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Brownian motion in attenuated or renormalized inverse-square Poisson potential

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Abstract. We consider the parabolic Anderson problem with random potentials having inverse-square singularities around the points of a standard Poisson point process in $\mathbb{R}^d$, $d \geq 3$. The potentials we consider are obtained via superposition of translations over the points of the Poisson point process of a kernel $\mathfrak{K}$ behaving as $\mathfrak{K}(x) \approx \theta|x|^{-2}$ near the origin, where $\theta \in (0, (d-2)^2/16]$. In order to make sense of the corresponding path integrals, we require the potential to be either *attenuated* (meaning that $\mathfrak{K}$ is integrable at infinity) or, when $d = 3$, *renormalized*, as introduced by Chen and Kulik in (*Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 631–660). Our main results include existence and large-time asymptotics of non-negative solutions via Feynman–Kac representation. In particular, we settle for the renormalized potential in $d = 3$ the existence problem with critical parameter $\theta = 1/16$, left open by Chen and Rosinski in (Chen and Rosinski (2011)).

Résumé. Nous considérons le problème parabolique d'Anderson avec potentiels aléatoires ayant des singularités en carré inverse autour des points d'un processus de Poisson standard dans $\mathbb{R}^d$, $d \geq 3$. Les potentiels sont obtenus par superposition de translations par les points du processus de Poisson d'un noyau $\mathfrak{K}$ satisfaisant $\mathfrak{K}(x) \approx \theta|x|^{-2}$ près de l'origine, où $\theta \in (0, (d-2)^2/16]$. Afin de pouvoir définir les intégrales de chemin correspondantes, nous demandons que le noyau soit ou bien *atténué* (intégrable à l'infini), ou, en $d = 3$, *renormalisé* au sens de Chen et Kulik (*Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 631–660). Nous montrons l'existence et le comportement en temps long des solutions positives par représentation de Feynman–Kac, en particulier dans le cas critique $\theta = 1/16$ laissé ouvert par Chen et Rosinski (Chen et Rosinski (2011)).

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1. Introduction and main results

Fix $d \in \mathbb{N}$ and let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$. We denote by $\mathbb{P}_x$ its law when started at x, and by $\mathbb{E}_x$ the corresponding expectation. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a random potential function, which we take independent of W. The integral $\int_0^t V(W_s)\,\mathrm{d}s$ represents the total potential energy along the Brownian path up to time t, and is used to define the *quenched Gibbs measure*

$$Q_{t,x}(\cdot) := \frac{1}{Z_{t,x}} \mathbb{E}_x\left[\exp\left\{\int_0^t V(W_s)\,\mathrm{d}s\right\} \mathbb{1}\{W \in \cdot\}\right], \quad \text{where } Z_{t,x} := \mathbb{E}_x\left[\exp \int_0^t V(W_s)\,\mathrm{d}s\right], \tag{1.1}$$

describing the behaviour of W under the influence of the random potential.

A main feature in the study of Brownian motion in random potential is the connection to the (continuous) *parabolic Anderson model*, i.e., the initial value problem

$$\begin{aligned} \partial_t u(t,x) &= \frac{1}{2}\Delta u(t,x) + V(x)u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.2}$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial^2 x_i}$ denotes the (weak) Laplacian in $L^2(\mathbb{R}^d)$, and $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is some initial data. When V is e.g. in the Kato class (cf. [24, page 8, equation (2.4)]), the unique mild solution to (1.2) (in the sense of [21, Definition 6.1.1]) is given by the classical *Feynman–Kac formula*

$$u(t,x) = \mathbb{E}_x\left[u_0(W_t)\exp\left\{\int_0^t V(W_s)\,\mathrm{d}s\right\}\right]. \tag{1.3}$$

In particular, $u(t,x) = Z_{t,x}$ in (1.1) solves (1.2) with $u_0 \equiv 1$.

In this paper, we are interested in *Poisson (or shot noise) potentials*, obtained by superposing translations of a fixed function over the points of a Poisson cloud. To describe them, let ω be a standard Poisson point process in $\mathbb{R}^d$, i.e., having the Lebesgue measure as its intensity measure. Denote by $\mathbf{P}$ the law of ω, and by $\mathcal{P} = \{y \in \mathbb{R}^d : \omega(\{y\}) > 0\}$ its support, which is almost surely discrete. For a Borel-measurable *shape function* (or *kernel*) $\mathfrak{K} : \mathbb{R}^d \to \mathbb{R}$, we define the Poisson potential

$$V(x) = V(x,\omega) = \sum_{y\in\mathcal{P}} \mathfrak{K}(x-y) = \int_{\mathbb{R}^d} \mathfrak{K}(x-y)\omega(\mathrm{d}y), \quad x \in \mathbb{R}^d.$$

Since the Gibbs measure in (1.1) favours paths with larger energy functional $\int_0^t V(W_s)\,\mathrm{d}s$, the points of ω will under it either attract the Brownian particle if $\mathfrak{K}$ is positive, or repel it if $\mathfrak{K}$ is negative.

The study of the model of Brownian motion in a random Poisson potential is motivated by various applications from physics and other fields. Think, e.g., of an electron moving in a crystal with impurities, cf. [5,17,19]. For an overview on the mathematical treatment of the subject and further references, we refer the reader to the monographs [18,24]. In [24], essentially two types of potentials are considered: the *soft* obstacle potential, where $\mathfrak{K}$ is assumed to be negative, bounded and compactly supported, and the *hard* obstacle potential, where formally $\mathfrak{K} = -\infty\mathbb{1}_C$ for some compact, nonpolar set $C \subset \mathbb{R}^d$, i.e., the Brownian particle is immediately killed when entering the C-neighbourhood of the Poisson cloud and moves freely up to the entrance time. The case of $\mathfrak{K}$ positive, bounded and continuous (and satisfying a decay property) has been considered in [6,15]. The works mentioned identify almost-sure large-time asymptotics for $Z_{t,x}$ in (1.1).

It is of natural concern to study shape functions that are neither bounded nor have compact support. Kernels of the form $\mathfrak{K}(x) = |x|^{-p}$ are physically motivated, e.g. $p = d-2$ corresponds to Newton's law of gravitation. The inverse-square case $p = 2$ is of special interest both in mathematics and physics (cf. e.g. [1,2,11,14,16,23]), and is related to the inverse-cube central force; in this case, $\mathfrak{K}$ is *not* in the Kato class (cf. [24, Example 2.3, page 9]). It turns out however that, when $p \le d$, the corresponding Poisson potential almost surely explodes, i.e.,

$$\int_{\mathbb{R}^d} |x-y|^{-p}\omega(\mathrm{d}y) = \infty \qquad \mathbf{P}\text{-a.s. for each } x \in \mathbb{R}^d, \tag{1.4}$$

cf. [8, Proposition 2.1]. Indeed, when $p \le d$, the integrability in (1.4) is obstructed by the slow decay of the function $|x|^{-p}$ at infinity. To solve this problem, Chen and Kulik have constructed a *renormalized* version $\overline{V}$ of the Poisson potential V, formally written as

$$\overline{V}(x) = \int_{\mathbb{R}^d} |x-y|^{-p}\big[\omega(\mathrm{d}y) - \mathrm{d}y\big], \quad x \in \mathbb{R}^d \ \big(\text{where } |\cdot| \text{ is the Euclidean norm}\big). \tag{1.5}$$

The mathematical definition of $\overline{V}$ is as limit in probability of the same expression with integrable approximating kernels, for which both integrals against $\mathrm{d}y$ and $\omega(\mathrm{d}y)$ are well defined; for details, we refer the reader to [8, Section 2]. This procedure is natural since, at each step of the approximation, both V and $\overline{V}$ give rise to the same quenched Gibbs measure. In [8, Corollary 1.3], it is shown that (1.5) is well-defined whenever $d/2 < p < d$, in particular when $p = 2$, $d = 3$.

Even if (1.5) is well defined, the exponential moment $Z_{t,x}$ in (1.1) (with $\overline{V}$ in place of V) may still be infinite. Indeed, Theorem 1.5 in [8] states that, for $d/2 < p < d$ and any $\theta, t > 0$,

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\begin{cases} < \infty & \mathbf{P}\text{-a.s. if } p < 2, \\ = \infty & \mathbf{P}\text{-a.s. if } p > 2. \end{cases}$$

In the critical case $p = 2$ (and necessarily $d = 3$), the integrability depends on the value of the parameter θ: according to [9, Theorem 2.1], for any $t > 0$,

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\begin{cases} < \infty & \mathbf{P}\text{-a.s. if } \theta < \frac{1}{16}, \\ = \infty & \mathbf{P}\text{-a.s. if } \theta > \frac{1}{16}. \end{cases} \tag{1.6}$$

The boundary case $\theta = \frac{1}{16}$ is not considered in [9], and is included in our Theorem 1.7 below. The fact that $\theta = \frac{1}{16}$ is critical is related to the celebrated Hardy inequality (in $d = 3$)

$$\frac{(d-2)^2}{8} \int_{\mathbb{R}^d} \frac{g^2(x)}{|x|^2}\,\mathrm{d}x \le \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)}, \quad g \in H^1\big(\mathbb{R}^d\big), \tag{1.7}$$

where $H^1(\mathbb{R}^d)$ is the Sobolev space of $L^2(\mathbb{R}^d)$ functions whose (weak) partial derivatives are also in $L^2(\mathbb{R}^d)$, and the constant $(d-2)^2/8$ is sharp.

Once finiteness of exponential moments is settled, our interest turns to large-time asymptotics. In the non-critical regime $d/2 < p < \min(2, d)$, $\theta > 0$, it is shown in [7, Theorem 2.2] that

$$\lim_{t\to\infty} \frac{1}{t}\left(\frac{\log\log t}{\log t}\right)^{\frac{2}{2-p}} \log \mathbb{E}_0\big[\mathrm{e}^{\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s}\big] = c(d, p, \theta) \quad \mathbf{P}\text{-a.s.}, \tag{1.8}$$

where $c(d, p, \theta)$ is an explicit deterministic constant depending only on d, p, θ. The case $p = 2$, $d = 3$, already considered in [9], turns out to be rather different: after suitable rescaling, the log of the exponential moment does *not* converge to a constant, but fluctuates randomly, cf. Theorem 1.10 below. Here we again extend the investigation to the boundary case $\theta = 1/16$.

Finally, we do not restrict our analysis to the renormalized potential $\overline{V}$, but also consider integrable versions of the inverse-square kernel. For this class of *attenuated potentials*, cf. Definition 1.1 below, we show similar results as outlined above in all dimensions $d \ge 3$; in fact, our asymptotic results for $\overline{V}$ in $d = 3$ are obtained via comparison to attenuated potentials, cf. Theorem 1.9 below.

1.1. *Main results*

Let $d \ge 3$. We define next the class $\mathcal{K}$ of potential kernels we are after, whose elements have an inverse-square singularity at the origin and are integrable at infinity.

Definition 1.1. We say that a measurable $\mathfrak{K} : \mathbb{R}^d \to [-\infty, \infty]$ belongs to the class $\mathcal{K}$ if and only if

$$y \mapsto \sup_{|x|\le 1} \big|\mathfrak{K}(x-y)\big| \wedge 1 \quad \text{belongs to } L^1\big(\mathbb{R}^d\big) \tag{1.9}$$

and

$$\limsup_{a\downarrow 0} \max\left\{a^2 \sup_{|x|>a} \big|\mathfrak{K}(x)\big|, \sup_{|x|\le a}\left|\mathfrak{K}(x) - \frac{1}{|x|^2}\right|\right\} < \infty. \tag{1.10}$$

We call $\mathcal{K}$ the class of *attenuated inverse-square potential kernels*.

Given $\mathfrak{K} \in \mathcal{K}$, we denote the Poisson potential with kernel $\mathfrak{K}$ by

$$V^{(\mathfrak{K})}(x) := \int_{\mathbb{R}^d} \mathfrak{K}(x-y)\omega(\mathrm{d}y), \quad x \in \mathbb{R}^d \setminus \mathcal{P}. \tag{1.11}$$

By [8, Proposition 2.1], $V^{(\mathfrak{K})}$ is a.s. well-defined and finite. Important examples are the truncated kernels $\mathfrak{K}_a(x) := |x|^{-2}\mathbb{1}_{\{|x|\le a\}}$, $a > 0$, in which case we abbreviate $V^{(a)} := V^{(\mathfrak{K}_a)}$.

To state our results for $V^{(\mathfrak{K})}$, denote by

$$h_d := \frac{(d-2)^2}{8} \tag{1.12}$$

as in the form (1.7) of Hardy's inequality, and set, for $\theta \in (0, h_d/2]$,

$$k_\theta := \left\lfloor \frac{h_d}{\theta} \right\rfloor \ge 2. \tag{1.13}$$

Our first two results show existence of solutions to (1.2) via Feynman–Kac representation.

Theorem 1.2. *For all $d \geq 3$, $\mathfrak{K} \in \mathcal{K}$ and $\theta \in (0, \frac{h_d}{2}]$, it holds* **P**-*almost surely that*

$$v_\theta^{(\mathfrak{K})}(t,x) := \mathbb{E}_x\left[\exp\left(\theta \int_0^t \left|V^{(\mathfrak{K})}\right|(W_s)\,\mathrm{d}s\right)\right] < \infty \quad \forall x \in \mathbb{R}^d \setminus \mathcal{P}, t \geq 0. \tag{1.14}$$

Moreover, $v_\theta^{(\mathfrak{K})}(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for all $t \geq 0$. As a consequence, for all $u_0 \in L^\infty(\mathbb{R}^d)$ the function

$$u_\theta^{(\mathfrak{K},u_0)}(t,x) := \mathbb{E}_x\left[u_0(W_t)\exp\left(\theta \int_0^t V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\right] \quad \textit{is well-defined for all } x \in \mathbb{R}^d \setminus \mathcal{P}, t \geq 0, \tag{1.15}$$

and $u_\theta^{(\mathfrak{K},u_0)}(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for all $t \geq 0$. The same is true with $|V^{(\mathfrak{K})}|$ in place of $V^{(\mathfrak{K})}$.

Theorem 1.3. *The function $u_\theta^{(\mathfrak{K},u_0)}$ defined in (1.15) is a mild solution to (1.2) with $V = \theta V^{(\mathfrak{K})}$. The analogous holds for $|V^{(\mathfrak{K})}|$ in place of $V^{(\mathfrak{K})}$.*

When $\theta > h_d/2$, (1.14) is infinite even with $V^{(\mathfrak{K})}$ in place of $|V^{(\mathfrak{K})}|$; a proof can be obtained as in [9, Theorem 2.1]. In the following, when $u_0 \equiv 1$ we write $u_\theta^{(\mathfrak{K})}$ instead of $u_\theta^{(\mathfrak{K},u_0)}$.

Our next three results concern large time asymptotics of $u_\theta^{(\mathfrak{K})}(t,0)$, starting with tightness.

Theorem 1.4. *Let $d \geq 3$, $\mathfrak{K} \in \mathcal{K}$ and $\theta \in (0, \frac{h_d}{2}]$. For any $t \mapsto g(t) > 0$ with $g(t) \xrightarrow{t\to\infty} \infty$,*

$$g(t)t^{-\frac{k_\theta+1}{k_\theta-1}} \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\right] \xrightarrow{t\to\infty} \infty \quad \textit{in probability} \tag{1.16}$$

and

$$g(t)^{-1}t^{-\frac{k_\theta+1}{k_\theta-1}} \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\right] \xrightarrow{t\to\infty} 0 \quad \textit{in probability}. \tag{1.17}$$

In other words, the process $t^{-\frac{k_\theta+1}{k_\theta-1}} \log u_\theta^{(\mathfrak{K})}(t,0)$, $t > 0$ is tight on the open interval $(0,\infty)$.

The following two theorems provide almost-sure lim sup and lim inf asymptotics.

Theorem 1.5. *Let $d \geq 3$, $\mathfrak{K} \in \mathcal{K}$ and $\theta \in (0, \frac{h_d}{2}]$. For any slowly varying $\ell\colon (0,\infty) \to (1,\infty)$,*

$$\limsup_{t\to\infty} t^{-\frac{k_\theta+1}{k_\theta-1}} \ell(t)^{-\frac{2}{d(k_\theta-1)}} \log u_\theta^{(\mathfrak{K})}(t,0) = \begin{cases} 0 & \mathbf{P}\text{-a.s. if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} < \infty, \\ \infty & \mathbf{P}\text{-a.s. if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} = \infty. \end{cases} \tag{1.18}$$

Theorem 1.6. *For any $d \geq 3$ and $\theta \in (0, \frac{h_d}{2}]$, there exist $0 < C_{\inf} < C^{\inf} < \infty$ such that, for all $\mathfrak{K} \in \mathcal{K}$,*

$$\liminf_{t\to\infty} t^{-\frac{k_\theta+1}{k_\theta-1}} (\log\log t)^{\frac{2}{d(k_\theta-1)}} \log u_\theta^{(\mathfrak{K})}(t,0) \in \left[C_{\inf}, C^{\inf}\right] \quad \mathbf{P}\text{-a.s.} \tag{1.19}$$

Corresponding results also hold for the renormalized potential $\overline{V}$ when $d = 3$. We start with the analogues of Theorems 1.2–1.3.

Theorem 1.7. *Let $d = 3$. For each $\theta \in (0, 1/16]$, it holds* **P**-*almost surely that*

$$\bar{v}_\theta(t,x) := \mathbb{E}_x\left[\exp\left(\theta \int_0^t |\overline{V}|(W_s)\,\mathrm{d}s\right)\right] < \infty \qquad \forall x \in \mathbb{R}^3 \setminus \mathcal{P}, t \geq 0. \tag{1.20}$$

Moreover, $\bar{v}_\theta(t,\cdot)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ for all $t \geq 0$. As a consequence, for all $u_0 \in L^\infty(\mathbb{R}^3)$ the function

$$\bar{u}_\theta^{(u_0)}(t,x) := \mathbb{E}_x\left[u_0(W_t)\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \quad \textit{is well-defined for all } x \in \mathbb{R}^3 \setminus \mathcal{P}, t \geq 0, \tag{1.21}$$

and $\bar{u}_\theta^{(u_0)}(t,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ for all $t \geq 0$. The analogous holds for $|\overline{V}|$ in place of $\overline{V}$.

Theorem 1.8. *The function $\bar{u}_\theta^{(u_0)}$ defined in (1.21) is a mild solution to (1.2) with $d = 3$, and $V = \theta\overline{V}$. The analogous holds for $|\overline{V}|$ in place of $\overline{V}$.*

When $u_0 \equiv 1$, we will write $\bar{u}_\theta$ instead of $\bar{u}_\theta^{(u_0)}$. Our next theorem provides a convenient comparison between potential kernels, allowing us to concentrate on the truncated case $\mathfrak{K}_a(x) = |x|^{-2}\mathbb{1}_{\{|x|\le a\}}$.

Theorem 1.9. *For any $\theta \in (0, h_d/2]$, any $a \in (0,\infty)$ and any $\mathfrak{K} \in \mathcal{K}$,*

$$\lim_{t\to\infty} \frac{\log u_\theta^{(\mathfrak{K})}(t,0)}{\log u_\theta^{(\mathfrak{K}_a)}(t,0)} = 1 \quad \mathbf{P}\text{-almost surely.} \tag{1.22}$$

The same is true with $v_\theta^{(\mathfrak{K})}$ in place of $u_\theta^{(\mathfrak{K})}$. When $d = 3$, (1.22) also holds with either $\bar{v}_\theta$ or $\bar{u}_\theta$ in place of $u_\theta^{(\mathfrak{K})}$.

Finally, using Theorem 1.9, we can transfer our results for $V^{(\mathfrak{K})}$ to $|V^{(\mathfrak{K})}|$, $\overline{V}$ and $|\overline{V}|$:

Theorem 1.10. *The statements of Theorems 1.4, 1.5 and 1.6 are true for $v_\theta^{(\mathfrak{K})}$ in place of $u_\theta^{(\mathfrak{K})}$. When $d = 3$, these statements also hold with either $\bar{v}_\theta$ or $\bar{u}_\theta$ in place of $u_\theta^{(\mathfrak{K})}$.*

We discuss next our results and provide some heuristics for the scale $t^{(k_\theta+1)/(k_\theta-1)}$.

1.2. *Discussion and heuristics*

(1) Theorems 1.4–1.6 imply that there is no rescaling under which $\log u_\theta^{(\mathfrak{K})}(t,0)$ converges almost surely as $t\to\infty$ to a non-trivial deterministic constant (and analogously for $\log \bar{u}_\theta(t,0)$ by Theorem 1.10). We conjecture that, after rescaling by $t^{(k_\theta+1)/(k_\theta-1)}$, it converges in distribution to a non-degenerate random variable. We also conjecture that the lim inf in Theorem 1.6 is deterministic.

(2) As already mentioned, our main contribution in Theorems 1.7, 1.8 and 1.10 is the boundary case $\theta = 1/16$, left open in [9]. The proof given in [9] that $\bar{u}_\theta$ is well-defined for $0 < \theta < \frac{1}{16}$ cannot be extended to the case $\theta = \frac{1}{16}$, as it is based on the following strategy. Decompose the Brownian path according to which of the cubes $Q_n = (-R_n, R_n)^3$ has been exited until time t, where $(R_n)_{n\in\mathbb{N}}$ is some properly chosen increasing sequence; i.e., setting $\tau_0 = 0$, $\tau_n = \inf\{s \ge 0\colon W_s \notin Q_n\}$, write

$$\begin{aligned}\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] &= \sum_{n=1}^\infty \mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{n-1}\le t<\tau_n\}}\right]\\ &\le \sum_{n=1}^\infty \mathbf{P}[\tau_{n-1}\le t]^{1/p}\,\mathbb{E}_0\left[\exp\left(q\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_n\}}\right]^{1/q}\end{aligned}$$

by Hölder's inequality, where $p, q > 0$, $p^{-1} + q^{-1} = 1$. The last expectation cannot be controlled if $q\theta > 1/16$, and thus $\theta < 1/16$ is required to use this argument. In order to overcome this, we develop for our proof a more careful decomposition of Brownian paths according to their excursions to and from certain *islands* whose principal eigenvalues are large, cf. Section 3 below.

(3) Even though we only prove Theorems 1.2–1.3 and 1.7–1.8 for initial data $u_0 \in L^\infty(\mathbb{R}^d)$, a close inspection of our proofs will show that they may be generalized to $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying e.g.

$$\limsup_{R\to\infty} \frac{\log\log \|u_0\|_{L^\infty(B_R)}}{\log R} < \frac{1}{k}.$$

The key step is to adapt the statement of Lemma 5.7 taking the above condition into account.

(4) Let $\mathbb{E}^t_{x,y}$ denote expectation under the law of a Brownian bridge between $x, y \in \mathbb{R}^d$ in time t, and write $p_t(x-y) = (2\pi t)^{-d/2}\mathrm{e}^{-|x-y|^2/(2t)}$ for the Brownian transition kernel. Since the law of the Brownian bridge is a regular conditional probability for the law of Brownian motion given $W_t = y$ (cf. e.g. [24, Appendix to Part I]), Theorem 1.2 implies that, $\mathbf{P}$-a.s., for all $R > 0$,

$$\int_{B_R\times\mathbb{R}^d} p_t(x-y)\mathbb{E}^t_{x,y}\left[\exp\left(\theta\int_0^t \left|V^{(\mathfrak{K})}\right|(W_s)\,\mathrm{d}s\right)\right]\mathrm{d}x\,\mathrm{d}y = \int_{B_R} v_\theta^{(\mathfrak{K})}(t,x)\,\mathrm{d}x < \infty$$

and thus, for all $t \geq 0$ and almost every $x, y \in \mathbb{R}^d$, the function

$$(t, x) \mapsto p_t(x-y)\mathbb{E}^t_{x,y}\left[\exp\left(\theta \int_0^t V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\right] \tag{1.23}$$

is well-defined and, by Theorem 1.3, solves (1.2) with initial condition $u_0 = \delta_y$. In $d = 3$, the analogous is true for $\overline{V}$ by Theorems 1.7–1.8. Using the techniques of Section 3 and Section 2.2, we could extend the definition of (1.23) to all $x, y \in \mathbb{R}^d \setminus \mathcal{P}$; in the interest of brevity, we will not pursue this here.

(5) We provide next some heuristics for the scale $t^{(k_\theta+1)/(k_\theta-1)}$ appearing in Theorem 1.4. The main point is that the logarithmic order of $u^{(\mathfrak{K})}_\theta(t, 0)$ is the same when restricting the expectation to Brownian paths that reach by time $s < t$ a region $D \subset \mathbb{R}^d$ containing precisely $k_\theta + 1$ Poisson points, and afterwards stay there until time t. Spectral methods show that the reward for staying in D for time $t - s$ is approximately $\mathrm{e}^{(t-s)\lambda_{\max}}$, where $\lambda_{\max}$ is the principal Dirichlet eigenvalue of $\frac{1}{2}\Delta + V^{(\mathfrak{K})}$ in D. Asymptotics for this eigenvalue may be estimated with the help of *multipolar Hardy inequalities* as in [4] (see also Section 2.4 below), yielding that its order roughly equals $\mathrm{diam}(D)^{-2}$. Now, if R is the distance of D to the origin, Poisson statistics dictate that it may be chosen with $\mathrm{diam}(D) \approx R^{-1/k_\theta}$, but not much smaller. On the other hand, the probabilistic cost for Brownian motion to reach D by time s is roughly $\mathrm{e}^{-R^2/s}$. The total contribution is thus about $\exp\{(t-s)R^{2/k_\theta} - R^2/s\}$; optimizing the exponent over s and R, we obtain $R = t^{k_\theta/(k_\theta+1)}$ and $\log u^{(\mathfrak{K})}_\theta(t, 0) \approx t^{(k_\theta+1)/(k_\theta-1)}$.

1.3. *Outline and notation*

The rest of the paper is organized as follows. After introducing some notation, we develop in Section 2 upper and lower spectral bounds on the Feynman–Kac functional (1.3) in the setting of deterministic point clouds. The upper bounds are extended in Section 3 using a path decomposition technique. Section 4 presents some elementary geometric properties of the standard Poisson point process. The proofs of the main theorems are completed in Section 5.

Notation and terminology. We write $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$ for the open ball with radius $r \in (0, \infty)$ around $x \in \mathbb{R}^d$ with respect to the Euclidean norm $|\cdot|$; when $x = 0$ we abbreviate $B_r := B_r(0)$. For $D \subset \mathbb{R}^d$, we write $B_r(D) = \{x \in \mathbb{R}^d : \exists y \in D, |x - y| < r\}$ for the r-neighbourhood of D. We denote by $|D|$ the volume of a Borel measurable subset $D \subset \mathbb{R}^d$, and by $\tau_D := \inf\{t \geq 0 : W_t \in D\}$ the entrance time of Brownian motion in D. A subset $D \subset \mathbb{R}^d$ is called a *domain* if it is open and connected. For a real-valued function f, a positive function g and $a \neq 0$, we write $f(x) \sim ag(x)$ as $x \to \infty$ to denote that $\lim_{x\to\infty} f(x)/g(x) = a$; when $a = 0$, we write $f = o(g)$ instead, or equivalently $|f| \ll g$ or $g \gg |f|$. We write $f = \mathcal{O}(g)$ as $x \to \infty$ if there exists a constant $C \in (0, \infty)$ such that $f(x) \leq Cg(x)$ for all large enough x. We write $\log^+ x := \log(x \vee \mathrm{e})$, $x \in \mathbb{R}$.

2. Deterministic spectral bounds

In this section, we consider Brownian motion in $\mathbb{R}^d$, $d \geq 3$, moving among a deterministic point cloud. Our goal is to obtain lower and upper spectral bounds in L^1 and L^∞ for relevant Feynman–Kac formulae. First we collect some basic tools from the theory of Schrödinger operators (Section 2.1), which are then applied to derive upper bounds on both time-dependent and stopped Feynman–Kac functionals (Section 2.2). After that, we obtain a lower bound for the time-dependent functional (Section 2.3), and conclude the section with a multipolar Hardy inequality (Section 2.4).

Define the family of non-empty, locally finite subsets of $\mathbb{R}^d$

$$\mathscr{Y} = \{\mathcal{Y} \subset \mathbb{R}^d : \mathcal{Y} \neq \varnothing, \#K \cap \mathcal{Y} < \infty \;\forall \text{ compact } K \subset \mathbb{R}^d\}, \tag{2.1}$$

as well as the family of non-empty, finite subsets

$$\mathscr{Y}_f = \{\mathcal{Y} \in \mathscr{Y} : \#\mathcal{Y} < \infty\}. \tag{2.2}$$

Note that the support $\mathcal{P} = \{x \in \mathbb{R}^d : \omega(\{x\}) = 1\}$ of the Poisson point process ω belongs almost surely to $\mathscr{Y}$. For $\mathcal{Y} \in \mathscr{Y}$ and $a \in (0, \infty]$ satisfying either $\mathcal{Y} \in \mathscr{Y}_f$ or $a < \infty$, let

$$V^{(a)}_{\mathcal{Y}}(x) = \sum_{y\in\mathcal{Y}} \frac{\mathbb{1}_{\{|x-y|\leq a\}}}{|x-y|^2}, \quad x \in \mathbb{R}^d \setminus \mathcal{Y}. \tag{2.3}$$

When $a < \infty$, $V^{(a)}_{\mathcal{P}} = V^{(\mathfrak{K}_a)}$ as in (1.11) with $\mathfrak{K}_a(x) = |x|^{-2}\mathbb{1}_{\{|x|\leq a\}}$. For $\mathcal{Y} \in \mathscr{Y}_f$, we write $V_{\mathcal{Y}} = V^{(\infty)}_{\mathcal{Y}}$.

2.1. *Preliminaries on Schrödinger operators and the Feynman–Kac formula*

The content of this section is classical and has been treated by many authors. Our major references here are the books [12] by Engel and Nagel and [10] by Chung and Zhao.

Let $D \subset \mathbb{R}^d$ be an open subset. By $H_0^1(D)$ we denote the Sobolev space on D with zero-boundary condition, i.e. the closure of the space $C_c^\infty(D)$ of smooth, compactly supported functions on D with respect to the Sobolev norm $\|f\|_{H^1(D)} = \sum_{1\le i\le d} \|\partial_i f\|_{L^2(D)}$, where "$\partial_i$" denotes differentiation with respect to the ith coordinate. For a potential $q \in L^1_{\mathrm{loc}}(D)$, we define

$$\lambda_{\max}(D,q) := \sup_{g\in H_0^1(D), \|g\|_{L^2(D)}=1} \left\{ \int_D q(x) g(x)^2\,\mathrm{d}x - \frac{1}{2}\int_D \left|\nabla g(x)\right|^2 \mathrm{d}x \right\} \in \mathbb{R}\cup\{+\infty\}. \tag{2.4}$$

Note that $\lambda_{\max}(D,q) \ge 0$ if $q \ge 0$; more generally, the following monotonicity property holds.

Remark 2.1. Let $D_1 \subset D_2 \subset \mathbb{R}^d$ be open and $q_1 \in L^1_{\mathrm{loc}}(D_1)$, $q_2 \in L^1_{\mathrm{loc}}(D_2)$ with $q_1 \le q_2$ on D_1. Then

$$\lambda_{\max}(D_1,q_1) \le \lambda_{\max}(D_2,q_2). \tag{2.5}$$

When q has some regularity (e.g. when it is in the Kato class), $\lambda_{\max}(D,q)$ is the supremum of the spectrum of the *Schrödinger operator* $\mathcal{H}_q = \Delta + q$ in $L^2(D)$ with zero Dirichlet boundary conditions, where Δ is the weak Laplacian whose domain is dense in $H_0^1(D)$. This holds in particular when

$$q \in L^\infty(D), \tag{2.6}$$

in which case $\lambda_{\max}(D,q) < \infty$ and $\mathcal{H}_q$ is a closed self-adjoint operator generating a strongly continuous semigroup $(T_t)_{t\ge0} = (\mathrm{e}^{t\mathcal{H}_q})_{t\ge0}$ on $L^2(D)$ (see e.g. [10, Proposition 3.29]). We will assume (2.6) in the remainder of this subsection.

An important fact about $\lambda_{\max}$ is that it controls the growth of T_t via the inequality

$$\|T_t f\|_{L^2(D)} \le \|f\|_{L^2(D)} \exp\{t\lambda_{\max}(D,q)\} \qquad \forall t\ge 0, \tag{2.7}$$

cf. e.g. [10, Equation (30), Section 8.3]. From this we get the basic but crucial bound for the resolvent of the operator $\mathcal{H}_q$ (cf. e.g. [12, Theorem II.1.10]): for $\gamma > \lambda_{\max}(D,q)$,

$$\left\|(\mathcal{H}_q - \gamma)^{-1}\right\|_{L^2(D)\to L^2(D)} \le \frac{1}{\gamma - \lambda_{\max}(D,q)}. \tag{2.8}$$

The semigroup $(T_t)_{t\ge0}$ can be used to solve the initial boundary value problem

$$\partial_t u(t,x) = \frac{\Delta}{2} u(t,x) + q(x)u(t,x), \quad (t,x) \in [0,\infty)\times D \tag{2.9}$$

$$u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\partial D \tag{2.10}$$

$$u(0,x) = u_0(x), \quad x \in D \tag{2.11}$$

with initial data $u_0 \in L^2(D)$, as follows. We want to consider solutions to (2.9)–(2.11) in the *mild* sense (cf. [21, Definition 6.1.1]), i.e., we demand that

$$\int_0^t \int_D p_{t-s}(x-y)\left|q(y)u(s,y)\right| \mathrm{d}y\,\mathrm{d}s < \infty \quad \forall x\in D, t>0 \tag{2.12}$$

and

$$u(t,x) = u_0(x) + \int_0^t \int_D p_{t-s}(x-y) q(y) u(s,y)\,\mathrm{d}y\,\mathrm{d}s \quad \forall x \in D, t>0, \tag{2.13}$$

where $p_t(x)$ is the Gaussian density

$$p_t(x) := (2\pi t)^{-d/2} \exp\{-|x|^2/(2t)\}, \tag{2.14}$$

i.e., the transition density of Brownian motion at time t started from 0.

The next proposition characterizes the mild solutions to (2.9)–(2.11), connecting Schrödinger semigroups and Brownian motion via the celebrated Feynman–Kac representation:

Proposition 2.2 (Feynman–Kac formula). *Under* (2.6), *the unique mild solution to* (2.9)–(2.11) *is given by*

$$u(t,x) = T_t u_0(x) = \mathbb{E}_x\left[u_0(W_t)\exp\left(\int_0^t q(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{D^c}>t\}}\right]. \tag{2.15}$$

Proof. Follows from e.g. [12, Proposition II.6.4]) and [10, Theorems 3.17 and 3.27]. □

Additionally to the time-dependent Feynman–Kac formula (2.15), we will use a *stopped* Feynman–Kac formula as follows. Consider the time-independent Schrödinger equation

$$\begin{aligned} &\frac{\Delta}{2}u(x) + q(x)u(x) = \gamma u(x), \quad x \in D,\\ &u(x) = f(x), \quad x \in \partial D, \end{aligned} \tag{2.16}$$

with $f : \partial D \to \mathbb{R}$ continuous and $\gamma \in \mathbb{R}$. A function $u \in L^1_{\mathrm{loc}}(D)$ is called a *weak solution* to (2.16) if

$$\int_D u(x)\Delta\phi(x)\,\mathrm{d}x = -2\int_D \big(q(x)-\gamma\big)u(x)\phi(x)\,\mathrm{d}x \tag{2.17}$$

for all $\phi \in C_c^\infty(D)$, and u is continuous on $\overline{D}$ with $u = f$ on ∂D. Recall that D is called *regular* if $\mathbb{P}_x(\tau_{D^c} = 0) = 1$ for all $x \in \partial D$. The next result follows from [10, Theorems 4.7 and 4.19].

Proposition 2.3. *Assume* (2.6). *If D is a bounded regular domain and $\gamma > \lambda_{\max}(D,q)$, then*

$$u(x) := \mathbb{E}_x\left[\exp\left(\int_0^{\tau_{D^c}} \big(q(W_s)-\gamma\big)\,\mathrm{d}s\right) f(W_{\tau_{D^c}})\right] \tag{2.18}$$

is the unique weak solution to the boundary value problem (2.16).

2.2. *Upper bounds*

Let $D \subset \mathbb{R}^d$ be a bounded regular domain. Recall $h_d = (d-2)^2/8$. Fix $\theta \in (0,h_d]$, $\mathcal{Y} \in \mathscr{Y}_f$ with $\mathcal{Y} \subset D$ and put $M := \#\mathcal{Y}$. We give next L^1 upper bounds for both stopped and time-dependent Feynman–Kac functionals with potentials of the form (2.3). We note that, by [4, Theorem 1], $\lambda_{\max}(\mathbb{R}^d, \theta V_{\mathcal{Y}}) < \infty$; by Remark 2.1, also $\lambda_{\max}(D, \theta V_{\mathcal{Y}}^{(a)}) < \infty$ for any $a > 0$.

Lemma 2.4. *For any measurable $D' \subset \mathbb{R}^d$, any $a \in (0,\infty]$ and any $\gamma > \lambda_{\max}(D, \theta V_{\mathcal{Y}}^{(a)})$,*

$$\int_{D'} \mathbb{E}_x\big[\mathrm{e}^{\int_0^t (\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{D^c}>t\}}\big]\,\mathrm{d}x \le \sqrt{|D'\cap D||D|}. \tag{2.19}$$

Moreover, there exists a constant $c = c(d) \in (0,\infty)$ independent of D, θ, $\mathcal{Y}$, γ, a, D' such that

$$\int_{D'} \mathbb{E}_x\big[\mathrm{e}^{\int_0^{\tau_{D^c}} (\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s}\big]\,\mathrm{d}x \le |D'| + c\sqrt{|D||D'\cap D|}\,\frac{\gamma + (M^2+\theta)\mathrm{dist}(D^c,\mathcal{Y})^{-2}}{\gamma - \lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}. \tag{2.20}$$

Proof. Fix $D' \subset \mathbb{R}^d$ measurable, $a > 0$ and $\gamma > \lambda_{\max}(D, \theta V_{\mathcal{Y}}^{(a)})$. Note that, when $x \in D^c$, the integrands in (2.19) and (2.20) are respectively equal to 0 and 1, and thus we may assume that $D' \subset D$.

We start with (2.19). For $m \in \mathbb{N}$, let $F_m = \min(V_{\mathcal{Y}}^{(a)}, m)$ and write $(T_t^{(m)})_{m\in\mathbb{N}}$ for the Schrödinger semigroup associated with the potential $q = \theta F_m$ as in (2.15). Note that, for all $m \in \mathbb{N}$,

$$\begin{aligned} \int_{D'} \mathbb{E}_x\big[\mathrm{e}^{\int_0^t (\theta F_m(W_s)-\gamma)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{D^c}>t\}}\big]\,\mathrm{d}x &= \mathrm{e}^{-t\gamma}\big\langle \mathbb{1}_{D'}, T_t^{(m)}\mathbb{1}_D\big\rangle_{L^2(D)}\\ &\le \mathrm{e}^{-t\gamma}\|\mathbb{1}_{D'}\|_{L^2(D)}\big\|T_t^{(m)}\big\|_{L^2(D)\to L^2(D)}\|\mathbb{1}_D\|_{L^2(D)}\\ &\le \mathrm{e}^{-t\gamma}\sqrt{|D'||D|}\mathrm{e}^{t\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)}\wedge m)} \le \sqrt{|D'||D|}, \end{aligned} \tag{2.21}$$

where we used the Cauchy–Schwarz inequality and $\lambda_{\max}(D,\theta F_m)\le\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})<\gamma$ by Remark 2.1. Letting $m\to\infty$, (2.19) follows by monotone convergence.

Consider now (2.20). By Proposition 2.3, the function $u_m(x)=\mathbb{E}_x[\exp\int_0^{\tau_{D^c}}(\theta F_m(W_s)-\gamma)\,\mathrm{d}s]$ is the unique weak solution to the boundary value problem

$$\begin{aligned}&\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)u(x)=0,\quad x\in D\\&u(x)=1,\quad x\in\partial D.\end{aligned}\tag{2.22}$$

Abbreviate $\delta:=\operatorname{dist}(D^c,\mathcal{Y})$, take $g\colon\mathbb{R}\to[0,1]$ smooth with $g(r)=0$ for $r\le 1/2$ and $g(r)=1$ for $r\ge 1$, and put $\phi(x):=\prod_{y\in\mathcal{Y}}g(|x-y|/\delta)$. We may check that $\phi\in C^2(\mathbb{R}^d)$, $0\le\phi\le 1$ on D, $\phi\equiv 1$ on D^c, and there exists a constant $c=c(d)\in(1,\infty)$, not depending on D,θ or $\mathcal{Y}$, such that $|\Delta\phi|\le 2cM^2\delta^{-2}$ and $\phi V_{\mathcal{Y}}\le c\delta^{-2}$ uniformly on $\mathbb{R}^d$. Moreover, $v_m:=u_m-\phi$ solves

$$\begin{aligned}&\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)v_m(x)=-\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi(x),\quad x\in D,\\&v_m(x)=0,\quad x\in\partial D,\end{aligned}\tag{2.23}$$

i.e., $v_m=-\mathcal{R}_\gamma^{(m)}(\frac{\Delta}{2}+\theta F_m-\gamma)\phi$ where $\mathcal{R}_\gamma^{(m)}$ is the resolvent of $\frac{1}{2}\Delta+\theta F_m$ at γ. Hence

$$\begin{aligned}\|v_m\|_{L^1(D')}&=\left\langle\left|-\mathcal{R}_\gamma^{(m)}\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi\right|,\mathbb{1}_{D'}\right\rangle_{L^2(D)}\\&\le\sqrt{|D'|}\,\|\mathcal{R}_\gamma^{(m)}\|_{L^2(D)\to L^2(D)}\left\|\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi\right\|_{L^2(D)}\\&\le\sqrt{|D'|}\frac{\gamma+c(M^2+\theta)\delta^{-2}}{\gamma-\lambda_{\max}(D,\theta F_m)}\sqrt{|D|}\end{aligned}\tag{2.24}$$

by the bound (2.8) on the resolvent and the pointwise bounds on ϕ, $\Delta\phi$ and $V_{\mathcal{Y}}\phi$. Noting now that, since $F_m\le V_{\mathcal{Y}}^{(a)}$, $\lambda_{\max}(D,\theta F_m)\le\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})$ by Remark 2.1, we obtain

$$\|u_m\|_{L^1(D')}\le\|v_m\|_{L^1(D')}+\|\phi\|_{L^1(D')}\le c\sqrt{|D'||D|}\frac{\gamma+(M^2+\theta)\delta^{-2}}{\gamma-\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}+|D'|.\tag{2.25}$$

Now (2.20) follows by monotone convergence since $F_m\uparrow V_{\mathcal{Y}}^{(a)}$ as $m\to\infty$. □

From the L^1-bound above we derive two pointwise estimates that will be useful in Section 3.

Lemma 2.5. *Fix $x\in D\backslash\mathcal{Y}$ and set $\varepsilon_x=\frac{1}{2}\operatorname{dist}(x,\mathcal{Y})$. Assume that $0<a<\varepsilon_x$ and $\gamma>\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})$, and let $c=c(d)$ be the constant from Lemma 2.4. Then*

$$\mathbb{E}_x\left[\exp\int_0^{\tau_{D^c}}\left(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right)\mathrm{d}s\right]\le 2+c\sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\frac{\gamma+(M^2+\theta)\operatorname{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}.\tag{2.26}$$

Moreover, for all $t\in(0,\infty)$,

$$\mathbb{E}_x\left[\mathbb{1}_{\{\tau_{D^c}>t\}}\exp\int_0^t\left(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right)\mathrm{d}s\right]\le 2+\sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\left(1+c\frac{\gamma+(M^2+\theta)\operatorname{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}\right).\tag{2.27}$$

Proof. Fix $0<r<\varepsilon_x$ and abbreviate $I_s^t:=\exp\int_s^t(\theta V_{\mathcal{Y}}^{(a)}(W_u)-\gamma)\,\mathrm{d}u$. We begin with the proof of (2.26). Since $V_{\mathcal{Y}}^{(a)}\equiv 0$ on $B_{\varepsilon_x}(x)$, using the strong Markov property we may write

$$\mathbb{E}_x\left[I_0^{\tau_{D^c}}\right]\le 1+\mathbb{E}_x\left[\mathbb{1}_{\{\tau_{B_r(x)^c}<\tau_{D^c}\}}I_{\tau_{B_r(x)^c}}^{\tau_{D^c}}\right]\le 1+\mathbb{E}_x\left[\mathbb{E}_{W_{\tau_{\partial B_r(x)}}}\left[I_0^{\tau_{D^c}}\right]\right].\tag{2.28}$$

Since $W_{\tau_{\partial B_r(x)}}$ is uniformly distributed on the sphere $\partial B_r(x)$,

$$\mathbb{E}_x\big[I_0^{\tau_{D^c}}\big] \le 1 + \frac{1}{\sigma_d r^{d-1}} \int_{\partial B_r(x)} \mathbb{E}_z\big[I_0^{\tau_{D^c}}\big]\sigma(\mathrm{d}z), \tag{2.29}$$

where σ denotes surface measure on $\partial B_r(x)$ and σ_d is the area of the d-dimensional unit sphere. Multiplying both sides of (2.29) by $\sigma_d r^{d-1}$ and integrating over r between 0 and ε_x leads to

$$|B_{\varepsilon_x}|\big(\mathbb{E}_x\big[I_0^{\tau_{D^c}}\big] - 1\big) \le \int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\big[I_0^{\tau_{D^c}}\big]\,\mathrm{d}z. \tag{2.30}$$

Now apply the L^1-bound from Lemma 2.4 to the right-hand side with $D' = B_{\varepsilon_x}(x)$, which gives

$$\int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\big[I_0^{\tau_{D^c}}\big]\,\mathrm{d}z \le |B_{\varepsilon_x}|\left\{1 + c\sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\left(\frac{\gamma + (M^2+\theta)\,\mathrm{dist}(D^c,\mathcal{Y})^{-2}}{\gamma - \lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}\right)\right\}. \tag{2.31}$$

This yields (2.26), and we continue with the proof of (2.27). Again, by the strong Markov property and since $V_{\mathcal{Y}}^{(a)} \equiv 0$ on $B_{\varepsilon_x}(x)$,

$$\mathbb{E}_x\big[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}\big] \le 1 + \mathbb{E}_x\Big[\mathrm{e}^{-\gamma\tau_{\partial B_r(x)}}\mathbb{1}_{\{\tau_{\partial B_r(x)}<t\}}\mathbb{E}_{W_{\tau_{\partial B_r(x)}}}\big[I_0^{t-s}\mathbb{1}_{\{\tau_{D^c}>t-s\}}\big]_{s=\tau_{\partial B_r(x)}}\Big]. \tag{2.32}$$

Split the event $\{\tau_{D^c} > t-s\}$ according to whether $\tau_{D^c} > t$ or not to write, using $\gamma \ge 0$, $V_{\mathcal{Y}}^{(a)} \ge 0$,

$$I_0^{t-s}\mathbb{1}_{\{\tau_{D^c}>t-s\}} = \mathrm{e}^{s\gamma}\mathrm{e}^{\int_0^{t-s}\theta V_{\mathcal{Y}}^{(a)}(W_s)\,\mathrm{d}s - t\gamma}\mathbb{1}_{\{\tau_{D^c}>t-s\}} \le \mathrm{e}^{s\gamma}\big\{I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}} + I_0^{\tau_{D^c}}\big\}. \tag{2.33}$$

Substituting this back into (2.32), we obtain

$$\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}}\big] \le 1 + \frac{1}{\sigma_d r^{d-1}}\int_{\partial B_r(x)} \mathbb{E}_z\big[I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}} + I_0^{\tau_{D^c}}\big]\sigma(\mathrm{d}z), \tag{2.34}$$

and the same calculation as between (2.29)–(2.30) gives

$$|B_{\varepsilon_x}|\big(\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}}\big] - 1\big) \le \int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\big[I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}}\big]\,\mathrm{d}z + \int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\big[I_0^{\tau_{D^c}}\big]\,\mathrm{d}z. \tag{2.35}$$

To conclude, apply (2.19) with $D' = B_{\varepsilon_x}(x)$ to the first integral above, and (2.31) to the second. $\square$

2.3. *Lower bound*

We derive here an L^1 lower bound (cf. Lemma 2.8 below) on the Feynman–Kac functional in (2.15) with $q = \theta V_{\mathcal{Y}}$, $\mathcal{Y} \in \mathscr{Y}_f$. Recall $h_d = (d-2)^2/8$. Define the truncated potential

$$\widetilde{V}(x) := \begin{cases} 1, & \text{if } |x| \le 1 \\ |x|^{-2}, & \text{else.} \end{cases} \tag{2.36}$$

Lemma 2.6. *For any $\varepsilon > 0$, there exists $K_\varepsilon \in [1,\infty)$ such that, for all $K \ge K_\varepsilon$,*

$$\sup_{g\in H_0^1(B_K),\,\|g\|_{L^2(B_K)}=1} (h_d+\varepsilon)\int_{B_K} g^2(x)\widetilde{V}(x)\,\mathrm{d}x - \frac{1}{2}\|\nabla g\|^2_{L^2(B_K)} > 0. \tag{2.37}$$

Proof. Taking, for $n \in \mathbb{N}$,

$$\tilde{g}_n(x) := \begin{cases} 1 & \text{when } |x| \le 1, \\ |x|^{-(d-2)/2} & \text{when } 1 < |x| \le n, \\ n^{-d/2}(2n - |x|) & \text{when } n < |x| \le 2n, \\ 0 & \text{when } |x| > 2n, \end{cases} \tag{2.38}$$

it follows that, for all $K > 2n$, $\tilde{g}_n \in H_0^1(B_K)$ and

$$(h_d+\varepsilon)\frac{\int_{B_K}\tilde{g}_n^2(x)\widetilde{V}(x)\,\mathrm{d}x}{\frac{1}{2}\int_{B_K}|\nabla\tilde{g}_n(x)|^2\,\mathrm{d}x} \geq \left(1+\frac{8\varepsilon}{(d-2)^2}\right)\left(1-\frac{c}{\log n}\right) \tag{2.39}$$

for some constant $c \in (0,\infty)$. Letting $g_n := \tilde{g}_n/\|\tilde{g}_n\|_{L^2(B_K)}$, we obtain

$$(h_d+\varepsilon)\int_{B_K} g_n^2(x)\widetilde{V}(x)\,\mathrm{d}x - \frac{1}{2}\|\nabla g_n\|^2_{L^2(B_K)} \geq \frac{2\varepsilon}{(d-2)^2}\|\nabla g_n\|^2_{L^2(B_K)} > 0 \tag{2.40}$$

for n large and $K > 2n$. □

Let $\mathcal{Y} \in \mathscr{Y}_f$ with $M = \#\mathcal{Y} \geq 2$ and fix $\theta \in (\frac{(d-2)^2}{8M}, \frac{(d-2)^2}{8}]$. We define

$$\delta_\star = \delta_\star(d,M,\theta) := \frac{1}{4}\left(1-\frac{h_d}{\theta M}\right). \tag{2.41}$$

Lemma 2.7. *If $|y| \leq \delta_\star$ for all $y \in \mathcal{Y}$, then*

$$\theta V_{\mathcal{Y}}(x) \geq (h_d + 2\theta M\delta_\star)\widetilde{V}(x) \quad \forall x \in \mathbb{R}^d \setminus \mathcal{Y}. \tag{2.42}$$

Proof. Follows from a simple computation using $|x-y|^2 \leq |x|^2 + 2|x||y| + |y|^2$. □

The following is the key lemma to obtain a lower bound on the total mass.

Lemma 2.8. *There exist constants $K > 1$ and $c_1, c_2 > 0$ depending on d, M, θ such that, for any $a \in (0,\infty)$ and any $x \in \mathbb{R}^d \setminus \mathcal{Y}$ such that $\mathcal{Y} \subset B_a(x)$,*

$$\int_{B_{Ka}(x)} \mathbb{E}_z\big[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(W_s)\,\mathrm{d}s}\mathbb{1}\{\tau_{B_{Ka}(x)^{\mathrm{c}}} > t\}\big]\,\mathrm{d}z \geq c_1 a^d \mathrm{e}^{c_2 t a^{-2}} \quad \forall t \geq 0. \tag{2.43}$$

Proof. *By translation invariance, we may suppose that $x = 0$ and $\mathcal{Y} \subset B_a$. Set $b = \delta_\star/a$, $K = K_\star/\delta_\star$, where $K_\star$ is given by Lemma 2.6 with $\varepsilon := 2\theta M\delta_\star$, and write*

$$\int_{B_{Ka}} \mathbb{E}_z\big[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(W_s)\,\mathrm{d}s}\mathbb{1}\{\tau_{B^{\mathrm{c}}_{Ka}} > t\}\big]\,\mathrm{d}z = b^{-d}\int_{B_{K_\star}} \mathbb{E}_{z/b}\big[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(W_s)\,\mathrm{d}s}\mathbb{1}\{\tau_{B^{\mathrm{c}}_{Ka}} > t\}\big]\,\mathrm{d}z. \tag{2.44}$$

By Brownian scaling, the integrand in the right-hand side of (2.44) equals

$$\mathbb{E}_z\big[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(b^{-1}W_{b^2 s})\,\mathrm{d}s}\mathbb{1}\big\{\tau_{B^{\mathrm{c}}_{K_\star}} > b^2 t\big\}\big] = \mathbb{E}_z\big[\mathrm{e}^{\int_0^{b^2 t} \theta V_{b\mathcal{Y}}(W_s)\,\mathrm{d}s}\mathbb{1}\big\{\tau_{B^{\mathrm{c}}_{K_\star}} > b^2 t\big\}\big] \tag{2.45}$$

where $b\mathcal{Y} := \{by\colon y \in \mathcal{Y}\}$. Since $|y| \leq \delta_\star$ for all $y \in b\mathcal{Y}$, (2.45) is at least

$$\mathbb{E}_z\left[\exp\left\{\int_0^{b^2 t}(h_d+\varepsilon)\widetilde{V}(W_s)\,\mathrm{d}s\right\}\mathbb{1}\big\{\tau_{B^{\mathrm{c}}_{K_\star}} > b^2 t\big\}\right] \tag{2.46}$$

by Lemma 2.6. Now using a Fourier expansion as in [15, Equation (2.33)], we obtain

$$\int_{B_{K_\star}} \mathbb{E}_z\left[\exp\left\{\int_0^{b^2 t}(h_d+\varepsilon)\widetilde{V}(W_s)\,\mathrm{d}s\right\}\mathbb{1}\big\{\tau_{B^{\mathrm{c}}_{K_\star}} > b^2 t\big\}\right]\mathrm{d}z \geq \mathrm{e}^{b^2 t\widetilde{\lambda}_{\max}}\|e_1\|^2_{L^1(B_{K_\star})} \tag{2.47}$$

where $\widetilde{\lambda}_{\max} := \lambda_{\max}(B_{K_\star}, (h_d+\varepsilon)\widetilde{V})$ is the principal Dirichlet eigenvalue of $\frac{1}{2}\Delta + (h_d+\varepsilon)\widetilde{V}$ in $B_{K_\star}$, and e_1 is the corresponding eigenfunction normalized so that $\|e_1\|_{L^2(B_{K_\star})} = 1$. Now (2.43) follows with $c_1 = \delta_\star^{-d}\|e_1\|^2_{L^1(B_{K_\star})}$ and $c_2 = \delta_\star^2\widetilde{\lambda}_{\max}$, which is strictly positive by Lemma 2.6. □

2.4. *Multipolar Hardy inequality*

We provide in this section upper bounds for $\lambda_{\max}(\mathbb{R}^d, q)$ in (2.4) with $q = \theta V_{\mathcal{Y}}$, $\mathcal{Y} \in \mathscr{Y}_f$ and $\theta \in (0, h_d]$ (recall $h_d = (d-2)^2/8$), which will be useful to control (2.26) and (2.27).

When $\#\mathcal{Y} = 1$, Hardy's inequality (1.7) states that

$$\lambda_{\max}\big(\mathbb{R}^d, \theta V_{\mathcal{Y}}\big) = 0 \quad \text{if } 0 \le \theta \le h_d, \tag{2.48}$$

which clearly extends to $\#\mathcal{Y} \ge 2$ in the sense that, with $M = \#\mathcal{Y}$,

$$\lambda_{\max}\big(\mathbb{R}^d, \theta V_{\mathcal{Y}}\big) = 0 \quad \text{if } 0 \le \theta \le \frac{h_d}{M}. \tag{2.49}$$

More general bounds, known as *multipolar Hardy inequalities*, are considered for example in [4]. The next proposition is obtained by combining results and methods from [4], and offers in some cases an improvement of Theorem 1 therein.

Proposition 2.9. *Fix $\mathcal{Y} \in \mathscr{Y}_f$. Assume that $M := \#\mathcal{Y} \ge 2$ and $\theta \in (\frac{h_d}{M}, \frac{h_d}{(M-1)}]$. Let*

$$\Gamma := \inf\big\{r > 0 \colon B_r(\mathcal{Y}) \text{ is connected}\big\}. \tag{2.50}$$

Then

$$\lambda_{\max}\big(\mathbb{R}^d, \theta V_{\mathcal{Y}}\big) \le \frac{M(\pi^2 + 3\theta)}{2\Gamma^2}. \tag{2.51}$$

Proof. Fix $r \in (0, \Gamma)$ and choose $\widehat{\mathcal{Y}} \subset \mathcal{Y}$ such that $\widehat{\mathcal{Y}} \neq \varnothing$, $N := \#\widehat{\mathcal{Y}} \le \lfloor M/2 \rfloor$,

$$B_r(\widehat{\mathcal{Y}}) \text{ is connected and } B_r(\widehat{\mathcal{Y}}) \cap B_r(\mathcal{Y} \setminus \widehat{\mathcal{Y}}) = \varnothing, \tag{2.52}$$

i.e., $B_r(\widehat{\mathcal{Y}})$ is a connected component of $B_r(\mathcal{Y})$ containing at most half of the points of $\mathcal{Y}$. Define a partition of unity (cf. definition after Theorem 1 in [4]) with 2 terms as follows. Set

$$J(t) := \begin{cases} 0, & t \in [0, 1/2], \\ -\cos(\pi t), & t \in [1/2, 1], \\ 1, & t \ge 1, \end{cases} \tag{2.53}$$

put $J_1(x) := \prod_{y \in \widehat{\mathcal{Y}}} J(|x-y|/r)$ and $J_2(x) := [1 - J_1(x)^2]^{1/2}$. By Lemma 2 in [4],

$$Q[u] := \int_{\mathbb{R}^d} \big\{\theta V_{\mathcal{Y}}(x) u(x)^2 - \big|\nabla u(x)\big|^2\big\}\,\mathrm{d}x = \sum_{i=1}^{2} Q[J_i u] + \int_{\mathbb{R}^d} u(x)^2 \sum_{i=1}^{2} \big|\nabla J_i(x)\big|^2\,\mathrm{d}x \tag{2.54}$$

for all $u \in H^1(\mathbb{R}^d)$. Note that, by (2.52) and the definition of J_1, J_2,

$$V_{\mathcal{Y}}(x) J_2(x)^2 \le V_{\widehat{\mathcal{Y}}}(x) J_2(x)^2 + \frac{M-N}{r^2} \quad \forall x \in \mathbb{R}^d \setminus \widehat{\mathcal{Y}} \tag{2.55}$$

while, for all $x \notin \widetilde{\mathcal{Y}} := \mathcal{Y} \setminus \widehat{\mathcal{Y}}$,

$$\begin{aligned} V_{\mathcal{Y}}(x) J_1(x)^2 &= \Big\{ V_{\widetilde{\mathcal{Y}}}(x) + \sum_{y \in \widehat{\mathcal{Y}}} \frac{\mathbb{1}_{\{|x-y| \ge r/2\}}}{|x-y|^2} \Big\} J_1(x)^2 \\ &\le V_{\widetilde{\mathcal{Y}}}(x) J_1(x)^2 + \frac{N}{r^2} \sup_{t \ge 1/2} \frac{J(t)^2}{t^2} \le V_{\widetilde{\mathcal{Y}}}(x) J_1(x)^2 + \frac{2N}{r^2}, \end{aligned} \tag{2.56}$$

where for the last step we used $\sup_{t \ge 1/2} J(t)^2/t^2 = \sup_{t \in [1/2,1]} \cos(\pi t)^2/t^2 < 2$ (see the proof of Lemma 3 in [4]). Applying (2.48)–(2.49), we obtain

$$\sum_{i=1}^{2} Q[J_i u] \le \theta \frac{M+N}{r^2} \|u\|^2_{L^2(\mathbb{R}^d)} \quad \forall u \in H^1\big(\mathbb{R}^d\big). \tag{2.57}$$

Next we claim that

$$\sum_{i=1}^{2}\big|\nabla J_i(x)\big|^2 \le N\frac{\pi^2}{r^2} \quad \forall x \in \mathbb{R}^d. \tag{2.58}$$

Indeed, we may restrict to $x \in B_r(\widehat{Y})$, in which case we note that

$$\sum_{i=1}^{2}\big|\nabla J_i(x)\big|^2 = \frac{|\nabla J_1(x)|^2}{1-J_1(x)^2} \le \frac{\pi^2}{r^2} \sup_{\eta\in[0,\pi/2)^N} F(\eta), \tag{2.59}$$

where, for $\eta = (\eta_1, \dots, \eta_N) \in [0, \pi/2)^N$,

$$F(\eta) := \left(1 - \prod_{i=1}^{N} \sin(\eta_i)^2\right)^{-1} \left(\sum_{i=1}^{N} \cos(\eta_i) \prod_{j\neq i} \sin(\eta_j)\right)^2. \tag{2.60}$$

Let us show that $\sup_{\eta\in[0,\pi/2)^N} F(\eta) \le N$. First note that, if $\min_i \eta_i = 0$, then $F(\eta) \le 1 < N$, and thus we may restrict to $\eta \in (0, \pi/2)^N$. In the latter set, $F = f/g$ where

$$f(\eta) := \left(\sum_{i=1}^{N} \cot(\eta_i)\right)^2, \qquad g(\eta) := \prod_{i=1}^{N} \csc(\eta_i)^2 - 1. \tag{2.61}$$

Using $\csc(\eta_i)^2 = 1 + \cot(\eta_i)^2$ and expanding the product in the definition of g, we obtain $g(\eta) \ge \sum_{i=1}^{N} \cot(\eta_i)^2$. On the other hand, by the Cauchy–Schwarz inequality, $f(\eta) \le N \sum_{i=1}^{N} \cot(\eta_i)^2 \le N g(\eta)$, finishing the proof of (2.58). As a consequence,

$$\int_{\mathbb{R}^d} u(x)^2 \sum_{i=1}^{2}\big|\nabla J_i(x)\big|^2 \,\mathrm{d}x \le \frac{\lfloor M/2 \rfloor \pi^2}{r^2} \|u\|^2_{L^2(\mathbb{R}^d)} \quad \forall u \in H^1\big(\mathbb{R}^d\big). \tag{2.62}$$

Collecting now (2.54), (2.57), (2.62) and letting $r \uparrow \Gamma$, we conclude (2.51). □

3. Path expansions

In this section, we provide an upper bound for the contribution to the Feynman–Kac formula of Brownian paths that leave a large ball. This is achieved by means of a path expansion technique that splits the Brownian path in excursions between neighbourhoods of the Poisson points, cf. Section 3.1 below.

Recall $h_d = (d-2)^2/8$ and fix $\mathcal{Y} \in \mathscr{Y}_f$ (cf. (2.2)). Given $r > 0$, we denote by $\mathfrak{C}_{\mathcal{Y}}^{(r)}$ the set of connected components of $B_r(\mathcal{Y})$. For $a \in (0, r)$, $\theta \in (0, h_d]$ and $\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)}$, let

$$N_{\mathcal{C}} := \#\mathcal{Y} \cap \mathcal{C}, \qquad \lambda_{\mathcal{C}} := \lambda_{\max}\big(\mathcal{C}, \theta V_{\mathcal{Y}}^{(a)}\big) = \lambda_{\max}\big(\mathcal{C}, \theta V_{\mathcal{Y}\cap\mathcal{C}}^{(a)}\big) \ge 0, \tag{3.1}$$

where $V_{\mathcal{Y}}^{(a)}$ is as in (2.3) and $\lambda_{\max}(D, V)$ as in (2.4). Note that $\lambda_{\mathcal{C}} < \infty$ by [4, Theorem 1]. Define

$$N_{\mathcal{Y}}^{(r)} := \max_{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}} N_{\mathcal{C}}, \qquad \Lambda_{\mathcal{Y}}^{(\theta,a,r)} := \max_{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}} \lambda_{\mathcal{C}} \tag{3.2}$$

and, for measurable $D' \subset \mathbb{R}^d$,

$$\mathfrak{N}_{\mathcal{Y}}^{(r)}\big(D'\big) := 1 \vee \sqrt{\#\big\{\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)} : D' \cap \mathcal{C} \neq \varnothing\big\}}. \tag{3.3}$$

The following is the main result of this section.

Theorem 3.1. *There exist constants $K \in [1,\infty)$ and $c, c_* \in (0,\infty)$ such that the following holds. Let $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0, h_d]$, $a \in (0,1]$ and $r > 4a$. For $\gamma > \Lambda_{\mathcal{Y}}^{(\theta,a,r)}$, let*

$$\begin{aligned} L = L(\mathcal{Y},\theta,a,r,\gamma) &:= K\big(N_{\mathcal{Y}}^{(r)}\big)^{5/2}\left(\frac{r}{a}\right)^{\frac{d}{2}}\left(1+\frac{\gamma+(1+\theta)r^{-2}}{\gamma-\Lambda_{\mathcal{Y}}^{(\theta,a,r)}}\right), \\ \varrho = \varrho(\mathcal{Y},\theta,a,r,\gamma) &:= L\exp\{-ac_*\sqrt{\gamma}\}. \end{aligned} \tag{3.4}$$

Assume that $\varrho \le 1/2$. Then

$$\sup_{z\in B_r(\mathcal{Y})^c}\sup_{t\ge 0}\mathbb{E}_z\left[\exp\int_0^t\left\{\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right\}\mathrm{d}s\right] \le \frac{1}{1-\varrho} \le 2 \tag{3.5}$$

and, for all measurable $D' \subset \mathbb{R}^d$,

$$\sup_{t\ge 0}\int_{D'}\mathbb{E}_z\left[\exp\int_0^t\left\{\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right\}\mathrm{d}s\right]\mathrm{d}z \le \frac{4L\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')}{1-\varrho}\big(|D'|\vee\sqrt{|D'|}\big). \tag{3.6}$$

Moreover, for all $R \ge 8rN_{\mathcal{Y}}^{(r)}$ and all $t > 0$,

$$\sup_{z\in B_r(\mathcal{Y})^c}\mathbb{E}_z\left[\mathbb{1}_{\{\tau_{B_R^c(z)}\le t\}}\exp\int_0^t\left\{\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right\}\mathrm{d}s\right] \le 2KL\left\{\frac{R}{r}\mathrm{e}^{-\frac{cR^2}{t}}+\varrho^{\frac{R}{4rN_{\mathcal{Y}}^{(r)}}}\right\} \tag{3.7}$$

and

$$\int_{D'}\mathbb{E}_z\big[\mathbb{1}_{\{\tau_{B_R^c(z)}\le t\}}\mathrm{e}^{\int_0^t\{\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\}\mathrm{d}s}\big]\mathrm{d}z \le 4KL\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')\big(|D'|\vee\sqrt{|D'|}\big)\left\{\frac{R}{r}\mathrm{e}^{-\frac{cR^2}{t}}+\varrho^{\frac{R}{4rN_{\mathcal{Y}}^{(r)}}}\right\}. \tag{3.8}$$

3.1. *Proof of Theorem 3.1*

We start with auxiliary results that will be needed in the following, and that will allow us to identify the constants in Theorem 3.1. The first lemma concerns standard bounds for Brownian motion.

Lemma 3.2. *There exist $K_* = K_*(d) \in [1,\infty)$ and $c_* = c_*(d) \in (0,\infty)$ such that*

$$\mathbb{P}_0\Big(\sup_{0\le s\le t}|W_s| > R\Big) \le K_*\mathrm{e}^{-\frac{c_*R^2}{t}} \quad \text{for all } t, R > 0, \tag{3.9}$$

and

$$\mathbb{E}_0\big[\mathrm{e}^{-u\tau_{B_a^c}}\big] \le K_*\mathrm{e}^{-c_*a\sqrt{u}} \quad \text{for all } a, u > 0. \tag{3.10}$$

Proof. Follows from union bounds and standard estimates for one-dimensional Brownian motion, e.g. Remark 2.22 and Exercise 2.18 in [20]. □

The next lemma is a consequence of the bounds in Lemma 2.4 and Lemma 2.5.

Lemma 3.3. *There exists a constant $K_1 \in [1,\infty)$ such that, for all $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0,h_d]$, $a \in (0,1]$, $r > 2a$, $\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)}$, $\gamma > \lambda_{\mathcal{C}}$, and all measurable $D' \subset \mathbb{R}^d$,*

$$\sup_{t\ge 0}\int_{D'}\mathbb{E}_x\big[\mathrm{e}^{\int_0^t(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{\mathcal{C}^c}>t\}}\big]\mathrm{d}x \le \sqrt{|D'|}K_1N_{\mathcal{C}}^{1/2}\left(\frac{r}{a}\right)^{\frac{d}{2}} \tag{3.11}$$

and

$$\int_{D'}\mathbb{E}_x\big[\mathrm{e}^{\int_0^{\tau_{\mathcal{C}^c}}(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s}\big]\mathrm{d}x \le \big(|D'|\vee\sqrt{|D'|}\big)K_1N_{\mathcal{C}}^{5/2}\left(\frac{r}{a}\right)^{\frac{d}{2}}\left(1+\frac{\gamma+(1+\theta)r^{-2}}{\gamma-\lambda_{\mathcal{C}}}\right). \tag{3.12}$$

Moreover, for all $x \in \mathcal{C} \setminus B_{2a}(\mathcal{Y})$,

$$\mathbb{E}_x\left[\exp \int_0^{\tau_{\mathcal{C}^c}} \left(\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma\right) \mathrm{d}s\right] \le K_1 N_{\mathcal{C}}^{5/2} \left(\frac{r}{a}\right)^{\frac{d}{2}} \left(1 + \frac{\gamma + (1+\theta) r^{-2}}{\gamma - \lambda_{\mathcal{C}}}\right) \tag{3.13}$$

and

$$\sup_{t \ge 0} \mathbb{E}_x\left[\mathrm{e}^{\int_0^t (\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma)\, \mathrm{d}s} \mathbb{1}_{\{\tau_{\mathcal{C}^c} > t\}}\right] \le K_1 N_{\mathcal{C}}^{5/2} \left(\frac{r}{a}\right)^{\frac{d}{2}} \left(1 + \frac{\gamma + (1+\theta) r^{-2}}{\gamma - \lambda_{\mathcal{C}}}\right). \tag{3.14}$$

Proof. By [10, Proposition 1.22], each $\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)}$ is a bounded regular domain. Noting that $V_{\mathcal{Y}}^{(a)}(x) = V_{\mathcal{Y} \cap \mathcal{C}}^{(a)}(x)$ for $x \in \mathcal{C} \setminus \mathcal{Y}$, apply Lemmas 2.4–2.5 with $D = \mathcal{C}$ and use $|\mathcal{C}| \le |B_1| N_{\mathcal{C}} r^d$, $N_{\mathcal{C}} \ge 1$. □

Corollary 3.4. *For any $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0, h_d]$, $a \in (0, \infty)$, $r > 4a$, $\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)}$, $\gamma > \lambda_{\mathcal{C}}$ and $x \in \mathcal{C} \cap B_{r-a}(\mathcal{Y}) \setminus B_{3a}(\mathcal{Y})$,*

$$\mathbb{E}_x\left[\exp \int_0^{\tau_{\mathcal{C}^c}} \left(\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma\right) \mathrm{d}s\right] \le K_* K_1 N_{\mathcal{C}}^{5/2} \left(\frac{r}{a}\right)^{\frac{d}{2}} \mathrm{e}^{-c_* a \sqrt{\gamma}} \left(1 + \frac{\gamma + (1+\theta) r^{-2}}{\gamma - \lambda_{\mathcal{C}}}\right) \tag{3.15}$$

where K_*, c_* are as in Lemma 3.2 and K_1 as in Lemma 3.3.

Proof. Use the strong Markov property at the exit time of $B_a(x)$ and apply Lemma 3.3 and (3.10). □

With these results in place, we may identify the constants K, c in Theorem 3.1 as

$$K := 2(K_*)^2 K_1, \qquad c := \frac{c_*}{16}, \tag{3.16}$$

where K_*, c_* are as in Lemma 3.2 and K_1 as in Lemma 3.3.

Fix now $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0, h_d]$, $a \in (0, 1]$, $r > 4a$ and $\gamma > \Lambda_{\mathcal{Y}}^{(\theta,a,r)}$. In the following, we fix K, c as in (3.16) and let L, ϱ be defined by (3.4).

The core of the proof of Theorem 3.1 is a decomposition of the Brownian path according to its excursions to and from neighbourhoods of $\mathcal{Y}$, which are marked by the following stopping times. Let $\check{\tau}_0 = \hat{\tau}_0 := 0$ and, recursively for $n \ge 0$,

$$\begin{aligned} \check{\tau}_{n+1} &:= \begin{cases} \infty & \text{if } \hat{\tau}_n = \infty, \\ \inf\{t > \hat{\tau}_n \colon W_t \in \overline{B_{3a}(\mathcal{Y})}\} & \text{otherwise,} \end{cases} \\ \hat{\tau}_{n+1} &= \begin{cases} \infty & \text{if } \check{\tau}_{n+1} = \infty, \\ \inf\{t > \check{\tau}_{n+1} \colon W_t \notin B_r(\mathcal{Y})\} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.17}$$

For $t \ge 0$, define

$$E_t := \inf\{n \ge 0 \colon \check{\tau}_{n+1} > t\}. \tag{3.18}$$

In the following we will abbreviate, for $0 \le s_1 \le s_2 \le \infty$,

$$I_{s_1}^{s_2} := \exp\left\{\int_{s_1}^{s_2} \left(\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma\right) \mathrm{d}s\right\}. \tag{3.19}$$

Lemma 3.5. *For all $n \in \mathbb{N}_0$,*

$$\sup_{x \notin B_r(\mathcal{Y})} \sup_{t \ge 0} \mathbb{E}_x\left[I_0^t \mathbb{1}_{\{E_t = n\}}\right] \le \varrho^n. \tag{3.20}$$

Moreover, for all measurable $D' \subset \mathbb{R}^d$,

$$\sup_{t \ge 0} \int_{D'} \mathbb{E}_z\left[I_0^t \mathbb{1}_{\{E_t = n\}}\right] \mathrm{d}z \le 2 \mathfrak{N}_{\mathcal{Y}}^{(r)}(D') \left(|D'| \vee \sqrt{|D'|}\right) L \varrho^{(n-1)^+}. \tag{3.21}$$

Proof. Note first that, if $n=0$, both (3.20) and (3.21) hold since then $V_{\mathcal{Y}}^{(a)}(W_s)=0$ for all $0\le s\le t$. Let us prove (3.20) by induction in n. To treat the case $n=1$, fix $x\notin B_r(\mathcal{Y})$ and $t>0$. There are two cases: either $\check{\tau}_1\le t<\hat{\tau}_1$, or $\hat{\tau}_1\le t<\check{\tau}_2$. Let $\check{\mathcal{C}}_1\in\mathfrak{C}_{\mathcal{Y}}^{(r)}$ such that $W_{\check{\tau}_1}\in\check{\mathcal{C}}_1$. Using the Markov property, $\gamma>\Lambda_{\mathcal{Y}}^{(\theta,a,r)}$ and Lemma 3.3, we may bound, $\mathbb{P}_x$-a.s. on the event $\{\check{\tau}_1\le t\}$,

$$\begin{aligned}\mathbb{E}_x\big[I_{\check{\tau}_1}^t\mathbb{1}_{\{\check{\tau}_1\le t<\hat{\tau}_1\}}|\check{\tau}_1,(W_s)_{s\le\check{\tau}_1}\big]&=\mathbb{E}_{W_{\check{\tau}_1}}\big[I_0^{t-s}\mathbb{1}_{\{\tau_{\mathcal{C}^c}>t-s\}}\big]_{s=\check{\tau}_1,\mathcal{C}=\check{\mathcal{C}}_1}\\&\le L/\big(2K_*^2\big)\le L/(2K_*)\end{aligned}\tag{3.22}$$

and, using that $V_{\mathcal{Y}}^{(a)}(W_s)=0$ for all $s\in[\hat{\tau}_1,t]$ when $\hat{\tau}_1\le t<\check{\tau}_2$ and Corollary 3.4,

$$\begin{aligned}\mathbb{E}_x\big[I_{\check{\tau}_1}^t\mathbb{1}_{\{\hat{\tau}_1\le t<\check{\tau}_2\}}|\check{\tau}_1,(W_s)_{s\le\check{\tau}_1}\big]&\le\mathbb{E}_x\big[I_{\check{\tau}_1}^{\hat{\tau}_1}|\check{\tau}_1,(W_s)_{s\le\check{\tau}_1}\big]\\&=\mathbb{E}_{W_{\check{\tau}_1}}\big[I_0^{\tau_{\mathcal{C}^c}}\big]_{\mathcal{C}=\check{\mathcal{C}}_1}\\&\le\varrho/(2K_*)<L/(2K_*).\end{aligned}\tag{3.23}$$

Since $r>4a$ and $x\notin B_r(\mathcal{Y})$, $\check{\tau}_1\ge\tau_{B_a^c(x)}$ and thus

$$\mathbb{E}_x\big[I_0^{\check{\tau}_1}\mathbb{1}_{\{\check{\tau}_1\le t\}}\big]\le\mathbb{E}_0\big[e^{-\gamma\tau_{B_a^c}}\big]\le K_*e^{-c_*a\sqrt{\gamma}}\tag{3.24}$$

by Lemma 3.2. This together with (3.22)–(3.23) gives

$$\begin{aligned}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1\}}\big]&=\mathbb{E}_x\big[I_0^{\check{\tau}_1}\mathbb{1}_{\{\check{\tau}_1\le t\}}\mathbb{E}_x\big[I_{\check{\tau}_1}^t\mathbb{1}_{\{E_t=1\}}|\check{\tau}_1,(W_s)_{s\le\check{\tau}_1}\big]\big]\\&\le Le^{-c_*a\sqrt{\gamma}}=\varrho\end{aligned}\tag{3.25}$$

by (3.4), concluding the case $n=1$. Suppose now by induction that (3.20) has been shown for some $n\ge1$. If $E_t=n+1$, then $\hat{\tau}_1\le t$ and we can write

$$\begin{aligned}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=n+1\}}\big]&=\mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\le t\}}\mathbb{E}_{W_{\hat{\tau}_1}}\big[I_0^{t-s}\mathbb{1}_{\{E_{t-s}=n\}}\big]_{s=\hat{\tau}_1}\big]\le\varrho^n\mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\le t\}}\big]\\&\le\varrho^n\mathbb{E}_x\big[I_{\check{\tau}_1}^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\le t\}}\big]\le\varrho^{n+1}/(2K_*)\end{aligned}\tag{3.26}$$

by the induction hypothesis, (3.23) and (3.4). This concludes the proof of (3.20).

We turn next to (3.21). Assume first that $n=1$. There are again two cases: either $\check{\tau}_1\le t<\hat{\tau}_1$, or $\hat{\tau}_1\le t<\check{\tau}_2$. In the first case,

$$\begin{aligned}\int_{D'\cap B_r(\mathcal{Y})}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1,\check{\tau}_1\le t<\hat{\tau}_1\}}\big]\,dx&=\sum_{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}}\int_{D'\cap\mathcal{C}}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{\tau_{\mathcal{C}^c}>t\}}\big]\,dx\\&\le\sqrt{\#\big\{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}:\,D\cap\mathcal{C}\neq\varnothing\big\}}\sqrt{\big|D'\cap B_r(\mathcal{Y})\big|}L/\big(2K_*^2\big)\end{aligned}\tag{3.27}$$

by (3.11) and the Cauchy–Schwarz inequality. In the second case,

$$\begin{aligned}&\int_{D'\cap B_r(\mathcal{Y})}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1,\hat{\tau}_1\le t<\check{\tau}_2\}}\big]\,dx\\&\quad\le\int_{D'\cap B_r(\mathcal{Y})}\mathbb{E}_x\big[I_0^{\hat{\tau}_1}\big]\,dx=\sum_{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}}\int_{D'\cap\mathcal{C}}\mathbb{E}_x\big[I_0^{\tau_{\mathcal{C}^c}}\big]\,dx\\&\quad\le\Big(\big|D'\cap B_r(\mathcal{Y})\big|+\sqrt{\#\big\{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}:\mathcal{C}\cap D'\neq\varnothing\big\}}\sqrt{\big|D'\cap B_r(\mathcal{Y})\big|}\Big)L/\big(2K_*^2\big)\end{aligned}\tag{3.28}$$

by (3.12). Combining (3.27), (3.28) and (3.20) we get

$$\begin{aligned}\int_{D'}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1\}}\big]\,dx&\le|D'\setminus B_r(\mathcal{Y})|\varrho+\big(\big|D'\cap B_r(\mathcal{Y})\big|+2\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')\sqrt{\big|D'\cap B_r(\mathcal{Y})\big|}\big)L/2\\&\le2\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')\big(\big|D'\big|\vee\sqrt{\big|D'\big|}\big)L.\end{aligned}\tag{3.29}$$

This concludes the case $n = 1$. To deal with $n+1$, $n \geq 1$, note that the first line of (3.26) is valid for any $x \in D'$. Then we may write

$$\int_{D'} \mathbb{E}_x\big[I_0^t \mathbb{1}_{\{E_t=n+1\}}\big]\,\mathrm{d}x \leq \varrho^{n+1}\big|D' \setminus B_r(\mathcal{Y})\big| + \varrho^n \sum_{\mathcal{C}\in\mathfrak{C}_{\mathcal{Y}}^{(r)}} \int_{D'\cap\mathcal{C}} \mathbb{E}_x\big[I_0^{\tau_{\mathcal{C}^c}}\big]\,\mathrm{d}x$$
$$\leq 2\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')\big(|D'| \vee \sqrt{|D'|}\big)L\varrho^n. \tag{3.30}$$

This finishes the proof (3.21). □

The next result is the key lemma for the proof of Theorem 3.1.

Lemma 3.6. *For all $R > 0$, $n \in \mathbb{N}_0$, and $z \in \mathbb{R}^d$,*

$$\mathbb{E}_x\big[I_0^t \mathbb{1}_{\{E_t=n,\tau_{B_R^c(z)}\leq t\}}\big] \leq 2L\varrho^{(n-1)^+}\mathbb{P}_x\Big(\sup_{0\leq s\leq t}|W_s - z| > R - 2N_{\mathcal{Y}}^{(r)}nr\Big) \quad \forall x \notin B_r(\mathcal{Y}), t \geq 0. \tag{3.31}$$

If moreover $R > 2rN_{\mathcal{Y}}$, then, for any measurable $D' \subset \mathbb{R}^d$,

$$\sup_{t\geq 0}\int_{D'} \mathbb{E}_x\big[I_0^t \mathbb{1}_{\{E_t=n,\tau_{B_R^c(x)}\leq t\}}\big]\,\mathrm{d}x$$
$$\leq 4L\mathfrak{N}_{\mathcal{Y}}^{(r)}(D')\big(|D'| \vee \sqrt{|D'|}\big)\varrho^{(n-1)^+}\mathbb{P}_0\Big(\sup_{0\leq s\leq t}|W_s| > R - 2N_{\mathcal{Y}}^{(r)}nr\Big). \tag{3.32}$$

Proof. Fix $z \in \mathbb{R}^d$ and $R > 0$. Note that, when $n = 0$, both inequalities hold since then $V_{\mathcal{Y}}^{(a)}(W_s) = 0$ for all $0 \leq s \leq t$. Let us prove (3.31) by induction in n. Define the events

$$\mathcal{E}_u^n(z) := \Big\{\sup_{0\leq s\leq u}|W_s - z| \geq R - 2nrN_{\mathcal{Y}}^{(r)}\Big\}, \quad n \in \mathbb{N}_0, u \geq 0. \tag{3.33}$$

For the case $n = 1$, fix $x \notin B_r(\mathcal{Y})$ and $t > 0$. Consider first the case $\tau_{B_R^c(z)} \leq \hat{\tau}_1$. We claim that, on this event, $\mathcal{E}_{\check{\tau}_1}^1(z)$ occurs. Indeed, if $\tau_{B_R^c(z)} \leq \check{\tau}_1$ this is clear, and if $\check{\tau}_1 < \tau_{B_R^c(z)} \leq \hat{\tau}_1$ then $|W_{\check{\tau}_1} - z| > R - 2rN_{\mathcal{Y}}$ as the diameter of any component $\mathcal{C} \in \mathfrak{C}_{\mathcal{Y}}^{(r)}$ is bounded by $2rN_{\mathcal{Y}}^{(r)}$. Thus

$$\mathbb{E}_x\big[I_0^t \mathbb{1}_{\{\tau_{B_R^c(z)}\leq\hat{\tau}_1,E_t=1\}}\big] \leq \mathbb{E}_x\big[\mathbb{1}_{\mathcal{E}_{\check{\tau}_1}^1(z)\cap\{\check{\tau}_1\leq t\}}\mathbb{E}_x\big[I_{\check{\tau}_1}^t \mathbb{1}_{\{E_t=1\}}|\check{\tau}_1,(W_s)_{s\leq\check{\tau}_1}\big]\big]$$
$$\leq L\mathbb{P}_x\big(\mathcal{E}_t^1(z)\big) \tag{3.34}$$

by (3.22)–(3.23) above. If $t \geq \tau_{B_R^c(z)} > \hat{\tau}_1$, then $\hat{\tau}_1 < t < \check{\tau}_2$, and thus

$$\mathbb{E}_x\big[I_0^t \mathbb{1}_{\{\hat{\tau}_1<\tau_{B_R^c(z)}\leq t,E_t=1\}}\big] \leq \mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\leq t\}}\mathbb{P}_{W_{\hat{\tau}_1}}\big(\mathcal{E}_{t-s}^0(z)\big)_{s=\hat{\tau}_1}\big]. \tag{3.35}$$

Note now that, since $\check{\tau}_1 \leq \hat{\tau}_1$ and $|W_{\hat{\tau}_1} - W_{\check{\tau}_1}| \leq 2rN_{\mathcal{Y}}$,

$$\mathbb{P}_{W_{\hat{\tau}_1}}\big(\mathcal{E}_{t-s}^0(z)\big)_{s=\hat{\tau}_1} \leq \mathbb{P}_{W_{\check{\tau}_1}}\big(\mathcal{E}_{t-s}^1(z)\big)_{s=\check{\tau}_1} \tag{3.36}$$

and thus (3.35) is at most

$$\mathbb{E}_x\big[\mathbb{1}_{\{\check{\tau}_1\leq t\}}\mathbb{P}_{W_{\check{\tau}_1}}\big(\mathcal{E}_{t-s}^1(z)\big)_{s=\check{\tau}_1}\mathbb{E}_{W_{\check{\tau}_1}}\big[I_0^{\tau_{\mathcal{C}^c}}\big]_{\mathcal{C}=\check{\mathcal{C}}_1}\big] \leq \frac{\varrho}{2}\mathbb{P}_x\big(\mathcal{E}_t^1(z)\big) < L\mathbb{P}_x\big(\mathcal{E}_t^1(z)\big) \tag{3.37}$$

by Corollary 3.4 and (3.4). Collecting (3.34)–(3.37), we conclude the case $n = 1$.

Assume now by induction that (3.31) holds for some $n \geq 1$. There are two possible cases: either $\tau_{B_R^c(z)} \leq \hat{\tau}_1$ or not. In the first case, we conclude as before that $\mathcal{E}_{\check{\tau}_1}^1(z)$ occurs. Then we may write

$$\mathbb{E}_x\big[I_0^t \mathbb{1}_{\{E_t=n+1,\tau_{B_R^c(z)}\leq\hat{\tau}_1\}}\big] \leq \mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\mathcal{E}_{\check{\tau}_1}^1(z)\cap\{\hat{\tau}_1\leq t\}}\mathbb{E}_{W_{\hat{\tau}_1}}\big[I_0^{t-s}\mathbb{1}_{\{E_{t-s}=n\}}\big]_{s=\hat{\tau}_1}\big]$$

$$\leq \varrho^n \mathbb{E}_x\big[\mathbb{1}_{\mathcal{E}^1_{\check{\tau}_1}(z)\cap\{\check{\tau}_1\leq t\}}\mathbb{E}_{W_{\check{\tau}_1}}\big[I_0^{\tau_{\mathcal{C}^c}}\big]_{\mathcal{C}=\check{\mathcal{C}}_1}\big]$$

$$\leq \varrho^n \frac{\varrho}{2}\mathbb{P}_x\big(\mathcal{E}^1_t(z)\big) < L\varrho^n \mathbb{P}_x\big(\mathcal{E}^1_t(z)\big) \tag{3.38}$$

by Lemma 3.5, Corollary 3.4 and (3.4). Consider now the case $\hat{\tau}_1 < \tau_{B^c_R(z)}$ and write

$$\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=n+1,\hat{\tau}_1<\tau_{B^c_R(z)}\leq t\}}\big] = \mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\leq t\}}\mathbb{E}_{W_{\hat{\tau}_1}}\big[I_0^{t-s}\mathbb{1}_{\{E_{t-s}=n,\tau_{B^c_R(z)}\leq t-s\}}\big]_{s=\hat{\tau}_1}\big]$$

$$\leq 2L\varrho^{n-1}\mathbb{E}_x\big[I_0^{\hat{\tau}_1}\mathbb{1}_{\{\hat{\tau}_1\leq t\}}\mathbb{P}_{W_{\hat{\tau}_1}}\big(\mathcal{E}^n_{t-s}(z)\big)_{s=\hat{\tau}_1}\big] \tag{3.39}$$

by the induction hypothesis. Reasoning as for (3.36), we see that

$$\mathbb{P}_{W_{\hat{\tau}_1}}\big(\mathcal{E}^n_{t-s}(z)\big)_{s=\hat{\tau}_1} \leq \mathbb{P}_{W_{\check{\tau}_1}}\big(\mathcal{E}^{n+1}_{t-s}(z)\big)_{s=\check{\tau}_1},$$

and hence (3.39) is at most

$$2L\varrho^{n-1}\mathbb{E}_x\big[\mathbb{1}_{\{\check{\tau}_1\leq t\}}\mathbb{P}_{W_{\check{\tau}_1}}\big(\mathcal{E}^{n+1}_{t-s}(z)\big)_{s=\check{\tau}_1}\mathbb{E}_{\check{\tau}_1}\big[I_0^{\tau_{\mathcal{C}^c}}\big]_{\mathcal{C}=\check{\mathcal{C}}_1}\big] \leq 2L\varrho^{n-1}(\varrho/2)\mathbb{P}_x\big(\mathcal{E}^{n+1}_t(z)\big) \tag{3.40}$$

by Corollary 3.4. Combining (3.38) and (3.40) we conclude the induction step and the proof of (3.31).

Fix now a measurable $D' \subset \mathbb{R}^d$ and consider (3.32). Let $n = 1$ and $x \in D' \cap B_r(\mathcal{Y})$. Since $R > 2rN_{\mathcal{Y}}$, we must have $\hat{\tau}_1 < \tau_{B^c_R(D')}$ $\mathbb{P}_x$-almost surely, and if $E_t = 1$ then $\hat{\tau}_1 < t < \check{\tau}_2$ as well. Note that (3.35) still holds in this case (with $z = x$). Since $|W_{\hat{\tau}_1} - x| < 2rN_{\mathcal{Y}}$ $\mathbb{P}_x$-a.s.,

$$\mathbb{P}_{W_{\hat{\tau}_1}}\big(\mathcal{E}^0_{t-s}(x)\big) \leq \mathbb{P}_x\big(\mathcal{E}^1_t(x)\big) = \mathbb{P}_0\big(\mathcal{E}^1_t(0)\big)$$

and thus

$$\int_{D'\cap B_r(\mathcal{Y})}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1,\tau_{B^c_R(D')}\leq t\}}\big]\,\mathrm{d}x$$

$$\leq \mathbb{P}_0\big(\mathcal{E}^1_t(0)\big)\sum_{\mathcal{C}\in\mathfrak{C}^{(r)}_{\mathcal{Y}}}\int_{D'\cap\mathcal{C}}\mathbb{E}_x\big[I_0^{\tau_{\mathcal{C}^c}}\big]\,\mathrm{d}x$$

$$\leq \mathbb{P}_0\big(\mathcal{E}^1_t(0)\big)\Big(\big|D'\cap B_r(\mathcal{Y})\big| + \sqrt{\#\big\{\mathcal{C}\in\mathfrak{C}^{(r)}_{\mathcal{Y}}:\, D'\cap\mathcal{C}\neq\varnothing\big\}}\sqrt{\big|D'\cap B_r(\mathcal{Y})\big|}\Big)L/\big(2K^2_*\big) \tag{3.41}$$

by (3.12) and the Cauchy–Schwarz inequality. Now (3.41) and (3.31) imply

$$\int_{D'}\mathbb{E}_x\big[I_0^t\mathbb{1}_{\{E_t=1,\tau_{B^c_R(D')}\leq t\}}\big]\,\mathrm{d}x$$

$$\leq 2L\mathbb{P}_0\big(\mathcal{E}^1_t(0)\big)\big\{\big|D'\setminus B_r(\mathcal{Y})\big| + \big|D'\cap B_r(\mathcal{Y})\big|/4 + \mathfrak{N}^{(r)}_{\mathcal{Y}}(D')\sqrt{\big|D'\cap B_r(\mathcal{Y})\big|}/4\big\}$$

$$\leq 4L\mathbb{P}_0\big(\mathcal{E}^1_t(0)\big)\mathfrak{N}^{(r)}_{\mathcal{Y}}(D')\big(|D'|\vee\sqrt{|D'|}\big), \tag{3.42}$$

finishing the proof of the case $n = 1$. The general case is analogous, using (3.39) instead of (3.35). □

We are now ready to finish the:

Proof of Theorem 3.1. Items (3.5)–(3.6) follow from Lemma 3.5. To show (3.7), fix $z \in B_r(\mathcal{Y})^c$ and write

$$\mathbb{E}_z\big[I_0^t\mathbb{1}_{\{\tau_{B^c_R(z)}\leq t\}}\big] = \sum_{n=0}^{\infty}\mathbb{E}_z\big[I_0^t\mathbb{1}_{\{\tau_{B^c_R(z)}\leq t, E_t=n\}}\big]$$

$$\leq 2L\sum_{n=0}^{\infty}\varrho^{(n-1)^+}\mathbb{P}_0\Big(\sup_{0\leq s\leq t}|W_s| \geq R - 2N^{(r)}_{\mathcal{Y}}nr\Big) \tag{3.43}$$

by Lemma 3.6 and the translation invariance of Brownian motion. Split the sum in (3.43) according to whether $4N^{(r)}_{\mathcal{Y}}(n-1)r > R$ or not to obtain

$$\frac{1}{2L}\mathbb{E}_z\big[I_0^t \mathbb{1}_{\{\tau_{B^c_R(z)}\le t\}}\big] \le 2\varrho^{\frac{R}{4rN^{(r)}_{\mathcal{Y}}}} + \Big(\frac{R}{4rN^{(r)}_{\mathcal{Y}}}+2\Big)\mathbb{P}_0\Big(\sup_{0\le s\le t}|W_s|\ge \frac{1}{4}R\Big)$$
$$\le K\Big\{\varrho^{\frac{R}{4rN_{\mathcal{Y}}}} + \frac{R}{r}\mathrm{e}^{-\frac{cR^2}{t}}\Big\} \tag{3.44}$$

using $\varrho \le 1/2$, $R \ge 8rN^{(r)}_{\mathcal{Y}}$, Lemma 3.2 and (3.16). This concludes the proof of (3.7).

For (3.8), we have instead

$$\int_{D'}\mathbb{E}_z\big[I_0^t \mathbb{1}_{\{\tau_{B^c_R(z)}\le t\}}\big]\,\mathrm{d}z = \sum_{n=0}^{\infty}\int_{D'}\mathbb{E}_z\big[I_0^t \mathbb{1}_{\{\tau_{B^c_R(z)}\le t, E_t=n\}}\big]\,\mathrm{d}z$$
$$\le 4L\mathfrak{N}^{(r)}_{\mathcal{Y}}(D')\big(|D'|\vee\sqrt{|D'|}\big)\sum_{n=0}^{\infty}\varrho^{(n-1)^+}\mathbb{P}_0\Big(\sup_{0\le s\le t}|W_s|\ge R-2N^{(r)}_{\mathcal{Y}}nr\Big)$$

so we may conclude as before. □

4. Small distances in Poisson clouds

We collect some elementary facts concerning the probability to find Poisson points close to each other. With the help of Proposition 2.9, this will allow us to control in Section 5.1 the growth of the maximal principal eigenvalue $\Lambda^{(\theta,a,r)}_{\mathcal{Y}}$ appearing in Theorem 3.1 with $\mathcal{Y} = \mathcal{P}\cap B_R$.

Lemma 4.1. *For any measurable $D\subset\mathbb{R}^d$, any $r\in(0,\infty)$ and any $k\in\mathbb{N}_0$,*

$$\mathbf{P}\Big(\exists\text{ distinct } y_0,\dots,y_k\in\mathcal{P}\colon\ y_0\in D,\ \max_{1\le i\le k}|y_i-y_{i-1}|\le r\Big)\le |D|\frac{|B_r|^k}{(k+1)!}. \tag{4.1}$$

Moreover,

$$\mathbf{P}\Big(\sup_{x\in D}\omega\big(B_r(x)\big)\ge k+1\Big)\le \big|B_r(D)\big|\frac{|B_{2r}|^k}{(k+1)!}. \tag{4.2}$$

Proof. We start with (4.1). We may assume that $|D|<\infty$. First note that, if $y_0\in D$ and $|y_i-y_{i-1}|<r$ for $1\le i\le k$, then $\{y_0,\dots,y_k\}\subset D_k := B_{kr}(D)$. Let $(X_i)_{i\ge 0}$ be i.i.d. random vectors, each uniformly distributed in D_k. Note that, for any fixed $N\in\mathbb{N}$, $D_k\cap\mathcal{P}$ has under its conditional law given that $\omega(D_k)=N$ the same distribution as $\{X_1,\dots,X_N\}$. For $N\ge k+1$, estimate with a union bound

$$\mathbf{P}\Big(\exists\text{ distinct } j_0,\dots,j_k\in\{1,\dots,N\}\colon\ X_{j_0}\in D,\ \max_{1\le i\le k}|X_{j_i}-X_{j_{i-1}}|\le r\Big)$$
$$\le\binom{N}{k+1}\mathbf{P}\Big(X_0\in D,\ \max_{1\le i\le k}|X_i-X_{i-1}|\le r\Big)$$
$$=\binom{N}{k+1}\frac{1}{|D_k|^{k+1}}\int_D\mathrm{d}x_0\int_{B_r(x_0)}\mathrm{d}x_1\cdots\int_{B_r(x_{k-1})}\mathrm{d}x_k=\binom{N}{k+1}\frac{|D||B_r|^k}{|D_k|^{k+1}}. \tag{4.3}$$

Since $|\mathcal{P}\cap D_k|$ has distribution Poisson($|D_k|$), splitting the left-hand side of (4.1) according to whether $|\mathcal{P}\cap D_k|=N\ge k+1$ and using (4.3), we get the bound

$$\sum_{N=k+1}^{\infty}\binom{N}{k+1}\frac{|D||B_r|^k}{|D_k|^{k+1}}\frac{|D_k|^N}{N!}\mathrm{e}^{-|D_k|}=|D|\frac{|B_r|^k}{(k+1)!} \tag{4.4}$$

as advertised. Now (4.2) follows from (4.1) with D, r substituted by $B_r(D)$, $2r$. □

Next we provide a lower bound on the probability to have close Poisson points.

Lemma 4.2. *For all measurable $D \subset \mathbb{R}^d$, all $k \in \mathbb{N}_0$ and all $r \in (0,\infty)$,*

$$\mathbf{P}\big(\exists x \in D\colon \omega\big(B_r(x)\big) = k+1\big) \ge 1 - \exp\bigg\{-|D|\frac{r^{kd}\mathrm{e}^{-|B_r|}}{2^d(k+1)!}\bigg\} \tag{4.5}$$

Proof. Note that there exists a finite $F \subset D$ such that $B_r(x) \cap B_r(y) = \varnothing$ for all distinct $x, y \in F$ and $\#F \ge \lceil |D|/|B_{2r}|\rceil$, which can be proved e.g. by induction on $\lceil |D|/|B_{2r}|\rceil$. Then the family $\omega(B_r(x))$, $x \in F$, is i.i.d., and we may estimate

$$\mathbf{P}\big(\forall x \in D\colon \omega\big(B_r(x)\big) \neq k+1\big) \le \big(1 - \mathbf{P}\big(\omega(B_r) = k+1\big)\big)^{\#F} \le \exp\bigg\{-|D|\frac{\mathbf{P}(\omega(B_r) = k+1)}{|B_{2r}|}\bigg\},$$

where we also used $1 - x \le \mathrm{e}^{-x}$. Since $\omega(B_r)$ has distribution Poisson($|B_r|$), (4.5) follows. □

We now apply the bounds in Lemmas 4.1–4.2 to derive several asymptotic results. As a first consequence of Lemma 4.1, we can show that, for fixed $a > 0$, the maximal number of Poisson points in a-neighbourhoods of points in B_R grows at most logarithmically in R:

Corollary 4.3. *For any $a \in (0,\infty)$,*

$$\lim_{R\to\infty} (\log R)^{-1} \sup_{x\in B_R} \omega\big(B_a(x)\big) = 0 \quad \mathbf{P}\text{-a.s.} \tag{4.6}$$

Proof. Fix $a \in (0,\infty)$ and $K > 1$. By (4.2), there exists a constant $c \in (0,\infty)$ such that

$$\mathbf{P}\Big(\sup_{x\in B_{K^n}} \omega\big(B_a(x)\big) \ge n\Big) \le c\frac{(Ka)^n}{n!}.$$

Since this is summable in n, the Borel–Cantelli lemma yields $\sup_{x\in B_{K^n}} \omega(B_a(x)) \le n$ a.s. eventually. For $R \in (1,\infty)$, take $n_R \in \mathbb{N}$ such that $K^{n_R-1} < R \le K^{n_R}$. Then $\lim_{R\to\infty} n_R = \infty$ and

$$\limsup_{R\to\infty} (\log R)^{-1} \sup_{x\in B_R} \omega\big(B_a(x)\big) \le \lim_{R\to\infty} \frac{n_R}{\log R} = (\log K)^{-1} \quad \mathbf{P}\text{-a.s.},$$

and we complete the proof letting $K \to \infty$. □

Next we show that the number of points in neighborhoods with radii decreasing sufficiently fast to 0 are bounded by a constant. Recall the notation $\mathcal{P} = \{x \in \mathbb{R}^d : \omega(\{x\}) = 1\}$ for the support of ω.

Lemma 4.4. *Fix $k \in \mathbb{N}$ and a function $g : (0,\infty) \to (0,\infty)$. Let $R(t), r(t) \in (0,\infty)$ satisfy*

$$R(t) \to \infty, \quad r(t) \to 0 \quad \text{and} \quad R(t)r(t)^k \sim g(t)^{-\frac{1}{d}} \quad \text{as } t \to \infty. \tag{4.7}$$

Assume that $R(t)$ is eventually non-decreasing, $r(t)$ is eventually non-increasing and $\sum_{n=1}^{\infty} g(2^n)^{-1} < \infty$. Then, for $N_{\mathcal{Y}}^{(r)}$ as in (3.2) and $\mathcal{P}_R = \mathcal{P} \cap B_R$,

$$\limsup_{t\to\infty} \sup_{|x|\le R(t)} \omega\big(B_{r(t)}(x)\big) \le k \quad \text{and} \quad \limsup_{t\to\infty} N^{(r(t))}_{\mathcal{P}_{R(t)}} \le k \quad \mathbf{P}\text{-a.s.} \tag{4.8}$$

Proof. Applying (4.2) and our assumptions we get, for some constant $c > 0$ and all n large enough,

$$\mathbf{P}\Big(\sup_{x\in B_{R(2^{n+1})}} \omega\big(B_{r(2^n)}(x)\big) \ge k+1\Big) \le cg\big(2^n\big)^{-1}. \tag{4.9}$$

Now the Borel–Cantelli lemma implies that $\sup_{x\in B_{R(2^{n+1})}} \omega(B_{r(2^n)}(x)) \le k$ almost surely for all large enough n, and the first inequality in (4.8) follows by interpolation and monotonicity. To see that the second inequality follows from the first, note that, for any $R, r > 0$,

$$\big\{\exists \mathcal{C} \in \mathfrak{C}^{(r)}_{\mathcal{P}_R} : N_{\mathcal{C}} \ge k+1\big\} \subset \big\{\exists x \in B_R : \omega\big(B_{2kr}(x)\big) \ge k+1\big\}. \qquad \square$$

The following corollary is immediate from (4.5).

Corollary 4.5. *Fix $n \in \mathbb{N}$ and let $R(t), r(t) \in (0,\infty)$ satisfy $r(t) \to 0$, $R(t)r(t)^k \to \infty$ as $t \to \infty$. Then*

$$\lim_{t\to\infty} \mathbf{P}\big(\exists x \in B_{R(t)} : \omega\big(B_{r(t)}(x)\big) = k+1\big) = 1. \tag{4.10}$$

The next lemma is needed for the results on the lim sup-asymptotic.

Lemma 4.6. *Fix $k \in \mathbb{N}$. Let $R(t), r(t), g(t) \in (0,\infty)$ satisfy (4.7) and $\sum_{n\geq 1} g(2^n)^{-1} = \infty$. Then*

$$\limsup_{t\to\infty} \sup_{|x|\leq R(t)} \omega\big(B_{r(t)}(x)\big) \geq k+1 \qquad \mathbf{P}\text{-}a.s. \tag{4.11}$$

Proof. Let $A_n := B_{R(2^n)-r(2^n)} \setminus B_{R(2^{n-1})+r(2^n)}$, $n \in \mathbb{N}$. Using (4.5), our assumptions on $R(t)$, $r(t)$ and $1 - \mathrm{e}^{-x} \sim x$ as $x \to 0$, we find a constant $c > 0$ such that

$$\sum_{n\in\mathbb{N}} \mathbf{P}\Big(\sup_{x\in A_n} \omega\big(B_{r(2^n)}(x)\big) \geq k+1\Big) \geq c \sum_{n=1}^{\infty} g\big(2^n\big)^{-1} = \infty.$$

Noting that $\sup_{x\in A_n} \omega(B_{r(\beta^n)}(x))$, $n \in \mathbb{N}$, are independent random variables, the second Borel–Cantelli lemma yields the result. □

In the remaining lemmata we investigate the lim inf behaviour.

Lemma 4.7. *Fix $k \in \mathbb{N}$. Let $R(t), r(t) \in (0,\infty)$ such that $R(t) \to \infty$ and $r(t) \to 0$ eventually monotonically as $t \to \infty$. Assume that*

$$c := \liminf_{t\to\infty} \frac{(R(t)r(t)^k)^d}{\log\log t} > \frac{2^d(k+1)!}{|B_1|}, \tag{4.12}$$

and furthermore that, for any $\varepsilon > 0$, there exists $\beta \in (1,\infty)$ such that $\inf_{n\in\mathbb{N}} \frac{R(\beta^n)}{R(\beta^{n+1})} \geq 1-\varepsilon$. Then

$$\liminf_{t\to\infty} \sup_{x\in B_{R(t)}} \omega\big(B_{r(t)}(x)\big) \geq k+1 \quad \mathbf{P}\text{-}a.s. \tag{4.13}$$

Proof. For $\varepsilon > 0$ satisfying $\delta := c(1-\varepsilon)^d|B_1|/(2^d(k+1)!) - 1 > 0$, choose $\beta \in (1,\infty)$ as in the statement. By (4.5) and our assumptions on R, r, for all n large enough,

$$\mathbb{P}\Big(\sup_{x\in B_{R(\beta^n)}} \omega\big(B_{r(\beta^{n+1})}(x)\big) \leq k\Big) \leq \exp\bigg\{-\frac{|B_1|(1-\varepsilon)^d}{2^d(k+1)!} R\big(\beta^{n+1}\big)^d r\big(\beta^{n+1}\big)^{kd} \mathrm{e}^{-|B_r(\beta^{n+1})|}\bigg\} \leq n^{1+\delta/2}.$$

Now (4.13) follows by the Borel–Cantelli lemma, interpolation and monotonicity. □

We state next an improvement of (4.2). For $D \subset \mathbb{R}^d$ and $r > 0$, we denote by

$$\vartheta_r(D) := \min\bigg\{n \in \mathbb{N} : \exists z_1,\dots,z_n \in \mathbb{R}^d, D \subset \bigcup_{i=1}^n \big(z_i + [0,r]^d\big)\bigg\} \tag{4.14}$$

the minimum number of boxes of side-length r needed to cover D.

Lemma 4.8. *For any $k, m \in \mathbb{N}$, any measurable $D_1,\dots,D_m \subset \mathbb{R}^d$, and any $r_1,\dots,r_m \in (0,\infty)$,*

$$\mathbf{P}\Big(\sup_{1\leq i\leq m} \sup_{x\in D_i} \omega\big(B_{r_i}(x)\big) \leq k\Big) \geq \prod_{i=1}^m \bigg(1 - \frac{(2r_i)^d|B_{2r_i}|^k}{(k+1)!}\bigg)^{\vartheta_{r_i}(D)}. \tag{4.15}$$

Proof. We first note that $\sup_{x\in D_i} \omega(B_{r_i}(x))$, $1 \leq i \leq m$, is a family of associated random variables (cf. [22, Proposition 4], see also [13, Theorem 5.1]), i.e.,

$$\mathbf{P}\Big(\sup_{1\leq i\leq m} \sup_{x\in D_i} \omega\big(B_{r_i}(x)\big) \leq k\Big) \geq \prod_{i=1}^m \mathbf{P}\Big(\sup_{x\in D_i} \omega\big(B_{r_i}(x)\big) \leq k\Big). \tag{4.16}$$

Consider the case $m = 1$, and write $D = D_1$, $r = r_1$. Then, with $z_1, \dots, z_{\widehat{m}} \in \mathbb{R}^d$ as in (4.14),

$$\mathbf{P}\Big(\sup_{x\in D} \omega\big(B_r(x)\big) \le k\Big) \ge \mathbf{P}\Big(\sup_{i=1}^{\vartheta_r(D)} \sup_{x\in z_i+[0,r]^d} \omega\big(B_r(x)\big) \le k\Big) \ge \bigg(1 - (2r)^d \frac{|B_{2r}|^k}{(k+1)!}\bigg)^{\vartheta_r(D)} \tag{4.17}$$

by (4.16) and (4.2). Now (4.15) follows from (4.16)–(4.17). □

The following lemma uses ideas from [9, Lemma 5.2].

Lemma 4.9. *Let $k \ge 2$ and $R(t), r(t) > 0$ satisfy*

$$R(t) \sim t^{\frac{k}{k-1}} (\log\log t)^{-\frac{1}{d(k-1)}}, \qquad r(t) \sim t^{-\frac{1}{k-1}} (\log\log t)^{\frac{1}{d(k-1)}} \quad \text{as } t \to \infty. \tag{4.18}$$

Let $b_n > 0$, $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} \big(2^{n-1} b_n^k\big)^d < \frac{(k+1)!}{(2^d|B_1|)^{k+1}}. \tag{4.19}$$

Let $\rho > 0$ and $z(t) := \lfloor \rho \log\log t \rfloor$. Then

$$\liminf_{t\to\infty} \sup_{n=1}^{z(t)} \sup_{x\in B_{2^{n-1}R(t)}} \omega\big(B_{b_n r(t)}(x)\big) \le k \quad \mathbf{P}\text{-a.s.} \tag{4.20}$$

Proof. We may assume that $\rho > 1$. Abbreviate $\ell(t) := \log\log t$. Take $t_0 \in (1, \infty)$ large enough such that $\ell(t_0) > 1$, and define a growing sequence $(t_j)_{j\in\mathbb{N}_0}$ recursively by

$$t_j = t_{j-1} \exp\big\{\rho\ell(t_{j-1})\big\}, \quad j \in \mathbb{N}. \tag{4.21}$$

For $j \in \mathbb{N}$ and $n \in \mathbb{N}$, set

$$A_{j,n} := B_{2^{n-1}R(t_j)} \setminus B_{R(t_{j-1})}, \qquad X_j := \sup_{n=1}^{z(t_j)} \sup_{x\in B_{2^{n-1}R(t_j)}} \omega\big(B_{b_n r(t_j)}(x)\big),$$

$$\hat{X}_j := \sup_{n=1}^{z(t_j)} \sup_{x\in A_{j,n}} \omega\big(B_{b_n r(t_j)}(x)\big), \qquad \check{X}_j := \sup_{x\in B_{R(t_{j-1})}} \omega\big(B_{b_1 r(t_j)}(x)\big).$$

Note that $X_j = \max(\check{X}_j, \hat{X}_j)$. Thus it will be sufficient to show that $\mathbf{P}$-a.s. both

$$\limsup_{j\to\infty} \check{X}_j \le k \quad \text{and} \quad \liminf_{j\to\infty} \hat{X}_j \le k. \tag{4.22}$$

To obtain the first inequality, note that by (4.2) there exists a constant $c \in (0, \infty)$ such that

$$\mathbf{P}(\check{X}_j \ge k+1) \le c\big(R(t_{j-1}) r(t_j)^k\big)^d \le 2c\mathrm{e}^{-\frac{k}{k-1}\{d\rho\ell(t_{j-1}) - \log\ell(t_{j-1})\}} \le 2c\mathrm{e}^{-\frac{dk\rho(1-\varepsilon_j)}{k-1}\ell(t_{j-1})} \tag{4.23}$$

for all large enough j, where we used $\ell(t_j) \le 2\ell(t_{j-1})$, and $\varepsilon_j \to 0$ as $j \to \infty$. To conclude with the Borel–Cantelli lemma, note that (4.23) is summable in j since, for any $\alpha > 1$,

$$\infty > \int_{t_0}^{\infty} \frac{1}{t}\mathrm{e}^{-\alpha\ell(t)}\,\mathrm{d}t = \sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} \frac{1}{t}\mathrm{e}^{-\alpha\ell(t)}\,\mathrm{d}t \ge \sum_{j=0}^{\infty} \log(t_{j+1}/t_j)\mathrm{e}^{-\alpha\ell(t_{j+1})} > \sum_{j=0}^{\infty} \mathrm{e}^{-\alpha\ell(t_{j+1})}.$$

Consider now the second inequality in (4.22). By (4.15), for all $j \in \mathbb{N}$,

$$\log \mathbf{P}(\hat{X}_j \le k) \ge \log \mathbf{P}(X_j \le k) \ge \sum_{n=1}^{z(t_j)} \vartheta_{b_n r(t_j)}(B_{2^{n-1}R(t_j)}) \log\bigg(1 - \frac{(2b_n r(t_j))^d |B_{2b_n r(t_j)}|^k}{(k+1)!}\bigg).$$

Using (4.18), $\log(1-x) \sim -x$ as $x \to 0$ and $\vartheta_r(B_R) \sim |B_R|/r^d$ as $r \downarrow 0$, $R \uparrow \infty$, we obtain

$$\log \mathbf{P}(\hat{X}_j \le k) \ge -(1+\varepsilon_j)\frac{(2^d|B_1|)^{k+1}}{(k+1)!}\ell(t_j)\sum_{n=1}^{\infty}\big(2^{n-1}b_n^k\big)^d > -(1-\delta)\ell(t_j)$$

for large j by (4.19), where $\varepsilon_j \to 0$ as $j \to \infty$ and $\delta \in (0,1)$. Since, for some $c \in (0,\infty)$,

$$\infty = \int_{t_0}^{\infty}\frac{1}{t}\mathrm{e}^{-\ell(t)}\,\mathrm{d}t \le \sum_{j=0}^{\infty}\rho\ell(t_j)\mathrm{e}^{-\ell(t_j)} \le c\sum_{j=0}^{\infty}\mathrm{e}^{-(1-\delta)\ell(t_j)},$$

we deduce $\sum_{j=0}^{\infty}\mathbf{P}(\hat{X}_j \le k) = \infty$. Note now that, since $R(t_{j+1}) \gg 2^{z(t_j)}R(t_j)$ as $j \to \infty$, there exists a $j_0 \in \mathbb{N}$ such that both $(\hat{X}_{2j})_{j\ge j_0}$ and $(\hat{X}_{2j+1})_{j\ge j_0}$ are families of independent random variables, allowing us to conclude the proof with an application of the second Borel–Cantelli lemma. □

5. Proof of the main theorems

Throughout this section, we fix $d \ge 3$ arbitrary in general, but $d = 3$ whenever we treat the renormalized potential $\overline{V}$. We also fix $\theta \in (0, \frac{h_d}{2}]$ and set $k = k_\theta = \lfloor \frac{h_d}{\theta} \rfloor$, where $h_d = (d-2)^2/8$.

The section is organized as follows. In Sections 5.1–5.2 below, we provide some preparatory results concerning respectively bounds for principal eigenvalues and estimates of the error introduced when substituting either $V^{(\mathfrak{K})}$ or $\overline{V}$ by a truncated potential $V^{(a)}$. Section 5.3 contains the proofs of Theorems 1.7 and 1.2 as well as of the upper bounds for Theorems 1.4, 1.5 and 1.6. Corresponding lower bounds are proved first in the special case of truncated potentials in Section 5.4. The proofs of Theorem 1.9 is given in Section Section 5.5, as well as the completion of the proofs of Theorems 1.4, 1.5, 1.6 and 1.10. Finally, Theorems 1.8 and 1.3 are proved in Section 5.6.

5.1. *Bounds for principal eigenvalues*

In order to make use of the upper bound given in Theorem 3.1, we study the almost-sure asymptotics as $R \to \infty$ of $\Lambda_{\mathcal{Y}}^{(\theta,a,r)}$ defined in (3.2) with $\mathcal{Y} = \mathcal{P}_R = \mathcal{P} \cap B_R$. To this end, we will combine the multipolar Hardy inequality from Section 2.4 and the Poissonian asymptotics stated in Section 4.

Fix $0 < a < r < \infty$ and recall (3.1)–(3.2). For $s > 0$, write

$$\big\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\big\} \subset \big\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s, N_{\mathcal{P}_R}^{(r)} \le k+1\big\} \cup \big\{N_{\mathcal{P}_R}^{(r)} \ge k+2\big\}. \tag{5.1}$$

The second event in (5.1) can be controlled by

$$\big\{N_{\mathcal{P}_R}^{(r)} \ge k+2\big\} \subset \big\{\exists x \in B_R\colon \omega\big(B_{(k+1)r}(x)\big) \ge k+2\big\}.$$

To control the first event in (5.1), write, for $\mathcal{C} \in \mathfrak{C}_{\mathcal{P}_R}^{(r)}$,

$$\Gamma(\mathcal{C}) := \inf\big\{s > 0\colon B_s(\mathcal{P}_R \cap \mathcal{C}) \text{ is connected}\big\}$$

and set

$$c_{\mathrm{mp}} := (k+1)\frac{\pi^2 + 3\theta}{2}. \tag{5.2}$$

Note that $\lambda_{\mathcal{C}} = 0$ for each $\mathcal{C} \in \mathfrak{C}_{\mathcal{P}_R}^{(r)}$ with $N_{\mathcal{C}} \le k$ due to the Hardy inequality (cf. (2.49)) and Remark 2.1. Then, by the multipolar Hardy inequality (2.51),

$$\begin{aligned}\big\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s, N_{\mathcal{P}_R}^{(r)} \le k+1\big\} &\subset \big\{\exists \mathcal{C} \in \mathfrak{C}_{\mathcal{P}_R}^{(r)}\colon \lambda_{\mathcal{C}} > s, N_{\mathcal{C}} = k+1\big\}\\ &\subset \big\{\exists \mathcal{C} \in \mathfrak{C}_{\mathcal{P}_R}^{(r)}\colon \Gamma(\mathcal{C})^2 < c_{\mathrm{mp}}/s, N_{\mathcal{C}} = k+1\big\}\\ &\subset \left\{\exists \text{ distinct } y_1,\dots,y_{k+1} \in \mathcal{P}_R\colon \bigcup_{i=1}^{k+1} B_{(c_{\mathrm{mp}}/s)^{1/2}}(y_i) \text{ is connected}\right\}\\ &\subset \big\{\exists x \in B_R\colon \omega\big(B_{2k(c_{\mathrm{mp}}/s)^{1/2}}(x)\big) \ge k+1\big\}.\end{aligned}$$

Combining these results, we get

$$\big\{\Lambda^{(\theta,a,r)}_{\mathcal{P}_R} > s\big\} \subset \Big\{\sup_{x\in B_R} \omega\big(B_{2k(c_{\mathrm{mp}}/s)^{1/2}}(x)\big) \geq k+1\Big\} \cup \Big\{\sup_{x\in B_R} \omega\big(B_{(k+1)r}(x)\big) \geq k+2\Big\}. \tag{5.3}$$

With this inclusion at hand, we derive next several consequences of the results from Section 4.

Lemma 5.1. *Let $0 < a < r < R < \infty$ and $\theta \in (0, \frac{h_d}{2}]$. There exists a constant $c \in (0,\infty)$ depending only on θ and d such that, for all $s > cR^{-2}$,*

$$\mathbf{P}\big(\Lambda^{(\theta,a,r)}_{\mathcal{P}_R} > s\big) \leq cR^d\big(s^{-\frac{d}{2}k} + r^{d(k+1)}\big). \tag{5.4}$$

Proof. We can assume $c \geq 4k^2 c_{\mathrm{mp}}$. Using (5.3), (4.2) and $2k(c_{\mathrm{mp}}/s)^{1/2} < R$, we get

$$\begin{aligned}\mathbf{P}\big(\Lambda^{(\theta,a,r)}_{\mathcal{P}_R} > s\big) &\leq \mathbf{P}\Big(\sup_{x\in B_R} \omega\big(B_{2k(c_{\mathrm{mp}}/s)^{1/2}}(x)\big) \geq k+1\Big) + \mathbf{P}\Big(\sup_{x\in B_R} \omega\big(B_{(k+1)r}(x)\big) \geq k+2\Big) \\ &\leq |B_1|(2R)^d \frac{(|B_1|(4k(c_{\mathrm{mp}}/s)^{1/2})^d)^k}{(k+1)!} + |B_1|\big(2(k+1)R\big)^d \frac{(|B_1|(2(k+1)r)^d)^{k+1}}{(k+2)!}.\end{aligned}$$

This shows (5.4). □

Lemma 5.2. *Fix $\alpha > (k+1)^{-1}$ and let $R(t) \to \infty$, $g(t) \to \infty$ as $t \to \infty$. For any $c_1, c_2 \in (0,\infty)$,*

$$\lim_{t\to\infty} \frac{\Lambda_{R(t)}}{g(t)R(t)^{2/k}} = 0 \quad \textit{in probability, where } \Lambda_R := \Lambda^{(\theta,c_1R^{-\alpha},c_2R^{-\alpha})}_{\mathcal{P}_R}. \tag{5.5}$$

If moreover $\sum_{n=1}^{\infty} g(2^n)^{-dk/2} < \infty$, R is regularly varying with positive index, and g is either eventually non-decreasing or slowly varying, then (5.5) holds almost surely.

Proof. (5.5) follows directly from (5.4). For the second statement, note that, for $n \in \mathbb{N}$, (5.3) yields

$$\big\{\Lambda_{R(t)} > n^{-1}g(t)R(t)^{2/k}\big\} \subset \Big\{\sup_{|x|\leq R(t)} \omega\big(B_{r(t)}(x)\big) \geq k+1\Big\} \cup \Big\{\sup_{|x|\leq R(t)} \omega\big(B_{c_2(k+1)R(t)^{-\alpha}}(x)\big) \geq k+2\Big\},$$

where $r(t) = 2k\sqrt{c_{\mathrm{mp}}n}\, g(t)^{-1/2}R(t)^{-1/k}$. By [3, Theorem 1.5.3], we may assume that $R(t)$ and $r(t)$ are eventually monotone. By (4.8), $\limsup_{t\to\infty} \Lambda_{R(t)}/(g(t)R(t)^{2/k}) \leq 1/n$ almost surely, and to conclude we let $n \uparrow \infty$. □

The following lemma will be used in the proof of Theorem 1.6.

Lemma 5.3. *Let $R(t)$ as in (4.18) and $\alpha > (k+1)^{-1}$. For $n \geq 1$, let $a_n(t) := (2^{n-1}R(t))^{-\alpha}$ and, for $A > 0$,*

$$\Lambda_{t,n} := \Lambda^{(\theta,a_n(t),5a_n(t))}_{\mathcal{P}_{2^{n-1}R(t)+1}}, \qquad \Theta_{t,n}(A) := \Lambda_{t,n} - A\mathbb{1}_{\{n\geq 2\}}4^{n-1}t^{\frac{2}{k-1}}(\log\log t)^{-\frac{2}{d(k-1)}}.$$

Let $\rho > 0$ and $z(t) := \lfloor \rho \log\log t\rfloor$. For any $A > 0$, there exists a $C = C(A,k,d) \in (0,\infty)$ such that

$$\liminf_{t\to\infty} t^{-\frac{2}{k-1}}(\log\log t)^{\frac{2}{d(k-1)}} \max_{n=1}^{z(t)} \Theta_{t,n}(A) \leq C \quad \mathbf{P}\textit{-almost surely.} \tag{5.6}$$

Proof. Fix $A, \rho > 0$. Let $\chi_k := (k+1)!/(|B_2|^{k+1})$ as in (4.19) and c_{mp} as in (5.2), and pick

$$C > (4A) \vee \left(\frac{(4k\sqrt{c_{\mathrm{mp}}})^k}{\sqrt{A}\chi_k^{1/d}}\right)^{2/(k-1)}. \tag{5.7}$$

Define $b_n > 0$, $n \in \mathbb{N}$ by setting

$$b_1 = 2k(c_{\mathrm{mp}}/C)^{1/2} \quad \text{and} \quad b_n = 2k(c_{\mathrm{mp}}/C)^{1/2}\big(1 + (A/C)4^{n-1}\big)^{-\frac{1}{2}}, \quad n \geq 2.$$

Let us verify that b_n satisfies (4.19). Indeed, setting $n_0 := \lfloor \log_4(C/A) \rfloor \geq 1$, we may write

$$\begin{aligned}
&\big(2k(c_{\mathrm{mp}}/C)^{1/2}\big)^{-kd} \sum_{n=1}^{\infty} \big(2^{n-1} b_n^k\big)^d \\
&\quad \leq \sum_{n=1}^{n_0+1} 2^{(n-1)d} + (C/A)^{kd/2} \sum_{n=n_0+2}^{\infty} 2^{-(k-1)d(n-1)} \\
&\quad \leq 2^{n_0 d+1} + 2(C/A)^{kd/2} 2^{-(k-1)dn_0} \leq 2^{kd}(C/A)^{d/2} \\
&\quad = \big(2k(c_{\mathrm{mp}}/C)^{1/2}\big)^{-kd} \big((4k\sqrt{c_{\mathrm{mp}}})^k A^{-1/2} C^{-(k-1)/2}\big)^d < \big(2k(c_{\mathrm{mp}}/C)^{1/2}\big)^{-kd} \chi_k
\end{aligned}$$

by our choice of C. This shows (4.19). Let now $r(t) := t^{-\frac{1}{k-1}} (\log\log t)^{\frac{1}{d(k-1)}}$ and use (5.3) to write

$$\begin{aligned}
&\Big\{ \max_{n=1}^{z(t)} \Theta_{t,n}(A) \leq C t^{\frac{2}{k-1}} (\log\log t)^{-\frac{2}{d(k-1)}} \Big\} \\
&\quad = \bigcap_{n=1}^{z(t)} \big\{ \Lambda_{t,n} \leq r(t)^{-2} (2k\sqrt{c_{\mathrm{mp}}})^2 b_n^{-2} \big\} \\
&\quad \supset \bigcap_{n=1}^{z(t)} \Big\{ \sup_{|x| \leq 2^{n-1} R(t)+1} \omega\big(B_{b_n r(t)}(x)\big) \leq k \Big\} \cap \Big\{ \sup_{|x| \leq 2^{n-1} R(t)+1} \omega\big(B_{5(1+k)a_n(t)}(x)\big) \leq k+1 \Big\}) \\
&\quad \supset \Big\{ \max_{n=1}^{z(t)} \sup_{|x| \leq 2^{n-1}(R(t)+1)} \omega\big(B_{b_n r(t)}(x)\big) \leq k \Big\} \cap \Big\{ \sup_{|x| \leq 2^{z(t)}(R(t)+1)} \omega\big(B_{5(1+k)a_0(t)}(x)\big) \leq k+1 \Big\}.
\end{aligned}$$

The first event on the right-hand side above occurs a.s. infinitely often by (4.20), and the second event occurs eventually by (4.8). This yields (5.6). □

5.2. *Truncation of Poisson potentials*

In this section, we control the error that occurs when replacing either an attenuated potential $V^{(\mathfrak{K})}$ as in (1.11) or the renormalized potential $\overline{V}$ by a truncated potential $V^{(a)} = V^{(\mathfrak{K}_a)}$, where $\mathfrak{K}_a(x) = |x|^{-2} \mathbb{1}_{\{|x| \leq a\}}$. We first state an auxiliary result.

Lemma 5.4. *Let* $R \mapsto \mathfrak{K}(R) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *satisfy*

$$C := \limsup_{R \to \infty} \big\| \mathfrak{K}(R) \big\|_{L^\infty(\mathbb{R}^d)} < \infty \quad \text{and} \quad \limsup_{R \to \infty} \int_{\mathbb{R}^d} \sup_{|x| \leq 1} \big| \mathfrak{K}(R)(x-y) \big| \, \mathrm{d}y < \infty. \tag{5.8}$$

Then

$$\limsup_{R \to \infty} \frac{\log\log R}{\log R} \max_{|x| \leq R} \big| V^{(\mathfrak{K}(R))}(x) \big| \leq dC \quad \mathbf{P}\text{-a.s.} \tag{5.9}$$

Proof. Using (5.8) and [8, Proposition 2.7], one can follow the proof of [15, Lemma 2.6]. □

Our comparison lemma reads as follows.

Lemma 5.5. *Let* $d \geq 3$, $\mathfrak{K} \in \mathcal{K}$ *and* $a \in (0, \infty)$. *Then,* $\mathbf{P}$*-almost surely for all bounded* $D \subset \mathbb{R}^d$,

$$\sup_{x \in D \setminus \mathcal{P}} \big| V^{(\mathfrak{K})}(x) - V^{(a)}(x) \big| < \infty. \tag{5.10}$$

Moreover, for any $R \mapsto a_R > 0$ *such that* $\limsup_{R \to \infty} a_R < \infty$,

$$\lim_{R \to \infty} \frac{a_R^2}{\log R} \sup_{|x| \leq R:\, x \notin \mathcal{P}} \big| V^{(\mathfrak{K})}(x) - V^{(a_R)}(x) \big| = 0 \quad \mathbf{P}\text{-a.s.} \tag{5.11}$$

When $d = 3$, (5.10)–(5.11) *hold with either* $\overline{V}$ *or* $|\overline{V}|$ *in place of* $V^{(\mathfrak{K})}$.

Proof. Note that, for all $x \in \mathbb{R}^d \setminus \mathcal{P}$ and all $a > 0$,

$$\big|V^{(\mathfrak{K})}(x) - V^{(a)}(x)\big| \le \omega\big(B_a(x)\big) \sup_{|z|\le a}\left|\mathfrak{K}(z) - \frac{1}{|z|^2}\right| + a^{-2}\int a^2\big|\mathfrak{K}(x-y)\big|\mathbb{1}_{\{|x-y|>a\}}\omega(\mathrm{d}y),$$

proving (5.10). With $a = a_R$ as in the statement, (5.11) follows by (1.10), Corollary 4.3 and Lemma 5.4.

Consider now $d = 3$. Fix $a > 0$ and let $\alpha : \mathbb{R}^+ \to [0,1]$ be a smooth truncation function with $\alpha(\lambda) = 1$ on $[0,1]$, $\alpha(\lambda) = 0$ for $\lambda \ge 3$ and $-1 \le \alpha'(\lambda) \le 0$. Decompose $\overline{V} = \overline{V}_1 + \overline{V}_2$ by setting

$$\overline{V}_1(x) := \int_{\mathbb{R}^3} \frac{1-\alpha(a^{-1}|x-y|)}{|x-y|^2}\big[\omega(\mathrm{d}y) - \mathrm{d}y\big], \qquad \overline{V}_2(x) := \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\big[\omega(\mathrm{d}y) - \mathrm{d}y\big]. \tag{5.12}$$

Note that $\overline{V}_1$ exactly matches $\overline{V}_{a,\varepsilon}$ in [9, Eq. (3.5)] with $\varepsilon = 1$. Thus, by [9, Eq. (3.6)],

$$\sup_{x\in D} \overline{V}_1(x) < \infty \quad \mathbf{P}\text{-a.s.} \tag{5.13}$$

for any bounded $D \subset \mathbb{R}^3$ while, by Lemma 3.3 in the same reference,

$$\lim_{R\to\infty} (\log R)^{-1} \sup_{|x|\le R} \big|\overline{V}_1(x)\big| = 0 \quad \mathbf{P}\text{-a.s.} \tag{5.14}$$

Furthermore, since the integrand in the definition of $\overline{V}_2$ is in $L^1(\mathbb{R}^3)$, we may separate the integration in terms of $\omega(\mathrm{d}y)$ and $\mathrm{d}y$ using [8, Proposition 2.5], i.e.,

$$\overline{V}_2(x) = \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\omega(\mathrm{d}y) - \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\,\mathrm{d}y. \tag{5.15}$$

The second integral above is a finite constant independent of x. For the first integral, we get

$$\int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\omega(\mathrm{d}y) = V^{(b)}(x) + \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\mathbb{1}_{\{|x-y|\ge b\}}\omega(\mathrm{d}y) \tag{5.16}$$

for any $b \in (0,a]$. Now note that, since $\alpha(\lambda) = 0$ for $\lambda \ge 3$,

$$\sup_{x\in D}\int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\mathbb{1}_{\{|x-y|\ge b\}}\omega(\mathrm{d}y) \le b^{-2}\sup_{x\in D}\omega\big(B_{3a}(x)\big) < \infty \quad \mathbf{P}\text{-a.s.} \tag{5.17}$$

Combining (5.13) and (5.15)–(5.17) with $b = a$, we obtain (5.10) with $\overline{V}$ in place of $V^{(\mathfrak{K})}$. To obtain (5.11), take $a > \limsup_{R\to\infty} a_R$, $b = a_R$, $D = B_R$ and apply additionally (5.14) and Corollary 4.3. The statement for $|\overline{V}|$ is obtained with the inequality $||x| - |y|| \le |x-y|$, $x, y \in \mathbb{R}$. □

5.3. *The upper bounds*

We introduce next some notation and a key result that will be used in the following proofs of the upper bounds. Fix $\alpha \in (\frac{1}{k+1}, \frac{1}{k})$ and recall (3.2). Throughout the section, we will use the notation

$$\Lambda_R := \Lambda^{(\theta, R^{-\alpha}, 5R^{-\alpha})}_{\mathcal{P}\cap B_{R+2}}, \qquad R > 0. \tag{5.18}$$

Note that, for any $a \in (0,1]$ and $z \in B_{R+1}$, $V^{(a)}(z) = V^{(a)}_{\mathcal{P}\cap B_{R+2}}(z)$.

In the proofs below, we will work with certain radii sequences $R_n(t) \in [1,\infty)$, $n \in \mathbb{N}$, $t > 0$, which we keep arbitrary for now. According to the choice of $R_n(t)$, we introduce

$$a_n(t) = R_n(t)^{-\alpha}, \qquad r_n(t) = 5a_n(t), \qquad R_0(t) = 8(k+1)r_1(t), \tag{5.19}$$

as well as the hitting times

$$\hat{\tau}_n(x) = \hat{\tau}_n(t,x) := \tau_{B^{\mathrm{c}}_{R_n(t)}(x)} = \inf\big\{s \ge 0\colon W_s \notin B_{R_n(t)}(x)\big\}, \qquad n \in \mathbb{N}_0, x \in \mathbb{R}^d. \tag{5.20}$$

Fix $\mathfrak{K} \in \mathcal{K}$ and define the error terms

$$S_n(t) := \sup_{z \in B_{R_n(t)+1}} \left| V^{(\mathfrak{K})}(z) - V^{(a_n(t))}(z) \right|, \qquad \overline{S}_n(t) := \sup_{z \in B_{R_n(t)+1}} \left| \overline{V}(z) - V^{(a_n(t))}(z) \right|. \tag{5.21}$$

Recall (3.2) and define, for $t > 0$,

$$\zeta_t^\circ := \inf\left\{ n \in \mathbb{N}\colon\ N^{(r_n(t))}_{\mathcal{P} \cap B_{R_n(t)+2}} \le k+1 \text{ and } R_{n-1}(t) \ge 8 r_n(t)(k+1) \right\} \tag{5.22}$$

and, for $x \in \mathbb{R}^d \setminus \mathcal{P}$,

$$\zeta_t(x) := \zeta_t^\circ \vee \inf\left\{ n \in \mathbb{N}\colon x \notin B_{r_n(t)}(\mathcal{P}) \right\}. \tag{5.23}$$

When $x = 0$, we write $\hat{\tau}_n$, ζ_t instead of $\hat{\tau}_n(0)$, $\zeta_t(0)$. The next lemma provides conditions on $R_n(t)$ guaranteeing the finiteness of ζ_t°, $\zeta_t(x)$.

Lemma 5.6. *Let $R_n(t) \ge 1$, $n \in \mathbb{N}$, $t > 0$ satisfy*

$$\forall t_2 > t_1 > 0\colon \quad \lim_{n \to \infty} R_n(t_1) = \infty \quad \textit{and} \quad \liminf_{n \to \infty} \inf_{t,s \in [t_1,t_2]} \frac{R_n(t)}{R_n(s)} > 0. \tag{5.24}$$

Then, $\mathbf{P}$*-almost surely for all $x \in \mathbb{R}^d \setminus \mathcal{P}$ and $t > 0$, $1 \le \zeta_t^\circ \le \zeta_t(x) < \infty$ and there exist $0 \le t_0^\circ \le t_0(x) < \infty$ such that $\zeta_t^\circ = 1$ for all $t \ge t_0^\circ$ and $\zeta_t(x) = 1$ for all $t \ge t_0(x)$.*

Proof. If (5.24) holds, then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} \sup_{\epsilon \le t \le \epsilon^{-1}} r_n(t) = 0, \qquad \lim_{n \to \infty} \inf_{\epsilon \le t \le \epsilon^{-1}} R_{n-1}(t) = \infty \quad \text{and} \quad \limsup_{n \to \infty} \sup_{\epsilon \le t \le \epsilon^{-1}} N^{(r_n(t))}_{\mathcal{P} \cap B_{2R_n(t)}} \le k$$

almost surely by (4.8) (with $R(t) = t$). Similar estimates hold when $n = 1$, $t \to \infty$. □

We are now ready to state the key estimate of the section.

Lemma 5.7. *There exist deterministic constants $\chi \in [1, \infty)$ and $c_1, c_2 \in (0, \infty)$ such that, for any $R_n(t) \ge 1$ satisfying* (5.24)*, the following holds* $\mathbf{P}$*-almost surely for all $t \ge 0$. Let*

$$\gamma_n(t) \ge \max\left\{ 2\Lambda_{R_n(t)}, \chi R_n(t)^{2\alpha} \right\}, \quad n \in \mathbb{N}. \tag{5.25}$$

Then, for all $\mathfrak{K} \in \mathcal{K}$ and all $0 \le A_1 < A_2 \le \infty$,

$$\begin{aligned} &\int_{B_1} \mathbb{E}_x\left[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \le t < \tau_{B^c_{A_2}}(x)\}} \right] \mathrm{d}x \\ &\quad \le \int_{B_1} \mathbb{E}_x\left[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \le t < \hat{\tau}_{\zeta_t^\circ - 1}(x)\}} \right] \mathrm{d}x \\ &\quad\quad + \sqrt{1 \vee |\mathcal{P} \cap B_1|} \sum_{\substack{n \ge \zeta_t^\circ: \\ A_1 < R_n(t) < A_2}} c_1 \mathrm{e}^{t\theta S_n(t) + t\gamma_n(t) + \log^+(\sqrt{t} R_n(t)^\alpha) - c_2 R_{n-1}(t) \min\{t^{-1} R_{n-1}(t), \sqrt{\gamma_n(t)}\}}. \end{aligned} \tag{5.26}$$

Moreover, for all $x \in B_1 \setminus \mathcal{P}$,

$$\begin{aligned} &\mathbb{E}_x\left[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \le t < \tau_{B^c_{A_2}}(x)\}} \right] \\ &\quad \le \mathbb{E}_x\left[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \le t < \hat{\tau}_{\zeta_t(x) - 1}(x)\}} \right] \\ &\quad\quad + \sum_{\substack{n \ge \zeta_t(x): \\ A_1 < R_n(t) < A_2}} c_1 \mathrm{e}^{t\theta S_n(t) + t\gamma_n(t) + \log^+(\sqrt{t} R_n(t)^\alpha) - c_2 R_{n-1}(t) \min\{t^{-1} R_{n-1}(t), \sqrt{\gamma_n(t)}\}}. \end{aligned} \tag{5.27}$$

The same bounds hold with $|V^{(\mathfrak{K})}|$ in place of $V^{\mathfrak{K}}$ and, when $d = 3$, with $|\overline{V}|$, $\overline{S}_n(t)$ in place of $V^{(\mathfrak{K})}$, $S_n(t)$.

Proof. Fix $x \in B_1$. Splitting according to whether $t \geq \hat{\tau}_{\zeta_t^\circ - 1}(x)$ or not and, if so, according to which $n \geq \zeta_t^\circ$ satisfies $\hat{\tau}_{n-1}(x) \leq t < \hat{\tau}_n(x)$, we may decompose

$$\begin{aligned}
&\mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \leq t < \tau_{B^c_{A_2}}(x)\}}\big] \\
&\quad \leq \mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \leq t < \hat{\tau}_{\zeta_t^\circ - 1}(x)\}}\big] \\
&\qquad + \sum_{\substack{n \geq \zeta_t^\circ : \\ A_1 < R_n(t) < A_2}} \mathbb{E}_x\left[\exp\left(\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\hat{\tau}_{n-1}(x) \leq t < \hat{\tau}_n(x)\}}\right].
\end{aligned} \tag{5.28}$$

Set $\mathcal{Y}_n(t) := \mathcal{P} \cap B_{R_n(t)+2}$ and note that, if $t < \hat{\tau}_n(x)$, then $V^{(a_n(t))}(W_s) = V^{(a_n(t))}_{\mathcal{Y}_n(t)}(W_s)$ for all $s \in [0,t]$. Recalling (5.21), we see that the integral over B_1 of the series in (5.28) is bounded by

$$\sum_{n \geq \zeta_t^\circ :\, A_1 < R_n(t) < A_2}^{\infty} \mathrm{e}^{\theta t S_n(t)} \int_{B_1} \mathbb{E}_x\left[\exp\left(\int_0^t \theta V^{(a_n(t))}_{\mathcal{Y}_n(t)}(W_s)\,\mathrm{d}s\right)\mathbb{1}\{\tau_{B^c_{R_{n-1}(t)}}(x) \leq t\}\right]\mathrm{d}x. \tag{5.29}$$

We wish to apply the bound (3.8) to the terms of (5.29), with parameters chosen as follows:

$$\mathcal{Y} = \mathcal{Y}_n(t), \qquad R = R_{n-1}(t), \qquad a = a_n(t), \qquad r = r_n(t), \qquad \gamma = \gamma_n(t).$$

It is straightforward to verify that we may (deterministically) choose $\chi \in [1, \infty)$ large enough such that, with this choice of parameters, whenever $\gamma_n(t)$ satisfies (5.25) and $n \geq \zeta_t^\circ$, the function $L = L(\mathcal{Y}_n(t), \theta, a_n(t), r_n(t), \gamma_n(t))$ in (3.4) is uniformly bounded by a deterministic constant, and

$$c_* a_n(t)\sqrt{\gamma_n(t)} > 2\log(2L) \quad \text{(in particular, } \varrho < 1/2\text{)}. \tag{5.30}$$

We may thus apply (3.8) to the terms in (5.29), obtaining

$$\begin{aligned}
&\int_{B_1} \mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(a_n(t))}_{\mathcal{Y}_n(t)}(W_s) - \gamma_n(t)\,\mathrm{d}s}\mathbb{1}\{\tau_{B^c_{R_{n-1}(t)}}(x) \leq t\}\big]\,\mathrm{d}x \\
&\quad \leq c_1 \mathfrak{N}^{(r_n(t))}_{\mathcal{Y}_n(t)}(B_1)\big\{\sqrt{t}R_n(t)^\alpha \mathrm{e}^{-\frac{c_2 R_{n-1}(t)^2}{t}} + \mathrm{e}^{-c_2\sqrt{\gamma_n(t)}R_{n-1}(t)}\big\}
\end{aligned}$$

for some deterministic constants $c_1, c_2 \in (0, \infty)$, where we also used $\sup_{x>0} x\mathrm{e}^{-x^2/b} \leq \sqrt{b/2}$ for any $b > 0$. Together with the bound (5.29) and $\mathfrak{N}^{(r_n(t))}_{\mathcal{Y}_n(t)}(B_1) \leq \sqrt{1 \vee |\mathcal{P} \cap B_1|}$, this shows (5.26).

To show (5.27), we split instead according to whether $\hat{\tau}_{\zeta_t(x)-1} \leq t$ or not and, if so, according to which $n \geq \zeta_t(x)$ satisfies $\hat{\tau}_{n-1}(x) \leq t < \hat{\tau}_n(x)$. Arguing analogously as before, we obtain

$$\begin{aligned}
&\mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \leq t < \tau_{B^c_{A_2}}(x)\}}\big] \\
&\quad \leq \mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{B^c_{A_1}}(x) \leq t < \hat{\tau}_{\zeta_t(x)-1}(x)\}}\big] \\
&\qquad + \sum_{\substack{n \geq \zeta_t(x) : \\ A_1 < R_n(t) < A_2}} \mathrm{e}^{\theta t S_n(t)}\mathbb{E}_x\left[\exp\left(\int_0^t \theta V^{(a_n(t))}_{\mathcal{Y}_n(t)}(W_s)\,\mathrm{d}s\right)\mathbb{1}\{\tau_{B^c_{R_{n-1}(t)}}(x) \leq t\}\right].
\end{aligned}$$

Note that, when $n \geq \zeta_t(x)$, $x \notin B_{r_n(t)}(\mathcal{Y}_n(t))$ and $R_{n-1}(t) \geq 8r_n(t)N^{(r_n(t))}_{\mathcal{Y}_n(t)}$. Applying (3.7) to the terms in (5.29), (5.27) follows analogously as for (5.26). The proofs for $|V^{(\mathfrak{K})}|$, $|\overline{V}|$ are identical. □

5.3.1. *Proof of Theorems 1.2 and 1.7*

Proof. We start with Theorem 1.2. Fix $\mathfrak{K} \in \mathcal{K}$. Note that $v_\theta^{(\mathfrak{K})}(t,x)$ is non-decreasing in t, and thus it will be sufficient to show that, for each $y \in \mathbb{R}^d$ and each $t > 0$, $\mathbf{P}$-almost surely,

$$\int_{B_1(y)} v_\theta^{(\mathfrak{K})}(t,x)\,\mathrm{d}x < \infty \quad \text{and} \quad v_\theta^{(\mathfrak{K})}(t,x) < \infty \ \forall x \in B_1(y) \setminus \mathcal{P}.$$

By the homogeneity of ω, it is enough to consider $y = 0$. To this end, we will apply Lemma 5.7 with

$$R_n(t) = 1 \vee \big(2^{n-1}t\big)^{\frac{k+1}{k-1}}, \qquad \gamma_n(t) = \max\big\{2\Lambda_{R_n(t)}, \chi R_n^{2/k}\big\}, \qquad A_1 = 0, A_2 = \infty. \tag{5.31}$$

Note that $R_n(t)$ satisfies (5.24) and, by (5.19) and the choice of α, (5.25) is fulfilled.

Let us first control the first term in the right-hand side of (5.26). Recall Lemma 5.6, (5.21) and write

$$\int_{B_1} \mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta|V^{(\mathfrak{K})}|(W_s)\,\mathrm{d}s}\mathbb{1}_{\{t<\hat{\tau}_{\zeta_t^\circ-1}(x)\}}\big]\,\mathrm{d}x \le \mathrm{e}^{\theta t S_{\zeta_t^\circ-1}(t)} \int_{B_1} \mathbb{E}_x\big[\mathrm{e}^{\int_0^t \theta V^{(a_t)}_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s}\mathbb{1}_{\{t<\hat{\tau}_{\zeta_t^\circ-1}(x)\}}\big]\,\mathrm{d}x, \tag{5.32}$$

where $\mathcal{Y}_t := \mathcal{P} \cap B_{R_{\zeta_t^\circ-1}(t)+2}$, $a_t = a_{\zeta_t^\circ-1}(t)$ and we used that, if $t < \hat{\tau}_{\zeta_t^\circ-1}(x)$, then $V^{(a_t)}(W_s) = V^{(a_t)}_{\mathcal{Y}_t}(W_s)$ for $0 \le s \le t$. Since $\mathcal{P} \in \mathscr{Y}$ a.s., the multipolar Hardy inequality in [4], Theorem 1, implies that $\mathbf{P}(\forall R > 0\colon \lambda_{\max}(\mathbb{R}^d, \theta V_{\mathcal{P}\cap B_R}) < \infty) = 1$, and thus (5.32) is finite by (5.10) and (2.19). For the first term in the right-hand side of (5.27), let $\varepsilon_x := \frac{1}{2}\operatorname{dist}(x, \mathcal{P})$ and fix $\hat{a}_x \in (0, \varepsilon_x)$ to write

$$\begin{aligned}
&\mathbb{E}_x\bigg[\exp\bigg\{\int_0^t \theta\big|V^{(\mathfrak{K})}\big|(W_s)\,\mathrm{d}s\bigg\}\mathbb{1}\big\{t < \hat{\tau}_{\zeta_t(x)-1}(x)\big\}\bigg] \\
&\quad \le \exp\bigg\{\theta t \sup_{z\in B_{R_{\zeta_t-1}(t)}(x)} \big|V^{(\mathfrak{K})}(z) - V^{(\hat{a}_x)}(z)\big|\bigg\}\mathbb{E}_x\bigg[\exp\bigg\{\int_0^t \theta V^{(\hat{a}_x)}_{\mathcal{Y}_t(x)}(W_s)\,\mathrm{d}s\bigg\}\mathbb{1}\big\{t < \hat{\tau}_{\zeta_t(x)-1}(x)\big\}\bigg],
\end{aligned}$$

where $\mathcal{Y}_t(x) := \mathcal{P} \cap B_{R_{\zeta_t-1}(t)+\hat{a}_x}(x)$, so the latter is again finite by (5.10) and (2.27).

Consider now the series in (5.26) and (5.27). The term for $n = 1$ is bounded by

$$c_1 \exp\big\{\theta t S_1(t) + t\gamma_1(t) + \log^+\big(\sqrt{t}R_1(t)^\alpha\big)\big\}. \tag{5.33}$$

For some constants $c_3, c_4 > 0$ and using $\gamma_n(t) > R_n(t)^{2/k}$, we bound the terms for $n \ge 2$ by

$$c_3 \exp\big\{\theta t S_n(t) + t\gamma_n(t) - c_4\big(2^n t\big)^{\frac{k+1}{k-1}}\big\} \tag{5.34}$$

Note that, $\mathbf{P}$-almost surely and as $n \to \infty$, $S_n(t) = o(R_n(t)^{2\alpha}\log R_n(t))$ by (5.11) and, for any $\beta > 1/k$, $\gamma_n(t) = o(R_n(t)^{2\beta})$ by Lemma 5.2 (applied with $R(t) = t$). Therefore, the sum over $n \ge 2$ of (5.34) is finite. This finishes the proof of Theorem 1.2. The proof of Theorem 1.7 is completely analogous. □

Lemma 5.7 and the estimates in the proof above also allow us to show the following.

Lemma 5.8. *For any $\mathfrak{K} \in \mathcal{K}$ and any $\gamma \in (0, \infty)$ such that $\gamma(k-1) > 2/d$,*

$$\lim_{t\to\infty} \mathbb{E}_0\bigg[\exp\bigg\{\int_0^t \big|V^{(\mathfrak{K})}\big|(W_s)\,\mathrm{d}s\bigg\}\mathbb{1}\bigg\{\sup_{0\le s\le t}|W_s| \ge (\log t)^\gamma t^{\frac{k}{k-1}}\bigg\}\bigg] = 0 \quad \mathbf{P}\text{-a.s.} \tag{5.35}$$

When $d = 3$, the same holds with $\overline{V}$ in place of $V^{(\mathfrak{K})}$.

Proof. Take $R_n(t)$, $\gamma_n(t)$ as in (5.31). Using the bound (5.34) for the nth term of the series in (5.27), we bound the expectation in (5.35) by

$$c_3 \sum_{n=n_t+1}^{\infty} \exp\big\{t\theta S_n(t) + t\gamma_n(t) - c_4\big(2^n t\big)^{\frac{k+1}{k-1}}\big\}, \qquad \text{where } n_t := \left\lfloor \frac{\gamma(k-1)\log_2\log t}{k} \right\rfloor. \tag{5.36}$$

Let $\beta \in (0, 1)$ with $2/d < k\beta < \gamma(k-1)$. Then $g(t) := (\log t)^\beta$ satisfies the conditions of Lemma 5.2, implying that $\gamma_n(t) = o(g(2^{n-1}t)R_n(t)^{2/k})$. Using additionally the bound $S_n(t) = o(R_n(t)^{2\alpha}\log R_n(t))$ which holds almost surely by Lemma 5.5, we may check that, when t is large enough, the exponents of the summands in (5.36) are smaller than $-c_3(2^n t)^{(k+1)/(k-1)}$ for some constant $c_3 > 0$, from which (5.35) follows. The statement for $\overline{V}$ is obtained analogously, considering $\overline{S}_n(t)$. □

5.3.2. *Upper bound in Theorem 1.4*

Proof of (1.17). Let $R_n(t)$, $\gamma_n(t)$ as in (5.31). Recall (5.23) and that, by Lemma 5.6, $\zeta_t = 1$ a.s. for all large enough t. Using Lemma 5.7, Lemma 5.8 and the estimates (5.33)–(5.34) for the terms of the series in (5.27), we see that it is enough to show that, as $t \to \infty$,

$$\log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\hat{\tau}_0\}}\right] = \mathcal{O}(t) \quad \mathbf{P}\text{-a.s.}, \tag{5.37}$$

$$g(t)^{-1}t^{-\frac{2}{k-1}}\{\theta S_1(t) + \gamma_1(t)\} \to 0 \quad \text{in probability,} \tag{5.38}$$

and that, for any $\rho > 0$,

$$g(t)^{-1}t^{-\frac{k+1}{k-1}} \max_{2\le n\le\lfloor\rho\log\log t\rfloor}\left\{t\theta S_n(t) + t\gamma_n(t) - c_4\left(2^n t\right)^{\frac{k+1}{k-1}}\right\} \to 0 \quad \text{in probability.} \tag{5.39}$$

We start with (5.37). When t is large, $R_0(t) < 1$. Let $\varepsilon_0 := \frac{1}{2}\operatorname{dist}(0,\mathcal{P})$ and set $\hat{a}_0 := \frac{1}{2}(\varepsilon_0 \wedge 1)$. Note that, when $t < \tau_{B_1^c}$, $V^{(\hat{a}_0)}(W_s) = V^{(\hat{a}_0)}_{\mathcal{P}_2}(W_s)$ for $0 \le s \le t$. Moreover, $\lambda_{\max}(B_2, \theta V^{(\hat{a}_0)}_{\mathcal{P}_2}) < \infty$ and $\operatorname{dist}(B_2^c, \mathcal{P}_1) > 0$. Applying Lemma 5.5 and (2.27), we find (random) constants $C_1, C_2 \in (0,\infty)$ such that, a.s. for all large enough t, the expectation in (5.37) is at most

$$\mathrm{e}^{\theta t \sup_{x\in B_2}|V^{(\mathfrak{K})}(x)-V^{(\hat{a}_0)}(x)|}\mathbb{E}_0\left[\exp\left\{\int_0^t \theta V^{(\hat{a}_0)}_{\mathcal{P}_2}(W_s)\,\mathrm{d}s\right\}\mathbb{1}_{\{t<\tau_{B_2^c}\}}\right] \le C_1\mathrm{e}^{C_2 t}.$$

Thus (5.37) follows, and we move to (5.38)–(5.39). Note first that, by (5.11),

$$\lim_{t\to\infty}\max_{1\le n\le\lfloor\rho\log\log t\rfloor} t^{-\frac{2}{k-1}}S_n(t) = 0 \quad \mathbf{P}\text{-a.s.}, \tag{5.40}$$

and (5.38) follows by Lemma 5.2. To control the remaining term in (5.39), fix $\varepsilon > 0$ and estimate with a union bound

$$\mathbf{P}\left(\max_{n\ge1}\frac{t\gamma_n(t) - c_4(2^n t)^{\frac{k+1}{k-1}}}{g(t)t^{\frac{k+1}{k-1}}} > \varepsilon\right) \le \sum_{n=1}^{\infty}\mathbf{P}\left(\gamma_n(t) > t^{\frac{2}{k-1}}\left(\varepsilon g(t) + c_4\left(2^n\right)^{\frac{k+1}{k-1}}\right)\right). \tag{5.41}$$

Now note that, since $g(t) \to \infty$, when t is large enough, it is impossible to have $\gamma_n(t) = \chi R_n(t)^{2/k}$ if $\gamma_n(t)$ satisfies the inequality in (5.41); thus in this case $\gamma_n(t) = 2\Lambda_{R_n(t)}$. Applying (5.4), we obtain deterministic constants $c_5, c_6 \in (0,\infty)$ such that (5.41) is at most

$$\begin{aligned}
&c_5\sum_{n=1}^{\infty}R_n(t)^d\left\{t^{-\frac{dk}{k-1}}\left(\varepsilon g(t) + c_4\left(2^n\right)^{\frac{k+1}{k-1}}\right)^{-\frac{dk}{2}} + R_n(t)^{-\alpha d(k+1)}\right\} \\
&\quad\le c_5\sum_{n=1}^{\infty}\left(\frac{(2^n)^{\frac{2}{k-1}}}{\varepsilon g(t) + c_4(2^n)^{1+\frac{2}{k-1}}}\right)^{\frac{dk}{2}} + c_6\sum_{n=1}^{\infty}\left(2^n t\right)^{-\frac{dk}{k-1}(\alpha(k+1)-1)} \xrightarrow{t\to\infty} 0
\end{aligned}$$

since $\alpha > (k+1)^{-1}$. Together with (5.40), this shows (5.39), completing the proof of (1.17). □

5.3.3. *Upper bound in Theorem 1.5*

Before we proceed to the proof, we recall that, when ℓ is slowly varying, $\ell(\lambda r) \sim \ell(r)$ as $r \to \infty$ uniformly over λ in compact subsets (cf. [3, Theorem 1.2.1]). It is then straightforward to translate the integrability condition in (1.18) into a summability condition, namely,

$$\int_1^{\infty}\frac{\mathrm{d}t}{t\ell(t)} < \infty \quad \text{if and only if} \sum_{n=0}^{\infty}\ell\left(2^n\right)^{-1} < \infty. \tag{5.42}$$

Proof of the upper bound in (1.18). Fix $t \mapsto \ell(t)$ slowly varying with $\int_1^{\infty}\frac{\mathrm{d}r}{r\ell(r)} < \infty$, and set

$$R_n(t) := \left(2^{n-1}t\right)^{\frac{k}{k-1}}\ell\left(2^{n-1}t\right)^{\frac{1}{d(k-1)}}, \qquad \gamma_n(t) := \max\left(2\Lambda_{R_n(t)}, R_n(t)^{\frac{2}{k}}\ell\left(2^{n-1}t\right)^{\frac{2}{dk}}\right). \tag{5.43}$$

When t is large, $R_n(t) \ge 1$, $\zeta_t = 1$ and (5.25) holds, so we may apply Lemma 5.7.

Note that (5.37) still holds as $R_0(t)$ is given by (5.19), and thus the first term in the right-hand side of (5.27) is controlled. For the term with $n=1$, note that, by Lemma 5.2 (with $g(t)=\ell(t)^{\frac{2}{dk}}$), (5.42) and (5.11), (5.38) holds almost surely with $g(t)=\ell(t)^{\frac{2}{d(k-1)}}$. It is thus enough to show that, $\mathbf{P}$-a.s.,

$$\limsup_{t\to\infty}\sum_{n\geq 2}\exp\Big(\theta t S_n(t)+t\gamma_n(t)+\log\big(tR_n(t)^\alpha\big)-c_2R_{n-1}(t)\min\Big\{\frac{R_{n-1}(t)}{t},\sqrt{\gamma_n(t)}\Big\}\Big)<\infty. \tag{5.44}$$

To this end, use the slow variation of ℓ to find a constant $c>0$ such that, for t large enough,

$$R_{n-1}(t)\min\Big\{\frac{R_{n-1}(t)}{t},\sqrt{\gamma_n(t)}\Big\}\geq c2^{n-1}t\big(2^{n-1}t\big)^{\frac{2}{k-1}}\ell\big(2^{n-1}t\big)^{\frac{2}{d(k-1)}},\quad n\geq 2. \tag{5.45}$$

Applying Lemma 5.2 (with t substituted by $2^{n-1}t$), we obtain $c'\in(0,\infty)$ such that

$$t\gamma_n(t)\leq tc'R_n(t)^{\frac{2}{k}}\ell\big(2^{n-1}t\big)^{\frac{2}{dk}}=c't\big(2^{n-1}t\big)^{\frac{2}{k-1}}\ell\big(2^{n-1}t\big)^{\frac{2}{d(k-1)}},\quad n\geq 2.$$

Noting that, by (5.11), the remaining terms are of lower order, we can choose $n_0=n_0(c,c')$ sufficiently large so that, for any $n\geq n_0$, the n-th term in the series in (5.44) is bounded by the exponential of $-c_3t(2^{n-1}t)^{\frac{2}{k-1}}\ell(2^{n-1}t)^{\frac{2}{d(k-1)}}$ for some constant $c_3>0$, showing (5.44). This finishes the proof. □

5.3.4. *Upper bound in Theorem 1.6*

Proof. Let $A:=c_2\wedge 1$ with c_2 as in Lemma 5.7, and set

$$R_n(t):=2^{n-1}t^{\frac{k}{k-1}}(\log\log t)^{-\frac{1}{d(k-1)}},\qquad \gamma_n(t):=\max\Big\{2\Lambda_{R_n(t)},\frac{A^2}{4}t^{-2}R_{n-1}(t)^2\Big\}.$$

Applying Lemma 5.7, Lemma 5.8, Lemma 5.5 and (5.37), we see that we only need to find a constant $C^{\inf}\in(0,\infty)$ such that, for all $\rho>0$,

$$\liminf_{t\to\infty}\max_{1\leq n\leq\lfloor\rho\log\log t\rfloor}\frac{\gamma_n(t)-A\mathbb{1}_{\{n\geq 2\}}t^{-1}R_{n-1}(t)\min\{t^{-1}R_{n-1}(t),\sqrt{\gamma_n(t)}\}}{(\log\log t)^{-\frac{2}{d(k-1)}}t^{\frac{2}{k-1}}}\leq C^{\inf}. \tag{5.46}$$

Abbreviate $s_t:=t^{\frac{2}{k-1}}(\log\log t)^{-\frac{2}{d(k-1)}}$. Note that $t^{-1}R_{n-1}(t)=2^{n-1}\sqrt{s_t}$ and, since $A\leq 1$,

$$\min\big\{t^{-1}R_{n-1}(t),\sqrt{\gamma_n(t)}\big\}\geq\frac{A}{2}t^{-1}R_{n-1}(t)=\frac{A}{2}2^{n-1}\sqrt{s_t}. \tag{5.47}$$

Hence, as $t\to\infty$,

$$\frac{A^2}{4}t^{-2}R_{n-1}(t)^2-A\mathbb{1}_{\{n\geq 2\}}t^{-1}R_{n-1}(t)\min\big\{t^{-1}R_{n-1}(t),\sqrt{\gamma_n(t)}\big\}\ll s_t. \tag{5.48}$$

On the other hand, Lemma 5.3 provides a constant $C\in(1,\infty)$ and a subsequence $t_j\to\infty$ as $j\to\infty$ such that, for all $j\in\mathbb{N}$ and all $1\leq n\leq\lfloor\rho\log\log t_j\rfloor$,

$$\Lambda_{R_n(t_j)}\leq\frac{A^2}{4}\mathbb{1}_{\{n\geq 2\}}t_j^{-2}R_{n-1}(t_j)^2+Cs_{t_j}. \tag{5.49}$$

Now (5.47)–(5.49) and the definition of $\gamma_n(t)$ imply (5.46) with $C^{\inf}=2C$. □

5.4. *The lower bounds*

In this section, we will prove the lower bounds in Theorems 1.4, 1.5 and 1.6 for the truncated potentials $V^{(a)}=V^{(\mathfrak{K}_a)}$ where $\mathfrak{K}_a(x)=|x|^{-2}\mathbb{1}_{\{|x|\leq a\}}$. The proof of the theorems will be finished in Section 5.5 after the proof of Theorem 1.9. The following lemma will be used in all the proofs of this section:

Lemma 5.9. *There exists $c \in (0,1]$ such that the following holds* $\mathbf{P}$*-almost surely. Fix $a \in (0,\infty)$ and let $R(t), r(t)$ satisfy $\mathrm{e}^{-t} \ll r(t) \ll 1 \ll R(t)$ as $t \to \infty$, and $R(t)r(t) \le \sqrt{ct}$ for all t large enough. Define*

$$\mathcal{A}_t := \big\{\exists x \in B_{R(t)} \colon \omega\big(B_{r(t)}(x)\big) \ge k+1\big\}. \tag{5.50}$$

Then, for all large enough t, on $\mathcal{A}_t$,

$$\log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \ge \frac{ct}{r(t)^2} - 2\sqrt{c}\frac{R(t)}{r(t)} - \mathcal{O}(t). \tag{5.51}$$

Proof. On $\mathcal{A}_t$, we pick $x_t \in B_{R(t)}$ such that $\omega(B_{r(t)}(x_t)) \ge k+1$. Choose distinct $y_1, \dots, y_{k+1} \in \mathcal{P} \cap B_{r(t)}(x_t)$ (for example, according to lexicographical order) and set $\mathcal{Y}_t := \{y_1, \dots, y_k\}$. Clearly $V^{(a)} \ge V^{(a)}_{\mathcal{Y}_t}$, and $V_{\mathcal{Y}_t} - V^{(a)}_{\mathcal{Y}_t} \le \#\mathcal{Y}_t a^{-2} = (k+1)a^{-2} =: c_0$. Thus

$$\log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \ge -\theta c_0 t + \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s\right)\right]. \tag{5.52}$$

Let now K, c_1, c_2 as in Lemma 2.8, and set $c := c_2 \wedge 1$. Write, for $0 \le t_0 \le t$,

$$\mathbb{E}_0\big[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s}\big] \ge \mathbb{E}_0\big[\mathrm{e}^{\int_{t_0}^t \theta V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{W_s \in B_{Kr(t)}(x_t) \forall s \in [t_0,t]\}}\big]. \tag{5.53}$$

Denote by $p(0,y,t) = (2\pi t)^{-d/2}\mathrm{e}^{-|y|^2/(2t)}$ the probability density of Brownian motion at time t started at 0. Applying the Markov property, we see that (5.53) equals

$$\begin{aligned}
&\int_{B_{Kr(t)}(x_t)} p(0,y,t_0)\mathbb{E}_y\big[\mathrm{e}^{\theta\int_0^{t-t_0} V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^{\mathrm{c}}_{Kr(t)}(x_t)} > t-t_0\}}\big]\,\mathrm{d}y \\
&\quad \ge (2\pi t)^{-\frac{d}{2}} \mathrm{e}^{-\frac{R(t)^2}{t_0}} \int_{B_{Kr(t)}(x_t)} \mathbb{E}_y\big[\mathrm{e}^{\int_0^{t-t_0} \theta V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^{\mathrm{c}}_{Kr(t)}(x_t)} > t-t_0\}}\big]\,\mathrm{d}y,
\end{aligned} \tag{5.54}$$

where we used $|y|^2 \le (|x_t| + Kr(t))^2 \le 2R(t)^2$ for large t. The integral above can be identified with the integral in (2.43) with $a = r(t)$, $x = x_t$, implying that (5.54) is at least

$$c_1 (2\pi t)^{-\frac{d}{2}} r(t)^d \exp\left\{-\frac{R(t)^2}{t_0} + c_2(t-t_0)r(t)^{-2}\right\}. \tag{5.55}$$

Maximizing the exponent over $t_0 \in (0,t)$ we obtain $t_0 = R(t)r(t)/\sqrt{c_2} \le t$, which yields

$$\log \mathbb{E}_0\big[\mathrm{e}^{\theta\int_0^t V_{\mathcal{Y}_t}(W_s)\,\mathrm{d}s}\big] \ge -2\sqrt{c_2}\frac{R(t)}{r(t)} + \frac{c_2 t}{r(t)^2} + \log\big(c_1(2\pi t)^{-\frac{d}{2}} r(t)^d\big). \tag{5.56}$$

Now (5.51) follows from (5.52), (5.56) and our assumptions on $R(t)$, $r(t)$. □

With Lemma 5.9 at hand, we are ready to complete the proofs of Theorems 1.4, 1.5 and 1.6 in the special case of the truncated kernels $\mathfrak{K} = \mathfrak{K}_a$.

Lemma 5.10. *For any $a \in (0,\infty)$, (1.16) holds with $\mathfrak{K}(x) = \mathfrak{K}_a(x) = |x|^{-2}\mathbb{1}_{\{|x| \le a\}}$.*

Proof. We may assume that $g(t) = t^{o(1)}$ as $t \to \infty$. Take c as in Lemma 5.9, let $A > 0$ and

$$R(t) = \frac{1}{2}\sqrt{\frac{1}{2}Ag(t)^{-1}}\, t^{\frac{k}{k-1}}, \qquad r(t) = \sqrt{\frac{1}{2}cA^{-1}g(t)t^{-\frac{1}{k-1}}}.$$

Lemma 5.9 implies that, on the event $\mathcal{A}_t$ defined in (5.50),

$$g(t)t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \ge A - o(1).$$

Since $\lim_{t\to\infty} \mathbf{P}(\mathcal{A}_t) = 1$ by Corollary 4.5 and A is arbitrary, we conclude (1.16). □

Lemma 5.11. *For any $a \in (0,\infty)$, the lower bound in (1.18) holds with $\mathfrak{K}(x) = \mathfrak{K}_a(x) = |x|^{-2}\mathbb{1}_{\{|x|\le a\}}$.*

Proof. Let $\ell(t) \ge 1$ be slowly varying with $\int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} = \infty$. Fix $A > 0$ and set $R(t) := \sqrt{\frac{A}{8}} t^{\frac{k}{k-1}} \ell(t)^{\frac{1}{d(k-1)}}$, $r(t) := \sqrt{\frac{1}{2} c A^{-1}} t^{-\frac{1}{k-1}} \ell(t)^{-\frac{1}{d(k-1)}}$ with c as in Lemma 5.9. On $\mathcal{A}_t$,

$$\ell(t)^{-\frac{2}{d(k-1)}} t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \ge A - o(1).$$

Now Lemma 4.6 and (5.42) provide a sequence $t_j \to \infty$ as $j \to \infty$ such that $\mathcal{A}_{t_j}$ occurs, and we conclude by taking $A \uparrow \infty$. □

Lemma 5.12. *For any $a \in (0,\infty)$, the lower bound in (1.19) holds with $\mathfrak{K}(x) = \mathfrak{K}_a(x) = |x|^{-2}\mathbb{1}_{\{|x|\le a\}}$.*

Proof. Let c as in Lemma 5.9 and pick $\mu, \nu > 0$ satisfying

$$\mu\nu < \sqrt{c} \quad \text{and} \quad \left(\mu\nu^k\right)^d > \frac{2^d (k+1)!}{|B_1|}. \tag{5.57}$$

Set $R(t) := \mu t^{\frac{k}{k-1}} (\log\log t)^{-\frac{1}{d(k-1)}}$ and $r(t) := \nu t^{-\frac{1}{k-1}} (\log\log t)^{\frac{1}{d(k-1)}}$. By Lemma 5.9, on $\mathcal{A}_t$,

$$(\log\log t)^{\frac{2}{d(k-1)}} t^{\frac{k+1}{k-1}} \log \mathbb{E}_0\left[\mathrm{e}^{\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s}\right] \ge \frac{c}{\nu^2} - 2\sqrt{c}\frac{\mu}{\nu} - o(1).$$

On the other hand, by Lemma 4.7, $\mathcal{A}_t$ occurs for all large enough t, and thus we may take $C_{\inf}$ as the maximum of $c\nu^{-2} - 2\sqrt{c}\mu/\nu$ over $\mu, \nu > 0$ satisfying (5.57), which turns out to be

$$C_{\inf} := \frac{c^{\frac{k}{k-1}}(k-1)}{(k+1)^{\frac{k+1}{k-1}}} \left(\frac{|B_1|}{2^d (k+1)!}\right)^{\frac{2}{d(k-1)}}.$$

□

5.5. *Proof of Theorems 1.9, 1.4, 1.5, 1.6, and 1.10*

Proof of Theorem 1.9. Let $I_t^{(\mathfrak{K})} := \exp \int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s$ and $I_t^{(a)} := \exp \int_0^t \theta V^{(a)}(W_s)\,\mathrm{d}s$. Let $R(t) := (\log t)^\gamma \times t^{k/(k+1)}$ with γ as in Lemma 5.8, and put $S_t := \sup_{x \in B_{R(t)} \setminus \mathcal{P}} |V^{(\mathfrak{K})}(x) - V^{(a)}(x)|$. Then

$$\log \mathbb{E}_0\left[I_t^{(a)} \mathbb{1}\{\tau_{B^{\mathrm{c}}_{R(t)}} \ge t\}\right] - \theta t S_t \le \log \mathbb{E}_0\left[I_t^{(\mathfrak{K})} \mathbb{1}\{\tau_{B^{\mathrm{c}}_{R(t)}} \ge t\}\right] \le \log \mathbb{E}_0\left[I_t^{(a)} \mathbb{1}\{\tau_{B^{\mathrm{c}}_{R(t)}} \ge t\}\right] + \theta t S_t.$$

Now, by Lemma 5.12 and (5.11) with $a_R \equiv a$, $t S_t / \log \mathbb{E}_0[I_t^{(a)}]$ tends a.s. to 0. To obtain (1.22), note that, by Lemma 5.8, $\mathbb{E}_0[I_t^{(a)} \mathbb{1}\{\tau_{B^{\mathrm{c}}_{R(t)}} \ge t\}] \sim \mathbb{E}_0[I_t^{(a)}]$, and the same can be concluded for $I_t^{(\mathfrak{K})}$ by taking into account the first inequality above. The proofs for $|V^{(\mathfrak{K})}|$, $\overline{V}$ or $|\overline{V}|$ are identical. □

As anticipated, this allows us to finally give the:

Proof of Theorems 1.4, 1.5 and 1.6. For $\mathfrak{K} = \mathfrak{K}_a$, the results follow from the upper bounds in Section 5.3 together with Lemmas 5.10–5.12. The other cases then follow from Theorem 1.9. □

Proof of Theorem 1.10. Follows from Theorems 1.9, 1.7, 1.2, 1.4, 1.5 and 1.6. □

5.6. *Proof of Theorems 1.3 and 1.8*

Proof. We follow [8, Proposition 1.6]. We start with Theorem 1.3. Fix $\mathfrak{K} \in \mathcal{K}$. Let $m \in \mathbb{N}$ and $F_m := |\theta V^{(\mathfrak{K})}| \wedge m \in L^\infty(\mathbb{R}^d)$. By Proposition 2.2, $w_m(t,x) := \mathbb{E}_x[\mathrm{e}^{\int_0^t F_m(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B^{\mathrm{c}}_m} > t\}}]$ satisfies

$$w_m(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) F_m(y) w_m(s,y)\,\mathrm{d}y\,\mathrm{d}s, \quad (t,x) \in (0,\infty) \times B_m \setminus \mathcal{P} \tag{5.58}$$

with $p_t(x)$ as in (2.14). Letting $m \uparrow \infty$ and applying the monotone convergence theorem, we see that (5.58) still holds true with F_m and v_m replaced by $|\theta V^{(\mathfrak{K})}|$ and $v_\theta^{(\mathfrak{K})}(t,x) = \mathbb{E}_x[\exp \int_0^t \theta |V^{\mathfrak{K}}|(W_s)\,\mathrm{d}s]$, both sides being finite almost surely by Theorem 1.2. In particular,

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\big|V^{(\mathfrak{K})}\big|(y)\big|v_\theta^{(\mathfrak{K},u_0)}(s,y)\big|\,\mathrm{d}y\,\mathrm{d}s < \infty, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}. \tag{5.59}$$

Noting that $t \mapsto \mathrm{e}^{\pm \int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s}$ are absolutely continuous, the fundamental theorem of calculus gives

$$\exp\left(\int_0^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}s\right) = 1 + \int_0^t \theta V^{(\mathfrak{K})}(W_s) \exp\left(\int_s^t \theta V^{(\mathfrak{K})}(W_s)\,\mathrm{d}u\right)\mathrm{d}s,$$

which we use to write, for all $(t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}$ and all $u_0 \in L^\infty(\mathbb{R}^d)$,

$$\begin{aligned} u_\theta^{(\mathfrak{K},u_0)}(t,x) &= \int p_t(x-y)u_0(y)\,\mathrm{d}y + \int_0^t \mathbb{E}_x\big[\theta V^{(\mathfrak{K})}(W_s)\mathrm{e}^{\int_s^t \theta V^{(\mathfrak{K})}(W_u)\,\mathrm{d}u}\big]\,\mathrm{d}s \\ &= \int p_t(x-y)u_0(y)\,\mathrm{d}y + \int_0^t \mathbb{E}_x\big[\theta V^{(\mathfrak{K})}(W_s)u_\theta^{(\mathfrak{K})}(t-s,y)\big]\,\mathrm{d}s \\ &= \int p_t(x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\theta V^{(\mathfrak{K})}(y)u_\theta^{(\mathfrak{K})}(s,y)\,\mathrm{d}y\,\mathrm{d}s, \end{aligned}$$

where we used Fubini's theorem (which is justified by (5.59)), the Markov property of W, and time reversal. The same argument works for $|V^{(\mathfrak{K})}|$, and thus we complete the proof of Theorem 1.3. Theorem 1.8 is proved analogously. □

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