

Limit fluctuations for density of asymmetric simple exclusion processes with open boundaries

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Abstract. We investigate the fluctuations of cumulative density of particles in the asymmetric simple exclusion process with respect to the stationary distribution (also known as the steady state), as a stochastic process indexed by $[0, 1]$. In three phases of the model and their boundaries within the fan region, we establish a complete picture of the scaling limits of the fluctuations of the density as the number of sites goes to infinity. In the maximal current phase, the limit fluctuation is the sum of two independent processes, a Brownian motion and a Brownian excursion. This extends an earlier result by Derrida et al. (*J. Statist. Phys.* **115** (2004) 365–382) for totally asymmetric simple exclusion process in the same phase. In the low/high density phases, the limit fluctuations are Brownian motion. Most interestingly, at the boundary of the maximal current phase, the limit fluctuation is the sum of two independent processes, a Brownian motion and a Brownian meander (or a time-reversal of the latter, depending on the side of the boundary). Our proofs rely on a representation of the joint generating function of the asymmetric simple exclusion process with respect to the stationary distribution in terms of joint moments of a Markov processes, which is constructed from orthogonality measures of the Askey–Wilson polynomials.

Résumé. Nous étudions les fluctuations de la densité de particules dans un processus d'exclusion simple asymétrique sous la distribution stationnaire (ou état stable), vues comme un processus stochastique indexé par $[0, 1]$. Pour trois des phases du modèle et à leurs frontières nous obtenons une description complète des limites d'échelles de ces fluctuations lorsque le nombre de sites tend vers l'infini. Dans la phase de courant maximal, la limite est la somme de deux processus indépendants : un mouvement brownien et une excursion brownienne. Ce résultat étend celui obtenu précédemment par Derrida et al. (*J. Statist. Phys.* **115** (2004) 365–382) pour le processus d'exclusion simple totalement asymétrique et dans la même phase. Dans les phases de fortes et faibles densités, les limites sont des mouvements browniens. De façon plus intéressante, à la frontière de la phase de courant maximal, la limite est la somme de deux processus indépendants : un mouvement brownien et un méandre brownien (ou, selon la partie de la frontière, un méandre brownien renversé en temps). Nos démonstrations reposent sur une représentation des fonctions génératrices des lois fini-dimensionnelles du processus d'exclusion simple asymétrique en termes de moments joints d'un processus de Markov construit à partir de mesures rendant orthogonaux les polynômes d'Askey–Wilson.

MSC: 60F05; 60K35

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1. Introduction and main results

1.1. Background

The asymmetric simple exclusion process (ASEP) with open boundaries in one dimension is one of the most widely investigated models for open non-equilibrium systems in the physics literature. The process models particles jumping

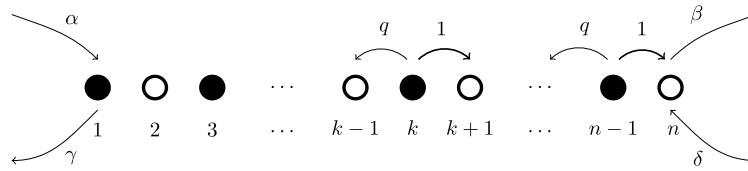


Fig. 1. Transition rates of the asymmetric simple exclusion process with open boundaries, with parameters $\alpha, \beta, \gamma, \delta, q$. Black disks represent occupied sites.

independently with hardcore repulsion over a one-dimensional lattice, which also has particles injected to the left end and removed from the right end, and an external field driving the particles towards the right direction. The ASEP, despite its simple definition, captures representative features of more complicated models, including in particular phase transitions. The model actually has its origin in modeling protein synthesis in biology [35]. In mathematics literature, the model was first investigated by Spitzer [43], see also Liggett [34, Section 3] for early developments. See more references on background, motivations and applications in the survey papers [4,15,16].

The ASEP with open boundaries is an irreducible finite-state Markov process on the state space $\{0, 1\}^n$ with parameters

$$\alpha > 0, \quad \beta > 0, \quad \gamma \geq 0, \quad \delta \geq 0, \quad \text{and} \quad 0 \leq q < 1. \tag{1.1}$$

Informally, the process models the evolution of the particles located at sites $1, \dots, n$ that can jump to the right with rate 1 and to the left with rate q , if the target site is unoccupied. Furthermore, particles arrive at site 1 (respectively, n), if empty, at rate α (respectively, δ), and leave site n (respectively, 1), if occupied, at rate β (respectively, γ). The transitions are summarized in Figure 1. For $q < 1$, particles move in an asymmetric way, with higher rate to the right than to the left; in the special case $q = 0$, particles move only to the right and the model is known as the totally asymmetric simple exclusion process (TASEP).

We let π_n denote the stationary distribution of the ASEP as a Markov process on $\{0, 1\}^n$, which is also called *the steady state* in the physics literature. We let τ_1, \dots, τ_n denote the occupations of each corresponding location: $\tau_j = 1$ if the j -th location is occupied by a particle, and $\tau_j = 0$ otherwise. All statistics of the ASEP are then expressed in terms of τ_1, \dots, τ_n .

Throughout we assume (1.1) and work with the following parameterization of the ASEP, which dates back at least to the 90s in the physics literature (e.g. [40]): set

$$\kappa_{x,y}^\pm = \frac{1}{2x} (1 - q - x + y \pm \sqrt{(1 - q - x + y)^2 + 4xy}),$$

and denote

$$A = \kappa_{\beta,\delta}^+, \quad B = \kappa_{\beta,\delta}^-, \quad C = \kappa_{\alpha,\gamma}^+, \quad \text{and} \quad D = \kappa_{\alpha,\gamma}^-. \tag{1.2}$$

By definition, $A, C \geq 0$ and it is easy to check that $-1 < B, D \leq 0$, compare [10,44].

The phase transition of the ASEP is known to be characterized by A and C only. For example, it has been known since Derrida et al. [17], Sandow [40] that the ASEP has the following three phases:

- (1) maximal current phase $A < 1, C < 1$,
- (2) low density phase $C > 1, C > A$,
- (3) high density phase $A > 1, A > C$.

Derrida et al. [21,22] distinguish also the two regions

- (1) fan region $AC < 1$,
- (2) shock region $AC > 1$.

Figure 2 illustrates the three phases and the two regions. In this paper, we restrict ourselves to the fan region and its boundary, as our approach does not work for the shock region. See [21,22] for more discussions of the properties of ASEP in the shock region.

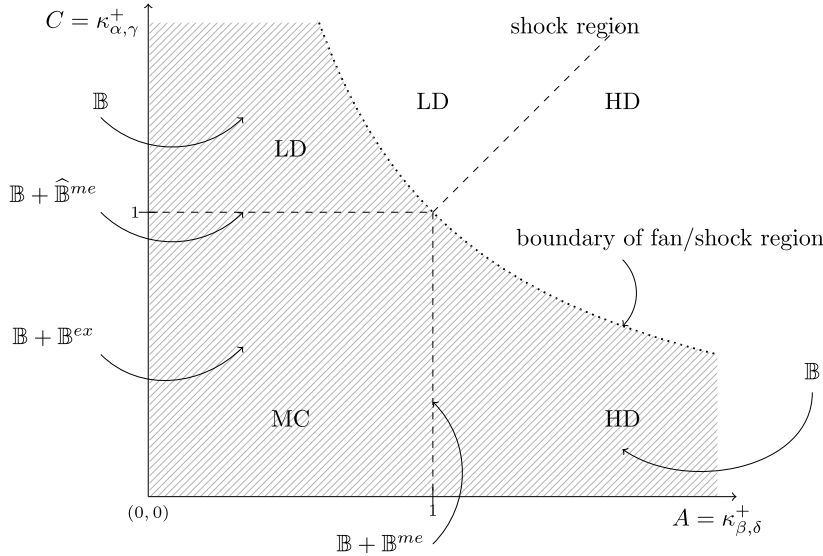


Fig. 2. A phase diagram for the limit fluctuations of density of the ASEP in the steady state. LD, HD, MC stand for low density, high density and maximal current (phases), respectively. The fan region consists of the shaded area. \mathbb{B} , \mathbb{B}^{ex} and \mathbb{B}^{me} stand for Brownian motion, excursion and meander, respectively, and $\widehat{\mathbb{B}}^{me}$ stands for the reversed Brownian meander defined by $\{\widehat{\mathbb{B}}_x^{me}\}_{x \in [0,1]} \stackrel{d}{=} \{\mathbb{B}_{1-x}^{me} - \mathbb{B}_1^{me}\}_{x \in [0,1]}$. Processes in the sums are assumed to be independent. The multiplicative constants (possibly depending on A, C) in front of processes are omitted.

Another set of commonly used parameters is the pair (ρ_a, ρ_b) with

$$\rho_a = \frac{1}{1+C} \quad \text{and} \quad \rho_b = \frac{A}{1+A}.$$

For example, the fan region and the shock region are often characterized equivalently by $\rho_a > \rho_b$ and $\rho_a < \rho_b$, respectively (e.g. [21] and [22, (1.1)]). From the point of view of modeling a non-equilibrium system with open boundaries, the two parameters represent the densities of the two reservoirs connected to the left and the right of the system. For convenience, we shall use A and C exclusively in the sequel.

We are interested in the *cumulative density function*

$$[0, 1] \ni x \mapsto \frac{1}{n} \sum_{j=1}^{\lfloor nx \rfloor} \tau_j,$$

which we consider as a random process under π_n . The following limits in probability for the cumulative density function are well known:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{\lfloor nx \rfloor} \tau_j = \begin{cases} \frac{1}{2}x, & A < 1, C < 1 \text{ (maximal current phase),} \\ \frac{1}{1+C}x, & C > 1, C > A \text{ (low density phase),} \\ \frac{A}{1+A}x, & A > 1, A > C \text{ (high density phase),} \end{cases} \quad (1.3)$$

see for example [21,42,44]. Phase diagram affects the behavior of many other statistics, including current [20,40], correlation functions of the density [25,45], and the large deviation functionals of the density [21] or the current [14]. See [16] and more references therein.

The fluctuations of the cumulative density function with appropriate normalization are easy to describe for the boundary of the fan region ($AC = 1$). Since it is known that in this case, τ_1, \dots, τ_n are i.i.d. Bernoulli random variables with mean $A/(1+A) = 1/(1+C)$ (see [24] and Remark 2.4), the scaling limit is the Brownian motion, an immediate consequence of Donsker's theorem [3].

Theorem 1.1 (Boundary of fan region). *When $AC = 1$,*

$$\frac{1}{\sqrt{n}} \left\{ \sum_{j=1}^{\lfloor nx \rfloor} \left(\tau_j - \frac{A}{1+A} \right) \right\}_{x \in [0,1]} \Rightarrow \frac{\sqrt{A}}{1+A} \{\mathbb{B}_x\}_{x \in [0,1]}$$

as $n \rightarrow \infty$ in the space $D([0, 1])$.

One might expect naturally that the Brownian-motion behavior at the boundary of the fan region persists if one looks at the phase that is close to the boundary. However, this intuition is not entirely correct, as the limit is non-Gaussian for $A < 1, C < 1$ arbitrarily close to the point $(A, C) = (1, 1)$ at the boundary, a remarkable result due to Derrida et al. [19], who showed this in the special case $q = 0, \gamma = \delta = 0$.

1.2. Main results

In this paper, we provide a complete picture of the limit fluctuations of the cumulative density function, that is, of the process $\{\sum_{j=1}^{\lfloor nx \rfloor} \tau_j\}_{x \in [0,1]}$ with appropriate normalization as $n \rightarrow \infty$, in the fan region. First, as conjectured in [19, Section 3], we show that the limit fluctuation for a full range of parameters (1.1) in the maximal current phase is the same as for the case of $q = 0, \gamma = \delta = 0$ studied in [19]. Second, and most interestingly, we identify two different limit fluctuations at the boundary of the maximal current phase. Third, in the low/high density phases in the fan region, we show that the scaling limit of fluctuations is a Brownian motion.

Our results are stated in terms of Brownian motion, Brownian excursion and Brownian meander, denoted by $\mathbb{B}, \mathbb{B}^{ex}$ and \mathbb{B}^{me} respectively throughout this paper. One may think of Brownian excursion as the Brownian bridge conditioned to stay strictly positive until time $t = 1$, and Brownian meander as the Brownian motion conditioned to stay strictly positive over time interval $(0, 1]$. See for example [23,32,37–39,47] for more background and applications.

We first state our results on the maximal current phase and its boundary. Introduce

$$h_n(x) = \sum_{j=1}^{\lfloor nx \rfloor} \left(\tau_j - \frac{1}{2} \right), \quad x \in [0, 1],$$

and view $\{h_n(x)\}_{x \in [0,1]}$ as a stochastic process with law induced by π_n . The following theorem extends the already mentioned result of Derrida et al. [19] to a larger range of parameters (1.1) confirming the conjecture in [19, Section 3]. We let $\xrightarrow{\text{f.d.d.}}$ denote convergence of finite-dimensional distributions. Recall that definition (1.2) gives $A \geq 0, C \geq 0$.

Theorem 1.2 (Maximal current phase). *If $A < 1, C < 1$ then*

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{1}{2\sqrt{2}} \{\mathbb{B}_x + \mathbb{B}_x^{ex}\}_{x \in [0,1]},$$

as $n \rightarrow \infty$, where the Brownian motion \mathbb{B} and the Brownian excursion \mathbb{B}^{ex} are independent stochastic processes.

The boundary of the maximal current phase, see Figure 2, splits into three regions with different limit fluctuations: the corner point where $A = 1, C = 1$ with asymptotically Brownian fluctuations described in Theorem 1.1, and two line-segments corresponding to $A < 1, C = 1$ and $A = 1, C < 1$ with the following fluctuations.

Theorem 1.3 (Boundary of maximal current phase). *We have,*

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \begin{cases} \frac{1}{2\sqrt{2}} \{\mathbb{B}_x + \mathbb{B}_x^{me}\}_{x \in [0,1]}, & A = 1, C < 1, \\ \frac{1}{2\sqrt{2}} \{\mathbb{B}_x + \mathbb{B}_{1-x}^{me} - \mathbb{B}_1^{me}\}_{x \in [0,1]}, & A < 1, C = 1 \end{cases}$$

as $n \rightarrow \infty$, where the Brownian motion \mathbb{B} and the Brownian meander \mathbb{B}^{me} are independent stochastic processes.

For the low/high density phases, we use centering as indicated in (1.3). For $x \in [0, 1]$, introduce

$$h_n^L(x) = \sum_{j=1}^{\lfloor nx \rfloor} \left(\tau_j - \frac{1}{1+C} \right), \quad h_n^H(x) = \sum_{j=1}^{\lfloor nx \rfloor} \left(\tau_j - \frac{A}{1+A} \right), \quad (1.4)$$

and view both as stochastic processes with laws induced by π_n .

Theorem 1.4 (Low/high density phases of fan region). *Suppose $AC < 1$. In the low density phase, $C > 1$, we have*

$$\frac{1}{\sqrt{n}} \{h_n^L(x)\}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{\sqrt{C}}{1+C} \{\mathbb{B}_x\}_{x \in [0,1]} \quad \text{as } n \rightarrow \infty.$$

In the high density phase, $A > 1$, we have

$$\frac{1}{\sqrt{n}} \{h_n^H(x)\}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{\sqrt{A}}{1+A} \{\mathbb{B}_x\}_{x \in [0,1]} \quad \text{as } n \rightarrow \infty.$$

The paper is organized as follows. In Section 1.3 we describe informally the basic ideas behind the proof. Section 2 provides technical background on Askey–Wilson processes and generating functions of ASEP. In Section 3 we prove Theorem 1.4. Section 4 presents proofs of Theorems 1.2 and 1.3. In the Appendix we discuss the Laplace transform criterion for weak convergence that we use.

1.3. Overview of the proof

Our starting point is the identity

$$\left\langle \prod_{j=1}^n t_j^{\tau_j} \right\rangle_n = \frac{\mathbb{E}[\prod_{j=1}^n (1 + t_j + 2\sqrt{t_j} Y_{t_j})]}{2^n \mathbb{E}(1 + Y_1)^n}, \quad \text{for all } 0 < t_1 \leq t_2 \leq \dots \leq t_n, \quad (1.5)$$

which expresses the probability generating function of ASEP on the left-hand side as a functional of an auxiliary Markov process $\{Y_t\}_{t \geq 0}$. The process $\{Y_t\}_{t \geq 0}$, introduced in [9], is an inhomogeneous Markov process with transition probabilities constructed from the Askey–Wilson laws, that is, from the “weight functions” of the Askey–Wilson polynomials [1], as described in Section 2.1. The parameters A, B, C, D introduced in (1.2) are the parameters of this process and our notation here is consistent with [9,10]. Identity (1.5) comes from [10] and is a new representation of the matrix ansatz, which is a powerful and commonly used method developed in the seminal work of Derrida et al. [20]. Our approach, however, is of an analytical nature that is different from most applications of the matrix ansatz to the ASEP in the literature (see Remark 2.3).

Theorems 1.2, 1.3 and 1.4 are established by representing the Laplace transforms of the finite-dimensional distributions of processes h_n , and h_n^H , normalized by \sqrt{n} , in terms of this auxiliary Markov process $\{Y_t\}_{t \geq 0}$. In this representation, which is a straightforward application of (1.5), see (3.2) and (4.18), the arguments of the Laplace transform become time arguments for the Markov process. This reduces the study of fluctuations of ASEP as the system size n increases to the analysis of asymptotic behavior of Markov process $\{Y_t\}_{t \geq 0}$ near $t = 1$. In the case of the low/high density regimes, this then leads to a quick proof for the limit fluctuations (Theorem 1.4). The proof for the maximal current phase and its boundary is more involved and requires two additional ingredients that we now explain.

The first ingredient is the so-called *tangent process* [28] at the upper boundary of the support of process $\{Y_t\}_{t \geq 0}$. The tangent process, denoted by $\{\mathbb{Z}_t\}_{t \geq 0}$, is a positive 2-self-similar Markov process with explicit transition probability density function (see Section 4.1) and arises as follows. Intuitively, the tangent process captures the asymptotic fluctuations of the process $\{Y_t\}_{t \geq 0}$, as the time parameter t is approaching 1 and Y_1 is approaching the upper boundary end of the support $[-1, 1]$. To utilize this concept, we introduce a sequence of Markov processes $\{\widehat{Y}_s^{(n)}\}_{s \geq 0}$ which up to a multiplicative constant behave roughly like $\{(1 - Y_{1-\varepsilon s})/\varepsilon^2\}_{s \geq 0}$ for $\varepsilon^2 \sim 1/n$, (for precise definition, see (4.6) below). In Proposition 4.1 we show that as $\varepsilon \rightarrow 0$ we have

$$\mathcal{L}(\{\widehat{Y}_s^{(n)}\}_{s \geq 0} | \widehat{Y}_0^{(n)} = u) \xrightarrow{\text{f.d.d.}} \mathcal{L}(\{\mathbb{Z}_s\}_{s \geq 0} | \mathbb{Z}_0 = u), \quad \text{for all } u > 0. \quad (1.6)$$

The left-hand side is interpreted as the law of the Markov process $\{\widehat{Y}_s^{(n)}\}_{s \geq 0}$ starting at $\widehat{Y}_0^{(n)} = u$; similar interpretation applies for the right-hand side. In the second part of Proposition 4.1 we show that under appropriate normalization, the density of $\widehat{Y}_0^{(n)}$ converges to an infinite measure $\nu(du)$ which is proportional to either $u^{1/2} du$ (in the case $A < 1, C < 1$) or $u^{-1/2} du$ (in the case $A = 1, C < 1$).

Up to a normalizing constant, the Laplace transform of the finite-dimensional distributions of $\{h_n(x)\}_{x \in [0,1]}$ takes the form of $\mathbb{E}G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, \widehat{Y}_0)$ for some $s_1 > \dots > s_d > 0$, where the sequence of functions $G_n : \mathbb{R}^{d+1} \rightarrow \mathbb{R}_+$ converges to function G defined in (4.22). Convergence in (1.6) is a key step to show that these Laplace transforms converge to the limit given by the functional

$$\int_{\mathbb{R}_+} \mathbb{E}(G(Z_{s_1}, \dots, Z_{s_d}, u) \mid Z_0 = u) \nu(du)$$

of the tangent process \mathbb{Z} , see (4.30).

The second ingredient of our proof consists of some recently developed duality formulas [8] that express the Laplace transforms of Brownian excursion and meander in terms of the tangent process $\{\mathbb{Z}_t\}_{t \geq 0}$ ((4.2) and (4.3)). We recognize that the integral above has two factors: the Laplace transform of the Brownian motion, and a functional of $\{\mathbb{Z}_s\}_{s \geq 0}$ which we identify as the Laplace transform of Brownian excursion ($A < 1, C < 1$), see (4.31), or of Brownian meander ($A = 1, C < 1$), see (4.32). A delicate issue actually arises here as due to the use of Markov process we establish convergence of the Laplace transforms only in an open region away from the origin in \mathbb{R}^d . We clarify how this leads to the desired weak convergence in the Appendix.

Technical difficulties arise in the above approach when transitions probabilities of $\{Y_t\}_{t \geq 0}$ are of mixed type near $t = 1$. We avoid this issue by applying the so-called particle-hole duality, which is a well known symmetry feature of the ASEP. In particular, the case $A < 1, C = 1$ and the low density phase will be derived from the case $A = 1, C < 1$ and the high density phase, respectively, by this duality.

Remark 1.5. In principle, our approach might work for the weakly asymmetric exclusion process, as in [18], where the authors consider the case $q \uparrow 1$ at a rate that may depend on $n \rightarrow \infty$ and show that the fluctuations are Gaussian. This would require to determine first the relevant tangent process as $q \uparrow 1$.

We also mention that there is a huge literature on the asymptotic behavior of ASEP as a temporal-spatial process, by letting the ASEP to evolve from a non-stationary distribution, and possibly with $q \uparrow 1$ at the rate that may depend on $n \rightarrow \infty$. See for example [12,26,27,30] and references therein. Such results are beyond the scope of our methods.

2. Askey–Wilson process and ASEP

2.1. Askey–Wilson process

Askey–Wilson processes are a family of Markov processes based on Askey–Wilson measures, which we recall first. The Askey–Wilson measures are the probability measures that make the Askey–Wilson polynomials orthogonal. We do not use these polynomials here, and instead we write directly the orthogonality measure as given in [1], see also [33, Section 3.1] where a typo to weight of higher atoms is corrected. The formulas below incorporate this correction and probabilistic normalization, and come from [9].

The Askey–Wilson probability measure $\nu(dy; a, b, c, d, q)$ depends on five parameters a, b, c, d, q . It is assumed that $q \in (-1, 1)$. For the parameters a, b, c, d , it is assumed that they are all real, or two of the parameters are real and the other two form a complex conjugate pair, or the parameters form two complex conjugate pairs, and in addition

$$ac, ad, bc, bd, qac, qad, qbc, qbd, abcd, qabcd \notin [1, \infty). \tag{2.1}$$

The Askey–Wilson measure is invariant with respect to permutations of a, b, c, d . More precisely, the measure is of mixed type

$$\nu(dy; a, b, c, d, q) = f(y; a, b, c, d, q) dy + \sum_{z \in F(a,b,c,d,q)} p(z) \delta_z(dy),$$

with the absolutely continuous part supported on $[-1, 1]$ and with the discrete part supported on a finite or empty set F . For certain choices of parameters, the measure can be only discrete or only absolutely continuous. The absolutely continuous part is

$$f(y; a, b, c, d, q) = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty \sqrt{1-y^2}} \left| \frac{(e^{2i\theta_y}; q)_\infty}{(ae^{i\theta_y}, be^{i\theta_y}, ce^{i\theta_y}, de^{i\theta_y}; q)_\infty} \right|^2, \tag{2.2}$$

where $y = \cos \theta_y$ (with the convention that $f(y; a, b, c, d, q) = 0$ when $|y| > 1$). Here and below, for complex α , $n \in \mathbb{N} \cup \{\infty\}$ and $|q| < 1$ we use the q -Pochhammer symbol

$$(\alpha; q)_n = \prod_{j=0}^{n-1} (1 - \alpha q^j), \quad (a_1, \dots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n. \tag{2.3}$$

The set $F = F(a, b, c, d, q)$ of atoms of $\nu(dy; a, b, c, d, q)$ is non-empty if there is a parameter $\alpha \in \{a, b, c, d\}$ with $|\alpha| > 1$. In this case, necessarily α is real and generates atoms: for example, if $|a| > 1$ then it generates the atoms

$$y_j = \frac{1}{2} \left(aq^j + \frac{1}{aq^j} \right) \quad \text{for } j = 0, 1, \dots \text{ such that } |aq^j| \geq 1, \tag{2.4}$$

and the corresponding masses are

$$p(y_0; a, b, c, d, q) = \frac{(a^{-2}, bc, bd, cd; q)_\infty}{(b/a, c/a, d/a, abcd; q)_\infty}, \tag{2.5}$$

$$p(y_j; a, b, c, d, q) = p(y_0; a, b, c, d, q) \frac{(a^2, ab, ac, ad; q)_j (1 - a^2 q^{2j})}{(q, qa/b, qa/c, qa/d; q)_j (1 - a^2)} \left(\frac{q}{abcd} \right)^j, \quad j \geq 1.$$

The formula of $p(y_j; a, b, c, d, q)$ given here only applies for $a, b, c, d \neq 0$, and takes a different form otherwise. We shall however only need $p(y_0; a, b, c, d, q)$ in this paper.

The Askey–Wilson process is a time-inhomogeneous Markov process introduced in Bryc and Wesolowski [9], based on Askey–Wilson measures. It is then explained in Bryc and Wesolowski [10] how each ASEP with parameters $\alpha, \beta > 0, \gamma, \delta \geq 0, q \in [0, 1)$ is associated to an Askey–Wilson process Y , the parameters of which are denoted by A, B, C, D, q , with A, B, C, D given in (1.2).

As we already noted, (1.2) implies $A, C \geq 0$ and $-1 < B, D \leq 0$. So for the Askey–Wilson process to exist, the restriction (2.1) becomes $AC < 1$, which we assume throughout in the sequel. Then, the Askey–Wilson process with parameters (A, B, C, D, q) is introduced as the Markov process with marginal distribution

$$\mathbb{P}(Y_t \in dy) = \nu(dy; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}, q), \quad 0 < t < \infty,$$

and the transition probabilities

$$\mathbb{P}(Y_t \in dz \mid Y_s = y) = \nu(dz; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}(y + \sqrt{y^2 - 1}), \sqrt{s/t}(y - \sqrt{y^2 - 1})), \tag{2.6}$$

for $0 < s < t, y, z > 0$. When $|y| < 1$, $y \pm \sqrt{y^2 - 1}$ is understood as $e^{\pm i\theta_y}$ with θ_y determined by $\cos \theta_y = y$. It was shown in [9] that the above marginal and transition laws are consistent and determine a Markov process indexed by $t \in [0, \infty)$. The Askey–Wilson process turned out to be closely related to a large family of Markov processes, the so-called quadratic harnesses [6] in the literature; see [10, Section 1.3] for more on this connection. More explicit expressions for the law of Y will appear below when they are needed in the proofs.

2.2. Generating function of ASEP via Askey–Wilson process

Let $\langle \cdot \rangle_n$ denote the expectation with respect to the invariant measure π_n of ASEP. Derrida et al. [20] derives the well known matrix ansatz method that provides an explicit expression of the joint generating function, which made many

calculations of the model possible. Formally, for any $t_1, \dots, t_n > 0$, from [20] one can write

$$\left\langle \prod_{j=1}^n t_j^{\tau_j} \right\rangle_n = \frac{\langle W|(E + t_1 D) \times \dots \times (E + t_n D)|V \rangle}{\langle W|(E + D)^n|V \rangle}, \tag{2.7}$$

for a pair of infinite matrices D, E , a row vector W and a column vector V , satisfying

$$\begin{aligned} DE - qED &= D + E, \\ \langle W|(\alpha E - \gamma D) &= \langle W|, \\ (\beta D - \delta E)|V \rangle &= |V \rangle. \end{aligned}$$

See [15,16] for reviews of literature. However, for our purpose, we shall apply an alternative expression developed recently in [10, Theorem 1], summarized in the following theorem.

Theorem 2.1. *Consider the parameterization A, B, C, D in (1.2) for an ASEP with parameters $\alpha, \beta > 0, \gamma, \delta \geq 0$. Suppose that $AC < 1$ and $q \in [0, 1)$. Then for $0 < t_1 \leq t_2 \leq \dots \leq t_n$, the joint generating function of the stationary distribution of the ASEP with parameters is*

$$\left\langle \prod_{j=1}^n t_j^{\tau_j} \right\rangle_n = \frac{\mathbb{E}[\prod_{j=1}^n (1 + t_j + 2\sqrt{t_j} Y_{t_j})]}{2^n \mathbb{E}(1 + Y_1)^n}, \tag{2.8}$$

where $\{Y_t\}_{t \geq 0}$ is the Askey–Wilson process with parameters (A, B, C, D, q) .

Now, to establish our limit theorems, it suffices to analyze the asymptotics of the two expectations that appear in the numerator and in the denominator on the right-hand side of (2.8). For this purpose, we shall see that asymptotically, only the law of $\{Y_t\}_{t \in [1-\varepsilon, 1]}$ matters for arbitrarily small $\varepsilon > 0$. We first proceed in Section 3 with the proof of the low/high density phases, in which case the law of Y_t near upper boundary of its support is easy to analyze.

Remark 2.2. The connection between Askey–Wilson polynomials and the ASEP has been known for a long time, see for example [41,44,45]. In [44,45], using Askey–Wilson polynomials and complex integrals, the asymptotics of most commonly investigated statistics are computed, including current, density, partition function and the multiple-point correlation function, for results in both fan and shock regions (except the case $A = C > 1$ where the steady state does not have constant density). The identification of the Askey–Wilson Markov process in Theorem 2.1 turned out to be convenient for our proofs, at the expense of restriction of parameters of ASEP to the fan region $AC < 1$. Notice that in general, Askey–Wilson polynomials do not necessarily admit a positive orthogonality measure, and conditions on the coefficients for its existence are subtle (see [9]).

Remark 2.3. The version of the matrix ansatz method that we use is more analytic so our method differs from the usual applications of the matrix ansatz that seem to have more combinatorial flavor. For example, a formula for the joint distribution of the increments of h_n is given in [24, Eq. (3.7)] and used to derive the large deviation principle via a combinatorial argument [24, Eq. (3.16), (3.17)], essentially by expressing the probability of interest as a sum of probabilities indexed by different paths and then counting the number of paths that asymptotically have the same order of probabilities. This argument is of a completely different nature of ours.

The combinatorial nature of the matrix ansatz method has also been exploited in applications to problems on combinatorial enumeration [11].

Remark 2.4. At the boundary of the fan region, $AC = 1$, one can read from [9, Eq. (2.14) and (2.15)] that

$$Y_t = \frac{[A + B - AB(C + D)]t + C + D - CD(A + B)}{2\sqrt{t}(1 - ABCD)} = \frac{1}{2} \left(A\sqrt{t} + \frac{1}{A\sqrt{t}} \right),$$

is deterministic. Now, from (2.8) we can read out that $\{\tau_j\}_{j=1,\dots,n}$ are independent, and $\langle t_j^{\tau_j} \rangle_n = t_j A / (1 + A) + 1 / (1 + A)$. So these are Bernoulli random variables with

$$\langle \tau_j \rangle_n = \frac{A}{1 + A}.$$

Theorem 1.1 now is a consequence of the well known Donsker’s theorem [3].

3. Proofs for low/high density phases

In this section, we investigate the case $A > 1, AC < 1$ and $C > 1, AC < 1$. In the representation (2.8), the law of the associated Askey–Wilson process with parameters $(A, B, C, D; q)$ may have atoms. It turns out that we will only need the point mass on the largest atom. We shall only use this representation for the high density phase ($A > 1, AC < 1$). For the low density phase, the result shall follow by the particle-hole duality.

Fix $A > 1$ and $C < 1/A$. Recall that $B, D \in (-1, 0]$ and atoms are only generated by parameters that have absolute value larger than 1, so possibly by $A\sqrt{t}, C/\sqrt{t}$ and D/\sqrt{t} . When $t \in (\max\{1/A^2, D^2\}, 1]$, all the atoms are generated by $A\sqrt{t}$ by (2.4) with $a = A\sqrt{t}$, and in this case we let $y_j(t)$ denote the $(j + 1)$ th largest atom of the law of Y_t . In particular, we have

$$y_0(t) = \frac{1}{2} \left(A\sqrt{t} + \frac{1}{A\sqrt{t}} \right) > 1 \quad \text{for } t \in \left(\max \left\{ \frac{1}{A^2}, D^2 \right\}, 1 \right].$$

We shall need the mass of Y_1 on $y_0(1)$, which is denoted by, recalling (2.5),

$$p_0 = p(y_0(1); A, B, C, D, q) = \frac{(1/A^2, BC, BD, CD; q)_\infty}{(B/A, C/A, D/A, ABCD; q)_\infty}.$$

In the sequel, we write $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

3.1. Proof of Theorem 1.4 for the high density phase $A > 1, AC < 1$

We prove the convergence of corresponding Laplace transform. We first recall the Laplace transform of the finite-dimensional distribution of the Brownian motion. For $x_0 = 0 < x_1 < \dots < x_d \leq x_{d+1} = 1, c_1, \dots, c_d > 0, s_k = c_k + \dots + c_d, k = 1, \dots, d$, and $s_{d+1} = 0$, we have

$$\begin{aligned} \mathbb{E} \exp \left(- \sum_{k=1}^d c_k \mathbb{B}_{x_k} \right) &= \mathbb{E} \exp \left(- \sum_{k=1}^d (s_k - s_{k+1}) \mathbb{B}_{x_k} \right) \\ &= \mathbb{E} \exp \left(- \sum_{k=1}^d s_k (\mathbb{B}_{x_k} - \mathbb{B}_{x_{k-1}}) \right) \\ &= \exp \left(\frac{1}{2} \sum_{k=1}^{d+1} s_k^2 (x_k - x_{k-1}) \right). \end{aligned} \tag{3.1}$$

For the ASEP in the high density phase, consider the centered cumulative density function (1.4) and its Laplace transform with argument $\mathbf{c} = (c_1, \dots, c_d) \in \mathbb{R}_+^d$ defined by

$$\varphi_{\mathbf{x},n}^H(\mathbf{c}) = \left\langle \exp \left(- \sum_{k=1}^d c_k h_n^H(x_k) \right) \right\rangle_n.$$

Note that in Theorem 1.4, the limit Brownian motion is scaled by $\sqrt{A}/(1 + A)$. Therefore, by Theorem A.1 to prove the high density phase of Theorem 1.4 it suffices to prove

$$\lim_{n \rightarrow \infty} \varphi_{x,n}^H \left(\frac{\mathbf{c}}{\sqrt{n}} \right) = \exp \left(\sum_{k=1}^{d+1} \frac{A}{2(1+A)^2} s_k^2 (x_k - x_{k-1}) \right).$$

To do so, we first write

$$\begin{aligned} \varphi_{x,n}^H(\mathbf{c}) &= \left\langle \exp \left(- \sum_{k=1}^d \sum_{j=\lfloor nx_{k-1} \rfloor + 1}^{\lfloor nx_k \rfloor} \left(\tau_j - \frac{A}{1+A} \right) (c_k + \dots + c_d) \right) \right\rangle_n \\ &= \exp \left(\sum_{k=1}^d \frac{A}{1+A} s_k (n_k - n_{k-1}) \right) \left\langle \prod_{k=1}^{d+1} \prod_{j=n_{k-1}+1}^{n_k} (e^{-s_k})^{\tau_j} \right\rangle_n, \end{aligned}$$

where $n_k = \lfloor nx_k \rfloor, k = 1, \dots, d + 1$. By (2.8),

$$\varphi_{x,n}^H(\mathbf{c}) = \frac{1}{2^n Z_n} \mathbb{E} \left[\prod_{k=1}^{d+1} \left(\frac{1 + e^{-s_k} + 2e^{-s_k/2} Y_{e^{-s_k}}}{e^{-s_k A/(1+A)}} \right)^{n_k - n_{k-1}} \right] \tag{3.2}$$

with $Z_n = \mathbb{E}(1 + Y_1)^n$.

Lemma 3.1. *If $A > 1$ and $AC < 1$, then*

$$Z_n \sim \frac{(1 + A)^{2n}}{2^n A^n} p_0.$$

This result has been known in the literature. See Remark 4.6. We provide a proof here for completeness.

Proof of Lemma 3.1. Let $y_1^*(1) = \max(y_1(1), 1)$ denote the upper bound of the support of the law of Y_1 on $\mathbb{R} \setminus \{y_0(1)\}$. Note that $y_1^*(1) < y_0(1)$ (recall that $y_0(1)$ and $y_1(1)$ are the locations of the point masses of the first and second atom of Y_1 , see (2.4), and that the first atom lies above 1 and above the second atom). It then follows that

$$Z_n = \int_{\{y_0(1)\}} (1 + y)^n v(dy; A, B, C, D, q) + \int_{[-1, y_1^*(1)]} (1 + y)^n v(dy; A, B, C, D, q).$$

The first term equals

$$p_0 (1 + y_0(1))^n = p_0 \frac{(1 + A)^{2n}}{2^n A^n},$$

and the second term is bounded from above by $(1 + y_1^*(1))^n$, which converges to 0 when divided by $(1 + y_0(1))^n$. \square

In view of the asymptotics of Z_n , we introduce

$$\psi(s, y) = \frac{1 + e^{-s} + 2e^{-s/2} y}{e^{-sA/(1+A)}} \frac{A}{(1 + A)^2},$$

and have

$$\varphi_{x,n}^H \left(\frac{\mathbf{c}}{\sqrt{n}} \right) \sim \frac{M_n}{p_0} \quad \text{with } M_n = \mathbb{E} \left[\prod_{k=1}^{d+1} \psi \left(\frac{s_k}{\sqrt{n}}, Y_{e^{-s_k/\sqrt{n}}} \right)^{n_k - n_{k-1}} \right].$$

The desired result now follows from the following.

Lemma 3.2. *With the notation above,*

$$\lim_{n \rightarrow \infty} M_n = \mathfrak{p}_0 \exp \left(\sum_{k=1}^d \frac{A}{2(1+A)^2} s_k^2 (x_k - x_{k-1}) \right).$$

Proof. We have that

$$\psi(s, y_0(e^{-s})) = \left(\frac{1}{1+A} e^{sA/(1+A)} + \frac{A}{1+A} e^{-s/(1+A)} \right) = 1 + \frac{As^2}{2(1+A)^2} + o(s^2) \tag{3.3}$$

as $s \downarrow 0$. Introduce $s_{k,n} = s_k/\sqrt{n}$ and $t_{k,n} = e^{-s_k/\sqrt{n}}$. Note that due to our choice of s_k , we have $t_{1,n} < t_{2,n} < \dots < t_{d,n} < 1 = t_{d+1,n}$.

We write $M_n = M_{n,1} + M_{n,2}$ with

$$M_{n,1} = \mathbb{E} \left[\prod_{k=1}^{d+1} \psi(s_{k,n}, Y_{t_{k,n}})^{n_k - n_{k-1}} \mathbf{1}_{\{Y_{t_{1,n}} = y_0(t_{1,n})\}} \right].$$

We shall show that $M_n \sim M_{n,1}$ as $n \rightarrow \infty$. Indeed, we have that $Y_{t_{k,n}} \leq y_0(t_{k,n})$ and hence $\psi(s_{k,n}, Y_{t_{k,n}}) \leq \psi(s_{k,n}, y_0(t_{k,n}))$ almost surely. First, observe that

$$\mathbb{P}(Y_{t_{1,n}} = y_0(t_{1,n})) = \mathbb{P}(Y_{t_{k,n}} = y_0(t_{k,n}), k = 1, \dots, d + 1).$$

That is, once the process Y_s reaches the highest point $y_0(s)$ at some time s , necessarily $s > 1/A^2$, Y_s stays on the deterministic trajectory $(y_0(t))_{t \geq s}$. This follows by computing $\mathbb{P}(Y_t = y_0(t) \mid Y_s = y_0(s))$ for $1/A^2 < s < t$. In this case, one has $y_0(s) > 1$,

$$y_0(s) + \sqrt{y_0(s)^2 - 1} = A\sqrt{s}, \quad \text{and} \quad y_0(s) - \sqrt{y_0(s)^2 - 1} = \frac{1}{A\sqrt{s}}.$$

So by (2.6) and (2.5),

$$\begin{aligned} \mathbb{P}(Y_t = y_0(t) \mid Y_s = y_0(s)) &= v(\{y_0(t)\}; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}A\sqrt{s}, \sqrt{s/t}/(A\sqrt{s})) \\ &= p(y_0(t); A\sqrt{t}, B\sqrt{t}, As/\sqrt{t}, 1/(A\sqrt{t}), q) \\ &= \frac{(1/(A^2t), ABs, B/A, s/t; q)_\infty}{(B/A, s/t, 1/(A^2t), ABs; q)_\infty} = 1. \end{aligned}$$

Introduce also $\mathfrak{p}_{0,n} = \mathbb{P}(Y_{t_{1,n}} = y_0(t_{1,n}))$. Recalling (2.5), we have

$$\begin{aligned} \mathfrak{p}_{0,n} &= p(y_0(t_{1,n}); A\sqrt{t_{1,n}}, B\sqrt{t_{1,n}}, C/\sqrt{t_{1,n}}, D/\sqrt{t_{1,n}}, q) \\ &= \frac{(1/(A^2t_{1,n}), BC, BD, CD/t_{1,n}; q)_\infty}{(B/A, C/(At_{1,n}), D/(At_{1,n}), ABCD; q)_\infty} \rightarrow \frac{(1/A^2, BC, BD, CD; q)_\infty}{(B/A, C/A, D/A, ABCD; q)_\infty} = \mathfrak{p}_0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore, by (3.3),

$$M_{n,1} = \mathfrak{p}_{0,n} \prod_{k=1}^{d+1} \psi(s_{k,n}, y_0(t_{k,n}))^{n_k - n_{k-1}} \rightarrow \mathfrak{p}_0 \exp \left(\sum_{k=1}^d \frac{A}{2(1+A)^2} s_k^2 (x_k - x_{k-1}) \right)$$

as $n \rightarrow \infty$. On the other hand, introducing $y_1^*(s) = \max\{y_1(s), 1\}$, on the event $\{Y_s \neq y_0(s)\}$, we have $Y_s \leq y_1^*(s)$ almost surely. Then,

$$M_{n,2} \leq (1 - \mathfrak{p}_{0,n}) \psi(s_{1,n}, y_1^*(t_{1,n}))^{n_1} \prod_{k=2}^{d+1} \psi(s_{k,n}, y_0(t_{k,n}))^{n_k - n_{k-1}}.$$

Now, one sees immediately that, since by continuity $\lim_{n \rightarrow \infty} y_1^*(t_{1,n})/y_0(t_{1,n}) = y_1^*(1)/y_0(1) \in (0, 1)$,

$$\frac{M_{n,2}}{M_n} \leq \frac{1 - p_{0,n}}{p_{0,n}} \left(\frac{\psi(s_{1,n}, y_1^*(t_{1,n}))}{\psi(s_{1,n}, y_0(t_{1,n}))} \right)^{n_1} = \frac{1 - p_{0,n}}{p_{0,n}} \left(\frac{1 + e^{-2s_{1,n}} + 2e^{-s_{1,n}} y_1^*(t_{1,n})}{1 + e^{-2s_{1,n}} + 2e^{-s_{1,n}} y_0(t_{1,n})} \right)^{n_1} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $M_n \sim M_{n,1}$, and the desired result follows. □

3.2. Proof of Theorem 1.4 for the low density phase $C > 1, AC < 1$

The result for the low density phase is an immediate consequence of the result for the high density phase, by the particle-hole duality which we now explain. We have seen the definition of an ASEP with parameters $(\alpha, \beta, \gamma, \delta, q)$. Instead of thinking of particles jumping around, we view the particles as background and allow the holes to jump around (viewing Figure 1 as white particles jumping around among black unoccupied sites). In this way, equivalently a hole jumps to the unoccupied left and right sites with rates 1 and q , respectively, and disappears at site 1 with rate α and at site n with rate δ , and enters site n if unoccupied with rate β and site 1 if unoccupied with rate γ . This is the ASEP with parameters $(\beta, \alpha, \delta, \gamma, q)$, if we relabel the sites $\{1, \dots, n\}$ by $\{n, \dots, 1\}$.

Fix $q \in [0, 1)$. Let $\pi_n^{A,B,C,D}$ denote the stationary distribution of the ASEP with parameters $(\alpha, \beta, \gamma, \delta, q)$. Let τ_1, \dots, τ_n be as before, and set $\varepsilon_j = 1 - \tau_{n-j+1}$. Introduce

$$\widehat{h}_n^L(x) = \sum_{j=1}^{\lfloor nx \rfloor} \left(\varepsilon_j - \frac{C}{1+C} \right).$$

The above argument shows that $\{\widehat{h}_n^L(x)\}_{x \in [0,1]}$ with respect to $\pi_n^{A,B,C,D}$ has the same law as $\{h_n^H(x)\}_{x \in [0,1]}$ (defined in (1.4)) with respect to $\pi_n^{C,D,A,B}$. Therefore, the high density phase of Theorem 1.4 tells that

$$\frac{1}{\sqrt{n}} \{ \widehat{h}_n^L(x) \}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{\sqrt{C}}{1+C} \{ \mathbb{B}_x \}_{x \in [0,1]}.$$

We are interested in $h_n^L(x) = \sum_{j=1}^{\lfloor nx \rfloor} (\tau_j - 1/(1+C))$ with respect to $\pi_n^{A,B,C,D}$. Observe that

$$\left(\varepsilon_j - \frac{C}{1+C} \right) + \left(\tau_{n-j+1} - \frac{1}{1+C} \right) = 0,$$

so

$$h_n^L(x) + (\widehat{h}_n^L(1) - \widehat{h}_n^L(1-x)) = \begin{cases} 0, & nx = \lfloor nx \rfloor, \\ \frac{C}{1+C} - \varepsilon_{\lfloor n(1-x) \rfloor + 1}, & nx \neq \lfloor nx \rfloor. \end{cases} \tag{3.4}$$

Since the error term is uniformly bounded, the finite-dimensional distributions of $n^{-1/2} \{h_n^L(x)\}_{x \in [0,1]}$ have the same limit as the finite-dimensional distributions of $n^{-1/2} \{ \widehat{h}_n^L(1-x) - \widehat{h}_n^L(1) \}_{x \in [0,1]}$, and we arrive at

$$\frac{1}{\sqrt{n}} \{ h_n^L(x) \}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{\sqrt{C}}{1+C} \{ \mathbb{B}_{1-x} - \mathbb{B}_1 \}_{x \in [0,1]} \stackrel{d}{=} \frac{\sqrt{C}}{1+C} \{ \mathbb{B}_x \}_{x \in [0,1]}.$$

This proves the low density phase of Theorem 1.4.

4. Proofs for maximal current phase and its boundary

The proofs for the two cases are very similar and are hence unified. We need some preparation for the proof. In Section 4.1 we review an important auxiliary Markov process \mathbb{Z} , and in particular how this Markov process shows up in the Laplace representations of Brownian excursion and meander. Another important role of this Markov process is that it is the tangent process of the Askey–Wilson process at the boundary. This result, playing a central role in the proof, will be established first in Proposition 4.1 in Section 4.2. The case $A \leq 1, C < 1$ is then proved in Theorem 4.4 in Section 4.3. The case $A < 1, C = 1$ is proved by the particle-hole duality in Section 4.4.

4.1. An auxiliary Markov process

An auxiliary Markov process, denoted by \mathbb{Z} in the rest of the paper, will play a crucial role in the proof for the maximal current phase and its boundary. This is a positive self-similar Markov process with values in $[0, \infty)$ and transition probability density function

$$g_{s,t}(x, y) = \frac{2(t-s)\sqrt{y}}{\pi[(t-s)^4 + 2(t-s)^2(x+y) + (x-y)^2]} \mathbf{1}_{\{x \geq 0, y \geq 0\}}, \quad s < t. \tag{4.1}$$

This process is self-similar in the sense that, letting \mathbb{P}_x denote the law of \mathbb{Z} starting at $\mathbb{Z}_0 = x$,

$$(\{\mathbb{Z}_{\lambda t}\}_{t \geq 0}, \mathbb{P}_x) \stackrel{d}{=} (\lambda^2 \{\mathbb{Z}_t\}_{t \geq 0}, \mathbb{P}_{x/\lambda^2}) \quad \text{for all } \lambda, x > 0.$$

This process has not been much investigated in the literature, except for a series of recent papers [7,8,46]. In [7], when investigating the path properties of so-called q -Gaussian processes, we proved that the process \mathbb{Z} arises as their tangent process at the boundary. (We also proved in Section 3 therein that the transformed process $\tilde{\mathbb{Z}}$ via $\tilde{\mathbb{Z}}_t = \mathbb{Z}_{t/2} + t^2/4$ has already shown up in the literature: in a general framework connecting non-commutative stochastic process and classical Markov process developed by Biane [2]. In this framework, $\tilde{\mathbb{Z}}$ as a classical Markov process corresponds to the free $1/2$ -stable process, the knowledge of which we do not need here.)

Recently, we also found out in [8] that the process \mathbb{Z} plays an intriguing role in representations of the Laplace transforms of finite-dimensional distributions of Brownian excursion and Brownian meander. Write $\mathbb{E}_u(\cdot) = \mathbb{E}(\cdot \mid \mathbb{Z}_0 = u)$. For all $d \in \mathbb{N}$, $s_1 > s_2 > \dots > s_d > s_{d+1} = 0$, and $0 = x_0 < x_1 < \dots < x_d \leq x_{d+1} = 1$, we have shown in [8] that

$$\mathbb{E} \exp\left(-\sum_{k=1}^d (s_k - s_{k+1}) \mathbb{B}_{x_k}^{ex}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} u^{1/2} du \mathbb{E}_u \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k} (x_k - x_{k-1})\right), \tag{4.2}$$

and

$$\mathbb{E} \exp\left(-\sum_{k=1}^d (s_k - s_{k+1}) \mathbb{B}_{x_k}^{me}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \frac{1}{u^{1/2}} du \mathbb{E}_u \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k} (x_k - x_{k-1})\right). \tag{4.3}$$

Note that the formulae here are obtained via the changes of variables

$$s_k = \widehat{s}_{d+1-k} \quad \text{and} \quad x_k = 1 - t_{d+1-k}, \quad k = 1, \dots, d+1,$$

and $x_0 = 0$, where \widehat{s}_k and t_k correspond to variables s_k, t_k used in [8].

The above identities can be obtained by direct computation using the joint density functions of Brownian excursion and meander. These explicit densities of the two processes will not be used in this paper. Standard references about Brownian excursions and meanders include [37–39].

4.2. Tangent process of Askey–Wilson process with $A \leq 1$ and $C < 1$

The proof is essentially based on the laws of the Askey–Wilson process $\{Y_t\}_{t \in [1-\varepsilon, 1]}$ for some $\varepsilon > 0$ small enough. With $A \leq 1$ and $C < 1$, for this range of t the marginal and transition probability laws are absolutely continuous with respect to the Lebesgue measure, with compact support on $[-1, 1]$. For the purpose of computing asymptotics, we express the formula by regrouping q -Pochhammer factors into those that tend to a non-zero constant as $t \uparrow 1$ and $x \uparrow 1$, and those that tend to zero. (Some factors go to zero only when $A = 1$, and we include them in the second group.) In particular, the Askey–Wilson process Y has the marginal probability density function

$$\begin{aligned} \pi_t(x) &= f(x; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}, q) \\ &= \frac{(q; q)_\infty (ABt, AC, AD, BC, BD, CD/t; q)_\infty}{2\pi(ABCD; q)_\infty |(B\sqrt{t}e^{i\theta_x}, Ce^{i\theta_x}/\sqrt{t}, De^{i\theta_x}/\sqrt{t}; q)_\infty|^2} \times \frac{|(e^{2i\theta_x}; q)_\infty|^2}{\sqrt{1-x^2} |(A\sqrt{t}e^{i\theta_x}; q)_\infty|^2}, \end{aligned} \tag{4.4}$$

with $x = \cos \theta_x$, for $x \in [-1, 1]$, and transition probability density function

$$\begin{aligned}
 p_{s,t}(x, y) &= f(y; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}e^{i\theta_x}, \sqrt{s/t}e^{-i\theta_x}) \\
 &= \frac{(q; q)_\infty (ABt; q)_\infty |(B\sqrt{s}e^{i\theta_x}; q)_\infty|^2}{2\pi (ABs; q)_\infty |(B\sqrt{t}e^{i\theta_y}; q)_\infty|^2} \\
 &\quad \times \frac{|(A\sqrt{s}e^{i\theta_x}, e^{2i\theta_y}; q)_\infty|^2 (s/t; q)_\infty}{\sqrt{1-y^2} |(A\sqrt{t}e^{i\theta_y}, \sqrt{s/t}e^{i(\theta_x+\theta_y)}, \sqrt{s/t}e^{i(-\theta_x+\theta_y)}; q)_\infty|^2},
 \end{aligned} \tag{4.5}$$

where $y = \cos \theta_y$, for $x, y \in [-1, 1]$. (Recall (2.2) and q -Pochhammer notation in (2.3).)

It turns out to be helpful to consider $\{Y_t\}_{t \in [1-\varepsilon, 1]}$ in the reversed time direction when Y_1 is close to 1. For this purpose, we introduce Markov process

$$\widehat{Y}_s^{(n)} = 2n(1 - Y_{e^{-2s/\sqrt{n}}}), \quad s \geq 0. \tag{4.6}$$

(The parameterization for $\widehat{Y}_s^{(n)}$ is chosen so that we have the corresponding convergence to the tangent process (4.7) below.) The following limit theorem of $\widehat{Y}^{(n)}$ is at the core of the proof of Theorems 1.2 and 1.3. Let $\widehat{\pi}_s^{(n)}$ denote the probability density function of $\widehat{Y}_s^{(n)}$ and $\widehat{p}_{s,t}^{(n)}$ the transition probability density of $\widehat{Y}^{(n)}$ from time s to t .

Proposition 4.1. *Under the notation above, for $A \leq 1$ and $C < 1$, $0 \leq s < t$ and $u, v \geq 0$ we have*

$$\lim_{n \rightarrow \infty} \widehat{p}_{s,t}^{(n)}(u, v) = g_{s,t}(u, v).$$

In particular, we have

$$\mathcal{L}(\{\widehat{Y}_s^{(n)}\}_{s \geq 0} \mid \widehat{Y}_0^{(n)} = u) \xrightarrow{\text{f.d.d.}} \mathcal{L}(\{\mathbb{Z}_s\}_{s \geq 0} \mid \mathbb{Z}_0 = u), \quad \text{for all } u > 0, \tag{4.7}$$

where the left-hand side is understood as the law of \widehat{Y} given $\widehat{Y}_0^{(n)} = u$, and similarly for the right-hand side above. Moreover, as $n \rightarrow \infty$

$$\widehat{\pi}_0^{(n)}(u) \sim \begin{cases} \frac{c_1}{n^{3/2}} \cdot u^{1/2}, & A < 1, C < 1, \\ \frac{c_2}{n^{1/2}} \cdot \frac{1}{u^{1/2}}, & A = 1, C < 1, \end{cases} \quad \text{for all } u > 0, \tag{4.8}$$

with

$$\begin{aligned}
 c_1 &= \frac{(q; q)_\infty^3 (AB, AC, AD, BC, BD, CD; q)_\infty}{\pi (ABCD; q)_\infty (A, B, C, D; q)_\infty^2} \\
 c_2 &= \frac{(q; q)_\infty (BC, BD, CD; q)_\infty}{\pi (BCD, B, C, D; q)_\infty},
 \end{aligned}$$

and there exists a constant c such that for all n large enough,

$$\widehat{\pi}_0^{(n)}(u) \leq \begin{cases} \frac{c}{n^{3/2}} \cdot u^{1/2}, & A < 1, C < 1, \\ \frac{c}{n^{1/2}} \cdot \frac{1}{u^{1/2}}, & A = 1, C < 1, \end{cases} \quad \text{for all } u > 0. \tag{4.9}$$

Remark 4.2. Recall that notation $a_n \sim b_n$ means that (4.8) reads as

$$\lim_{n \rightarrow \infty} n^{3/2} \widehat{\pi}_0^{(n)}(u) = c_1 \cdot u^{1/2}$$

if $A < 1, C < 1$, and

$$\lim_{n \rightarrow \infty} n^{1/2} \widehat{\pi}_0^{(n)}(u) = c_2 \cdot u^{-1/2}$$

if $A = 1, C < 1$.

The first part of the proposition implies that the (time-reversed) tangent process of Y at the upper boundary of the support of Y_1 is process \mathbb{Z} . The role of the second part will become clear soon. It is remarkable that for different choices of A and C , the tangent processes are the same, but the initial laws $(\widehat{\pi}_0^{(n)})$ in the limit are different and of different normalization orders. Similar results on tangent processes have been known for closely related processes [7,46]. We expect that the finite-dimensional convergence can be strengthened to weak convergence in $D([0, 1])$ by a similar treatment as in [46], but this is not needed in this paper. We first compute some asymptotics of the Askey–Wilson process Y .

Lemma 4.3. *For $u, v, s, t > 0, s < t$ and*

$$x_n = 1 - \frac{u}{2n}, \quad y_n = 1 - \frac{v}{2n}, \quad s_n = e^{-2s/\sqrt{n}}, \quad t_n = e^{-2t/\sqrt{n}},$$

$$\pi_{s_n}(x_n) \sim \begin{cases} \frac{2c_1}{\sqrt{n}} \cdot \sqrt{u}, & A < 1, C < 1, \\ 2c_2\sqrt{n} \cdot \frac{\sqrt{u}}{s^2+u}, & A = 1, C < 1, \end{cases} \tag{4.10}$$

and there exists some constant c such that

$$\pi_{s_n}(x_n) \leq \begin{cases} c\sqrt{\frac{u}{n}}, & A < 1, C < 1, \\ c\sqrt{\frac{n}{u}}, & A = 1, C < 1, \end{cases} \tag{4.11}$$

for all n large enough. Moreover,

$$p_{t_n, s_n}(y_n, x_n) \sim \begin{cases} 2n \cdot g_{s,t}(v, u), & A < 1, C < 1, \\ 2n \cdot g_{s,t}(v, u) \frac{t^2+v}{s^2+u}, & A = 1, C < 1. \end{cases} \tag{4.12}$$

Proof. We first establish

$$|(\sqrt{t_n/s_n} e^{\pm i\theta_{x_n}} e^{i\theta_{y_n}}; q)_\infty|^2 \sim \frac{1}{n} [(t-s)^2 + (\sqrt{u} \pm \sqrt{v})^2] (q; q)_\infty^2. \tag{4.13}$$

For this, write

$$|(\sqrt{t_n/s_n} e^{\pm i\theta_{x_n}} e^{i\theta_{y_n}}; q)_\infty|^2 = |1 - \sqrt{t_n/s_n} e^{i(\pm\theta_{x_n} + \theta_{y_n})}|^2 |(\sqrt{t_n/s_n} e^{i(\pm\theta_{x_n} + \theta_{y_n})} q; q)_\infty|^2. \tag{4.14}$$

Since $t_n \rightarrow 1, s_n \rightarrow 1, \theta_{x_n} \rightarrow 0, \theta_{y_n} \rightarrow 0$, the second factor above is asymptotically equivalent to $(q; q)_\infty^2$. For the first factor,

$$\begin{aligned} |1 - \sqrt{t_n/s_n} e^{i(\pm\theta_{x_n} + \theta_{y_n})}|^2 &= 1 + t_n/s_n - 2\sqrt{t_n/s_n} (x_n y_n \mp \sqrt{1-x_n^2} \sqrt{1-y_n^2}) \\ &\sim \frac{1}{n} [(t-s)^2 + (\sqrt{u} \pm \sqrt{v})^2]. \end{aligned} \tag{4.15}$$

This proves (4.13). As special cases, we have

$$|(\sqrt{s_n} e^{i\theta_{x_n}}; q)_\infty|^2 \sim \frac{1}{n} (s^2 + u) (q; q)_\infty^2, \tag{4.16}$$

$$(t_n/s_n; q)_\infty \sim \frac{2(t-s)}{\sqrt{n}} (q; q)_\infty,$$

$$|(e^{i\theta_{x_n}}; q)_\infty|^2 \sim \frac{u}{n} (q; q)_\infty^2,$$

$$|(e^{2i\theta_{x_n}}; q)_\infty|^2 \sim \frac{4u}{n} (q; q)_\infty^2,$$

as $n \rightarrow \infty$.

We now examine $\pi_{s_n}(x_n)$ using (4.4). The pointwise asymptotics (4.10) are straightforward to obtain. We prove the upper bounds. Recall that π_{s_n} has support on $[-1, 1]$. Therefore from now on we assume $x_n \in [-1, 1]$, or equivalently $u \in [0, 4n]$. We first focus on the second fraction of $\pi_{s_n}(x_n)$ in the expression (4.4),

$$\tilde{\pi}_{s_n}(x_n) = \frac{|(e^{2i\theta_{x_n}}; q)_\infty|^2}{\sqrt{1-x_n^2} |(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2}.$$

Let c denote a constant independent of n , but may change from line to line. Suppose $A < 1$ first. Observe that $|(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2 \geq (A; q)_\infty^2$. Furthermore, we have $|(e^{2i\theta_{x_n}}; q)_\infty|^2 = 4(1-x_n^2)|(e^{2i\theta_{x_n}}q; q)_\infty|^2$ (see e.g. (4.14) and (4.15)). Therefore,

$$\tilde{\pi}_{s_n}(x_n) \leq c \frac{|(e^{2i\theta_{x_n}}; q)_\infty|^2}{\sqrt{1-x_n^2}} \leq c\sqrt{1-x_n^2} \leq c\sqrt{\frac{u}{n}}.$$

Now suppose $A = 1$. Then

$$\tilde{\pi}_{s_n}(x_n) \leq c \frac{|(e^{2i\theta_{x_n}}; q)_\infty|^2}{\sqrt{1-x_n^2} |(\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} \leq c \frac{\sqrt{1-x_n^2}}{|1-\sqrt{s_n}e^{i\theta_{x_n}}|^2}.$$

By (4.15), $|1-\sqrt{s_n}e^{i\theta_{x_n}}|^2 = 1+s_n-2\sqrt{s_n}x_n = (1-\sqrt{s_n})^2 + 2\sqrt{s_n}(1-x_n) > 2\sqrt{s_1}(1-x_n)$. So, we see that $\tilde{\pi}_{s_n}(x_n)$ is bounded from above by,

$$c \frac{\sqrt{1-x_n^2}}{1-x_n} \leq c\sqrt{\frac{n}{u}}.$$

So we see that $\tilde{\pi}_{s_n}(x_n)$ can be controlled by the bounds in (4.11). For the first fraction of $\pi_{s_n}(x_n)$ in (4.4), it suffices to control

$$\frac{(CD/s_n; q)_\infty}{(Ce^{i\theta_{x_n}}/\sqrt{s_n}, De^{i\theta_{x_n}}/\sqrt{s_n}; q)_\infty}.$$

For the numerator, since $CD \in (-1, 0]$, we have $(CD/s_n; q)_\infty \leq (CD/s_1; q)_\infty$. For the denominator, for n large enough so that $C/\sqrt{s_n} < (1+C)/2 < 1$, we have $|(Ce^{i\theta_{x_n}}/\sqrt{s_n}; q)_\infty|^2 \geq ((1+C)/2; q)_\infty^2$. Similarly, $|(De^{i\theta_{x_n}}/\sqrt{s_n}; q)_\infty|^2 > ((1-D)/2; q)_\infty$ for n large enough so that $-D/\sqrt{s_n} < (1-D)/2 < 1$. So the first fraction can be bounded by some constant c . This completes the proof of (4.11).

Now to show (4.12), observe first that by (4.13) applied to each factor,

$$\begin{aligned} & |(\sqrt{t_n/s_n}e^{i(\theta_{x_n}+\theta_{y_n})}; q)_\infty (\sqrt{t_n/s_n}e^{i(-\theta_{x_n}+\theta_{y_n})}; q)_\infty|^2 \\ & \sim \frac{1}{n^2} (q; q)_\infty^4 [(t-s)^2 + (\sqrt{u} + \sqrt{v})^2][(t-s)^2 + (\sqrt{u} - \sqrt{v})^2] \\ & = \frac{1}{n^2} (q; q)_\infty^4 [(t-s)^4 + 2(t-s)^2(u+v) + (u-v)^2]. \end{aligned}$$

Recalling that

$$\frac{(ABt_n; q)_\infty}{(ABs_n; q)_\infty} \sim 1 \quad \text{and} \quad \frac{|(B\sqrt{t_n}e^{i\theta_{y_n}}; q)_\infty|^2}{|(B\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} \sim 1,$$

expression (4.5) now yields

$$\begin{aligned}
 p_{t_n, s_n}(y_n, x_n) &\sim \frac{(q; q)_\infty |(A\sqrt{t_n}e^{i\theta_{y_n}}; q)_\infty|^2}{2\pi |(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} \frac{|(e^{2i\theta_{x_n}}; q)_\infty|^2 (t_n/s_n; q)_\infty}{\sqrt{1-x_n^2} |(\sqrt{t_n/s_n}e^{i(\theta_{y_n}+\theta_{x_n})}; q)_\infty (\sqrt{t_n/s_n}e^{i(-\theta_{y_n}+\theta_{x_n})}; q)_\infty|^2} \\
 &\sim \frac{(q; q)_\infty |(A\sqrt{t_n}e^{i\theta_{y_n}}; q)_\infty|^2}{2\pi |(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} \frac{\frac{4u}{n}(q; q)_\infty^2 \frac{2(t-s)}{\sqrt{n}}(q; q)_\infty}{\sqrt{\frac{u}{n} \frac{1}{n^2}}(q; q)_\infty^4 [(t-s)^4 + 2(t-s)^2(u+v) + (u-v)^2]} \\
 &= \frac{|(A\sqrt{t_n}e^{i\theta_{y_n}}; q)_\infty|^2}{|(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} \cdot 2n \cdot g_{s,t}(v, u).
 \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} \frac{|(A\sqrt{t_n}e^{i\theta_{y_n}}; q)_\infty|^2}{|(A\sqrt{s_n}e^{i\theta_{x_n}}; q)_\infty|^2} = \begin{cases} 1, & A < 1, \\ \frac{t^2+v}{s^2+u}, & A = 1, \end{cases}$$

where we used (4.16) when $A = 1$. This completes the proof. □

Proof of Proposition 4.1. Recall the relation between \widehat{Y} and Y in (4.6). We have that $\widehat{\pi}_s^{(n)}(u) = \pi_{s_n}(x_n)/(2n)$, and so the second part of the proposition follows from (4.10) and (4.11). Fix $0 < s < t$ now, so that $0 < t_n < s_n < 1$. For the transition probability density of \widehat{Y} , we have

$$\widehat{p}_{s,t}^{(n)}(u, v) = p_{s_n, t_n}(x_n, y_n) \frac{1}{2n} = p_{t_n, s_n}(y_n, x_n) \frac{\pi_{t_n}(y_n)}{\pi_{s_n}(x_n)} \frac{1}{2n},$$

where p_{s_n, t_n} denotes the transition probability of Y in the reversed time direction. In the case $A < 1, C < 1$, from (4.10) and (4.12) we get

$$\widehat{p}_{s,t}^{(n)}(u, v) \sim g_{s,t}(v, u) \sqrt{\frac{v}{u}} = g_{s,t}(u, v)$$

directly. In the case $A = 1, C < 1$, from (4.10) and (4.12) we get

$$\widehat{p}_{s,t}^{(n)}(u, v) \sim g_{s,t}(v, u) \frac{t^2+v}{s^2+u} \cdot \frac{\sqrt{v}/(t^2+v)}{\sqrt{u}/(s^2+u)} = g_{s,t}(u, v).$$

Since the transition densities determine conditional finite-dimensional densities, the finite-dimensional (conditional) densities converge. Therefore, by Scheffé’s Theorem the finite-dimensional (conditional) distributions converge weakly for every $u > 0$, which completes the proof for the first part of the proposition. □

4.3. Proof of Theorems 1.2 and 1.3 for the case $A \leq 1, C < 1$

For $d \in \mathbb{N}, c_1, \dots, c_d > 0, x_0 = 0 < x_1 < \dots < x_d \leq x_{d+1} = 1$, introduce the Laplace transform

$$\varphi_{x,n}(\mathbf{c}) = \left\langle \exp\left(-\sum_{k=1}^d c_k h_n(x_k)\right) \right\rangle_n = \left\langle \exp\left(-\sum_{k=1}^d \sum_{j=n_{k-1}+1}^{n_k} \left(\tau_j - \frac{1}{2}\right)(c_k + \dots + c_d)\right) \right\rangle_n,$$

where $n_k = \lfloor nx_k \rfloor, k = 1, \dots, d + 1$. This time it will be more convenient to write

$$s_k = \frac{1}{2}(c_k + \dots + c_d), \quad k = 1, \dots, d \tag{4.17}$$

(notice the extra $1/2$ compared to s_k in previous sections), and $s_{d+1} = 0$. We have, by Theorem 2.1 again,

$$\begin{aligned} \varphi_{x,n}(\mathbf{c}) &= \exp\left(\sum_{k=1}^d s_k(n_k - n_{k-1})\right) \left\langle \prod_{k=1}^{d+1} \prod_{j=n_{k-1}+1}^{n_k} (e^{-2s_k})^{\tau_j} \right\rangle_n \\ &= \exp\left(\sum_{k=1}^d s_k(n_k - n_{k-1})\right) \frac{\mathbb{E}[\prod_{k=1}^{d+1} (1 + e^{-2s_k} + 2e^{-s_k} Y_{e^{-2s_k}})^{n_k - n_{k-1}}]}{2^n \mathbb{E}(1 + Y_1)^n} \\ &= \frac{\mathbb{E}[\prod_{k=1}^{d+1} (\cosh(s_k) + Y_{e^{-2s_k}})^{n_k - n_{k-1}}]}{Z_n}, \end{aligned}$$

where we write $Z_n = \mathbb{E}(1 + Y_1)^n$. In order to establish the convergence of finite-dimensional distributions of h_n , we compute the limit of

$$\varphi_{x,n}\left(\frac{\mathbf{c}}{\sqrt{n}}\right) = \frac{\mathbb{E}[\prod_{k=1}^{d+1} (\cosh(s_k/\sqrt{n}) + Y_{e^{-2s_k/\sqrt{n}}})^{n_k - n_{k-1}}]}{Z_n}, \tag{4.18}$$

as $n \rightarrow \infty$, and identify the limit with the Laplace transform of the corresponding process. Now in view of Theorem A.1, Theorems 1.2 and 1.3 are a consequence of the following.

Theorem 4.4. For all $d \in \mathbb{N}$, $0 < x_1 < \dots < x_d \leq 1$ and $\mathbf{c} = (c_1, \dots, c_d) \in (0, \infty)^d$,

$$\lim_{n \rightarrow \infty} \varphi_{x,n}\left(\frac{\mathbf{c}}{\sqrt{n}}\right) = \begin{cases} \mathbb{E} \exp(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}) \mathbb{E} \exp(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}^{ex}), & A < 1, C < 1, \\ \mathbb{E} \exp(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}) \mathbb{E} \exp(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}^{me}), & A = 1, C < 1. \end{cases} \tag{4.19}$$

We first determine the asymptotics of $\mathbb{E}(1 + Y_1)^n$ in (4.18).

Lemma 4.5. We have

$$Z_n = \mathbb{E}(1 + Y_1)^n \sim \begin{cases} \frac{2^n}{n^{3/2}} \cdot 4\sqrt{\pi} \cdot \mathbf{c}_1, & A < 1, C < 1, \\ \frac{2^n}{n^{1/2}} \cdot 2\sqrt{\pi} \cdot \mathbf{c}_2, & A = 1, C < 1. \end{cases}$$

where $\mathbf{c}_1, \mathbf{c}_2$ are defined in Proposition 4.1.

Remark 4.6. The quantity $Z_n = \mathbb{E}(1 + Y_1)^n$ is closely related to the *partition function* in the literature, denoted by \mathcal{Z}_n for the discussion here, via (see explanation in [10, Remark 5]),

$$\mathcal{Z}_n = Z_n \frac{2^n \langle W | V \rangle}{(1 - q)^n},$$

where the factor $\langle W | V \rangle$ depends on the choice of the vectors in (2.7) and needs to be taken into account when comparing Lemma 4.5 with the literature. Partial results on asymptotics of partition function, including also low/high density phases, have been known. See for example [20, (52),(53) and (55)] for the case $q = 0, \gamma = \delta = 0$ (with $\langle W | V \rangle = 1$) and [5, (56)] for $A < 1, C < 1, B = D = 0$ with $\langle W | V \rangle = 1/(AC; q)_\infty$. In more generality, Uchiyama et al. [44, (6.6) and (6.9)] compute \mathcal{Z}_n for $A > 1, A > C$ and $A, C < 1$ (with $\langle W | V \rangle = (ABCD; q)_\infty / (q, AB, AC, AD, BC, BD, CD; q)_\infty$). We do not find general results on asymptotic of Z_n for $A = 1, C < 1$ in the literature.

Proof of Lemma 4.5. It follows from (4.10) that, taking $s = 0$ therein, as $n \rightarrow \infty$, for $u > 0$,

$$\pi_1\left(1 - \frac{u}{2n}\right) \sim \begin{cases} 2\mathbf{c}_1 \sqrt{\frac{u}{n}}, & A < 1, C < 1, \\ 2\mathbf{c}_2 \sqrt{\frac{u}{n}}, & A = 1, C < 1. \end{cases} \tag{4.20}$$

Consider first the case $A = 1$. Then

$$2^{-n} n^{1/2} \mathbb{E}(1 + Y_1)^n = 2^{-n} n^{1/2} \int_{-1}^1 (1 + y)^n \pi_1(y) dy = \int_{\mathbb{R}_+} \mathbf{1}_{\{u \leq 4n\}} \left(1 - \frac{u}{4n}\right)^n \pi_1\left(1 - \frac{u}{2n}\right) \frac{du}{2n^{1/2}}. \quad (4.21)$$

In view of (4.20), the integrand on the right-hand side of (4.21) converges to $c_2 e^{-u/4} / \sqrt{u}$. Since $\int_0^\infty e^{-u/4} / \sqrt{u} du = 2\sqrt{\pi}$, to conclude the proof by the dominated convergence theorem, we now give an integrable bound for the integrands. Recalling (4.11), and noting that $\pi_1(1 - u/(2n)) = 0$ for $u > 2n$, we have, for n large enough,

$$\left(1 - \frac{u}{4n}\right)^n \pi_1\left(1 - \frac{u}{2n}\right) \leq c e^{-u/4} \sqrt{\frac{n}{u}}, \quad \text{for all } u > 0$$

for some constant c that depends on C and $q < 1$.

The proof for $A < 1$, $C < 1$ is similar, starting from (4.21) and eventually by the dominated convergence theorem. In this process, the above bound is replaced by, for n large enough,

$$\left(1 - \frac{u}{4n}\right)^n \pi_1\left(1 - \frac{u}{2n}\right) \leq c e^{-u/4} \sqrt{\frac{u}{n}}, \quad \text{for all } u > 0.$$

Details are omitted. □

Next, we look at the numerator in (4.18). We write, recalling \widehat{Y} in (4.6),

$$2^{-n} \mathbb{E} \left[\prod_{k=1}^{d+1} \left(\cosh\left(\frac{s_k}{\sqrt{n}}\right) + Y_{e^{-2s_k/\sqrt{n}}} \right)^{n_k - n_{k-1}} \right] = \mathbb{E} G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d+1}}^{(n)})$$

with

$$\begin{aligned} G_n(\mathbf{u}) &= 2^{-n} \prod_{k=1}^{d+1} \left(\cosh\left(\frac{s_k}{\sqrt{n}}\right) + 1 - \frac{u_k}{2n} \right)^{n_k - n_{k-1}} \mathbf{1}_{\{u \leq 4n\}} \\ &= \prod_{k=1}^{d+1} \left(1 + \sinh^2\left(\frac{s_k}{2\sqrt{n}}\right) - \frac{u_k}{4n} \right)^{n_k - n_{k-1}} \mathbf{1}_{\{u \leq 4n\}}. \end{aligned}$$

Recall that \widehat{Y} takes values from $[0, 4n]$. We introduce the indicator function above for the convenience in later analysis ($u \leq 4n$ stands for $\max_{k=1, \dots, d+1} u_k \leq 4n$ here). Since $(n_k - n_{k-1})/n \rightarrow x_k - x_{k-1}$, we have

$$\lim_{n \rightarrow \infty} G_n(\mathbf{u}) = \exp\left(\frac{1}{4} \sum_{k=1}^{d+1} (s_k^2 - u_k)(x_k - x_{k-1})\right) =: G(\mathbf{u}). \quad (4.22)$$

The key step is to show the following. Recall that $s_1 > \dots > s_d > s_{d+1} = 0$.

Proposition 4.7. *With the notations above and $\mathbb{E}_u(\cdot) = \mathbb{E}(\cdot \mid \mathbb{Z}_0 = u)$,*

$$\mathbb{E} G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, \widehat{Y}_0^{(n)}) \sim \begin{cases} \frac{c_1}{n^{3/2}} \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) u^{1/2} du, & A < 1, C < 1 \\ \frac{c_2}{n^{1/2}} \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) \frac{1}{u^{1/2}} du, & A = 1, C < 1. \end{cases}$$

Proof. We start with some properties of G_n as preparations. First, since $(1 + x)^m \leq \exp(mx)$ for $m \in \mathbb{N}$, $x \geq -1$, we get an exponential bound on G_n :

$$G_n(\mathbf{u}) \leq \prod_{k=1}^{d+1} \exp\left(\frac{n_k - n_{k-1}}{4n} \left[4n \sinh^2\left(\frac{s_k}{2\sqrt{n}}\right) - u_k\right]\right) \mathbf{1}_{\{u \leq 4n\}} \leq c \prod_{k=1}^{d+1} \exp\left(-\frac{n_k - n_{k-1}}{4n} u_k\right). \quad (4.23)$$

Here and below, c denotes a constant that does not depend on \mathbf{u} and n , but may vary from line to line. This inequality also shows that $G_n(\mathbf{u})$ is uniformly bounded for all $\mathbf{u} \in \mathbb{R}_+^{d+1}$ and $n \in \mathbb{N}$.

Next, by the inequality $|\prod_{k=1}^n a_k - \prod_{k=1}^n b_k| \leq M^n \sum_{k=1}^n |a_k - b_k|$ provided that $|a_k|, |b_k| \leq M$, we have, for all $\mathbf{u}, \mathbf{u}' \in [0, 4n]$,

$$|G_n(\mathbf{u}) - G_n(\mathbf{u}')| \leq \left(1 + \sinh^2\left(\frac{s_1}{2\sqrt{n}}\right)\right)^n \sum_{k=1}^{d+1} (n_k - n_{k+1}) \frac{|u_k - u'_k|}{4n} \leq c \sum_{k=1}^{d+1} |u_k - u'_k|.$$

It follows that for all $\mathbf{u}_n, \mathbf{u} \in \mathbb{R}_+^{d+1}$,

$$\lim_{n \rightarrow \infty} G_n(\mathbf{u}_n) = G(\mathbf{u}), \quad \text{if } \mathbf{u}_n \rightarrow \mathbf{u} \in \mathbb{R}_+^{d+1} \text{ as } n \rightarrow \infty. \tag{4.24}$$

In the remaining part of the proof with a little abuse of notation we write $\mathbb{E}_u(\cdot)$ which is to be understood as $\mathbb{E}(\cdot | \widehat{Y}_0^{(n)} = u)$ when dealing with $\widehat{Y}^{(n)}$ and as $\mathbb{E}(\cdot | \mathbb{Z}_0 = u)$ when dealing with \mathbb{Z} . Conditioning on the value of $\widehat{Y}_{s_{d+1}}^{(n)} = \widehat{Y}_0^{(n)}$, we write

$$\mathbb{E}G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d+1}}^{(n)}) = \int_0^\infty \mathbb{E}_u G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, u) \widehat{\pi}_0^{(n)}(u) du. \tag{4.25}$$

Because of the uniform boundedness, (4.24) and the weak convergence of the tangent process established in Proposition 4.1, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_u G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, u) = \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) \quad \text{for all } u > 0 \tag{4.26}$$

(see e.g., [3, Exercise 6.6]).

By (4.8) the integrands on the right hand side of (4.25) converge pointwise under appropriate normalization. That is, when $A < 1, C < 1$ we have

$$\lim_{n \rightarrow \infty} \frac{n^{3/2}}{c_1} \mathbb{E}_u G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, u) \widehat{\pi}_0^{(n)}(u) = \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) u^{1/2} \tag{4.27}$$

and when $A = 1, C < 1$ we have

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{c_2} \mathbb{E}_u G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}, u) \widehat{\pi}_0^{(n)}(u) = \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) \frac{1}{u^{1/2}}. \tag{4.28}$$

To conclude the proof we now apply the dominated convergence theorem. We see that, if $x_d < 1$, (4.23) yields $G_n(\mathbf{u}) \leq c \exp(-(1 - x_d)u_{d+1}/4)$. Recall upper bounds on $\widehat{\pi}_0^{(n)}$ in (4.9). Therefore, the functions of u that appear on the left-hand side of (4.27) and (4.28) are bounded by the integrable functions $cu^{1/2} \exp(-(1 - x_d)u/4)$ and $cu^{-1/2} \exp(-(1 - x_d)u/4)$, respectively, for n large enough. This proves the case $x_d < 1$.

To prove the case $x_d = 1$, we need to work a little harder, although the approach is very similar. Notice that when $x_d = 1$, both $G_n(\mathbf{u})$ and its limit $G(\mathbf{u})$ do not depend on the last coordinate u_{d+1} . Therefore we introduce

$$G_n^*(u_1, \dots, u_d) = G_n(u_1, \dots, u_d, 0) \quad \text{and} \quad G^*(u_1, \dots, u_d) = G(u_1, \dots, u_d, 0)$$

(the choice of 0 in the last variable of G_n and G is irrelevant for the definitions), and write $\mathbb{E}_u^*(\cdot)$ as either $\mathbb{E}(\cdot | \widehat{Y}_{s_d}^{(n)} = u)$ or $\mathbb{E}(\cdot | \mathbb{Z}_{s_d} = u)$, depending on whether the conditional expectation is for $\widehat{Y}^{(n)}$ or \mathbb{Z} . By conditioning on the value of $\widehat{Y}_{s_d}^{(n)}$, we write

$$\begin{aligned} \mathbb{E}G_n(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d+1}}^{(n)}) &\equiv \mathbb{E}G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}) \\ &= \int_0^\infty \mathbb{E}_u^* G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d-1}}^{(n)}, u) \widehat{\pi}_{s_d}^{(n)}(u) du. \end{aligned}$$

The same argument in the proof of Proposition 4.1 for the convergence in law of tangent process leads to

$$\mathcal{L}((\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d-1}}^{(n)}) \mid \widehat{Y}_{s_d}^{(n)} = u) \Rightarrow \mathcal{L}((\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}) \mid \mathbb{Z}_{s_d} = u).$$

Therefore, by the same argument for (4.26), we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_u^* G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d-1}}^{(n)}, u) = \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u),$$

and instead of (4.27) and (4.28), we have, for $A < 1, C < 1$,

$$\lim_{n \rightarrow \infty} \frac{n^{3/2}}{c_1} \mathbb{E}_u^* G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d-1}}^{(n)}, u) \widehat{\pi}_{s_d}^{(n)}(u) = \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) u^{1/2}$$

and for $A = 1, C < 1$,

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{c_2} \mathbb{E}_u^* G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_{d-1}}^{(n)}, u) \widehat{\pi}_{s_d}^{(n)}(u) = \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) \frac{u^{1/2}}{u + s_d^2}.$$

One can show that for n large enough, $\widehat{\pi}_{s_d}^{(n)}$ has the same upper bounds as $\widehat{\pi}_0^{(n)}$ in (4.9) (independent of s_d), by applying (4.11) to $\widehat{\pi}_{s_d}^{(n)}(u) = \pi_{s_d}(1 - u/(2n))/2n$. By (4.23), for n large enough, $G_n(\mathbf{u}) \leq c \exp(-(x_d - x_{d-1})u_d/8)$ (we cannot use $(x_d - x_{d-1})/4$ as before, because of the rounding issue caused by $n_k = \lfloor nx_k \rfloor$). So the dominated convergence theorem applies, and we arrive at

$$\mathbb{E} G_n^*(\widehat{Y}_{s_1}^{(n)}, \dots, \widehat{Y}_{s_d}^{(n)}) \sim \begin{cases} \frac{c_1}{n^{3/2}} \int_{\mathbb{R}_+} \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) u^{1/2} du, & A < 1, C < 1, \\ \frac{c_2}{n^{1/2}} \int_{\mathbb{R}_+} \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) \frac{u^{1/2}}{u + s_d^2} du, & A = 1, C < 1. \end{cases}$$

These are however not the same expressions as desired yet, and we need to rewrite them. For the case $A < 1, C < 1$, since \mathbb{Z} is stationary with respect to the distribution $u^{1/2} du$, we have (recalling that $\mathbb{E}_u(\cdot) = \mathbb{E}(\cdot \mid \mathbb{Z}_{s_{d+1}} = u)$)

$$\int_{\mathbb{R}_+} \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) u^{1/2} du = \int_{\mathbb{R}_+} \mathbb{E}_u G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, \mathbb{Z}_{s_d}) u^{1/2} du.$$

For the case $A = 1, C < 1$, because of the fact that

$$\int_0^\infty \frac{1}{x^{1/2}} g_{0,t}(x, y) dx = \frac{\sqrt{y}}{y + t^2}, \tag{4.29}$$

(recall (4.1)) we have

$$\int_{\mathbb{R}_+} \mathbb{E}_u^* G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, u) \frac{u^{1/2}}{u + s_d^2} du = \int_{\mathbb{R}_+} \mathbb{E}_u G^*(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_{d-1}}, \mathbb{Z}_{s_d}) \frac{1}{u^{1/2}} du.$$

So to complete the proof it remains to show (4.29). For this purpose, introduce change of variables $x = u^2, y = v^2$, and we get elementary integrals

$$\begin{aligned} \int_0^\infty \frac{1}{x^{1/2}} g_{0,t}(x, y) dx &= 2 \int_0^\infty g_{0,t}(u^2, v^2) du \\ &= \int_0^\infty \frac{1}{\pi u} \left(\frac{t}{t^2 + (u - v)^2} - \frac{t}{t^2 + (u + v)^2} \right) du \\ &= \frac{t}{2\pi(t^2 + v^2)} \log \frac{t^2 + (u + v)^2}{t^2 + (u - v)^2} \Big|_{u=0}^{u=\infty} \end{aligned}$$

$$\begin{aligned}
 & + \frac{v}{\pi(t^2 + v^2)} \left(\arctan\left(\frac{u-v}{t}\right) + \arctan\left(\frac{u+v}{t}\right) \right) \Big|_{u=0}^{u=\infty} \\
 & = \frac{v}{t^2 + v^2} = \frac{\sqrt{y}}{y + t^2}.
 \end{aligned}$$

□

Proof of Theorem 4.4. By Lemma 4.5 and Proposition 4.7, we have

$$\lim_{n \rightarrow \infty} \varphi_{x,n} \left(\frac{\mathbf{c}}{\sqrt{n}} \right) = \begin{cases} \frac{1}{4\sqrt{\pi}} \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) u^{1/2} du, & A < 1, C < 1, \\ \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) \frac{du}{u^{1/2}}, & A = 1, C < 1. \end{cases} \tag{4.30}$$

We first prove the case of $A < 1, C < 1$. Observe that

$$\begin{aligned}
 & \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) u^{1/2} du \\
 & = \exp\left(\frac{1}{4} \sum_{k=1}^{d+1} s_k^2 (x_k - x_{k-1})\right) \times \int_{\mathbb{R}_+} \mathbb{E}_u \exp\left(-\frac{1}{4} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k} (x_k - x_{k-1})\right) u^{1/2} du.
 \end{aligned} \tag{4.31}$$

The first exponential function on the right-hand side above corresponds to the Laplace transform of the scaled Brownian motion $8^{-1/2}\mathbb{B}$ (see (3.1) and recall that in (3.1) we used $s_k = c_k + \dots + c_d$, but we have been using $s_k = (c_k + \dots + c_d)/2$ since (4.17)). The integral on the right-hand side of (4.31) corresponds to the Laplace transform of a Brownian excursion. Indeed, by self-similarity of the process \mathbb{Z} , it equals

$$\begin{aligned}
 & \int_{\mathbb{R}_+} u^{1/2} du \mathbb{E}_u \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \frac{1}{2} \mathbb{Z}_{s_k} (x_k - x_{k-1})\right) \\
 & = \int_{\mathbb{R}_+} u^{1/2} du \mathbb{E}_{u/2} \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k/\sqrt{2}} (x_k - x_{k-1})\right) \\
 & = \sqrt{8} \int_{\mathbb{R}_+} u^{1/2} du \mathbb{E}_u \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k/\sqrt{2}} (x_k - x_{k-1})\right).
 \end{aligned}$$

By the duality expression in (4.2), this becomes

$$4\sqrt{\pi} \mathbb{E} \exp\left(-\sum_{k=1}^d \frac{s_k - s_{k+1}}{\sqrt{2}} \mathbb{B}_{x_k}^{ex}\right) = 4\sqrt{\pi} \mathbb{E} \exp\left(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}^{ex}\right).$$

Combining all the identities together we have proved (4.19) for $A < 1, C < 1$.

Now we prove the case of $A = 1, C < 1$. This time we have

$$\begin{aligned}
 \int_{\mathbb{R}_+} \mathbb{E}_u G(\mathbb{Z}_{s_1}, \dots, \mathbb{Z}_{s_d}, u) \frac{du}{u^{1/2}} & = \mathbb{E} \exp\left(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}\right) \\
 & \times \int_{\mathbb{R}_+} \mathbb{E}_u \exp\left(-\frac{1}{4} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k} (x_k - x_{k-1})\right) \frac{du}{u^{1/2}}.
 \end{aligned} \tag{4.32}$$

Again by self-similarity and duality (4.3), we rewrite the integral on the right-hand side above as

$$\begin{aligned} \sqrt{2} \int_{\mathbb{R}_+} \mathbb{E}_u \exp\left(-\frac{1}{2} \sum_{k=1}^{d+1} \mathbb{Z}_{s_k/\sqrt{2}}(x_k - x_{k-1})\right) \frac{du}{u^{1/2}} &= 2\sqrt{\pi} \mathbb{E} \exp\left(-\sum_{k=1}^d \frac{s_k - s_{k+1}}{\sqrt{2}} \mathbb{B}_{x_k}^{me}\right) \\ &= 2\sqrt{\pi} \mathbb{E} \exp\left(-\sum_{k=1}^d \frac{c_k}{2\sqrt{2}} \mathbb{B}_{x_k}^{me}\right). \end{aligned}$$

This completes the proof for the case $A = 1$ and $C < 1$. □

4.4. Proof for the case $A < 1, C = 1$

This case of Theorem 1.2 follows from the case $A = 1, C < 1$ again by the particle-hole duality, which we have explained in Section 3.2. Write $\varepsilon_j = 1 - \tau_{n-j+1}$, and define

$$\widehat{h}_n(x) = \sum_{j=1}^{\lfloor nx \rfloor} \left(\varepsilon_j - \frac{1}{2}\right).$$

Similarly as in (3.4), this time we have

$$h_n(x) + (\widehat{h}_n(1) - \widehat{h}_n(1-x)) = \begin{cases} 0, & nx = \lfloor nx \rfloor, \\ -(\varepsilon_{\lfloor n(1-x) \rfloor + 1} - \frac{1}{2}), & nx \neq \lfloor nx \rfloor, \end{cases}$$

and $\{\widehat{h}_n(x)\}_{x \in [0,1]}$ with respect to $\pi_n^{A,B,C,D}$ has the same law as $\{h_n(x)\}_{x \in [0,1]}$ with respect to $\pi_n^{C,D,A,B}$. Therefore, finite-dimensional distributions of the process $n^{-1/2}\{h_n(x)\}_{x \in [0,1]}$ have the same limit as the finite-dimensional distributions of the process $n^{-1/2}\{\widehat{h}_n(1-x) - \widehat{h}_n(1)\}_{x \in [0,1]}$. We get

$$\frac{1}{\sqrt{n}} \{h_n(x)\}_{x \in [0,1]} \xrightarrow{\text{f.d.d.}} \frac{1}{2\sqrt{2}} \{\mathbb{B}_{1-x} + \mathbb{B}_{1-x}^{me} - (\mathbb{B}_1 + \mathbb{B}_1^{me})\}_{x \in [0,1]} \stackrel{d}{=} \frac{1}{2\sqrt{2}} \{\mathbb{B}_x + \mathbb{B}_{1-x}^{me} - \mathbb{B}_1^{me}\}_{x \in [0,1]}.$$

This completes the proof of Theorem 1.2.

Appendix: Weak convergence from convergence of Laplace transforms

It is well known that convergence of Laplace transforms in a neighborhood of $\mathbf{0} \in \mathbb{R}^d$ implies weak convergence, but it is less known under what conditions convergence on an open set away from the origin suffices. Hoffmann-Jørgensen [31, Section 5.14, page 378, (5.14.8)] and Mukherjea, Rao, and Suen [36, Theorem 2] independently discovered the pertinent result in the univariate case, and the argument in [36] generalizes to the multivariate setting, compare also [29, Theorem 2.1].

Let $\mathbf{X}^{(n)} = (X_1^{(n)}, X_2^{(n)}, \dots, X_d^{(n)})$ be a sequence of random vectors with Laplace transform

$$L_n(\mathbf{z}) = L_n(z_1, \dots, z_d) = \mathbb{E} \exp\left(\sum_{j=1}^d z_j X_j^{(n)}\right).$$

Theorem A.1. *Suppose that $L_n(\mathbf{z})$ are finite and converge pointwise to a function $L(\mathbf{z})$ for all \mathbf{z} from an open set in \mathbb{R}^d . If on this open set $L(\mathbf{z})$ is the Laplace transform of a random variable $\mathbf{Y} = (Y_1, \dots, Y_d)$, then $\mathbf{X}^{(n)}$ converges in distribution to \mathbf{Y} .*

Proof. The proof is a modification of [36, proof of Theorem 2]. Denote by P_n the law of $\mathbf{X}^{(n)}$ and by P_∞ the law of \mathbf{Y} . Choose $\varepsilon > 0$ and $\mathbf{c} \in \mathbb{R}^d$ such that $L_n(\mathbf{z}) \rightarrow L(\mathbf{z})$ for all \mathbf{z} from the open ball $\|\mathbf{z} - \mathbf{c}\| < \varepsilon$. Set $C_n = L_n(\mathbf{c})$, $C_\infty = L(\mathbf{c})$ and consider probability measures

$$Q_n(d\mathbf{x}) = \frac{1}{C_n} e^{\mathbf{x} \cdot \mathbf{c}} P_n(d\mathbf{x}),$$

with $n = \infty$ standing for the limiting measure. Due to our choice of \mathbf{c} ,

$$\int_{\mathbb{R}^d} e^{t \cdot \mathbf{x}} Q_n(d\mathbf{x}) = \frac{L_n(\mathbf{t} + \mathbf{c})}{C_n} \rightarrow \frac{L(\mathbf{t} + \mathbf{c})}{C_\infty} = \int_{\mathbb{R}^d} e^{t \cdot \mathbf{x}} Q_\infty(d\mathbf{x})$$

for all $\|\mathbf{t}\| < \varepsilon$. By the ‘‘usual form’’ of the Laplace criterion (for example, by the Cramér–Wold device and Curtiss [13, Theorem 3]), this implies weak convergence of Q_n to Q_∞ , i.e. for every bounded continuous function h ,

$$\int_{\mathbb{R}^d} h(\mathbf{x}) Q_n(d\mathbf{x}) \rightarrow \int_{\mathbb{R}^d} h(\mathbf{x}) Q_\infty(d\mathbf{x}).$$

Mimicking [36] we take $0 \leq h \leq 1$, $\varepsilon > 0$ and note that $f(\mathbf{x}) = e^{\mathbf{x} \cdot \mathbf{c}}$ is strictly positive, so

$$h_\varepsilon(\mathbf{x}) = \frac{h(\mathbf{x})f(\mathbf{x})}{f(\mathbf{x}) + \varepsilon} \nearrow h(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ as } \varepsilon \searrow 0.$$

Therefore,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int h(\mathbf{x}) P_n(d\mathbf{x}) &\geq \liminf_{n \rightarrow \infty} \int h_\varepsilon(\mathbf{x}) P_n(d\mathbf{x}) \\ &= \lim_{n \rightarrow \infty} C_n \int \frac{h(\mathbf{x})}{f(\mathbf{x}) + \varepsilon} Q_n(d\mathbf{x}) \\ &= C_\infty \int \frac{h(\mathbf{x})}{f(\mathbf{x}) + \varepsilon} Q_\infty(d\mathbf{x}) = \int h_\varepsilon(\mathbf{x}) P_\infty(d\mathbf{x}). \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0$, we get

$$\liminf_{n \rightarrow \infty} \int h(\mathbf{x}) P_n(d\mathbf{x}) \geq \int h(\mathbf{x}) P_\infty(d\mathbf{x}).$$

Applying the above to $1 - h$, we see that

$$\lim_{n \rightarrow \infty} \int h(\mathbf{x}) P_n(d\mathbf{x}) = \int h(\mathbf{x}) P_\infty(d\mathbf{x})$$

for all continuous functions $0 \leq h \leq 1$, which ends the proof. □

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