1/m$ for $m \in \mathbb{N}$, that is, $1/\alpha$ is a positive integer. Existence of such α-permanental point processes is guaranteed by [52], Theorem 1.2. The property of these point processes most important to us is that an α-permanental point process with kernel K is a superposition of $1/\alpha$ i.i.d. copies of a permanental point process with kernel αK (see [24], Section 4.10). Also from Definition (2.9), we obtain $\rho^{(k)}(x_1, \dots, x_k) \le \|K\|^k \alpha^k \sum_{\pi \in S_k} (\alpha^{-1})^{\nu(\pi)}$, and so we can take $\kappa_k = \prod_{i=0}^{k-1} (j\alpha + 1)\|K\|^k$ for an α-permanental point process. The following result is a consequence of the upcoming Proposition 2.3 and the identity (2.8) for decay constants of a permanental point process with kernel αK.

PROPOSITION 2.1. *Let $\alpha = 1/m$ for some $m \in \mathbb{N}$ and let $\mathcal{P}_\alpha$ be the stationary α-permanental point process with a kernel K which is Hermitian, nonnegative definite and locally trace class. Assume also that $|K(x, y)| \le \omega(|x - y|)$ for some fast-decreasing ω. Then $\mathcal{P}_\alpha$ is an admissible point process having fast decay of correlations with correlation decay constants $C_k = km^{1-k(m-1)} m!(k!)^m \|K\|^{km-1}$, $c_k = 1$ and decay function $\phi = \omega$.*

Zero set of Gaussian entire function (GEF). A Gaussian entire function $f(z)$ is the sum $\sum_{j \ge 0} X_j \frac{z^j}{\sqrt{j!}}$, where X_j are i.i.d. with the standard normal density on the complex plane. The zero set $f^{-1}(\{0\})$ gives rise to the point process $\mathcal{P}_{\mathrm{GEF}} := \sum_{x \in f^{-1}(\{0\})} \delta_x$ on $\mathbb{R}^2$. The point process $\mathcal{P}_{\mathrm{GEF}}$ is an admissible point process having fast decay of correlations [37], Theorem 1.4, and exhibits local repulsion of points. Though $\mathcal{P}_{\mathrm{GEF}}$ satisfies condition (1.17), it is unclear whether (1.16) holds. By [26], Theorem 1, $\mathcal{P}_{\mathrm{GEF}}(B_r(\mathbf{0}))$ has exponential moments.

Moment conditions. For $p \in [1, \infty)$, we show that the p-moment condition (1.19) holds when ξ is such that the pair $(\xi, \mathcal{P}_{\mathrm{GEF}})$ is of class (A1). By [37], The-

[7] In contrast to terminology in [24, 52], here we distinguish the two cases (i) $\alpha \ge 0$ (α-permanental) and (ii) $\alpha \le 0$ (α-determinantal).

orem 1.3, given $\mathcal{P} := \mathcal{P}_{\mathrm{GEF}}$, there exist constants $\tilde{D}_k$ such that

$$
\begin{aligned}
\tilde{D}_k^{-1} \prod_{i<j} \min\{|y_i - y_j|^2, 1\} &\le \rho^{(k)}(y_1, \dots, y_k) \\
&\le \tilde{D}_k \prod_{i<j} \min\{|y_i - y_j|^2, 1\}.
\end{aligned} \tag{2.10}
$$

Recall from [52], Lemma 6.4 (see also [20], Theorem 1, [8], Proposition 2.5), that the existence of correlation functions of any point process implies existence of reduced Palm correlation functions $\rho^{(k)}_{x_1,\dots,x_p}(y_1, \dots, y_k)$, which satisfy the following useful multiplicative identity: For Lebesgue a.e. $(x_1, \dots, x_p)$ and $(y_1, \dots, y_k)$, all distinct,

$$
\rho^{(p)}(x_1, \dots, x_p)\rho^{(k)}_{x_1,\dots,x_p}(y_1, \dots, y_k) = \rho^{(p+k)}(x_1, \dots, x_p, y_1, \dots, y_k). \tag{2.11}
$$

Combining (2.10) and (2.11), we get for Lebesgue a.e. $(x_1, \dots, x_p)$ and $(y_1, \dots, y_k)$, that

$$
\rho^{(k)}_{x_1,\dots,x_p}(y_1, \dots, y_k) \le D_{p+k}\rho^{(k)}(y_1, \dots, y_k), \tag{2.12}
$$

where $D_{p+k} := \tilde{D}_{p+k}\tilde{D}_p\tilde{D}_k$. Thus we have shown there exist constants D_j, $j \in \mathbb{N}$, such that for any bounded Borel subset B, $k \in \mathbb{N}$ and Lebesgue a.e. $(x_1, \dots, x_p) \in (\mathbb{R}^d)^p$, we have

$$
\mathbb{E}^!_{x_1,\dots,x_p}(\mathcal{P}^{(k)}(B^k)) \le D_{p+k}\mathbb{E}(\mathcal{P}^{(k)}(B^k)). \tag{2.13}
$$

By (2.5), (2.13) and (2.6) in this order, along with stationarity of $\mathcal{P}_{\mathrm{GEF}}$, we have for any $p \in [1, \infty)$,

$$
\begin{aligned}
&\sup_{1\le n\le\infty} \sup_{1\le p'\le\lfloor p\rfloor} \sup_{x_1,\dots,x_{p'}\in W_n} \mathbb{E}_{x_1,\dots,x_{p'}}\big|\xi(x_1, \mathcal{P}_n)\big|^p \\
&\quad \le \left(\frac{\|h\|_\infty}{k}\right)^p \kappa_{(k-1)p} D_{(k-1)p}\mathbb{E}\big[\big(\mathrm{Po}\big(\mathrm{Vol}_d\big(B_r(\mathbf{0})\big)\big) + p\big)^{(k-1)p}\big] < \infty,
\end{aligned} \tag{2.14}
$$

where as before $\mathrm{Po}(\lambda)$ denotes a Poisson random variable with mean λ and where we have assumed without loss of generality that the constants D_k are increasing in k. Thus the p-moment condition (1.19) holds for pairs $(\xi, \mathcal{P}_{\mathrm{GEF}})$ of class (A1) for all $p \in [1, \infty)$.

2.2.2. *Class A2 input*.

Determinantal input. The point process $\mathcal{P}$ is determinantal if its correlation functions are defined by $\rho^{(k)}(x_1, \dots, x_k) = \det(K(x_i, x_j))_{1\le i,j\le k}$, where $K(\cdot,\cdot)$ is again the Hermitian kernel of a locally trace class integral operator $\mathcal{K} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Determinantal point processes exhibit local repulsivity and their structure is preserved when restricting to subsets of $\mathbb{R}^d$ and as well as when considering their reduced Palm versions. These facts facilitate our analysis of determinantal input; the Supplementary Material [11] provides lemmas further illustrating their tractability. If a stationary determinantal point process has a fast-decreasing kernel as at (2.7), then [11], Lemma 1.3, in the Supplementary Material shows that it is an admissible point process having fast decay of correlations satisfying (1.16) with decay function $\phi = \omega$, with ω as at (2.7), and correlation decay constants

$$C_k := k^{1+(k/2)} \|K\|^{k-1}, \qquad c_k \equiv 1. \tag{2.15}$$

Consequently, ϕ satisfies the requisite exponential decay (1.17) whenever ω itself satisfies (1.17).

The Ginibre ensemble of eigenvalues of $N \times N$ matrices with independent standard complex Gaussian entries is a leading example of a determinantal point process. The limit of the Ginibre ensemble as $N \to \infty$ is the Ginibre point process (or the infinite Ginibre ensemble), here denoted $\mathcal{P}_{\mathrm{GIN}}$. It is the prototype of a stationary determinantal point process and has the following kernel: For $z_1, z_2 \in \mathbb{C}$,

$$K(z_1, z_2) := \exp(z_1 \bar{z_2}) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2}\right) = \exp\left(i \mathrm{Im}(z_1 \bar{z_2}) - \frac{|z_1 - z_2|^2}{2}\right).$$

More generally, for $0 < \beta \le 1$, the β-Ginibre (determinantal) point process (see [17]) has kernel

$$K_\beta(z_1, z_2) := \exp\left(\frac{1}{\beta} z_1 \bar{z_2}\right) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2\beta}\right), \qquad z_1, z_2 \in \mathbb{C}.$$

When $\beta = 1$, we obtain $\mathcal{P}_{\mathrm{GIN}}$ and as $\beta \to 0$ we obtain the Poisson point process. Thus the β-Ginibre point process interpolates between the Ginibre and Poisson point processes. Identifying the complex plane with $\mathbb{R}^2$, we see that all β-Ginibre point processes are admissible point processes having fast decay of correlations satisfying (1.16) and (1.17).

Moment conditions. Let $p \in [1, \infty)$ and let $\mathcal{P}$ be a stationary determinantal point process with a continuous and fast-decreasing kernel. We now show that the p-moment condition (1.19) holds for pairs $(\xi, \mathcal{P})$ of class (A1) or (A2), provided ξ has a deterministic radius of stabilization, say $r_0 \in [1, \infty)$. First, for all $(x_1, \dots, x_p) \in (\mathbb{R}^d)^p$, all increasing $F : \mathbb{N} \to \mathbb{R}^+$ and all bounded Borel sets B we have [17], Theorem 2, $\mathbb{E}^!_{x_1,\dots,x_p}(F(\mathcal{P}(B))) \le \mathbb{E}(F(\mathcal{P}(B)))$. Thus using (2.4), the above inequality and stationarity of $\mathcal{P}$, we get that for any bounded stabilizing score function ξ of class (A2),

$$\begin{aligned} &\sup_{1 \le n \le \infty} \sup_{1 \le p' \le \lfloor p \rfloor} \sup_{x_1, \dots, x_{p'} \in W_n} \mathbb{E}_{x_1,\dots,x_{p'}} \left|\xi(x_1, \mathcal{P}_n)\right|^p \\ &\qquad \le \mathbb{E}(\hat{c} r_0)^{p\mathcal{P}(B_{r_0}(\mathbf{0})) + p^2} < \infty. \end{aligned} \tag{2.16}$$

The finiteness of the last term follows from the fact that determinantal input considered here is of class (A2) and, by Remark (i) at the beginning of Section 2.1, such input has finite exponential moments.

α-Determinantal point processes. Similar to permanental point processes, we generalize determinantal point processes to include their α-determinantal versions, by requiring that the correlation functions satisfy (2.9) for some $\alpha \leq 0$. In what follows, we shall assume that $\alpha = -1/m, m \in \mathbb{N}$. Existence of such α-determinantal point processes again follows from [52], Theorem 1.2. Likewise, an α-determinantal point process with kernel K is a superposition of $-1/\alpha$ i.i.d. copies of a determinantal point process with kernel $-\alpha K$ ([24], Section 4.10). By [52], Proposition 4.3, we can take $\kappa_k = K(\mathbf{0}, \mathbf{0})^k$ for an α-determinantal point process. Analogously to Proposition 2.1, the next result follows from Proposition 2.3 below and the identity (2.15) for correlation decay constants of a determinantal point process with kernel $-\alpha K$.

PROPOSITION 2.2. *Let $\alpha = -1/m$ for some $m \in \mathbb{N}$ and $\mathcal{P}_\alpha$ be the stationary α-determinantal point process with a kernel K which is Hermitian, nonnegative definite and locally trace class. Assume also that $|K(x, y)| \leq \omega(|x - y|)$ for some fast-decreasing function ω. Then $\mathcal{P}_\alpha$ is an admissible point process having fast decay of correlations with decay function $\phi = \omega$ and correlation decay constants $C_k = m^{1-k(m-1)} m! K(\mathbf{0}, \mathbf{0})^{k(m-1)} k^{1+(k/2)} \|K\|^{k-1}$, $c_k = 1$. Further, if w satisfies (1.17), then $\mathcal{P}_\alpha$ is an admissible input of type* (A2).

From (2.16) and [11], (1.11), in the Supplementary Material, we have that for $\mathcal{P}_\alpha$, $-1/\alpha \in \mathbb{N}$ as above, and for any bounded stabilizing score function ξ of class (A2),

$$\sup_{1 \leq n \leq \infty} \sup_{1 \leq p' \leq \lfloor p \rfloor} \sup_{x_1, \ldots, x_{p'} \in W_n} \mathbb{E}_{x_1, \ldots, x_{p'}} \left| \xi(x_1, \mathcal{P}_n) \right|^p < \infty. \tag{2.17}$$

Rarified Gibbsian input. Consider the class Ψ of Hamiltonians consisting of pair potentials without negative part, area interaction Hamiltonians, hard core Hamiltonians and potentials generating a truncated Poisson point process (see [50] for further details of such potentials). For $\Psi \in \mathbf{\Psi}$ and $\beta \in (0, \infty)$, let $\mathcal{P}^{\beta\Psi}$ be the Gibbs' point process having Radon–Nikodym derivative $\exp(-\beta\Psi(\cdot))$ with respect to a reference homogeneous Poisson point process on $\mathbb{R}^d$ of intensity $\tau \in (0, \infty)$. There is a range of inverse temperature and activity parameters (β and τ) such that $\mathcal{P}^{\beta\Psi}$ has fast decay of correlations; see the introduction to Section 3 and [50] for further details. These rarified Gibbsian point processes are admissible point processes having fast decay of correlations and satisfy the input conditions (1.16) and (1.17) of class (A2). Setting $\xi(\cdot, \cdot) \equiv 1$ in Lemma 3.4 of [50] shows that (1.10) holds with C_k a scalar multiple of k and c_k a constant.

2.2.3. *Additional input examples.* For additional examples of admissible point processes having fast decay of correlations, we refer to the arxiv version of this paper [10], Section 2.3. We shall discuss but one example here.

Superpositions of i.i.d. point processes. A natural operation on point processes generating new point processes consists of independent superposition. We show that this operation preserves fast decay of correlations.

Let $\mathcal{P}_1, \dots, \mathcal{P}_m, m \in \mathbb{N}$, be i.i.d. copies of an admissible point process $\mathcal{P}$ with correlation functions ρ and having fast decay of correlations. Let ρ_0 denote the correlation functions of the point process $\mathcal{P}_0 := \bigcup_{i=1}^m \mathcal{P}_i$. For any $k \geq 1$ and distinct $x_1, \dots, x_k \in \mathbb{R}^d$, the following relation holds:

$$\rho_0^{(k)}(x_1, \dots, x_k) = \sum_{\sqcup_{i=1}^m S_i = [k]} \prod_{i=1}^m \rho(S_i), \tag{2.18}$$

where $\sqcup$ stands for disjoint union and where we abbreviate $\rho^{(|S_i|)}(x_j : j \in S_i)$ by $\rho(S_i)$. Here S_i may be empty, in which case we set $\rho(\varnothing) = 1$. From (2.18), we have that $\mathcal{P}_0$ is an admissible point process with intensity $m\rho^{(1)}(\mathbf{0})$. Further, we take $\kappa_k(\mathcal{P}_0) = (\kappa_k)^m m^k$. The proof of the next proposition, which shows that $\mathcal{P}_0$ has fast decay of correlations, is in the Supplementary Material (cf. [11], Proposition 1.8).

PROPOSITION 2.3. *Let $m \in \mathbb{N}$ and $\mathcal{P}_1, \dots, \mathcal{P}_m$ be i.i.d. copies of an admissible point process $\mathcal{P}$ having fast decay of correlations with decay function ϕ and correlation decay constants C_k and c_k. Then $\mathcal{P}_0 := \bigcup_{i=1}^m \mathcal{P}_i$ is an admissible point process having fast decay of correlations with decay function ϕ and correlation decay constants $m^k m!(\kappa_k)^{m-1} C_k$ and c_k. Further, if $\mathcal{P}$ is admissible input of type* (A2) *with $\kappa_k \leq \lambda^k$ for some $\lambda \in (0, \infty)$, then $\mathcal{P}_0$ is also admissible input of type* (A2).

We have already used this proposition in the context of fast decay of correlations of α-permanental and determinantal point processes.

2.3. *Applications.* Having provided examples of admissible point processes, one may use Theorems 1.12 and 1.14 to deduce the limit theory for geometric and topological statistics of these point processes. Examples include statistics arising in combinatorial and differential topology, integral geometry and computational geometry. As fully explained in Section 2.3 of [10], one may deduce expectation and variance asymptotics and central limit theorems for statistics of random Čech complexes, Morse critical points, as well as statistics of germ-grain models generated by admissible point processes. The results described in Section 2.3 of [10] are not exhaustive and include functionals in stochastic geometry already discussed in, for example, [5, 45]. There are further applications to (i) random packing models on input having fast decay of correlations (extending [43]), (ii) statistics of percolation models (extending, e.g., [31, 42]) and (iii) statistics of extreme points of input having fast decay of correlations (extending [2, 54]). Details are left to the reader.

Here we focus on two examples and in doing so, we use the full force of Theorems 1.12 and 1.14, applying them to sums of score functions whose radius of stabilization has either a bounded or exponentially decaying tail.

k-covered region of the germ-grain model. The following is a statistic of interest in coverage processes [19]. For locally-finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, define the score function

$$\beta^{(k)}(x, \mathcal{X}) := \int_{y \in B_r(x)} \frac{\mathbf{1}[\mathcal{X}(B_r(y)) \geq k]}{\mathcal{X}(B_r(y))} \, \mathrm{d}y.$$

Clearly, $\beta^{(k)}$ is an exponentially stabilizing score function as in Definition 1.1 with stabilization radius $2r$. Define the k-covered region of the germ-grain model by $\mathcal{C}_B^k(\mathcal{P}_n, r) = \{y : \mathcal{P}_n(B_r(y)) \geq k\}$. Thus $H_n^{\beta^{(k)}}(\mathcal{P})$ is the volume of $\mathcal{C}_B^k(\mathcal{P}_n, r)$. When $k = 1$, $H_n^{\beta^{(k)}}(\mathcal{P})$ is the volume of the germ-grain model having germs in $\mathcal{P}_n$. Clearly, $\beta^{(k)}$ is bounded by the volume of a radius r ball and so ξ satisfies the power growth condition (1.18). The following is an immediate consequence of Theorems 1.12 and 1.14 and the fact that if $\mathcal{P}$ is of class (A2) then the input pair $(\beta^{(k)}, \mathcal{P})$ is also of class (A2).

THEOREM 2.4. *For all $k \in \mathbb{N}$ and any point process $\mathcal{P}$ of class* (A2) *with the pair $(\beta^{(k)}, \mathcal{P})$ satisfying the moment condition* (1.19) *for all $p \in (1, \infty)$, we have*

$$\left| n^{-1} \mathbb{E} \operatorname{Vol}_d\left(\mathcal{C}_B^k(\mathcal{P}_n, r)\right) - \mathbb{E}_{\mathbf{0}} \beta^{(k)}(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \right| = O\left(n^{-1/d}\right)$$

and

$$\lim_{n \to \infty} n^{-1} \operatorname{Var} \operatorname{Vol}_d\left(\mathcal{C}_B^k(\mathcal{P}_n, r)\right) = \sigma^2\left(\beta^{(k)}\right).$$

Moreover, if $\operatorname{Var} \operatorname{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) = \Omega(n^{\nu})$ *for some $\nu \in (0, \infty)$, then as $n \to \infty$,*

$$\frac{\operatorname{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r)) - \mathbb{E} \operatorname{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r))}{\sqrt{\operatorname{Var} \operatorname{Vol}_d(\mathcal{C}_B^k(\mathcal{P}_n, r))}} \xrightarrow{\mathcal{D}} N. \tag{2.19}$$

In the case of Poisson input and $k = 1$, [19] establishes a central limit theorem for $\mathcal{C}_B^1(\mathcal{P}_n, r)$. For general k, the central limit theorem for Poisson input can be deduced from the general results in [5, 42] with presumably optimal bounds following from [29], Proposition 1.4.

Edge-lengths of k-nearest neighbor graphs. Statistics of the Voronoi tessellation as well as of graphs in computational geometry such as the k-nearest neighbors graph and sphere of influence graph may be expressed as sums of exponentially stabilizing score functionals [42], and hence via Theorems 1.12 and 1.14, we may deduce the limit theory for these statistics. To illustrate, we establish a weak law of large numbers, variance asymptotics and a central limit theorem for the total edge-length of the k-nearest neighbors graph on a α-determinantal point process

$\mathcal{P} := \mathcal{P}_\alpha$ with $-1/\alpha \in \mathbb{N}$ and a fast-decreasing kernel as in (2.7). As noted in Proposition 2.2, such an α-determinantal point process is of class (A2) as in Definition 1.7.

As shown in [11], Corollary 1.10, of the Supplementary Material, we may explicitly upper bound void probabilities for $\mathcal{P}$, allowing us to deduce exponential stabilization for score functions on $\mathcal{P}$. This is a recurring phenomena, and it is often the case that to show exponential stabilization of statistics, it suffices to control the Palm probability content of large Euclidean balls. This opens the way towards showing that other relevant statistics of random graphs exhibit exponential stabilization on $\mathcal{P}$. This includes intrinsic volumes of faces of Voronoi tessellations [49], Section 10.2, edge-lengths in a radial spanning tree [51], Lemma 3.2, proximity graphs including the Gabriel graph and Morse critical points.

Given locally finite $\mathcal{X} \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, the (undirected) k-nearest neighbors graph $\mathrm{NG}(\mathcal{X})$ is the graph with vertex set $\mathcal{X}$ obtained by including an edge $\{x, y\}$ if y is one of the k nearest neighbors of x and/or x is one of the k nearest neighbors of y. In the case of a tie, we may break the tie via some pre-defined total order (say lexicographic order) on $\mathbb{R}^d$. For any finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, we let $\mathcal{E}(x)$ be the edges e in $\mathrm{NG}(\mathcal{X})$ which are incident to x. Defining $\xi_L(x, \mathcal{X}) := \frac{1}{2}\sum_{e \in \mathcal{E}(x)} |e|$, we write the total edge length of $\mathrm{NG}(\mathcal{X})$ as $L(\mathrm{NG}(\mathcal{X})) = \sum_{x \in \mathcal{X}} \xi_L(x, \mathcal{X})$. Let $\sigma^2(\xi_L)$ be as at (1.22), with ξ put to be ξ_L.

THEOREM 2.5. *Let $\mathcal{P} := \mathcal{P}_\alpha$ be a stationary α-determinantal point process on $\mathbb{R}^d$ with $-1/\alpha \in \mathbb{N}$ and a fast-decreasing kernel K as at* (2.7). *We have*

$$\left| \frac{\mathbb{E}L(\mathrm{NG}(\mathcal{P}_n))}{n} - \mathbb{E}_{\mathbf{0}}\xi_L(\mathbf{0}, \mathcal{P})K(\mathbf{0}, \mathbf{0}) \right| = O(n^{-1/d})$$

and

$$\lim_{n \to \infty} \frac{\mathrm{Var}\, L(\mathrm{NG}(\mathcal{P}_n))}{n} = \sigma^2(\xi_L).$$

If $\mathrm{Var}\, L(\mathrm{NG}(\mathcal{P}_n)) = \Omega(n^\nu)$ *for some* $\nu \in (0, \infty)$ *then as* $n \to \infty$

$$\frac{L(\mathrm{NG}(\mathcal{P}_n)) - \mathbb{E}L(\mathrm{NG}(\mathcal{P}_n))}{\sqrt{\mathrm{Var}\, L(\mathrm{NG}(\mathcal{P}_n))}} \xrightarrow{\mathcal{D}} N.$$

Remark. Theorem 2.5 extends Theorem 6.4 of [40] which is confined to Poisson input. In this context, the work [29] provides a rate of normal approximation.

PROOF OF THEOREM 2.5. We want to show that $(\xi_L, \mathcal{P})$ is an admissible score and input pair of type (A2) and then apply Theorem 1.14. Note that $\mathcal{P}$ is an admissible point process which has fast decay of correlations satisfying (1.16) and (1.17). Thus we only need to show that ξ_L is exponentially stabilizing, that ξ_L satisfies the power growth condition (1.18), and the p-moment condition (1.19). When $d = 2$, we show exponential stabilization of ξ_L by closely following the proof of Lemma 6.1 of [44]. This goes as follows. For each $t > 0$, construct six disjoint congruent equilateral triangles $T_j(t)$, $1 \le j \le 6$, such that x is a vertex of each triangle and each edge has length t. Let the random variable R be the minimum t such that $\mathcal{P}_n(T_j) \ge k+1$ for all $1 \le j \le 6$. Notice that $R \in [r, \infty)$ implies that there is a ball inscribed in some $T_j(t)$ with center c_j of radius γr which does not contain $k+1$ points. Combining [11], Corollary 1.10, in the Supplementary Material and the fact that $\mathcal{P}$ has kernel K, the probability of this event satisfies

$$\begin{aligned}\mathbb{P}_{x_1,\dots,x_p}[R > r] &\le 6\mathbb{P}_{x_1,\dots,x_p}\big[\mathcal{P}\big(B_{\gamma r}(c_1)\big) \le k-1\big]\\ &\le 6\mathbb{P}^!_{x_1,\dots,x_p}\big[\mathcal{P}\big(B_{\gamma r}(c_1)\big) \le k-1\big]\\ &\le 6e^{m(2k+p-2)/8}e^{-K(\mathbf{0},\mathbf{0})\pi\gamma^2 r^2/8},\end{aligned}$$

that is to say that R has exponentially decaying tails. As in Lemma 6.1 of [44], we find that $R^\xi(x, \mathcal{P}_n) := 4R$ is a radius of stabilization for ξ_L, showing that (1.15) holds with $c = 2$. For $d > 2$, we may extend these geometric arguments (cf. the proof of Theorem 6.4 of [40]) to define a random variable R serving as a radius of stabilization. Mimicking the above arguments, we may likewise show that R has exponentially decaying tails.

For all $r \in (0, \infty)$ and $l \in \mathbb{N}$, we notice that (1.18) holds because $|\xi_L(x, \mathcal{X} \cap B_r(x))|\mathbf{1}[\mathcal{X}(B_r(x)) = l] \le r \cdot \min(l, 6) \le (cr)^l$. Since vertices in the k-nearest neighbors graph have degree bounded by $kC(d)$ as in Lemma 8.4 of [57], and since each edge incident to x has length at most $4R$, it follows that $|\xi_L(x, \mathcal{P}_n)| \le k \cdot C(d) \cdot 4R$. Since R has moments of all orders, $(\xi_L, \mathcal{P})$ satisfies the p-moment condition (1.19) for all $p \ge 1$. Thus ξ_L satisfies all conditions of Theorem 1.14 and we deduce Theorem 2.5 as desired. □

3. Proof of the fast decay (1.21) for correlations of the ξ-weighted measures. We show the decay bound (1.21) via a factorial moment expansion for the expectation of functionals of point processes. Notice that (1.21) holds for any exponentially stabilizing score function ξ satisfying the p-moment condition (1.19) for all $p \in [1, \infty)$ on a Poisson point process $\mathcal{P}$. Indeed if $x, y \in \mathbb{R}^d$ and $r_1, r_2 > 0$ satisfy $r_1 + r_2 < |x - y|$ then $\xi(x, \mathcal{P})\mathbf{1}[R^\xi(x, \mathcal{P}) \le r_1]$ and $\xi(y, \mathcal{P})\mathbf{1}[R^\xi(y, \mathcal{P}) \le r_2]$ are independent random variables. This yields the fast decay (1.21) with $k_1 = \cdots = k_{p+q} = 1$ and $\tilde{C}_n \le c_1^n$ with c_1 a constant, as in [5], Lemma 5.2. On the other hand, if $\mathcal{P}$ is rarified Gibbsian input and ξ is exponentially stabilizing, then [50], Lemma 3.4, shows the fast decay bound (1.21) with $k_1 = \cdots = k_{p+q} = 1$. These methods depend on quantifying the region of spatial dependencies of Gibbsian points via exponentially decaying diameters of their ancestor clans. Such methods apparently neither extend to determinantal input nor to the zero set $\mathcal{P}_{\text{GEF}}$ of a Gaussian entire function. On the other hand, for $\mathcal{P}_{\text{GEF}}$ and for $\xi \equiv 1$, the paper [37] uses the Kac–Rice–Hammersley formula and complex analysis tools to show (1.21) with $k_1 = \cdots = k_{p+q} = 1$. All three proofs are specific to either the underlying point process or to the score function ξ. The following more general and considerably different approach includes these results as special cases.

3.1. *Difference operators and factorial moment expansions.* We introduce some notation and collect auxiliary results required for an application of the much-needed factorial moment expansions [8, 9] for general point processes. Equip $\mathbb{R}^d$ with a total order $\prec$ defined using the lexicographical ordering of the polar coordinates. For $\mu \in \mathcal{N}$ and $x \in \mathbb{R}^d$, define the measure $\mu_{|x}(\cdot) := \mu(\cdot \cap \{y : y \prec x\})$. Note that since μ is a locally finite measure and the ordering is defined via polar coordinates $\mu_{|x}$ is a finite measure for all $x \in \mathbb{R}^d$. Let o denote the null-measure, that is, $o(B) = 0$ for all Borel subsets B of $\mathbb{R}^d$. For a measurable function $\psi : \mathcal{N} \to \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$ and $x_1, \dots, x_l \in \mathbb{R}^d$, we define the factorial moment expansion (FME) kernels [8, 9] as follows. For $l \geq 1$,

$$
\begin{aligned}
D^l_{x_1,\dots,x_l}\psi(\mu) &= \sum_{i=0}^{l}(-1)^{l-i} \sum_{J \subset \binom{[l]}{i}} \psi\Bigl(\mu_{|x_*} + \sum_{j\in J}\delta_{x_j}\Bigr) \\
&= \sum_{J\subset[l]}(-1)^{l-|J|}\psi\Bigl(\mu_{|x_*} + \sum_{j\in J}\delta_{x_j}\Bigr),
\end{aligned} \tag{3.1}
$$

where $\binom{[l]}{j}$ denotes the collection of all subsets of $[l] := \{1, \dots, l\}$ with cardinality j and $x_* := \min\{x_1, \dots, x_l\}$, with the minimum taken with respect to the order $\prec$. For $l = 0$, put $D^0\psi(\mu) := \psi(o)$. Note that $D^{(l)}_{x_1,\dots,x_l}\psi(\mu)$ is a symmetric function of $x_1, \dots, x_l$.[8]

We say that ψ is $\prec$-continuous at ∞ if for all $\mu \in \mathcal{N}$ we have $\lim_{x\uparrow\infty}\psi(\mu_{|x}) = \psi(\mu)$. We first recall the FME expansion proved in [8] (cf. Theorem 3.2) for dimension one and then extended to higher-dimensions [9] (cf. Theorem 3.1). Recall that $\mathbb{E}^!_{y_1,\dots,y_l}$ denote expectations with respect to reduced Palm probabilities.

THEOREM 3.1. *Let $\mathcal{P}$ be a simple point process and let $\psi : \mathcal{N} \to \mathbb{R}$ be $\prec$-continuous at ∞. Assume that for all $l \geq 1$*

$$
\int_{\mathbb{R}^{dl}} \mathbb{E}^!_{y_1,\dots,y_l}\bigl[\bigl|D^l_{y_1,\dots,y_l}\psi(\mathcal{P})\bigr|\bigr]\rho^{(l)}(y_1, \dots, y_l)\,\mathrm{d}y_1 \dots \mathrm{d}y_l < \infty \tag{3.2}
$$

and

$$
\frac{1}{l!}\int_{\mathbb{R}^{dl}} \mathbb{E}^!_{y_1,\dots,y_l}\bigl[D^l_{y_1,\dots,y_l}\psi(\mathcal{P})\bigr]\rho^{(l)}(y_1, \dots, y_l)\,\mathrm{d}y_1 \dots \mathrm{d}y_l \to 0 \quad \textit{as } l \to \infty. \tag{3.3}
$$

[8]For $x_l \prec x_{l-1} \prec \dots \prec x_1$, the functional $D^l_{x_1,\dots,x_l}\psi(\mu)$ is equal to the iterated difference operator: $D^1_{x_1}\psi(\mu) = \psi(\mu_{|x_1} + \delta_{x_1}) - \psi(\mu_{|x_1})$, $D^l_{x_1,\dots,x_l}\psi(\mu) = D^1_{x_l}(D^{l-1}_{x_1,\dots,x_{l-1}}\psi(\mu))$.

Then $\mathbb{E}[\psi(\mathcal{P})]$ *has the following* factorial moment expansion:

$$\mathbb{E}\big[\psi(\mathcal{P})\big] = \psi(o) + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^{dl}} D^l_{y_1,\dots,y_l}\psi(o)\rho^{(l)}(y_1,\dots,y_l)\,\mathrm{d}y_1\dots\mathrm{d}y_l. \tag{3.4}$$

Consider now admissible pairs $(\xi,\mathcal{P})$ of type (A1) or (A2) and $x_1,\dots,x_p \in \mathbb{R}^d$. The proof of (1.21) given in the next subsection is based on the FME expansion for $\mathbb{E}_{x_1,\dots,x_p}[\psi(\mathcal{P}_n)]$, where $\psi(\mu)$ is the product of score functions

$$\psi(\mu) := \psi_{k_1,\dots,k_p}(x_1,\dots,x_p;\mu) := \prod_{i=1}^{p} \xi(x_i,\mu)^{k_i} \tag{3.5}$$

with $k_1,\dots,k_p \ge 1$. However, under $\mathbb{P}_{x_1,\dots,x_p}$ the point process $\mathcal{P}_n$ has fixed atoms at $x_1,\dots,x_p$, which complicates the form of its factorial moment measures. It is more handy to consider these points as parameters of the following modified functional:

$$\psi^!(\mu) := \psi^!_{k_1,\dots,k_p}(x_1,\dots,x_p;\mu) := \prod_{i=1}^{p} \xi\left(x_i,\mu + \sum_{j=1}^{p}\delta_{x_j}\right)^{k_i} \tag{3.6}$$

and to not count points $x_1,\dots,x_p$ in $\mathcal{P}$, that is, to consider $\mathcal{P}$ under the reduced Palm probabilities $\mathbb{P}^!_{x_1,\dots,x_p}$. Obviously, $\mathbb{E}_{x_1,\dots,x_p}[\psi(\mathcal{P}_n)] = \mathbb{E}^!_{x_1,\dots,x_p}[\psi^!(\mathcal{P}_n)]$ and the latter expectation is more suitable for FME expansion with respect to the correlation functions $\rho^{(l)}_{x_1,\dots,x_p}(y_1,\dots,y_l)$ of $\mathcal{P}$ with respect to the Palm probabilities $\mathbb{P}^!_{x_1,\dots,x_p}$. The following consequence of Theorem 3.1 allows us to use FME expansions to prove (1.21).

LEMMA 3.2. *Assume that either* (i) $(\xi,\mathcal{P})$ *is an admissible score and input pair of type* (A1) *or* (ii) $(\xi,\mathcal{P})$ *satisfies the power growth condition* (1.18), *with* ξ *having a radius of stabilization satisfying* $\sup_{x\in\mathcal{P}} R^{\xi}(x,\mathcal{P}) \le r$ *a.s. for some* $r \in (1,\infty)$ *and with* $\mathcal{P}$ *having exponential moments. Then for distinct* $x_1,\dots,x_p \in \mathbb{R}^d$, *nonnegative integers* $k_1,\dots,k_p$ *and* $n \le \infty$ *the functional* $\psi^!$ *at* (3.6) *admits the FME*

$$\begin{aligned}
&\mathbb{E}_{x_1,\dots,x_p}\big[\psi_{k_1,\dots,k_p}(x_1,\dots,x_p;\mathcal{P}_n)\big] \\
&\quad = \mathbb{E}^!_{x_1,\dots,x_p}\big[\psi^!_{k_1,\dots,k_p}(x_1,\dots,x_p;\mathcal{P}_n)\big] \\
&\quad = \psi^!_{k_1,\dots,k_p}(x_1,\dots,x_p;o) \\
&\qquad + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathbb{R}^{dl}} D^l_{y_1,\dots,y_l}\psi^!_{k_1,\dots,k_p}(x_1,\dots,x_p;o) \\
&\qquad \times \rho^{(l)}_{x_1,\dots,x_p}(y_1,\dots,y_l)\,\mathrm{d}y_1\dots\mathrm{d}y_l.
\end{aligned} \tag{3.7}$$

When $(\xi, \mathcal{P})$ *is of type* (A1), *the series* (3.7) *has at most* $(k-1)\sum_{i=1}^{p} k_i$ *nonzero terms*, *where* k *is as in* (1.13).

PROOF. Throughout we fix nonnegative integers $k_1, \dots, k_p$ and suppress them when writing $\psi^!$; that is, $\psi^!(x_1, \dots, x_p; \mathcal{P}_n) := \psi^!_{k_1,\dots,k_p}(x_1, \dots, x_p; \mathcal{P}_n)$. The bounded radius of stabilization for ξ implies $\psi^!$ is $\prec$-continuous at ∞.

Consider first $\psi^!$ at (3.6) with ξ as in case (ii); later we consider the simpler case (i). We show the validity of the expansion (3.7) as follows. Let $y_1, \dots, y_l \in \mathbb{R}^d$. The difference operator $D^l_{y_1,\dots,y_l}$ vanishes as soon as $y_k \notin \bigcup_{i=1}^{p} B_r(x_i)$ for some $k \in \{1, \dots, l\}$, that is to say

$$D^l_{y_1,\dots,y_l}\psi^!(x_1, \dots, x_p; \mu) = 0. \tag{3.8}$$

To prove this, set $\mu_J := \mu|_{y_*} + \sum_{j \in J} \delta_{y_j}$ for $J \subset [l]$ and $y_* := \min\{y_1, \dots, y_l\}$, with the minimum taken with respect to $\prec$ order. From (3.1), we obtain

$$\begin{aligned} &D^l_{y_1,\dots,y_l}\psi^!(x_1, \dots, x_p; \mu) \\ &\quad = \sum_{J \subset [l], k \notin J} (-1)^{l-|J|}\psi^!(x_1, \dots, x_p; \mu_J) \\ &\qquad + \sum_{J \subset [l], k \notin J} (-1)^{l-|J|-1}\psi^!(x_1, \dots, x_p; \mu_{J\cup\{k\}}) = 0, \end{aligned}$$

where the last equality follows by noting that for $J \subset [l]$ with $k \notin J$, $\psi^!(x_1, \dots, x_p; \mu_J) = \psi^!(x_1, \dots, x_p; \mu_{J\cup\{k\}})$ because $R^\xi(x, \mathcal{P}) \in [1, r]$ by assumption.

Henceforth we put

$$K_p := \sum_{i=1}^{p} k_i, \qquad K_q := \sum_{i=1}^{q} k_{p+i}, \qquad K := \sum_{i=1}^{p+q} k_i. \tag{3.9}$$

Consider now $y_1, \dots, y_l \in \bigcup_{i=1}^{p} B_r(x_i)$. For $J \subset [l]$, from $1 \le R^\xi(x, \mathcal{P}) \le r$ and (1.18) we have

$$\psi^!(x_1, \dots, x_p; \mu_J) \le (\hat{c}r)^{K_p|J| + pK_p + \sum_{i=1}^{p} k_i \mu(B_r(x_i))}. \tag{3.10}$$

The term pK_p in the exponent of (3.10) is due to $\sum_{j=1}^{p} \delta_{x_j}$ in the argument of ξ in (3.6). Substituting this bound in (3.1) yields

$$\begin{aligned} &\left|D^l_{y_1,\dots,y_l}\psi^!(x_1, \dots, x_p; \mu)\right| \\ &\quad \le (\hat{c}r)^{pK_p + \sum_{i=1}^{p} k_i \mu(B_r(x_i))} \sum_{J \subset [l]} (\hat{c}r)^{K_p|J|} \\ &\quad = (\hat{c}r)^{pK_p + \sum_{i=1}^{p} k_i \mu(B_r(x_i))}\left(1 + (\hat{c}r)^{K_p}\right)^l. \end{aligned} \tag{3.11}$$

Consider $\psi^!(x_1,\dots,x_p;\mathcal{P}_n)$, with $\mathcal{P}_n := \mathcal{P}\cap W_n$ and $\psi^!$ defined as above. The bound (3.11) yields

$$\begin{aligned}
&\frac{1}{l!}\int_{\mathbb{R}^{dl}} (\mathbb{E}^!_{x_1,\dots,x_p})^!_{y_1,\dots,y_l}\big[\big|D^l_{y_1,\dots,y_l}\psi^!(x_1,\dots,x_p;\mathcal{P}_n)\big|\big] \\
&\quad\times \rho^{(l)}_{x_1,\dots,x_p}(y_1,\dots,y_l)\,\mathrm{d}y_1\cdots\mathrm{d}y_l \\
&\qquad = \frac{1}{l!}\int_{\mathbb{R}^{dl}} \mathbb{E}^!_{x_1,\dots,x_p,y_1,\dots,y_l}\big[\big|D^l_{y_1,\dots,y_l}\psi^!(x_1,\dots,x_p;\mathcal{P}_n)\big|\big] \\
&\qquad\quad \times \rho^{(l)}_{x_1,\dots,x_p}(y_1,\dots,y_l)\,\mathrm{d}y_1\cdots\mathrm{d}y_l \\
&\qquad \le \frac{(1+(\hat{c}r)^{K_p})^l(\hat{c}r)^{pK_p}}{l!} \\
&\qquad\quad \times \mathbb{E}^!_{x_1,\dots,x_p}\Bigg[\bigg(\mathcal{P}_n\bigg(\bigcup_{i=1}^p B_r(x_i)\bigg)\bigg)^l(\hat{c}r)^{\sum_{i=1}^p k_i\mathcal{P}_n(B_r(x_i))}\Bigg] \\
&\qquad \le \frac{(1+(\hat{c}r)^{K_p})^l(\hat{c}r)^{pK_p}}{l!} \\
&\qquad\quad \times \mathbb{E}^!_{x_1,\dots,x_p}\Bigg[\bigg(\mathcal{P}_n\bigg(\bigcup_{i=1}^p B_r(x_i)\bigg)\bigg)^l(\hat{c}r)^{K_p\mathcal{P}_n(\bigcup_{i=1}^p B_r(x_i))}\Bigg] \\
&\qquad \le \frac{(1+(\hat{c}r)^{K_p})^l}{l!} \\
&\qquad\quad \times \mathbb{E}_{x_1,\dots,x_p}\Bigg[\bigg(\mathcal{P}_n\bigg(\bigcup_{i=1}^p B_r(x_i)\bigg)\bigg)^l(\hat{c}r)^{K_p\mathcal{P}_n(\bigcup_{i=1}^p B_r(x_i))}\Bigg],
\end{aligned}\tag{3.12}$$

where the last inequality follows since the distribution of $\mathcal{P}$ under $\mathbb{P}_{x_1,\dots,x_p}$ is equal to that of $\mathcal{P}+\sum_{i=1}^p\delta_{x_i}$ under $\mathbb{P}^!_{x_1,\dots,x_p}$. Defining $N := \mathcal{P}_n(\bigcup_{i=1}^p B_r(x_i))$, we bound (3.12) by

$$\mathbb{E}_{x_1,\dots,x_p}\Bigg[(\hat{c}r)^{K_pN}\sum_{m=l}^{\infty}\frac{(1+(\hat{c}r)^{K_p})^l}{l!}N^l\Bigg] \le \mathbb{E}_{x_1,\dots,x_p}\big[(\hat{c}r)^{(1+(\hat{c}r)^{K_p}+K_p)N}\big] < \infty,$$

where the last inequality follows since $\mathcal{P}$ has exponential moments under the Palm measure as well (see Remark (i) at the beginning of Section 2.1). Consequently, by the Lebesgue dominated convergence theorem, the expression (3.12) converges to 0 as $l\to\infty$. Thus conditions (3.2) and (3.3) hold and (3.7) follows by Theorem 3.1.

Now we consider case (i), that is to say $\psi^!$ is as at (3.6) with ξ a U-statistic of type (A1). By [11], Lemma 1.1, in the Supplementary Material, with k as in (1.13), $\psi^!$ is a sum of U-statistics of orders not larger than $K_p(k-1)$. Consequently, for

$l \in (K_p(k-1), \infty)$ we have

$$D^l_{y_1,\dots,y_l} \psi^!(x_1, \dots, x_p; \mu) = 0 \qquad \forall y_1, \dots, y_l \in \mathbb{R}^d, \tag{3.13}$$

as shown in [48], Lemma 3.3, for Poisson point processes (the proof for general simple counting measures μ is identical). This implies that conditions (3.2) for $l \in (K_p(k-1), \infty)$ and (3.3) are trivially satisfied for $\psi^!$ as at (3.6). Now, we need to verify the condition (3.2) for $l \in [1, K_p(k-1)]$. For $y_1, \dots, y_l \in \mathbb{R}^d$, set as before $\mu_J = \mu|_{y_*} + \sum_{j \in J} \delta_{y_j}$ for $J \subset [l]$ and $y_* := \min\{y_1, \dots, y_l\}$, with the minimum taken with respect to the order $\prec$. Since ξ has a bounded stabilization radius, by (3.8) and (2.5), we have

$$\begin{aligned}
&\psi^!(x_1, \dots, x_p; \mu_J) \\
&\quad \le \prod_{i=1}^p \|h\|_\infty^{k_i} \left(\mu\left(\bigcup_{i=1}^p B_r(x_i) \right) + |J| + p \right)^{k_i(k-1)} \\
&\quad \le \|h\|_\infty^{K_p} \left(\mu\left(\bigcup_{i=1}^p B_r(x_i) \right) + |J| + p \right)^{K_p(k-1)}.
\end{aligned} \tag{3.14}$$

The number of subsets of $[l]$ is 2^l and so by (3.1), we obtain

$$\begin{aligned}
&\left| D^l_{y_1,\dots,y_l} \psi^!(x_1, \dots, x_p; \mu) \right| \\
&\quad \le \|h\|_\infty^{K_p} \sum_{J \subset [l]} \left(\mu\left(\bigcup_{i=1}^p B_r(x_i) \right) + |J| + p \right)^{K_p(k-1)} \\
&\quad \le \|h\|_\infty^{K_p} 2^l \left(\mu\left(\bigcup_{i=1}^p B_r(x_i) \right) + l + p \right)^{K_p(k-1)}.
\end{aligned} \tag{3.15}$$

Consider $\psi^!(x_1, \dots, x_p; \mathcal{P}_n)$ with $\psi^!$ defined as above. Using the refined Campbell theorem (1.9), the bound (3.15), and following the calculations as in (3.12), we obtain

$$\begin{aligned}
&\frac{1}{l!} \int_{\mathbb{R}^{dl}} (\mathbb{E}^!_{x_1,\dots,x_p})^!_{y_1,\dots,y_l} \left[\left| D^l_{y_1,\dots,y_l} \psi^!(x_1, \dots, x_p; \mathcal{P}_n) \right| \right] \\
&\quad \times \rho^{(l)}_{x_1,\dots,x_p}(y_1, \dots, y_l) \, \mathrm{d}y_1 \cdots \mathrm{d}y_l \\
&\qquad \le \|h\|_\infty^{K_p} 2^l \mathbb{E}_{x_1,\dots,x_p} \left[\mathcal{P}\left(\bigcup_{i=1}^p B_r(x_i) \right)^l \left(\mathcal{P}\left(\bigcup_{i=1}^p B_r(x_i) \right) + l + p \right)^{K_p(k-1)} \right].
\end{aligned}$$

Since $\mathcal{P}$ has all moments under the Palm measure (see Remark (ii) at the beginning of Section 2.1), the finiteness of the last term, and hence the validity of the condition (3.2) for $l \in [1, K_p(k-1)]$ follows. This justifies the FME expansion (3.7), with finitely many nonzero terms, when $\psi^!$ is the product of score functions of class (A1). $\square$

3.2. *Proof of Theorem* 1.11. First, assume that $(\xi, \mathcal{P})$ is of class (A2). Later we consider the simpler case that $(\xi, \mathcal{P})$ is of class (A1). For fixed $p, q, k_1, \dots, k_{p+q} \in \mathbb{N}$, consider correlation functions $m^{(k_1,\dots,k_{p+q})}(x_1, \dots, x_{p+q}; n)$, $m^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n)$ and $m^{(k_{p+1},\dots,k_{p+q})}(x_{p+1}, \dots, x_{p+q}; n)$ of the ξ-weighted measures at (1.6). We abbreviate $\psi_{k_1,\dots,k_p}(x_1, \dots, x_p; \mu)$ by $\psi(x_1, \dots, x_p; \mu)$ as at (3.5), and similarly for $\psi(x_{p+1}, \dots, x_{p+q}; \mu)$ and $\psi(x_1, \dots, x_{p+q}; \mu)$.

Given $x_1, \dots, x_{p+q} \in W_n$, we recall $s := d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$. Without loss of generality, we assume $s \in (4, \infty)$. Recalling the definition of b at (1.17) and that of K at (3.9), we may assume without loss of generality that $b \in (0, d)$. Put

$$t := t(s) := \left(\frac{s}{4}\right)^{b(1-a)/(2(K+d))}, \tag{3.16}$$

where $a \in [0, 1)$ is at (1.16). Since $s \in (4, \infty)$ and $K \geq 2$, we easily have $t \in (1, s/4)$. Given stabilization radii $R^\xi(x_i, \mathcal{P}_n)$, $1 \leq i \leq p+q$, we put

$$\tilde{\xi}(x_i, \mathcal{P}_n) := \xi\big(x_i, \mathcal{P}_n \cap B_{R^\xi(x_i, \mathcal{P}_n)}(x)\big)\mathbf{1}\big[R^\xi(x_i, \mathcal{P}_n) \leq t\big]$$

considered under $\mathbb{E}_{x_1,\dots,x_p}$. We denote by $\tilde{m}^{(k_1,\dots,k_p)}$ the correlation functions of the $\tilde{\xi}$-weighted atomic measure, that is,

$$\begin{aligned} &\tilde{m}^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n) \\ &\quad := \mathbb{E}_{x_1,\dots,x_p}\big[\tilde{\xi}(x_1, \mathcal{P}_n)^{k_1} \dots \tilde{\xi}(x_p, \mathcal{P}_n)^{k_p}\big]\rho^{(p)}(x_1, \dots, x_p). \end{aligned}$$

Write

$$\begin{aligned} \tilde{\psi}(x_1, \dots, x_p; \mathcal{P}_n) &= \psi(x_1, \dots, x_p; \mathcal{P}_n)\mathbf{1}\Big[\max_{i \leq p} R^\xi(x_i, \mathcal{P}_n) \leq t\Big] \\ &= \prod_{i=1}^p \tilde{\xi}(x_i, \mathcal{P}_n)^{k_i}. \end{aligned} \tag{3.17}$$

Next, write $\mathbb{E}_{x_1,\dots,x_p}\psi(x_1, \dots, x_p; \mathcal{P}_n)$ as a sum of

$$\mathbb{E}_{x_1,\dots,x_p}\Big[\psi(x_1, \dots, x_p; \mathcal{P}_n)\mathbf{1}\Big[\max_{i \leq p} R^\xi(x_i, \mathcal{P}_n) \leq t\Big]\Big]$$

and

$$\mathbb{E}_{x_1,\dots,x_p}\Big[\psi(x_1, \dots, x_p; \mathcal{P}_n)\mathbf{1}\Big[\max_{i \leq p} R^\xi(x_i, \mathcal{P}_n) > t\Big]\Big].$$

The bounds (1.11), (1.14), the moment condition (1.19), Hölder's inequality and $p \leq \sum_{i=1}^p k_i = K_p$ give for Lebesgue almost all $x_1, \dots, x_p$:

$$\begin{aligned} &\big|\mathbb{E}_{x_1,\dots,x_p}\psi(x_1, \dots, x_p; \mathcal{P}_n) - \mathbb{E}_{x_1,\dots,x_p}\tilde{\psi}(x_1, \dots, x_p; \mathcal{P}_n)\big| \\ &\quad \times \rho^{(p)}(x_1, \dots, x_p) \end{aligned}$$

$$
\begin{aligned}
&\leq p\kappa_p(\tilde{M}_{K_p+1})^{K_p/(K_p+1)}\varphi(a_p t)^{1/(K_p+1)} \\
&\leq K_p\kappa_{K_p}(\tilde{M}_{K_p+1})^{K_p/(K_p+1)}\varphi(a_{K_p} t)^{1/(K_p+1)} \\
&\leq c_1(K_p)\varphi(a_{K_p} t)^{1/(K_p+1)}.
\end{aligned}
\tag{3.18}
$$

Here $c_1(m) := m\kappa_m\tilde{M}_{m+1} \geq m\kappa_m(\tilde{M}_{m+1})^{m/(m+1)}$, as $\tilde{M}_m \geq 1$ by assumption. Similarly, condition (1.19) yields $|\mathbb{E}_{x_1,\dots,x_p}\psi(x_1,\dots,x_p;\mathcal{P}_n)|\rho^{(p)}(x_1,\dots,x_p) \leq c_1(K_p)$. Using (3.18) with p replaced by $p+q$, we find $m^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n)$ differs from $\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n)$ by $c_1(K)\varphi(a_K t)^{1/(K+1)}$, which is fast-decreasing by (1.15).

For any reals $A, B, \tilde{A}, \tilde{B}$, with $|\tilde{B}| \leq |B|$, we have $|AB - \tilde{A}\tilde{B}| \leq |A(B-\tilde{B})| + |(A-\tilde{A})\tilde{B}| \leq (|A|+|B|)(|B-\tilde{B}|+|A-\tilde{A}|)$. Hence, it follows that

$$
\begin{aligned}
&\big|m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n)m^{(k_{p+1},\dots,k_q)}(x_{p+1},\dots,x_{p+q};n) \\
&\quad - \tilde{m}^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n)\tilde{m}^{(k_{p+1},\dots,k_q)}(x_{p+1},\dots,x_{p+q};n)\big| \\
&\quad\leq \big(c_1(K_p)+c_1(K_q)\big) \\
&\qquad \times \big(c_1(K_p)\varphi(a_{K_p}t)^{1/(K_p+1)} + c_1(K_q)\varphi(a_{K_q}t)^{1/(K_q+1)}\big) \\
&\quad\leq c_2(K)\varphi(a_K t)^{1/(K+1)},
\end{aligned}
$$

with $c_2(m) := 4(c_1(m))^2$ and where we note that $\varphi(a_m t)^{1/(m+1)}$ is also fast-decreasing by (1.15). The difference of correlation functions of the ξ-weighted measures is thus bounded by

$$
\begin{aligned}
&\big|m^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n) - m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n) \\
&\quad\times m^{(k_{p+1},\dots,k_{p+q})}(x_{p+1},\dots,x_{p+q};n)\big| \\
&\quad\leq \big(c_1(K)+c_2(K)\big)\varphi(a_k t)^{1/(K+1)} + \big|\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q};n) \\
&\qquad - \tilde{m}^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n)\tilde{m}^{(k_{p+1},\dots,k_{p+q})}(x_{p+1},\dots,x_{p+q};n)\big|.
\end{aligned}
\tag{3.19}
$$

The rest of the proof consists of bounding $|\tilde{m}^{(k_1,\dots,k_{p+q})} - \tilde{m}^{(k_1,\dots,k_p)}\tilde{m}^{(k_{p+1},\dots,k_{p+q})}|$ by a fast-decreasing function of s. In this regard, we will consider the expansion (3.7) with $\psi(x_1,\dots,x_p;\mathcal{P}_n)$ replaced by $\tilde{\psi}(x_1,\dots,x_p;\mathcal{P}_n)$ as at (3.17) and similarly for $\tilde{\psi}(x_{p+1},\dots,x_{p+q};\mathcal{P}_n)$ and $\tilde{\psi}(x_1,\dots,x_{p+q};\mathcal{P}_n)$. By [11], Lemma 1.2, in the Supplementary Material, $\tilde{\xi}(x_i,\mathcal{P}_n)$, $1 \leq i \leq p$, have radii of stabilization bounded above by t and also satisfy the power-growth condition (1.18) since $|\tilde{\xi}| \leq |\xi|$. Thus the pair $(\tilde{\xi},\mathcal{P})$ satisfies the assumptions of Lemma 3.2. The corresponding version of $\tilde{\psi}$, accounting for the fixed atoms of $\mathcal{P}_n$ is $\tilde{\psi}^!(x_1,\dots,x_p;\mu) := \prod_{i=1}^p \tilde{\xi}(x_i,\mu+\sum_{i=1}^p \delta_{x_i})^{k_i}$ and similarly for $\tilde{\psi}^!(x_{p+1},\dots,x_q;\mathcal{P}_n)$ and $\tilde{\psi}^!(x_1,\dots,x_{p+q};\mathcal{P}_n)$.

Put $B_{t,n}(x_i) := B_t(x_i) \cap W_n$. Applying (3.7), the multiplicative identity (2.11) and (3.8), we obtain

$$
\begin{aligned}
&\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q}) \\
&\quad = \mathbb{E}^!_{x_1,\dots,x_{p+q}}\big[\tilde{\psi}^!(x_1,\dots,x_{p+q};\mathcal{P}_n)\big]\rho^{(p+q)}(x_1,\dots,x_{p+q}) \\
&\quad = \sum_{l=0}^{\infty}\frac{1}{l!}\int_{(W_n)^l} D^l_{y_1,\dots,y_l}\tilde{\psi}^!(o) \\
&\qquad \times \rho^{(l+p+q)}(x_1,\dots,x_{p+q},y_1,\dots,y_l)\,\mathrm{d}y_1\dots\mathrm{d}y_l \\
&\quad = \sum_{l=0}^{\infty}\frac{1}{l!}\int_{(\bigcup_{i=1}^{p+q} B_{t,n}(x_i))^l} D^l_{y_1,\dots,y_l}\tilde{\psi}^!(o) \\
&\qquad \times \rho^{(l+p+q)}(x_1,\dots,x_{p+q},y_1,\dots,y_l)\,\mathrm{d}y_1\cdots\mathrm{d}y_l.
\end{aligned}
$$

Applying (3.1) when μ is the null measure, this gives for $\alpha^{(p+q)}$ almost all $x_1,\dots,x_{p+q}$

$$
\begin{aligned}
&\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q}) \\
&\quad = \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!} \\
&\qquad \times \int_{(\bigcup_{i=1}^{p} B_{t,n}(x_i))^j\times(\bigcup_{i=1}^{q} B_{t,n}(x_{p+i}))^{l-j}} D^l_{y_1,\dots,y_l}\tilde{\psi}^!(x_1,\dots,x_{p+q};o) \\
&\qquad \times \rho^{(l+p+q)}(x_1,\dots,x_{p+q},y_1,\dots,y_l)\,\mathrm{d}y_1\dots\mathrm{d}y_l \\
&\quad = \sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{1}{j!(l-j)!}\int_{(\bigcup_{i=1}^{p} B_{t,n}(x_i))^j\times(\bigcup_{i=1}^{q} B_{t,n}(x_{p+i}))^{l-j}}\mathrm{d}y_1\dots\mathrm{d}y_l \\
&\qquad \times \sum_{J\subset[l]}(-1)^{l-|J|}\tilde{\psi}^!\Big(x_1,\dots,x_{p+q};\sum_{j\in J}\delta_{y_j}\Big) \\
&\qquad \times \rho^{(l+p+q)}(x_1,\dots,x_{p+q},y_1,\dots,y_l).
\end{aligned} \tag{3.20}
$$

To compare the $(p+q)$th correlation functions of the ξ-weighted measures with the product of their pth and qth correlation functions, we shall use the fact that $R^{\tilde{\xi}}(x_i,\mathcal{P}_n)\in(0,t]$ (cf. Supplementary Material [11], Lemma 1.2) implies the following factorization, which holds for $y_1,\dots,y_j\in\bigcup_{i=1}^{p}B_t(x_i)$ and $y_{j+1},\dots,y_l\in\bigcup_{i=1}^{q}B_t(x_{p+i})$, with $t\in(1,s/4)$ (making $\bigcup_{i=1}^{p}B_t(x_i)$ and $\bigcup_{i=1}^{q}B_t(x_{p+i})$ disjoint):

$$\tilde{\psi}^!\left(x_1,\ldots,x_{p+q};\sum_{i=1}^{l}\delta_{y_i}\right) = \tilde{\psi}^!\left(x_1,\ldots,x_p;\sum_{i=1}^{j}\delta_{y_i}\right)\tilde{\psi}^!\left(x_{p+1},\ldots,x_{p+q};\sum_{i=j+1}^{l}\delta_{y_i}\right). \tag{3.21}$$

Using the expansion (3.7) along with (3.21), we next derive an expansion for the product of pth and qth correlation functions of the ξ-weighted measures. Recalling the multiplicative identity (2.11) as well as the identity $\mathbb{E}_{x_1,\ldots,x_p}[\psi(\mathcal{P}_n)] = \mathbb{E}^!_{x_1,\ldots,x_p}[\psi^!(\mathcal{P}_n)]$ (cf. (3.6)), we obtain

$$\begin{aligned}
&\tilde{m}^{(k_1,\ldots,k_p)}(x_1,\ldots,x_p)\tilde{m}^{(k_{p+1},\ldots,k_q)}(x_{p+1},\ldots,x_{p+q}) \\
&\quad = \mathbb{E}^!_{x_1,\ldots,x_p}[\tilde{\psi}^!(x_1,\ldots,x_p;\mathcal{P}_n)]\mathbb{E}^!_{x_{p+1},\ldots,x_{p+q}}[\tilde{\psi}^!(x_{p+1},\ldots,x_{p+q};\mathcal{P}_n)] \\
&\qquad \times \rho^{(p)}(x_1,\ldots,x_p)\rho^{(q)}(x_{p+1},\ldots,x_{p+q}) \\
&\quad = \sum_{l_1,l_2=0}^{\infty}\frac{1}{l_1!l_2!}\int_{(\bigcup_{i=1}^p B_{t,n}(x_i))^{l_1}\times(\bigcup_{i=1}^q B_{t,n}(x_{p+i}))^{l_2}} D^{l_1}_{y_1,\ldots,y_{l_1}}\tilde{\psi}^!(x_1,\ldots,x_p;o) \\
&\qquad \times D^{l_2}_{z_1,\ldots,z_{l_2}}\tilde{\psi}^!(x_{p+1},\ldots,x_{p+q};o)\rho^{(l_1+p)}(x_1,\ldots,x_p,y_1,\ldots,y_{l_1}) \\
&\qquad \times \rho^{(l_2+q)}(x_{p+1},\ldots,x_{p+q},z_1,\ldots,z_{l_2})\,\mathrm{d}y_1\ldots\mathrm{d}y_{l_1}\,\mathrm{d}z_1\ldots\mathrm{d}z_{l_2}.
\end{aligned} \tag{3.22}$$

Applying (3.1) once more for μ the null measure, this gives

$$\begin{aligned}
&\tilde{m}^{(k_1,\ldots,k_p)}(x_1,\ldots,x_p)\tilde{m}^{(k_{p+1},\ldots,k_q)}(x_{p+1},\ldots,x_{p+q}) \\
&\quad = \sum_{l_1,l_2=0}^{\infty}\frac{1}{l_1!l_2!}\int_{(\bigcup_{i=1}^p B_{t,n}(x_i))^{l_1}\times(\bigcup_{i=1}^q B_{t,n}(x_{p+i}))^{l_2}} \mathrm{d}y_1\cdots\mathrm{d}y_{l_1}\,\mathrm{d}z_1\cdots\mathrm{d}z_{l_2} \\
&\qquad \times \sum_{J_1\subset[l_1],J_2\subset[l_2]}(-1)^{l_1+l_2-|J_1|-|J_2|} \\
&\qquad \times \tilde{\psi}^!\left(x_1,\ldots,x_p;\sum_{i\in J_1}\delta_{y_i}\right)\tilde{\psi}^!\left(x_{p+1},\ldots,x_{p+q};\sum_{i\in J_2}\delta_{z_i}\right) \\
&\qquad \times \rho^{(l_1+p)}(x_1,\ldots,x_p,y_1,\ldots,y_{l_1})\rho^{(l_2+q)}(x_{p+1},\ldots,x_{p+q},z_1,\ldots,z_{l_2}) \\
&\quad = \sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{1}{j!(l-j)!} \\
&\qquad \times \int_{(\bigcup_{i=1}^p B_{t,n}(x_i))^{j}\times(\bigcup_{i=1}^q B_{t,n}(x_{p+i}))^{l-j}} \sum_{J_1\subset[j],J_2\subset[l]\setminus[j]}(-1)^{l-|J_1|-|J_2|}
\end{aligned}$$

$$\times \tilde{\psi}^!\bigg(x_1,\dots,x_p;\sum_{i\in J_1}\delta_{y_i}\bigg)\tilde{\psi}^!\bigg(x_{p+1},\dots,x_{p+q};\sum_{i\in J_2}\delta_{y_i}\bigg)$$
$$\times \rho^{(j+p)}(x_1,\dots,x_p,y_1,\dots,y_j)$$
$$\times \rho^{(l-j+q)}(x_{p+1},\dots,x_{p+q},y_{j+1},\dots,y_l)\,\mathrm{d}y_1\cdots\mathrm{d}y_l$$
$$=\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{1}{j!(l-j)!}\int_{(\bigcup_{i=1}^p B_{t,n}(x_i))^j\times(\bigcup_{i=1}^q B_{t,n}(x_{p+i}))^{l-j}}\sum_{J\subset[l]}(-1)^{l-|J|}$$
$$\times \tilde{\psi}^!\bigg(x_1,\dots,x_{p+q};\sum_{i\in J}\delta_{y_i}\bigg)$$
$$\times \rho^{(j+p)}(x_1,\dots,x_p,y_1,\dots,y_j)$$
$$\times \rho^{(l-j+q)}(x_{p+1},\dots,x_{p+q},y_{j+1},\dots,y_l)\,\mathrm{d}y_1\cdots\mathrm{d}y_l,$$

where we have used (3.21) in the last equality.

Now we estimate the difference of (3.20) and (3.23). Applying (1.10) and replacing $B_{t,n}(x_i)$ with $B_t(x_i)$, we obtain

$$\begin{aligned}&\big|\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q})\\&\quad-\tilde{m}^{(k_1,\dots,k_p)}(x_1,\dots,x_p)\tilde{m}^{(k_{p+1},\dots,k_q)}(x_{p+1},\dots,x_{p+q})\big|\\&\quad\le\phi\Big(\frac{s}{2}\Big)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{C_{l+p+q}}{j!(l-j)!}\\&\qquad\times\int_{(\bigcup_{i=1}^p B_t(x_i))^j\times(\bigcup_{i=1}^q B_t(x_{p+i}))^{l-j}}\sum_{J\subset[l]}\bigg|\tilde{\psi}^!\bigg(x_1,\dots,x_{p+q};\\&\qquad\sum_{i\in J}\delta_{y_i}\bigg)\bigg|\,\mathrm{d}y_1\cdots\mathrm{d}y_l.\end{aligned}\tag{3.23}$$

Recalling (3.21), (3.10) and the definitions of K_p, K_q and K at (3.9), we bound $\sum_{J\subset[l]}|\tilde{\psi}^!(x_1,\dots,x_{p+q};\sum_{i\in J}\delta_{y_i})|$ by $2^l(\hat{c}t)^{jK_p+(l-j)K_q+K}$, where $\hat{c}t\in[1,\infty)$ holds since $\hat{c}\in[1,\infty)$ in (1.18). This gives

$$\begin{aligned}&\big|\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q})\\&\quad-\tilde{m}^{(k_1,\dots,k_p)}(x_1,\dots,x_p)\tilde{m}^{(k_{p+1},\dots,k_q)}(x_{p+1},\dots,x_{p+q})\big|\\&\le\phi\Big(\frac{s}{2}\Big)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{C_{l+p+q}}{j!(l-j)!}\\&\quad\times\int_{(\bigcup_{i=1}^p B_t(x_i))^j\times(\bigcup_{i=1}^q B_t(x_{p+i}))^{l-j}}2^l(\hat{c}t)^{jK_p+(l-j)K_q+K}\,\mathrm{d}y_1\cdots\mathrm{d}y_l.\end{aligned}\tag{3.24}$$

Consequently, bounding K_p and K_q by K we obtain

$$\begin{aligned}
&\big|\tilde{m}^{(k_1,\dots,k_{p+q})}(x_1,\dots,x_{p+q}) \\
&\quad - \tilde{m}^{(k_1,\dots,k_p)}(x_1,\dots,x_p)\tilde{m}^{(k_{p+1},\dots,k_q)}(x_{p+1},\dots,x_{p+q})\big| \\
&\quad \le \phi\left(\frac{s}{2}\right)\sum_{l=0}^{\infty} C_{l+p+q}2^l(\hat{c}t)^{(l+1)K}\big((p+q)\theta_d t^d\big)^l \sum_{j=0}^{l}\frac{1}{j!(l-j)!} \qquad (3.25)\\
&\quad \le \phi\left(\frac{s}{2}\right)\sum_{l=0}^{\infty}\frac{C_{l+p+q}}{l!}4^l(\hat{c}t)^{(l+1)K}\big((p+q)\theta_d t^d\big)^l \\
&\quad \le \phi\left(\frac{s}{2}\right)\sum_{l=0}^{\infty}\frac{C_{l+K}}{l!}4^l(\hat{c}t)^{(l+1)K}\big(K\theta_d t^d\big)^l,
\end{aligned}$$

where $\theta_d := \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the unit ball in $\mathbb{R}^d$ and where the last inequality uses $p+q \le K$. The bound (1.16) yields $C_{l+K} = O((l+K)^{a(l+K)})$. Thus there are constants c_1, c_2 and c_3 depending only on a, d and K such that

$$\sum_{l=0}^{\infty}\frac{C_{l+K}}{l!}4^l(\hat{c}t)^{(l+1)K}\big(K\theta_d t^d\big)^l \le t^K \sum_{l=0}^{\infty}\frac{c_1 c_2^l l^{c_3}(t^{K+d})^l \cdot l^{al}}{l!}.$$

By Stirling's formula, there are constants c_4, c_5 and c_6 depending only on a, d and K such that

$$t^K\sum_{l=0}^{\infty}\frac{C_{l+K}}{l!}4^l(\hat{c}t)^{lK}\big(K\theta_d t^d\big)^l \le t^K \sum_{l=0}^{\infty}\frac{c_4 c_5^l l^{c_6}(t^{K+d})^l}{(\lfloor l(1-a)\rfloor)!},$$

where for $r \in \mathbb{R}$, $\lfloor r \rfloor$ is the greatest integer less than r. We compute

$$\begin{aligned}
&t^K\sum_{l=0}^{\infty}\frac{C_{l+K}}{l!}4^l(\hat{c}t)^{lK}\big(K\theta_d t^d\big)^l \\
&\quad \le t^K\sum_{n=0}^{\infty}\sum_{\{l:\lfloor l(1-a)\rfloor=n\}}\frac{c_4 c_5^l l^{c_6}(t^{K+d})^l}{n!} \qquad (3.26)\\
&\quad \le t^K\sum_{n=0}^{\infty}\frac{c_4 c_5^n n^{c_6}(t^{K+d})^{(n+1)/(1-a)}}{(1-a)n!} \le c_7\exp\big(c_8 t^{(K+d)/(1-a)}\big),
\end{aligned}$$

where c_7 and c_8 depend only on a, d and K.

Recalling from (3.16) that $t := (s/4)^{b(1-a)/(2(K+d))}$ we obtain

$$\sum_{l=0}^{\infty}\frac{C_{l+K}}{l!}4^l(\hat{c}t)^{(l+1)K}\big(K\theta_d t^d\big)^l \le c_7\exp\left(c_8\left(\frac{s}{4}\right)^{\frac{b}{2}}\right).$$

By (1.17), there is a constant c_9 depending only on a such that for all s we have $\phi(s) \le c_9 \exp(-s^b/c_9)$. Combining this with (3.25) and (3.26) gives

$$\left|\tilde{m}^{(k_1,\dots,k_{p+q})} - \tilde{m}^{(k_1,\dots,k_p)}\tilde{m}^{(k_{p+1},\dots,k_{p+q})}\right| \le c_7 c_9 \exp\left(\frac{-(s/2)^b}{c_9} + c_8\left(\frac{s}{4}\right)^{\frac{b}{2}}\right).$$

This along with (3.19) shows (1.21) when $(\xi, \mathcal{P})$ is an admissible pair of class (A2).

Now we establish (1.21) when $(\xi, \mathcal{P})$ is of class (A1). Let k be as in (1.13). Follow the arguments for case (A2) word for word using that $\sup_{x\in\mathcal{P}} R^\xi(x, \mathcal{P}) \le r$. Notice that for $l \in ((k-1)K, \infty)$ the summands in (3.20) vanish. Likewise, when $l_1 \in ((k-1)K_p, \infty)$ and $l_2 \in ((k-1)K_q, \infty)$, the respective summands in (3.22) vanish. It follows that for $l \in ((k-1)K, \infty)$ the summands in (3.25) all vanish. The finiteness of $\tilde{C}_K$ in expression (1.21) is immediate, without requiring decay rates for ϕ or growth bounds on C_k. Thus (1.21) holds when $(\xi, \mathcal{P})$ is of class (A1).

4. Proof of main results. We provide the proofs of Theorems 1.12, 1.15 and 1.13 in this order.

4.1. *Proof of Theorem* 1.12.

4.1.1. *Proof of expectation asymptotics* (1.23). The definition of the Palm probabilities gives $\mathbb{E}\mu_n^\xi(f) = \int_{W_n} f(n^{-1/d}u)\mathbb{E}_u\xi(u, \mathcal{P}_n)\rho^{(1)}(u)\,\mathrm{d}u$. As $\mathcal{P}$ is stationary and ξ is translation invariant, we have $\mathbb{E}_{\mathbf{0}}\xi(\mathbf{0}, \mathcal{P}) = \mathbb{E}_u\xi(u, \mathcal{P})$. So,

$$\begin{aligned}
&\left|n^{-1}\mathbb{E}\mu_n^\xi(f) - \mathbb{E}_{\mathbf{0}}\xi(\mathbf{0}, \mathcal{P})\rho^{(1)}(\mathbf{0})\int_{W_1} f(x)\,\mathrm{d}x\right| \\
&\quad = \left|n^{-1}\int_{W_n} f(n^{-1/d}u)\{\mathbb{E}_u\xi(u, \mathcal{P}_n)\rho^{(1)}(u) - \mathbb{E}_{\mathbf{0}}\xi(\mathbf{0}, \mathcal{P})\rho^{(1)}(\mathbf{0})\}\,\mathrm{d}u\right| \\
&\quad = \left|n^{-1}\int_{W_n} f(n^{-1/d}u)\mathbb{E}_u\left[\left(\xi(u, \mathcal{P}_n) - \xi(u, \mathcal{P})\right)\rho^{(1)}(u)\right]\mathrm{d}u\right| \\
&\quad \le \|f\|_\infty n^{-1}\int_{W_n} \mathbb{E}_u\big[\left|\xi(u, \mathcal{P}_n) - \xi(u, \mathcal{P})\right| \\
&\qquad \times \mathbf{1}\left[\max\left(R^\xi(u, \mathcal{P}), R^\xi(u, \mathcal{P}_n)\right) \ge d(u, \partial W_n)\right]\big]\rho^{(1)}(u)\,\mathrm{d}u \\
&\quad \le \|f\|_\infty n^{-1}\int_{W_n} \mathrm{d}u\rho^{(1)}(u)\mathbb{E}_u\big[\left|\xi(u, \mathcal{P}_n) - \xi(u, \mathcal{P})\right| \\
&\qquad \times \left(\mathbf{1}\left[R^\xi(u, \mathcal{P}) \ge d(u, \partial W_n)\right] + \mathbf{1}\left[R^\xi(u, \mathcal{P}_n) \ge d(u, \partial W_n)\right]\right)\big] \\
&\quad \le 4\kappa_1\|f\|_\infty n^{-1}\tilde{M}_p\int_{W_n}\left(\varphi\left(a_1 d(u, \partial W_n)\right)\right)^{1/q}\mathrm{d}u,
\end{aligned}$$

where the last inequality follows from the Hölder inequality, (1.14), the bound (1.11), the p-moment condition (1.19) (recall $p \in (1, \infty)$ and $\tilde{M}_p \in [1, \infty)$) and where $1/p + 1/q = 1$. By (1.15), the bound (1.23) follows at once from

$$\int_{W_n} \bigl(\varphi\bigl(a_1 d(u, \partial W_n)\bigr)\bigr)^{1/q}\, \mathrm{d}u = O\bigl(n^{(d-1)/d}\bigr).$$

If ξ satisfies (1.14), but not (1.15), then by the bounded convergence theorem, we have

$$\begin{aligned}\limsup_{n\to\infty} n^{-1} &\int_{W_n} \bigl(\varphi\bigl(a_1 d(u, \partial W_n)\bigr)\bigr)^{1/q}\, \mathrm{d}u \\ &= \limsup_{n\to\infty} \int_{W_1} \bigl(\varphi\bigl(a_1 n^{1/d} d(z, \partial W_1)\bigr)\bigr)^{1/q}\, \mathrm{d}z = 0.\end{aligned}$$

Consequently, we have expectation asymptotics under (1.14) as follows:

$$\left| n^{-1}\mathbb{E}\mu_n^{\xi}(f) - \mathbb{E}_{\mathbf{0}}\xi(\mathbf{0}, \mathcal{P})\rho^{(1)}(\mathbf{0}) \int_{W_1} f(x)\, \mathrm{d}x \right| = o(1).$$

4.1.2. *Proof of variance asymptotics* (1.24). Recall the definition of correlation functions (1.6) of the ξ-weighted measures. We have

$$\begin{aligned}\operatorname{Var} \mu_n^{\xi}(f) &= \mathbb{E} \sum_{x \in \mathcal{P}_n} f\bigl(n^{-1/d}x\bigr)^2 \xi^2(x, \mathcal{P}_n) \\ &\quad + \mathbb{E} \sum_{x, y \in \mathcal{P}_n, x \neq y} f\bigl(n^{-1/d}x\bigr) f\bigl(n^{-1/d}y\bigr) \xi(x, \mathcal{P}_n)\xi(y, \mathcal{P}_n) \qquad (4.1) \\ &\quad - \Bigl(\mathbb{E} \sum_{x \in \mathcal{P}_n} f\bigl(n^{-1/d}x\bigr)\xi(x, \mathcal{P}_n)\Bigr)^2 \\ &= \int_{W_n} f\bigl(n^{-1/d}u\bigr)^2 \mathbb{E}_u\bigl(\xi^2(u, \mathcal{P}_n)\bigr)\rho^{(1)}(u)\, \mathrm{d}u \\ &\quad + \int_{W_n \times W_n} f\bigl(n^{-1/d}u\bigr) f\bigl(n^{-1/d}v\bigr) \qquad (4.2) \\ &\quad \times \bigl(m_{(2)}(u, v; n) - m_{(1)}(u; n)m_{(1)}(v; n)\bigr)\, \mathrm{d}u\, \mathrm{d}v.\end{aligned}$$

Since ξ satisfies the p-moment condition (1.19) for $p > 2$, we have that ξ^2 satisfies the p-moment condition for $p > 1$. Also, ξ and ξ^2 have the same radius of stabilization. Thus, the proof of expectation asymptotics, with ξ replaced by ξ^2, shows that the first term in (4.1), multiplied by n^{-1}, converges to

$$\mathbb{E}_{\mathbf{0}}\xi^2(\mathbf{0}, \mathcal{P})\rho^{(1)}(\mathbf{0}) \int_{W_1} f(x)^2\, \mathrm{d}x;$$

cf. expectation asymptotics (1.23). Setting $x = n^{-1/d}u$ and $z = v - u = v - n^{1/d}x$, the second term in (4.2), multiplied by n^{-1}, may be rewritten as

$$\begin{aligned}
&\int_{W_1}\int_{W_n - n^{1/d}x} f(x + n^{-1/d}z) f(x) \\
&\quad\times \bigl[m_{(2)}(n^{1/d}x, n^{1/d}x + z; n) \\
&\quad- m_{(1)}(n^{1/d}x; n) m_{(1)}(n^{1/d}x + z; n)\bigr]\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned} \tag{4.3}$$

Setting $\mathcal{P}_n^x := \mathcal{P} \cap (W_n - n^{1/d}x)$, the translation invariance of ξ and stationarity of $\mathcal{P}$ yields

$$\begin{aligned}
m_{(2)}(n^{1/d}x, n^{1/d}x + z; n) &= m_{(2)}(\mathbf{0}, z; \mathcal{P}_n^x), \\
m_{(1)}(n^{1/d}x; n) &= m_{(1)}(\mathbf{0}; \mathcal{P}_n^x), \\
m_{(1)}(n^{1/d}x + z; n) &= m_{(1)}(z; \mathcal{P}_n^x).
\end{aligned}$$

Putting aside for the moment technical details, one expects that the above moments converge to $m_{(2)}(\mathbf{0}, z)$, $m_{(1)}(\mathbf{0})$ and $m_{(1)}(z) = m_{(1)}(\mathbf{0})$, respectively, when $n \to \infty$. Moreover, splitting the inner integral in (4.3) into two terms

$$\begin{aligned}
&\int_{W_n - n^{1/d}x} (\cdots)\,\mathrm{d}z \\
&\quad= \int_{W_n - n^{1/d}x} \mathbf{1}\bigl[|z| \le M\bigr](\cdots)\,\mathrm{d}z + \int_{W_n - n^{1/d}x} \mathbf{1}\bigl[|z| > M\bigr](\cdots)\,\mathrm{d}z
\end{aligned} \tag{4.4}$$

for any $M > 0$, we see (at least when f is continuous) that the first term in the right-hand side of (4.4) converges to the desired value

$$\int_{\mathbb{R}^d} f(x)^2 \bigl[m_{(2)}(\mathbf{0}, z) - m_{(1)}(\mathbf{0})^2\bigr]\,\mathrm{d}z$$

when first $n \to \infty$ and then $M \to \infty$. By the fast decay of the second-order correlations of the ξ-weighted measures, that is, by (1.21) with $p = q = k_1 = k_2 = 1$ and all $n \in \mathbb{N} \cup \{\infty\}$, the absolute value of the second term in (4.4) can be bounded uniformly in n by

$$\|f\|_\infty^2 \tilde{C}_2 \int_{|z|>M} \tilde{\phi}(\tilde{c}_2 z)\,\mathrm{d}z,$$

which goes to 0 when $M \to \infty$ since $\tilde{\phi}(\cdot)$ is fast-decreasing (and thus integrable).

To formally justify the above statements, we need the following lemma. Denote

$$h_n^\xi(x, z) := m_{(2)}(\mathbf{0}, z; \mathcal{P}_n^x) - m_{(1)}(\mathbf{0}; \mathcal{P}_n^x) m_{(1)}(z; \mathcal{P}_n^x).$$

LEMMA 4.1. *Assume that translation invariant score function ξ on the input process $\mathcal{P}$ satisfies* (1.14) *and the p-moment condition* (1.19) *for $p \in (2, \infty)$. Then $h_n^\xi(x, z)$ is uniformly bounded*

$$\sup_{n \le \infty} \sup_{x \in W_1} \sup_{z \in W_n - n^{1/d}x} \bigl|h_n^\xi(x, z)\bigr| \le C_h < \infty$$

for some constant C_h *and*

$$\lim_{n\to\infty} h_n^\xi(x,z) = h_\infty^\xi(x,z) = m_{(2)}(\mathbf{0},z) - \big(m_{(1)}(\mathbf{0})\big)^2.$$

PROOF. Denote $X_n := \xi(\mathbf{0},\mathcal{P}_n^x)$, $Y_n := \xi(z,\mathcal{P}_n^x)$, $X := \xi(\mathbf{0},\mathcal{P})$, and $Y := \xi(z,\mathcal{P})$. We shall prove first that all expectations $\mathbb{E}_{\mathbf{0},z}(X_n^2)$, $\mathbb{E}_{\mathbf{0},z}(Y_n^2)$, $\mathbb{E}_{\mathbf{0},z}(X^2)$ $\mathbb{E}_{\mathbf{0},z}(Y^2)$, $\mathbb{E}_{\mathbf{0}}|X_n|$, $\mathbb{E}_z|Y_n|$, $\mathbb{E}_{\mathbf{0}}|X|$ and $\mathbb{E}_z|Y|$ are uniformly bounded. Indeed, by the Hölder inequality

$$\mathbb{E}_{\mathbf{0},z}(X_n^2) \le \big(\mathbb{E}_{\mathbf{0},z}|X_n|^p\big)^{2/p} = \big(\mathbb{E}_{n^{1/d}x,z}\big|\xi\big(n^{1/d}x,\mathcal{P}_n\big)\big|^p\big)^{2/p} \le \tilde{M}_p^{2/p}, \tag{4.5}$$

where in the last inequality we have used p-moment condition (1.19) for $p > 2$. Similarly, $\mathbb{E}_{\mathbf{0},z}(Y_n^2)$ and $\mathbb{E}_{\mathbf{0},z}(X^2)$, $\mathbb{E}_{\mathbf{0},z}(Y^2)$ are bounded by $\tilde{M}_p^{2/p}$. Again using p-moment condition (1.19), we obtain

$$\mathbb{E}_{\mathbf{0}}|X_n| \le \big(\mathbb{E}_{\mathbf{0}}(X_n)^2\big)^{1/2} \le \big(\mathbb{E}_{n^{1/d}x}\big|\xi^2\big(n^{1/d}x,\mathcal{P}_n\big)\big|\big)^{1/2} \le \tilde{M}_p^{1/2}$$

and similarly for $\mathbb{E}_z|Y_n|$, $\mathbb{E}_{\mathbf{0}}|X|$ and $\mathbb{E}_z|Y|$. This proves the uniform bound of $|h_n^\xi(x,z)|$. To prove the convergence notice that

$$\begin{aligned}
&\big|m_{(2)}(\mathbf{0},z;\mathcal{P}_n^x) - m_{(2)}(\mathbf{0},z)\big| \qquad (4.6)\\
&\quad = \big|\mathbb{E}_{\mathbf{0},z}(X_nY_n) - \mathbb{E}_{\mathbf{0},z}(XY)\big|\rho^{(2)}(\mathbf{0},z)\\
&\quad \le \kappa_2\big(\mathbb{E}_{\mathbf{0},z}|X_nY_n - X_nY| + \mathbb{E}_{\mathbf{0},z}|X_nY - XY|\big)\\
&\quad \le \kappa_2\big(\mathbb{E}_{\mathbf{0},z}(X_n^2)\mathbb{E}_{\mathbf{0},z}(Y_n - Y)^2\big)^{1/2} + \kappa_2\big(\mathbb{E}_{\mathbf{0},z}(Y^2)\mathbb{E}_{\mathbf{0},z}(X_n - X)^2\big)^{1/2}, \qquad (4.7)
\end{aligned}$$

where κ_2 bounds the second-order correlation function as at (1.11). We have already proved that $\mathbb{E}_{\mathbf{0},z}(X_n^2)$, $\mathbb{E}_{\mathbf{0},z}(Y^2)$ are bounded. Moreover,

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0},z}(X_n - X)^2\\
&\quad = \mathbb{E}_{\mathbf{0},z}\big((X_n - X)^2\mathbf{1}[X_n \ne X]\big)\\
&\quad \le \mathbb{E}_{\mathbf{0},z}\big(X_n^2\mathbf{1}[X_n \ne X]\big) + 2\mathbb{E}_{\mathbf{0},z}\big(|X_nX|\mathbf{1}[X_n \ne X]\big) + \mathbb{E}_{\mathbf{0},z}\big(X^2\mathbf{1}[X_n \ne X]\big).
\end{aligned}$$

The Hölder inequality gives for $p > 2$ and $2/p + 1/q = 1$,

$$\begin{aligned}
\mathbb{E}_{\mathbf{0},z}\big(X_n^2\mathbf{1}[X_n \ne X]\big) &\le \big(\mathbb{E}_{\mathbf{0},z}\big(X_n^p\big)\big)^{2/p}\big(\mathbb{P}_{\mathbf{0},z}(X_n \ne X)\big)^{1/q},\\
\mathbb{E}_{\mathbf{0},z}\big(|X_nX|\mathbf{1}[X_n \ne X]\big) &\le \big(\mathbb{E}_{\mathbf{0},z}\big(X_n^p\big)\mathbb{E}_{\mathbf{0},z}\big(X^p\big)\big)^{1/p}\big(\mathbb{P}_{\mathbf{0},z}(X_n \ne X)\big)^{1/q},\\
\mathbb{E}_{\mathbf{0},z}\big(X^2\mathbf{1}[X_n \ne X]\big) &\le \big(\mathbb{E}_{\mathbf{0},z}\big(X^p\big)^{2/p}\big(\mathbb{P}_{\mathbf{0},z}(X_n \ne X)\big)^{1/q}.
\end{aligned}$$

The pth moment of X_n and X under $\mathbb{E}_{\mathbf{0},z}$ can be bounded by $\tilde{M}_p$ using the p-moment condition (1.19) with $p > 2$ as in (4.5). Stabilization (1.14) with $l = 2$ gives

$$\begin{aligned}
\mathbb{P}_{\mathbf{0},z}(X_n \ne X) &\le \mathbb{P}_{\mathbf{0},z}\big(\big(\max\big(R^\xi(u,\mathcal{P}), R^\xi(u,\mathcal{P}_n)\big) > n^{1/d}d(x,\partial W_1)\big) \qquad (4.8)\\
&\le 2\varphi\big(a_2n^{1/d}d(x,\partial W_1)\big) \qquad (4.9)
\end{aligned}$$

with the right-hand side converging to 0 for all $x \notin \partial W_1$. This proves that $\mathbb{E}_{\mathbf{0},z}(X_n - X)^2$ and (by the very same arguments) $\mathbb{E}_{\mathbf{0},z}(Y_n - Y)^2$ converge to 0 as $n \to \infty$ for all $x \notin \partial W_1$. Concluding this part of the proof, we have shown that the expression in (4.6) converges to 0, and thus $m_{(2)}(\mathbf{0}, z; \mathcal{P}_n^x)$ converges to $m_{(2)}(\mathbf{0}, z)$. Using similar arguments, we derive

$$\begin{aligned}\big|m_{(1)}(\mathbf{0}, \mathcal{P}_n^x) - m_{(1)}(\mathbf{0})\big| &= \big|\mathbb{E}_{\mathbf{0}}(X_n) - \mathbb{E}_{\mathbf{0}}(X)\big|\rho^{(1)}(\mathbf{0}) \\ &\le \kappa_1\big(\big(\mathbb{E}_{\mathbf{0}}(X_n)^2\big)^{1/2} + \big(\mathbb{E}_{\mathbf{0}}(X^2)\big)^{1/2}\big)\big(\mathbb{P}_{\mathbf{0}}(X_n \ne X)\big)\big)^{1/2},\end{aligned}$$

by the p-moment condition (1.19) and the stabilization property (1.14) for $p = 1$ one can show that $m_{(1)}(\mathbf{0}, \mathcal{P}_n^x)$ converges to $m_{(1)}(\mathbf{0})$ uniformly in x for all $x \in W_1 \setminus \partial W_1$. Exactly the same arguments assure convergence of $m_{(1)}(z, \mathcal{P}_n^x)$ to $m_{(1)}(z) = m_{(1)}(\mathbf{0})$. This concludes the proof of Lemma 4.1. □

In order to complete the proof of variance asymptotics for general $f \in \mathcal{B}(W_1)$ (not necessarily continuous), we use arguments borrowed from the proof of [40], Theorem 2.1. Recall that $x \in W_1$ is a Lebesgue point for f if $(\mathrm{Vol}_d B_\varepsilon(x))^{-1} \int_{B_\varepsilon(x)} |f(z) - f(x)| \,\mathrm{d}z \to 0$ as $\varepsilon \to 0$. Denote by $\mathcal{C}_f$ all Lebesgue points of f in W_1. By the Lebesgue density theorem, almost every $x \in W_1$ is a Lebesgue point of f, and thus for any $M > 0$ and n large enough the double integral in (4.3) is equal to

$$\begin{aligned}&\int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{W_n - n^{1/d}x} f\big(x + n^{-1/d}z\big) h_n^\xi(x, z) \,\mathrm{d}z \,\mathrm{d}x \\ &\quad = \int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{|z| \le M} f\big(x + n^{-1/d}z\big) h_n^\xi(x, z) \,\mathrm{d}z \,\mathrm{d}x \\ &\qquad + \int_{W_1} \mathbf{1}[x \in \mathcal{C}_f] f(x) \int_{W_n - n^{1/d}x} 1\big(|z| > M\big) f\big(x + n^{-1/d}z\big) h_n^\xi(x, z) \,\mathrm{d}z \,\mathrm{d}x.\end{aligned}$$

As already explained, by the fast decay of the second-order correlations of the ξ-weighted measures, the second term converges to 0 as first $n \to \infty$ and then $M \to \infty$. Considering the first term, by the uniform boundedness of $h_n^\xi(x, z)$, using the dominated convergence theorem, it is enough to prove for any Lebesgue point x of f and fixed M that

$$\lim_{n\to\infty} \int_{|z|<M} h_n^\xi(x, z) f\big(x + n^{-1/d}z\big) \,\mathrm{d}z = f(x) \int_{|z|<M} h_\infty^\xi(x, z) \,\mathrm{d}z.$$

In this regard, notice that

$$\begin{aligned}&\int_{|z|<M} \big|h_n^\xi(x, z) f\big(x + n^{-1/d}z\big) - h_\infty^\xi(x, z) f(x)\big| \,\mathrm{d}z \\ &\qquad \le \int_{|z|<M} C_h \times \big|f\big(x + n^{-1/d}z\big) - f(x)\big|\end{aligned}$$

$$
\begin{aligned}
&+ \left|h_n^\xi(x,z) - h_\infty^\xi(x,z)\right| \times \|f\|_\infty \,\mathrm{d}z \\
&\le C_h n \int_{|z|<n^{-1/d}M} \left|f(x+z) - f(x)\right| \mathrm{d}z + \|f\|_\infty \\
&\quad \times \int_{|z|<M} \left|h_n^\xi(x,z) - h_\infty^\xi(x,z)\right| \mathrm{d}z.
\end{aligned}
$$

Both terms converge to 0 as $n \to \infty$: the first since x is a Lebesgue point of x, the second by the dominated convergence of $h_n^\xi(x,z)$; cf. Lemma 4.1. Note that

$$\int_{W_1} \int_{\mathbb{R}^2} \left|h_\infty^\xi(x,z)\right| \mathrm{d}z\,\mathrm{d}x < \infty,$$

which follows again from the fast decay of the second-order correlations of the ξ-weighted measure ((1.21) with $p = q = k_1 = k_2 = 1$ and all $n \in \mathbb{N} \cup \{\infty\}$). Letting M go to infinity in $\int_{W_1} f^2(x) \int_{|z|<M} h_\infty^\xi(x,z)\,\mathrm{d}z\,\mathrm{d}x$ completes the proof of variance asymptotics.

4.2. *Proof of Theorem* 1.15. The proof is inspired by the proofs of [34], Propositions 1 and 2. By the refined Campbell theorem and stationarity of $\mathcal{P}$, we have

$$
\begin{aligned}
&n^{-1} \operatorname{Var} \widehat{H}_n^\xi(\mathcal{P}) \\
&\quad = \int_{W_n} \mathbb{E}_x \xi^2(x;\mathcal{P}) \rho^{(1)}(x)\,\mathrm{d}x \\
&\qquad + \int_{W_n} \int_{W_n} \left[m_{(2)}(x,y) - m_{(1)}(x) m_{(1)}(y)\right] \mathrm{d}y\,\mathrm{d}x \qquad (4.10)\\
&\quad = \mathbb{E}_{\mathbf{0}} \xi^2(\mathbf{0},\mathcal{P}) \rho^{(1)}(\mathbf{0}) \\
&\qquad + n^{-1} \int_{W_n} \int_{W_n} \left(m_{(2)}(x,y) - m_{(1)}(x) m_{(1)}(y)\right) \mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Writing $c(x,y) := m_{(2)}(x,y) - m_{(1)}(x) m_{(1)}(y)$, the double integral in (4.10) becomes ($z = y - x$)

$$
\begin{aligned}
&n^{-1} \int_{W_n} \int_{W_n} \left(m_{(2)}(x,y) - m_{(1)}(x) m_{(1)}(y)\right) \mathrm{d}y\,\mathrm{d}x \\
&\quad = n^{-1} \int_{W_n} \int_{\mathbb{R}^d} c(\mathbf{0},z) \mathbf{1}[x + z \in W_n]\,\mathrm{d}z\,\mathrm{d}x \\
&\quad = n^{-1} \int_{W_n} \int_{\mathbb{R}^d} c(\mathbf{0},z) \mathbf{1}[x \in W_n - z]\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}
$$

Write $\mathbf{1}[x \in W_n - z]$ as $1 - \mathbf{1}[x \in (W_n - z)^c]$ to obtain

$$
\begin{aligned}
&n^{-1} \int_{W_n} \int_{W_n} \left(m_{(2)}(x,y) - m_{(1)}(x) m_{(1)}(y)\right) \mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} c(\mathbf{0},z)\,\mathrm{d}z - n^{-1} \int_{\mathbb{R}^d} \int_{W_n} c(\mathbf{0},z) \mathbf{1}\left[x \in \mathbb{R}^d \setminus (W_n - z)\right] \mathrm{d}x\,\mathrm{d}z.
\end{aligned}
$$

From (1.29), we have that $\gamma_{W_n}(z) := \mathrm{Vol}_d(W_n \cap (\mathbb{R}^d \setminus (W_n - z)))$, and thus rewrite (4.10) as

$$
\begin{aligned}
n^{-1} \operatorname{Var} \widehat{H}_n^{\xi}(\mathcal{P}) &= \mathbb{E}_{\mathbf{0}} \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) \\
&\quad + \int_{\mathbb{R}^d} c(\mathbf{0}, z)\, \mathrm{d}z - n^{-1} \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma_{W_n}(z)\, \mathrm{d}z.
\end{aligned}
\tag{4.11}
$$

Now we claim that

$$
\lim_{n\to\infty} n^{-1} \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma_{W_n}(z)\, \mathrm{d}z = 0.
$$

Indeed, as noted in Lemma 1 of [34], for all $z \in \mathbb{R}^d$ we have $\lim_{n\to\infty} n^{-1} \gamma_{W_n}(z) = 0$. Since $n^{-1} c(\mathbf{0}, z) \gamma_{W_n}(z)$ is dominated by the fast-decreasing function $c(\mathbf{0}, z)$, the dominated convergence theorem gives the claimed limit. Letting $n \to \infty$ in (4.11) gives

$$
\lim_{n\to\infty} n^{-1} \operatorname{Var} \widehat{H}_n^{\xi}(\mathcal{P}) = \mathbb{E}_{\mathbf{0}} \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) + \int_{\mathbb{R}^d} c(\mathbf{0}, z)\, \mathrm{d}z = \sigma^2(\xi),
\tag{4.12}
$$

where the last equality follows by the definition of $\sigma^2(\xi)$ in (1.22) and the finiteness follows by the fast-decreasing property of $c(\mathbf{0}, z, \mathcal{P})$ (which follows from the assumption of fast decay of the second mixed moment density).

Now if $\sigma^2(\xi) = 0$ then the right-hand side of (4.12) vanishes, that is,

$$
\mathbb{E}_{\mathbf{0}} \xi^2(\mathbf{0}, \mathcal{P}) \rho^{(1)}(\mathbf{0}) + \int_{\mathbb{R}^d} c(\mathbf{0}, z)\, \mathrm{d}z = 0.
$$

Applying this identity to the right-hand side of (4.11), then multiplying (4.11) by $n^{1/d}$ and taking limits we obtain

$$
\lim_{n\to\infty} n^{-(d-1)/d} \operatorname{Var} \widehat{H}_n^{\xi}(\mathcal{P}) = - \lim_{n\to\infty} n^{-(d-1)/d} \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma_{W_n}(z)\, \mathrm{d}z.
\tag{4.13}
$$

As in [34], we have $n^{-(d-1)/d} \gamma_{W_n}(z) \le C|z|$ and, therefore, again, by the fast-decreasing property of $c(\mathbf{0}, z)$ we conclude that $n^{-(d-1)/d} c(\mathbf{0}, z) \gamma_{W_n}(z)$ is dominated by an integrable function of z. Also, as in [34], Lemma 1, for all $z \in \mathbb{R}^d$ we have $\lim_{n\to\infty} n^{-(d-1)/d} \gamma_{W_n}(z) = \gamma(z)$. The dominated convergence theorem yields (1.31) as desired,

$$
\lim_{n\to\infty} n^{-(d-1)/d} \operatorname{Var} \widehat{H}_n^{\xi}(\mathcal{P}) = - \int_{\mathbb{R}^d} c(\mathbf{0}, z) \gamma(z)\, \mathrm{d}z.
$$

4.3. *First proof of the central limit theorem.*

4.3.1. *The method of cumulants.* We use the method of cumulants to prove Theorem 1.13. We shall define cumulants precisely in Section 4.3.2. Write $\overline{\mu_n^{\xi}}$ for the centered measure $\mu_n^{\xi} - \mathbb{E}\mu_n^{\xi}$ and recall that we write $\langle f, \mu \rangle$ for

$\int f\,d\mu$. The guiding principle is that as soon as the kth order cumulants C_n^k for $\langle f, \overline{\mu}_n^\xi\rangle/\sqrt{\operatorname{Var}\langle f, \mu_n^\xi\rangle}$ vanish as $n \to \infty$ for k large, then

$$\frac{\langle f, \overline{\mu}_n\rangle}{\sqrt{\operatorname{Var}\langle f, \mu_n^\xi\rangle}} \xrightarrow{\mathcal{D}} N. \tag{4.14}$$

We establish the vanishing of C_n^k for k large by showing that the fast decay of correlation functions for the ξ-weighted measures at (1.5) implies volume order growth (i.e., growth of order $O(n)$) for the kth order cumulant for $\langle f, \overline{\mu}_n^\xi\rangle$, $k \geq 2$, and then use the assumption $\operatorname{Var}\langle f, \mu_n^\xi\rangle = \Omega(n^\nu)$.

Our approach. The $O(n)$ growth of the kth order cumulant for $\langle f, \overline{\mu}_n^\xi\rangle$ is established by controlling the growth of kth order cumulant measures for μ_n^ξ, here denoted by c_n^k, and which are defined analogously to moment measures. We first prove a general result (see (4.19) and (4.20) below) showing that integrals of the cumulant measures c_n^k may be controlled by a finite sum of integrals of so-called (S,T) semi-cluster measures, where (S,T) is a generic partition of $\{1,\dots,k\}$. This result holds for any μ_n^ξ of the form (1.4) and depends neither on choice of input $\mathcal{P}$ nor on the localization properties of ξ. Semi-cluster measures for μ_n^ξ have the appealing property that they involve differences of measures on product spaces with product measures, and thus their Radon–Nikodym derivatives involve differences of correlation functions of the ξ-weighted measures.

In general, bounds on cumulant measures in terms of semi-cluster measures are not terribly informative. However, when ξ, together with $\mathcal{P}$, satisfy moment bounds and fast decay of correlations (1.21), then the situation changes. First, integrals of (S,T) semi-cluster measures on properly chosen subsets $W(S,T)$ of W_n^k, with (S,T) ranging over partitions of $\{1,\dots,k\}$, exhibit $O(n)$ growth. This is because the subsets $W(S,T)$ are chosen so that the Radon–Nikodym derivative of the (S,T) semi-cluster measure, being a difference of the correlation functions of the ξ-weighted measures, may be controlled by (1.21) for points $(v_1,\dots,v_k) \in W(S,T)$. Second, it conveniently happens that W_n^k is precisely the union of $W(S,T)$, as (S,T) ranges over partitions of $\{1,\dots,k\}$. Therefore, combining these observations, we see that every cumulant measure on W_n^k is a sum ranging over partitions (S,T) of $\{1,\dots,k\}$ of linear combinations of (S,T) semi-cluster measures on $W(S,T)$, each of which exhibits $O(n)$ growth.

Thus cumulant measures c_n^k exhibit growth *proportional to* $\operatorname{Vol}_d(W_n)$ carrying $\mathcal{P}_n$, namely

$$\langle f^k, c_n^k\rangle = O(n), \qquad f \in \mathcal{B}(W_1), k = 2, 3, \dots \tag{4.15}$$

The remainder of Section 4.3 provides the details justifying (4.15).

Remarks on related work. (a) The estimate (4.15) first appeared in [5], Lemma 5.3, but the work of [14] (and to some extent [56]) was the first to rigorously control the growth of c_n^k on the diagonal subspaces, where two or more coordinates coincide. In fact, Section 3 of [14] shows the estimate $\langle f^k, c_n^k\rangle \leq L^k (k!)^\beta n$, where L and β are constants independent of n and k. We assert that the arguments behind (4.15) are not restricted to Poisson input, but depend only on the fast decay of correlations (1.21) of the ξ-weighted measures and moment bounds (1.19). Since these arguments are not well known, we present them in a way which is hopefully accessible and reasonably self-contained. Since we do not care about the constants in (4.15), we shall suitably adopt the arguments of [5], Lemma 5.3 and [56], taking the opportunity to make those arguments more rigorous. Indeed those arguments did not adequately explain the fast decay of the correlations of the ξ-wighted measures of the ξ-weighted measures on diagonal subspaces.

(b) The breakthrough paper [37] shows that the kth order cumulant for the *linear statistic* $\langle f, \sum_x \delta_{n^{-1/d}x}\rangle/\sqrt{\mathrm{Var}\langle f, \sum_x \delta_{n^{-1/d}x}\rangle}$ vanishes as $n \to \infty$ and k large. This approach is extended to $\langle f, \mu_n^\xi\rangle$ in Section 4.4 thereby giving a second proof of the central limit theorem.

4.3.2. *Properties of cumulant and semi-cluster measures.*

Moments and cumulants. For a random variable Y with all finite moments, expanding the logarithm of the Laplace transform (in the negative domain) in a formal power series gives

$$\log \mathbb{E}(e^{tY}) = \log\left(1 + \sum_{k=1}^{\infty} \frac{M_k t^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{S_k t^k}{k!}, \tag{4.16}$$

where $M_k = \mathbb{E}(Y^k)$ is the kth moment of Y and $S_k = S_k(Y)$ denotes the k th cumulant of Y. Both series in (4.16) can be considered as formal ones and no additional condition (on exponential moments of Y) are required for the cumulants to exist. Explicit relations between cumulants and moments may be established by formal manipulations of these series; see, for example, [13], Lemma 5.2.VI. In particular,

$$S_k = \sum_{\gamma \in \Pi[k]} (-1)^{|\gamma|-1}\big(|\gamma| - 1\big)! \prod_{i=1}^{|\gamma|} M^{|\gamma(i)|}, \tag{4.17}$$

where $\Pi[k]$ is the set of all unordered partitions of the set $\{1, \dots, k\}$, and for a partition $\gamma = \{\gamma(1), \dots, \gamma(l)\} \in \Pi[k]$, $|\gamma| = l$ denotes the number of its elements, while $|\gamma(i)|$ the number of elements of subset $\gamma(i)$. (Although elements of $\Pi[k]$ are unordered partitions, we need to adopt some convention for the labeling of their elements: let $\gamma(1), \dots, \gamma(l)$ correspond to the ordering of the smallest elements in the partition sets.) In view of (4.17), the existence of the kth cumulant S_k follows from the finiteness of the moment M_k.

Moment measures. Given a random measure μ on $\mathbb{R}^d$, the kth moment measure $M^k = M^k(\mu)$ is the one (Section 5.4 and Section 9.5 of [13]) satisfying

$$\begin{aligned}\big\langle f_1 \otimes \cdots \otimes f_k, M^k(\mu)\big\rangle &= \mathbb{E}\big[\langle f_1, \mu\rangle \cdots \langle f_k, \mu\rangle\big] \\ &= \mathbb{E}\bigg[\sum_{x \in \mathcal{P}_n} f_1\bigg(\frac{x}{n^{1/d}}\bigg)\xi(x, \mathcal{P}_n) \cdots \sum_{x \in \mathcal{P}_n} f_k\bigg(\frac{x}{n^{1/d}}\bigg)\xi(x, \mathcal{P}_n)\bigg]\end{aligned}$$

for all $f_1, \ldots, f_k \in \mathbb{B}(\mathbb{R}^d)$, where $f_1 \otimes \cdots \otimes f_k : (\mathbb{R}^d)^k \to \mathbb{R}$ is given by $f_1 \otimes \cdots \otimes f_k(x_1, \ldots, x_k) = f_1(x_1) \cdots f_k(x_k)$.

As on page 143 of [13], when μ is a counting measure, M^k may be expressed as a sum of factorial moment measures $M_{[j]}$, $1 \le j \le k$, (as defined on page 133 of [13]):

$$M^k\big(\mathrm{d}(x_1 \times \cdots \times x_k)\big) = \sum_{j=1}^{k} \sum_{\mathcal{V}} M_{[j]}\left(\prod_{i=1}^{j} \mathrm{d}y_i(\mathcal{V})\right)\delta(\mathcal{V}),$$

where, to quote from [13], the inner sum is taken over all partitions $\mathcal{V}$ of the k coordinates into j nonempty disjoint subsets, the $y_i(\mathcal{V})$, $1 \le i \le j$, constitute an arbitrary selection of one coordinate from each subset and $\delta(\mathcal{V})$ is a δ function which equals zero unless equality holds among the coordinates in each nonempty subset of $\mathcal{V}$.

When μ is the atomic measure μ_n^{ξ}, we write M_n^k for $M^k(\mu_n^{\xi})$. By the Campbell formula, considering repetitions in the k-fold product of $\mathbb{R}^d$, and putting $\tilde{y}_i := y_i(\mathcal{V})$ and $\mathcal{V} := (\mathcal{V}_1, \ldots, \mathcal{V}_j)$ we have that

$$\begin{aligned}
&\langle f \otimes \cdots \otimes f, M_n^k \rangle \\
&\quad = \mathbb{E}[\langle f, \mu_n^{\xi}\rangle \cdots \langle f, \mu_n^{\xi}\rangle] \\
&\quad = \sum_{j=1}^{k} \sum_{\mathcal{V}} \int_{(W_n)^j} \prod_{i=1}^{k} f\left(\frac{y_i}{n^{1/d}}\right) \mathbb{E}_{\tilde{y}_1 \cdots \tilde{y}_j}\left[\prod_{i=1}^{j} \xi^{|\mathcal{V}_i|}(\tilde{y}_i, \mathcal{P}_n)\right] \\
&\qquad \times \rho^{(j)}(\tilde{y}_1, \ldots, \tilde{y}_j) \prod_{i=1}^{j} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}).
\end{aligned}$$

In other words, recalling Lemma 9.5.IV of [13] we get

$$dM_n^k(y_1, \ldots, y_k) = \sum_{j=1}^{k} \sum_{\mathcal{V}} m^{(|\mathcal{V}_1|, \ldots, |\mathcal{V}_j|)}(\tilde{y}_1, \ldots, \tilde{y}_j; n) \prod_{i=1}^{j} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}). \tag{4.18}$$

Cumulant measures. The kth cumulant measure $c_n^k := c^k(\mu_n)$ is defined analogously to the kth moment measure via

$$\big\langle f_1 \otimes \cdots \otimes f_k, c^k(\mu_n)\big\rangle = c\big(\langle f_1, \mu_n\rangle \cdots \langle f_k, \mu_n\rangle\big),$$

where $c(X_1, \ldots, X_k)$ denotes the joint cumulant of the random variables $X_1, \ldots, X_k$.

The existence of the cumulant measures c_n^l, $l = 1, 2, \ldots$ follows from the existence of moment measures in view of the representation (4.17). Thus, we have the following representation for cumulant measures:

$$c_n^l = \sum_{T_1, \ldots, T_p} (-1)^{p-1}(p-1)! M_n^{T_1} \cdots M_n^{T_p},$$

where $T_1, \dots, T_p$ ranges over all unordered partitions of the set $1, \dots, l$ (see page 30 of [33]). Henceforth for $T_i \subset \{1, \dots, l\}$, let $M_n^{T_i}$ denote a copy of the moment measure $M^{|T_i|}$ on the product space W^{T_i}. Multiplication denotes the usual product of measures: For T_1, T_2 disjoint sets of integers and for measurable $B_1 \subset (\mathbb{R}^d)^{T_1}$, $B_2 \subset (\mathbb{R}^d)^{T_2}$, we have $M_n^{T_1} M_n^{T_2}(B_1 \times B_2) = M_n^{T_1}(B_1) M_n^{T_2}(B_2)$. The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the covariance measure.

Cluster and semi-cluster measures. We show that every cumulant measure c_n^k is a linear combination of products of moment and cluster measures. We first recall the definition of cluster and semi-cluster measures. A cluster measure $U_n^{S,T}$ on $W_n^S \times W_n^T$ for nonempty $S, T \subset \{1, 2, \dots\}$ is defined by

$$U_n^{S,T}(B \times D) = M_n^{S \cup T}(B \times D) - M_n^S(B) M_n^T(D)$$

for Borel sets B and D in W_n^S and W_n^T, respectively, and where multiplication means product measure.

Let S_1, S_2 be a partition of S and let T_1, T_2 be a partition of T. A product of a cluster measure $U_n^{S_1,T_1}$ on $W_n^{S_1} \times W_n^{T_1}$ with products of moment measures $M_n^{|S_2|}$ and $M_n^{|T_2|}$ on $W_n^{S_2} \times W_n^{T_2}$ is an (S, T) *semi-cluster measure*.

For each nontrivial partition (S, T) of $\{1, \dots, k\}$, the kth cumulant c_n^k measure is represented as

$$c_n^k = \sum_{(S_1,T_1),(S_2,T_2)} \alpha\big((S_1, T_1), (S_2, T_2)\big) U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|}, \tag{4.19}$$

where the sum ranges over partitions of $\{1, \dots, k\}$ consisting of pairings (S_1, T_1), (S_2, T_2), where $S_1, S_2 \subset S$ and $T_1, T_2 \subset T$, where S_1 *and* T_1 *are nonempty*, and where $\alpha((S_1, T_1), (S_2, T_2))$ are integer valued pre-factors. In other words, for any non-trivial partition (S, T) of $\{1, \dots, k\}$, c_n^k is a linear combination of (S, T) semi-cluster measures. We prove this exactly as in the proof of Lemma 5.1 of [5], as that proof involves only combinatorics and does not depend on the nature of the input. For an alternate proof, with good growth bounds on the integer pre-factors $\alpha((S_1, T_1), (S_2, T_2))$, we refer to Lemma 3.2 of [14].

Let $\Xi(k)$ be the collection of partitions of $\{1, \dots, k\}$ into two subsets S and T. Whenever W_n^k may be expressed as the union of sets $W(S, T)$, $(S, T) \in \Xi(k)$, then we may write

$$\begin{aligned} \big|\langle f^k, c_n^k\rangle\big| &\le \sum_{(S,T)\in\Xi(k)} \int_{W(S,T)} \big|f(v_1)\cdots f(v_k)\big| \big|dc_n^k(v_1, \dots, v_k)\big| \\ &\le \|f\|_\infty^k \sum_{(S,T)\in\Xi(k)} \sum_{(S_1,T_1),(S_2,T_2)} \big|\alpha\big((S_1, T_1), (S_2, T_2)\big)\big| \\ &\quad \times \int_{W(S,T)} d\big(U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|}\big)(v_1, \dots, v_k), \end{aligned} \tag{4.20}$$

where the last inequality follows by (4.19). As noted at the outset, this bound is valid for any $f \in \mathbb{B}(\mathbb{R}^d)$ and any measure μ_n^ξ of the form (1.4).

We now specify the collection of sets $W(S,T), (S,T) \in \Xi(k)$, to be used in (4.20) as well as in all that follows. Given $v := (v_1, \dots, v_k) \in W_n^k$, let

$$D_k(v) := D_k(v_1, \dots, v_k) := \max_{i \le k}\big(|v_1 - v_i| + \cdots + |v_k - v_i|\big)$$

be the l^1 diameter for v. For all such partitions, consider the subset $W(S,T)$ of $W_n^S \times W_n^T$ having the property that $v \in W(S,T)$ implies $d(v^S, v^T) \ge D_k(v)/k^2$, where v^S and v^T are the projections of v onto W_n^S and W_n^T, respectively, and where $d(v^S, v^T)$ is the minimal Euclidean distance between pairs of points from v^S and v^T.

It is easy to see that for every $v := (v_1, \dots, v_k) \in W_n^k$, there is a partition (S,T) of $\{1, \dots, k\}$ such that $d(v^S, v^T) \ge D_k(v)/k^2$. If this were not the case, then given $v := (v_1, \dots, v_k)$, the distance between any two components of v must be strictly less than $D_k(v)/k^2$ and we would get $\max_{i \le k} \sum_{j=1}^k |v_i - v_j| \le (k-1)kD_k/k^2 < D_k$, a contradiction. Thus W_n^k is the union of sets $W(S,T), (S,T) \in \Xi(k)$, as asserted. We next describe the behavior of the differential $d(U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|})$ on $W(S,T)$.

Semi-cluster measures on $W(S,T)$. Next, given $S_1 \subset S$ and $T_1 \subset T$, notice that $d(v^{S_1}, v^{T_1}) \ge d(v^S, v^T)$ where v^{S_1} denotes the projection of v^S onto $W_n^{S_1}$ and v^{T_1} denotes the projection of v^T onto $W_n^{T_1}$. Let $\Pi(S_1, T_1)$ be the partitions of S_1 into j_1 sets $\mathcal{V}_1, \dots, \mathcal{V}_{j_1}$, with $1 \le j_1 \le |S_1|$, and the partitions of T_1 into j_2 sets $\mathcal{V}_{j_1+1}, \dots, \mathcal{V}_{j_1+j_2}$, with $1 \le j_2 \le |T_1|$. Thus an element of $\Pi(S_1, T_1)$ is a partition of $S_1 \cup T_1$.

If a partition $\mathcal{V}$ of $S_1 \cup T_1$ does not belong to $\Pi(S_1, T_1)$, then there is a partition element of $\mathcal{V}$ containing points in S_1 and T_1, and thus, recalling (4.18), we have $\delta(\mathcal{V}) = 0$ on the set $W(S,T)$. Thus we make the crucial observation that, *on the set $W(S,T)$ the differential $d(M_n^{S_1 \cup T_1})$ collapses into a sum over partitions in* $\Pi(S_1, T_1)$. Thus $d(M_n^{S_1 \cup T_1})$ and $d(M_n^{S_1} M_n^{T_1})$ both involve sums of measures on common diagonal subspaces, as does their difference, made precise as follows.

LEMMA 4.2. *On the set $W(S,T)$, we have*

$$d\big(U_n^{S_1,T_1}\big) = \sum_{j_1=1}^{|S_1|} \sum_{j_2=1}^{|T_1|} \sum_{\mathcal{V} \in \Pi(S_1,T_1)} [\cdots] \Pi_{i=1}^{j_1+j_2} \mathrm{d}y_i(\mathcal{V}) \delta(\mathcal{V}), \tag{4.21}$$

where

$$\begin{aligned} [\cdots] := {} & m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_1}|, |\mathcal{V}_{j_1+1}|, \dots, |\mathcal{V}_{j_1+j_2}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_1} \tilde{y}_{j_1+1}, \dots \tilde{y}_{j_1+j_2}; n) \\ & - m^{(|\mathcal{V}_1|, \dots, |\mathcal{V}_{j_1}|)}(\tilde{y}_1, \dots, \tilde{y}_{j_1}; n) m^{(|\mathcal{V}_{j_1+1}|, \dots, |\mathcal{V}_{j_1+j_2}|)}(\tilde{y}_{j_1+1}, \dots \tilde{y}_{j_1+j_2}; n). \end{aligned}$$

The representations of $dM_n^{|S_2|}$ and $dM_n^{|T_2|}$ follow from (4.18), that is to say

$$dM_n^{|S_2|} = \sum_{j_3=1}^{|S_2|} \sum_{\mathcal{V}\in\Pi(S_2)} m^{(|\mathcal{V}_1|,\dots,|\mathcal{V}_{j_3}|)}(\tilde{y}_1,\dots,\tilde{y}_{j_3};n) \prod_{i=1}^{j_3} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}), \tag{4.22}$$

where $\Pi(S_2)$ runs over partitions of S_2 into j_3 sets, $1 \le j_3 \le |S_2|$. Similarly,

$$dM_n^{|T_2|} = \sum_{j_4=1}^{|T_2|} \sum_{\mathcal{V}\in\Pi(T_2)} m^{(|\mathcal{V}_1|,\dots,|\mathcal{V}_{j_4}|)}(\tilde{y}_1,\dots,\tilde{y}_{j_4};n) \prod_{i=1}^{j_4} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}), \tag{4.23}$$

where $\Pi(T_2)$ runs over partitions of T_2 into j_4 sets, $1 \le j_4 \le |T_2|$.

4.3.3. *Fast decay of correlations and semi-cluster measures*. The previous section established properties of semi-cluster and cumulant measures valid for any μ_n^ξ of the form (1.4). If ξ with $\mathcal{P}$ exhibit fast decay of correlations (1.21) of the ξ-weighted measures and satisfies moment bounds, we now assert that each integral in (4.20) is $O(n)$.

LEMMA 4.3. *Assume ξ satisfies moment bounds* (1.19) *for all $p \ge 1$ and exhibits fast decay of correlations* (1.21) *in its ξ-weighted measure. For each partition element (S,T) of $\Xi(k)$, we have*

$$\int_{W(S,T)\subset W_n^S\times W_n^T} \left|d\left(U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|}\right)\right| = O(n). \tag{4.24}$$

PROOF. The differential $d(U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|})$ is a sum of products of three factors $\sum_{j_1=1}^{|S_1|}\sum_{j_2=1}^{|T_1|}\sum_{j_3=1}^{|S_2|}\sum_{j_4=1}^{|T_2|}[\cdots][\cdots][\cdots]$, one factor coming from each of the summands in (4.21)–(4.23). By Theorem 1.11, on the set $W(S,T)$ the factor arising from (4.21) is bounded in absolute value by

$$\tilde{C}_k \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right).$$

By the moment bound (1.19), the two remaining factors arising from summands in (4.22)–(4.23) are bounded by a constant $M'(k)$ depending only on k.

Thus we have

$$\begin{aligned}
&\int_{W(S,T)} \left|d\left(U_n^{S_1,T_1} M_n^{|S_2|} M_n^{|T_2|}\right)\right| \\
&\quad \le \tilde{C}_k \left(M'(k)\right)^2 \sum_{j=1}^{k} \sum_{\mathcal{V}} \int_{W(S,T)} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \prod_{i=1}^{j} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}) \\
&\quad \le \tilde{C}_k \left(M'(k)\right)^2 \sum_{j=1}^{k} \sum_{\mathcal{V}} \int_{(W_n)^j} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \prod_{i=1}^{j} \mathrm{d}y_i(\mathcal{V})\delta(\mathcal{V}).
\end{aligned}$$

Here $\mathcal{V}$ runs over all partitions of the k coordinates into j nonempty disjoint subsets. We assert that all summands are $O(n)$. We show this when $j = k$, as the proof for the remaining indices $j \in \{1, \dots, k-1\}$ is similar. Write

$$\begin{aligned}
&\int_{y_1 \in W_n} \cdots \int_{y_k \in W_n} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_k \\
&\quad = \int_{y_1 \in W_n} \int_{w_2 \in W_n - y_1} \cdots \\
&\qquad \times \int_{w_k \in W_n - y_1} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(\mathbf{0}, w_2, \dots, w_k)}{k^2}\right) \mathrm{d}y_1 \, \mathrm{d}w_2 \cdots \mathrm{d}w_k.
\end{aligned}$$

Now $D_k(\mathbf{0}, w_2, \dots, w_k) \geq \sum_{i=2}^k |w_i|$. Letting $e_k := \tilde{c}_k (k-1)/k^2$ gives

$$\begin{aligned}
&\int_{y_1 \in W_n} \cdots \int_{y_k \in W_n} \tilde{\phi}\left(\frac{\tilde{c}_k D_k(y)}{k^2}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_k \\
&\quad \leq n \int_{w_2 \in \mathbb{R}^d} \cdots \int_{w_k \in \mathbb{R}^d} \tilde{\phi}\left(\frac{e_k}{k-1} \sum_{i=2}^k |w_i|\right) \mathrm{d}w_2 \cdots \mathrm{d}w_k \\
&\quad \leq n \int_{w_2 \in \mathbb{R}^d} \cdots \int_{w_k \in \mathbb{R}^d} \tilde{\phi}\left(\prod_{i=2}^k |w_i|^{1/(k-1)}\right) \mathrm{d}w_2 \cdots \mathrm{d}w_k = O(n),
\end{aligned}$$

where the first inequality follows from the decreasing behavior of $\tilde{\phi}$, the second inequality follows from the arithmetic geometric mean inequality and the last equality follows since $\tilde{\phi}$ is decreasing faster than any polynomial. We similarly bound the other summands for $j \in \{1, \dots, k-1\}$, completing the proof of Lemma 4.3. $\square$

4.3.4. *Proof of Theorem* 1.13. By the bound (4.20) and Lemma 4.3, we obtain (4.15). Letting C_n^k be the kth cumulant for $\langle f, \mu_n^\xi \rangle / \sqrt{\mathrm{Var}\langle f, \mu_n^\xi \rangle}$, we obtain $C_n^1 = 0$, $C_n^2 = 1$, and for all $k = 3, 4, \dots$.

$$C_n^k = O\big(n\big(\mathrm{Var}\langle f, \mu_n^\xi \rangle\big)^{-k/2}\big).$$

Since $\mathrm{Var}\langle f, \mu_n^\xi \rangle = \Omega(n^\nu)$ by assumption, it follows that if $k \in (2/\nu, \infty)$, then the kth cumulant tends C_n^k to zero as $n \to \infty$. By a classical result of Marcinkiewicz (see, e.g., [53], Lemma 3), we get that all cumulants $C_n^k, k \geq 3$, converge to zero as $n \to \infty$. This gives (4.14) as desired and completes the proof of Theorem 1.13.

4.4. *Second proof of the central limit theorem.* We now give a second proof of the central limit theorem which we believe is of independent interest. Even though this proof is also based on the cumulant method used in Section 4.3.1, we shall bound the cumulants using a different approach, using Ursell functions of the ξ-weighted measure and establishing a property equivalent to Brillinger mixing; see Remarks at the end of Section 4.4.2. Though much of this proof can be read independently of the proof in Section 4.3, we repeatedly use the definition of moments and cumulants from Section 4.3.2.

Our approach. We shall adapt the approach in [37], Section 4, replacing $\mathcal{P}_{\mathrm{GEF}}$ by our ξ-weighted measures, which are purely atomic measures. As noted in Section 1.3, the correlation functions of the ξ-weighted measure are generalizations of the correlations functions of the simple point process, but the extension of the approach used in [37], Section 4, requires some care regarding the repeated arguments captured by general exponents k_i in (1.6).

4.4.1. *Ursell functions of the ξ-weighted measures.* Recall the definition of the correlation functions (1.6) of the ξ-weighted measures

$$m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n) := \mathbb{E}_{x_1,\dots,x_p}\big(\big(\xi(x_1,\mathcal{P}_n)\big)^{k_1}\cdots\big(\xi(x_p,\mathcal{P}_n)\big)^{k_p}\big)\rho^{(p)}(x_1,\dots,x_p).$$

We will drop dependence on n, that is, $m^{(k_1,\dots,k_p)}(x_1,\dots,x_p;n) = m^{(k_1,\dots,k_p)}(x_1,\dots,x_p)$ unless asymptotics in n is considered.

Inspired by the approach in [6], Section 2, we now introduce *Ursell functions* $m_\top^{(k_1,\dots,k_p)}$ (sometimes called *truncated correlation function*) of the ξ-weighted measures. Define $m_\top^{(k_1,\dots,k_p)}$ by taking $m_\top^{(k)}(x) := m^{(k)}(x)$ for all $k \in \mathbb{N}$ and inductively

$$m_\top^{(k_1,\dots,k_p)}(x_1,\dots,x_p) := m^{(k_1,\dots,k_p)}(x_1,\dots,x_p) - \sum_{\substack{\gamma\in\Pi[p]\\|\gamma|>1}}\prod_{i=1}^{|\gamma|} m_\top^{(k_j:j\in\gamma(i))}\big(x_j : j\in\gamma(i)\big) \tag{4.25}$$

for distinct $x_1,\dots,x_p \in W_n$ and all integers $k_1,\dots,k_p$, $p \ge 1$, and (implicitly) $n \le \infty$. It is straightforward to prove that these functions satisfy the following relations. They extend the known relations for point processes, where $m^{(k_1,\dots,k_p)}(x_1,\dots,x_p) = \rho^{(p)}(x_1,\dots,x_p)$ depend only on p, but we were unable to find them in the literature for (signed) purely atomic random measures, as our ξ-weighted measures. Assuming $1 \in \gamma(1)$ in (4.25) and summing over partitions of $\{1,\dots,p\}\setminus\gamma(1)$, we get the following relation:

$$m^{(k_1,\dots,k_p)}(x_1,\dots,x_p) \tag{4.26}$$

$$= m_\top^{(k_1,\dots,k_p)}(x_1,\dots,x_p) + \sum_{\substack{I\subsetneq\{1,\dots,p\}\\1\in I}} m_\top^{(k_j:j\in I)}(x_j : j\in I)\, m^{(k_j:j\in I^c)}\big(x_j : j\in I^c\big), \tag{4.27}$$

where $I^c := \{1, \dots, p\} \setminus I$. Using (4.27), by induction with respect to p, one obtains the direct relation to the correlation functions

$$
\begin{aligned}
&m_{\top}^{(k_1,\dots,k_p)}(x_1, \dots, x_p) \\
&\quad = \sum_{\gamma \in \Pi[p]} (-1)^{|\gamma|-1}\bigl(|\gamma|-1\bigr)! \prod_{i=1}^{|\gamma|} m^{(k_j : j \in \gamma(i))}\bigl(x_j : j \in \gamma(i)\bigr).
\end{aligned} \tag{4.28}
$$

This extends the relation [37], (27), valid for point processes. We say that a partition $\gamma = \{\gamma(1), \dots, \gamma(l)\} \in \Pi(p)$ *refines* partition $\sigma = \{\sigma(1), \dots, \sigma(l_1)\} \in \Pi(p)$ if for all $i \in \{1, \dots, l\}$, $\gamma(i) \subset \sigma(j)$ for some $j \in \{1, \dots, l_1\}$. Otherwise, the partition γ is said to *mix* partition σ. Now using (4.25) we get for any $I \subsetneqq \{1, \dots, p\}$

$$
\begin{aligned}
&m^{(k_j : j \in I)}(x_j : j \in I) m^{(k_j : j \in I^c)}\bigl(x_j : j \in I^c\bigr) \\
&\quad = \sum_{\substack{\gamma \in \Pi[p] \\ \gamma \text{refines}\{I, I^c\}}} \prod_{i=1}^{|\gamma|} m_{\top}^{(k_j : j \in \gamma(i))}\bigl(x_j : j \in \gamma(i)\bigr)
\end{aligned} \tag{4.29}
$$

and, therefore, again in view of (4.25),

$$
\begin{aligned}
&m_{\top}^{(k_1,\dots,k_p)}(x_1, \dots, x_p) \\
&\quad = \sum_{\substack{\gamma \in \Pi[p], |\gamma|>1 \\ \gamma \text{ mixes } \{I, I^c\}}} \prod_{i=1}^{|\gamma|} m_{\top}^{(k_j : j \in \gamma(i))}\bigl(x_j : j \in \gamma(i)\bigr) \\
&\qquad + m^{(k_1,\dots,k_p)}(x_1, \dots, x_p) \\
&\qquad - m^{(k_j : j \in I)}(x_j : j \in I) m^{(k_j : j \in I^c)}\bigl(x_j : j \in I^c\bigr).
\end{aligned} \tag{4.30}
$$

This extends the relation [37], last displayed formula in the proof of Claim 4.1, valid for point processes.

4.4.2. *Fast decay of correlations and bounds for Ursell functions.* We show now that fast decay of correlations (1.21) of the ξ-weighted measures implies some bounds on the Ursell functions of these measures. Since $m^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n)$ is invariant with respect to any joint permutation of its arguments $(k_1, \dots, k_p)$ and $(x_1, \dots, x_p)$, fast decay of correlations (1.21) of the ξ-weighted measures may be rephrased as follows: There exists a fast-decreasing function $\tilde{\phi}$ and constants $\tilde{C}_k$, $\tilde{c}_k$, such that for any collection of positive integers $k_1, \dots, k_p$, $p \geq 2$, satisfying $k_1 + \cdots + k_p = k$, for any nonempty, proper subset $I \subsetneqq \{1, \dots, p\}$, for all $n \leq \infty$ and all configurations $x_1, \dots, x_p \in W_n$ of distinct points we have

$$
\begin{aligned}
&\bigl|m^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n) \\
&\quad - m^{(k_j : j \in I)}(x_j : j \in I; n) m^{(k_j : j \in I^c)}\bigl(x_j : j \in I^c; n\bigr)\bigr| \\
&\quad \leq \tilde{C}_k \tilde{\phi}(\tilde{c}_k s),
\end{aligned} \tag{4.31}
$$

where $s := d(\{x_j : j \in I\}, \{x_j : j \in I^c\})$.

Now we consider the bounds of Ursell functions of the ξ-weighted measures. Following the idea of [37], Claim 4.1, one proves that fast decay of correlations (1.21) of the ξ-weighted measures and the p-moment condition (1.19) imply that there exists a fast-decreasing function $\tilde{\phi}_\top$ and constants $\tilde{C}_k^\top$, $\tilde{c}_k^\top$, such that for any collection of positive integers $k_1, \dots, k_p$, $p \geq 2$, satisfying $k_1 + \cdots + k_p = k$, for all $n \leq \infty$ and all configurations $x_1, \dots, x_p \in W_n$ of distinct points we have

$$|m_\top^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n)| \leq \tilde{C}_k^\top \tilde{\phi}_\top(c_k^\top \operatorname{diam}(x_1, \dots, x_p)), \tag{4.32}$$

where $\operatorname{diam}(x_1, \dots, x_p) := \max_{i,j=1\dots p}(|x_i - x_j|)$. The proof uses the representation (4.30), fast decay of correlations (1.21) of the ξ-weighted measures, together with the fact that there exist constants $c_p^\top$ (depending on the dimension d) such that for each configuration $x_1, \dots, x_p \in W_n$, there exists a partition $\{I, I^c\}$ of $\{1, \dots, p\}$ such that $d(\{x_j : j \in I\}, \{x_j : j \in I^c\}) \geq \tilde{c}_p^\top \operatorname{diam}(x_1, \dots, x_p)$.

Next, inequality (4.32) allows one to bound integrals

$$\sup_{n \leq \infty} \sup_{x_1 \in W_n} \sup_{\substack{k_1+\cdots+k_p=k \\ k_i>0}} \int_{(W_n)^{p-1}} |m_\top^{(k_1,\dots,k_p)}(x_1, \dots, x_p; n)| \, \mathrm{d}x_2 \cdots \mathrm{d}x_p < \infty. \tag{4.33}$$

Indeed, for a fixed point $x_1 \in W_n$, we split $(W_n)^{p-1}$ into disjoint sets:

$$G_0 := \{(x_2, \dots, x_p) \in (W_n)^{p-1} : \operatorname{diam}(x_1, \dots, x_p) \leq 1\},$$
$$G_l := \{(x_2, \dots, x_p) \in (W_n)^{p-1} : 2^{l-1} < \operatorname{diam}(x_1, \dots, x_p) \leq 2^l\}, \qquad l \geq 1$$

and use estimate (4.32) to bound the integral on the left-hand side of (4.33) by

$$\tilde{C}_k^\top + \tilde{C}_k^\top \sum_{l=1}^{\infty} 2^{dl(k-1)} \tilde{\phi}_\top(\tilde{c}_k^\top 2^{l-1}) < \infty$$

since $\tilde{\phi}_\top$ is fast-decreasing; cf. [37], Claim 4.2.

Remarks (i) A careful inspection of the relation (4.30) shows that in fact the fast decay of correlations (1.21) of the ξ-weighted measures *is equivalent* to the bound (4.32) on Ursell functions of these measures.

(ii) Condition (4.33), implied by (4.32), can be interpreted as the Brillinger mixing condition of the ξ-weighted measures. In fact, it is slightly stronger in the sense that the bound on the Ursell functions integrated over $\mathrm{d}x_2 \cdots \mathrm{d}x_p$ in the entire space (corresponding to the total reduced cumulant measures) is uniform for the whole family of the ξ-weighted measuress considered on W_n, parametrized by $n \leq \infty$ and, for $n < \infty$ the bound is also uniform over $x_1 \in W_n$ (which is immediate for reduced cumulant measures in the stationary case $n = \infty$).

4.4.3. *Proof of Theorem* 1.13. The cumulant of order one is equal to the expectation and hence disappears for the considered (centered) random variable $\overline{\mu}_n^{\xi}(f)$. The cumulant of order 2 is equal to the variance and hence equal to 1 in our case. For $k \geq 2$, note the following relation between the normalized and the unnormalized cumulants:

$$S_k\big(\big(\operatorname{Var}\overline{\mu}_n^{\xi}(f)\big)^{-1/2}\mu_n^{\xi}(f)\big) = \big(\operatorname{Var}\mu_n^{\xi}(f)\big)^{-k/2} \times S_k\big(\mu_n^{\xi}(f)\big). \tag{4.34}$$

We establish the vanishing of (4.34) for k large by showing that the kth order cumulant $S_k(\mu_n^{\xi}(f)$ is of order $O(n)$, $k \geq 2$, and then use assumption (1.26), that is, $\operatorname{Var}\langle f, \mu_n^{\xi}\rangle = \Omega(n^{\nu})$. We have

$$M_n^k := \mathbb{E}(\langle f, \mu_n^{\xi}\rangle)^k = \mathbb{E}\bigg(\sum_{x_i \in \mathcal{P}_n} f_n(x_i)\xi(x_i, \mathcal{P}_n)\bigg)^k,$$

where $f_n(\cdot) = f(\cdot/n^{1/d})$. Considering appropriately the repetitions of points x_i in the kth product of the sum and using the Campbell theorem at (1.9), one obtains

$$M_n^k = \sum_{\sigma \in \Pi[k]} \bigg\langle \bigotimes_{i=1}^{|\sigma|} f_n^{|\sigma(i)|} m^{(\sigma)}, \lambda_n^{|\sigma|}\bigg\rangle, \tag{4.35}$$

where λ_n^l denotes the Lebesgue measure on $(W_n)^l$ and $\bigotimes$ denotes the tensor product of functions

$$\bigg(\bigotimes_{i=1}^{p} f_n^{k_j}\bigg)(x_1, \dots, x_p) = \prod_{i=1}^{p} (f_n)^{k_j}(x_j), m^{(\sigma)}(x_1, \dots, x_{|\sigma|}; n)$$
$$:= m^{(|\sigma(1)|, \dots, |\sigma(|\sigma|)|)}(x_1, \dots, x_{|\sigma|}; n).$$

Using the above representation and (4.17), the kth cumulant $S_k(\mu_n^{\xi}(f))$ can be expressed as follows:

$$\begin{aligned}
&S_k\big(\mu_n^{\xi}(f)\big) \\
&\quad = \sum_{\gamma \in \Pi[k]} (-1)^{|\gamma|-1}\big(|\gamma| - 1\big)! \\
&\qquad \times \sum_{\substack{\sigma \in \Pi[k] \\ \sigma \text{ refines } \gamma}} \prod_{i=1}^{|\gamma|} \bigg\langle \bigotimes_{j=1}^{|\gamma(i)/\sigma|} f_n^{(\gamma(i)/\sigma)(j)} m^{(\gamma(i)/\sigma)}, \lambda_n^{|\gamma(i)/\sigma|}\bigg\rangle \qquad (4.36) \\
&\quad = \sum_{\sigma \in \Pi[k]} \sum_{\substack{\gamma \in \Pi[k] \\ \sigma \text{ refines } \gamma}} (-1)^{|\gamma|-1}\big(|\gamma| - 1\big)! \\
&\qquad \times \prod_{i=1}^{|\gamma|} \bigg\langle \bigotimes_{j=1}^{|\gamma(i)/\sigma|} f_n^{(\gamma(i)/\sigma)(j)} m^{(\gamma(i)/\sigma)}, \lambda_n^{|\gamma(i)/\sigma|}\bigg\rangle,
\end{aligned}$$

where $\gamma(i)/\sigma$ is the partition of $\gamma(i)$ induced by σ. Note that for any partition $\sigma \in \Pi[k]$, with $|\sigma(j)| = k_j$, $j = 1, \dots, |\sigma| = p$, the inner sum in (4.36) can be rewritten as follows:

$$
\begin{aligned}
&\sum_{\gamma \in \Pi[p]} (-1)^{|\gamma|-1}\big(|\gamma|-1\big)! \prod_{i=1}^{|\gamma|} \Big\langle \bigotimes_{j\in\gamma(i)} f_n^{k_j} m^{(k_j : j \in \gamma(i))}, \lambda_n^{|\gamma(i)|} \Big\rangle \\
&\quad = \Big\langle \bigotimes_{j=1}^{p} f_n^{k_j} m_{\top}^{(k_1,\dots,k_p)}, \lambda_n^p \Big\rangle,
\end{aligned} \tag{4.37}
$$

where the equality is due to (4.28). Consequently,

$$
S_k\big(\mu_n^{\xi}(f)\big) = \sum_{\sigma\in\Pi[k]} \Big\langle \bigotimes_{j=1}^{|\sigma|} f_n^{|\sigma(j)|} m_{\top}^{(|\sigma(1)|,\dots,|\sigma(|\sigma|)|)}, \lambda_n^{|\sigma|} \Big\rangle, \tag{4.38}
$$

which extends the relation [37], Claim 4.3, valid for point processes. The formula (4.38), which expresses the kth cumulant in terms of the Ursell functions, is the counterpart to the standard formula (4.35) expressing kth moments in terms of correlation functions. Now, using (4.33) and denoting the supremum therein by $\hat{C}_k$, we have that

$$
\begin{aligned}
&\Big| \Big\langle \bigotimes_{j=1}^{p} f_n^{k_j} m_{\top}^{(k_1,\dots,k_p)}, \lambda_n^p \Big\rangle \Big| \\
&\quad \le \int_{W_n^p} \Big| \bigotimes_{j=1}^{p} f_n^{k_j} \Big| \big| m_{\top}^{(k_1,\dots,k_p)}(x_1,\dots,x_p) \big| \, \mathrm{d}x_1 \dots \mathrm{d}x_p \\
&\quad \le \|f\|_{\infty}^{k} \int_{W_n} \mathrm{d}x_1 \int_{W_n^{p-1}} \big| m_{\top}^{(k_1,\dots,k_p)}(x_1,\dots,x_p) \big| \, \mathrm{d}x_2 \dots \mathrm{d}x_p \\
&\quad \le \|f\|_{\infty}^{k} \hat{C}_k \operatorname{Vol}_d(W_n).
\end{aligned}
$$

So, the above bound along with (4.36) and (4.37) gives us that $S_k(\mu_n^{\xi}(f)) = O(n)$ for all $k \ge 2$. Thus, using the variance lower bound condition (1.26) and the relation (4.34), we get for large enough k, that $S_k((\operatorname{Var} \mu_n^{\xi}(f))^{-1/2}\mu_n^{\xi}(f)) \to 0$ as $n \to \infty$. Now, as discussed in (4.14), this suffices to guarantee normal convergence.

Acknowledgments. The work benefitted from DY's visits to Lehigh University and IMA, Minneapolis, supported in part by the respective institutions. Part of this work was done when DY was a post-doc at Technion, Israel. He is thankful to the institute for its support and to his host, Robert Adler, for many discussions. The authors thank Manjunath Krishnapur for numerous inputs, especially those related to determinantal point processes and Gaussian analytic functions. The authors thank Jesper Møller and Günter Last for useful comments on the first draft of this article and Christophe Biscio for discussions related to Brillinger mixing. Finally, this paper benefitted from a thorough reading by an anonymous referee, whose numerous comments improved the exposition.

SUPPLEMENTARY MATERIAL

Supplement to "Limit theory for geometric statistics of point processes having fast decay of correlations" (DOI: 10.1214/18-AOP1273SUPP; .pdf). This supplement contains various auxiliary facts needed in the proofs. These facts, some of which are of independent interest, may also be found in the arXiv version [10] of this paper.

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