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Matrix Dirichlet processes

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Abstract. Matrix Dirichlet processes, in reference to their reversible measure, appear in a natural way in many different models in probability. Applying the language of diffusion operators and the theory of boundary equations, we describe Dirichlet processes on the matrix simplex and provide two models of matrix Dirichlet processes, which can be realized by various projections, through the Brownian motion on the special unitary group and also through Wishart processes.

Résumé. Les processus de Dirichlet matriciels, en référence à leur mesure réversible, apparaissent de manière naturelle dans de nombreux modèles différents en probabilité. En utilisant la langage des opérateurs de diffusion et la théorie des équations de bord, nous décrivont les processus de Dirichlet sur le simplexe matriciel et proposont deux modèles pour les processus de Dirichlet matriciels, qui peuvent être réalisés par les projections diverses, par le movement brownien sur le groupe unitaire spécial et par les processus de Wishart.

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1. Introduction

The complex matrix simplex $\Delta_{n,d}$ is the set of the sequences $(Z^{(1)},\dots,Z^{(n)})$ of non negative $d\times d$ Hermitian matrices such that $\sum_{i=1}^n Z^{(i)}\leq \mathrm{Id}$, where the inequality is understood in the sense of Hermitian matrices. On the matrix simplex, there exist natural probability measures, with densities $C\det(Z^{(1)})^{a_1-1}\cdots\det(Z^{(n)})^{a_n-1}\times \det(\mathrm{Id}-\sum_{i=1}^n Z^{(i)})^{a_{n+1}-1}$ (see Section 4). As the natural extensions of the Dirichlet measures on the simplex, they are called matrix Dirichlet measures.

It turns out that on matrix simplex there exist many diffusion processes which admit matrix Dirichlet measures as reversible ones, and their generators may be diagonalized by a sequence of orthogonal polynomials whose variables are the entries of the matrices. Therefore the matrix simplex appears to be a polynomial domain as described in [4], see Section 2.

The purpose of this paper is to describe these diffusion processes that we call matrix Dirichlet processes. They appear in a natural way in different models in probability: in the projection of Brownian motions on $SU(N)$, and also in the projection of Wishart matrices, as we will see below.

To begin with, we deal with diffusion processes on the simplex which have Dirichlet measures as reversible ones. Dirichlet measures are multivariate generalizations of the beta distribution, and play an important role in statistics and population biology, for example the Wright–Fisher model. In this paper, we talk about Dirichlet processes, by

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which we mean that these diffusion processes on the simplex are polynomial models with Dirichlet measures as their reversible measures.

The matrix Dirichlet measures, being analogues of Dirichlet distributions, were first introduced by Gupta and Richards [12], as special cases of matrix Liouville measures. They have been deeply studied, for example by Olkin and Rubin [16], Gupta and Nagar [11], see also Letac [14,15] and references therein. They not only provide models for multiple random matrices, which are related with orthogonal polynomials, integration formulas, etc., but also reflect the geometry of spaces of matrices, see [11,13]. Therefore, it is natural to consider their corresponding diffusion processes. It is worth to mention that matrix Jacobi processes, which can be considered as a one matrix case of matrix Dirichlet processes, were introduced by Doumerc in [10]. Demni studied the large size limit of matrix Jacobi processes in two interesting papers [7,8].

Our interest of this topic not only lies in its importance in statistics and random matrices, but also in the fact that it provides a polynomial model of multiple matrices. In a polynomial model, the diffusion operator (the generator) can be diagonalized by orthogonal polynomials, and this leads to the algebraic description of the boundary. Such polynomial models are quite rare: up to affine transformation, there are 3 models in $\mathbb{R}$ [3] and 11 models on compact domains in $\mathbb{R}^2$ [4]. More recently, Bakry and Bressaud [1] provided new models in dimensions two and three, by relaxing the hypothesis in [4] that polynomials are ranked with respect to their natural degree, and investigating the finite groups of $O(3)$ and their invariant polynomials.

As we will see in this paper, the simplex and the matrix simplex are both polynomial domains where there exist many different polynomial models, which is a situation even more rare: in general, there is just one polynomial diffusion process on a given domain, up to scaling. The situation here is quite complicated, since we are dealing with a family of matrices. By applying the theory of boundary equations, as introduced in [5], we are able to describe Dirichlet processes on the simplex, and we provide two models of such matrix Dirichlet processes. Our two models are found to be realized, via various projections, through the Brownian motion on the special unitary group and through the Wishart processes. This leads to some efficient ways to describe image measures in such projections.

This paper is organized as follows. In Section 2, we present the basics on diffusion operators and polynomial models that we will use in this paper. In Section 3, we introduce the Dirichlet process on the simplex, and describe its realizations in the special cases from spherical Laplacian, and also from Ornstein–Uhlenbeck or Laguerre processes. In Section 4, we give the description of two models. In Section 5, we present the realizations of matrix Dirichlet processes through projections of the Brownian motion on special unitary group and of Wishart processes which are extensions of their scalar counterparts.

2. Symmetric diffusion operators and polynomial models

We present in this section a brief introduction to symmetric diffusion processes and operators, in particular to those diffusion operators which may be diagonalized in a basis of orthogonal polynomials.

2.1. *Symmetric diffusion operators*

Symmetric diffusion operators are described in [2], which we refer the reader to for further details. Moreover, for those associated with orthogonal polynomials, we refer to the paper [4]. Although the description that we provide below is quite similar to that in [1], we choose to present it here for completeness.

Diffusion processes are Markov processes with continuous trajectories in some open set of $\mathbb{R}^n$ or on some manifold, usually given as solutions of stochastic differential equations. They are described by their infinitesimal generators, which are called diffusion operators.

Diffusion operators are second order differential operators with no zero order terms. When those operators have smooth coefficients, they are given in some open subset Ω of $\mathbb{R}^d$ by their action on smooth, compactly supported function f on Ω,

$$\mathrm{L}(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{2.1}$$

where the symmetric matrix $(g^{ij})(x)$ is everywhere non negative, i.e. the operator L is semi-elliptic.

A Markov process $(\xi_t)_{t\geq 0}$ is associated to such a diffusion operator through the requirement that the process $f(\xi_t) - \int_0^t \mathrm{L}(f)(\xi_s)\,ds$ is a local martingale for any function f in the domain of the operator L.

In this paper we will concentrate on the elliptic case (that is when the matrix (g^{ij}) is everywhere non degenerate), and on the case where this operator is symmetric with respect to some probability measure μ: for any smooth functions f, g, compactly supported in Ω, we have

$$\int_\Omega f\mathrm{L}(g)\,d\mu = \int_\Omega g\mathrm{L}(f)\,d\mu. \tag{2.2}$$

We say that μ is a reversible measure for L when the associated stochastic process $(\xi_t)_{t\geq 0}$ has a law which is invariant under time reversal, provided that the law at time 0 of the process is μ. In particular, the measure is invariant: when the associated process $(\xi_t)_{t\geq 0}$ is such that the law of ξ_0 is μ, the law of $(\xi_t)_{t\geq 0}$ is μ for any time $t > 0$.

When μ has a smooth positive density ρ with respect to the Lebesgue measure, the symmetry property (2.2) is equivalent to

$$b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}\partial_j \log \rho, \tag{2.3}$$

where $b^i(x)$ is the drift coefficient appearing in equation (2.1). In general by equation (2.3) we are able to completely determine μ up to some normalizing constant.

Now we introduce the carré du champ operator Γ. Suppose that we have some dense algebra $\mathcal{A}$ of functions in $\mathcal{L}^2(\mu)$ which is stable under the operator L and contains the constant functions. Then for $(f, g) \in \mathcal{A}$ we define

$$\Gamma(f, g) = \frac{1}{2}\big(\mathrm{L}(fg) - f\mathrm{L}(g) - g\mathrm{L}(f)\big). \tag{2.4}$$

If L is given by equation (2.1), and the elements of $\mathcal{A}$ are at least $\mathcal{C}^2$, we have

$$\Gamma(f, g) = \sum_{ij} g^{ij}\partial_i f \partial_j g,$$

so that Γ describes in fact the second order part of L. The semi-ellipticity of L gives rise to the fact that $\Gamma(f, f) \geq 0$, for any $f \in \mathcal{A}$.

Applying formula (2.2) with $g = 1$, we obtain $\int_\Omega \mathrm{L}f\,d\mu = 0$ for any $f \in \mathcal{A}$. Then with (2.2) again, we see immediately that for any $(f, g) \in \mathcal{A}$

$$\int_\Omega f\mathrm{L}(g)\,d\mu = -\int_\Omega \Gamma(f, g)\,d\mu, \tag{2.5}$$

so that the knowledge of Γ and μ describe entirely the operator L. Such a triple (Ω, Γ, μ) is called a Markov triple in [2].

By (2.1), we see that $\mathrm{L}(x^i) = b^i$ and $\Gamma(x^i, x^j) = g^{ij}$. The operator Γ is called the co-metric, and in our system of coordinates is described by a matrix $\Gamma = (\Gamma(x^i, x^j)) = (g^{ij})$.

In our setting, we will always assume that Ω is bounded and choose $\mathcal{A}$ to be the set of polynomials. Since polynomials are not compactly supported in Ω, the validity of equation (2.2) requires extra conditions on the coefficients (g^{ij}) at the boundary of Ω, which we will describe below.

The fact that L is a second order differential operator implies the change of variable formulas. Whenever $f = (f_1, \dots, f_n) \in \mathcal{A}^n$ and $\Phi(f_1, \dots, f_n) \in \mathcal{A}$, for some smooth function $\Phi : \mathbb{R}^n \mapsto \mathbb{R}$, we have

$$\mathrm{L}\big(\Phi(f)\big) = \sum_i \partial_i\Phi(f)\mathrm{L}(f_i) + \sum_{ij}\partial^2_{ij}\Phi(f)\Gamma(f_i, f_j) \tag{2.6}$$

and also

$$\Gamma\big(\Phi(f), g\big) = \sum_i \partial_i\Phi(f)\Gamma(f_i, g). \tag{2.7}$$

When $\mathcal{A}$ is the algebra of polynomials, properties (2.6) and (2.7) are equivalent.

An important feature in the examples described in this paper is the notion of image. Whenever we have a diffusion operator L on some set Ω, it may happen that we find some functions $(X_1, \dots, X_p)$ in the algebra $\mathcal{A}$ such that $\mathrm{L}(X_i) = B^i(X)$ and $\Gamma(X_i, X_j) = G^{ij}(X)$ where $X = (X_1, \dots, X_p)$. Then we say that we have a closed system. If $(\xi_t)_{t\geq 0}$ is the Markov diffusion process with generator L, $(\zeta_t) = X(\xi_t)$ is again a diffusion process, with its generator expressed in coordinates $(X_1, \dots, X_p)$

$$\hat{\mathrm{L}} = \sum_{ij} G^{ij}(X)\partial^2_{ij} + B^i(X)\partial_i.$$

Moreover, when L has a reversible probability measure μ, $\hat{\mathrm{L}}$ has the image measure of μ through the map X as its reversible measure. And equation (2.3) is an efficient way to determine image measure, which will be used many times in this paper.

2.2. *Polynomial models*

As mentioned above, we will restrict our attention to the elliptic case. Here we expect L to have a self adjoint extension (not unique in general), thus it has a spectral decomposition. Also we expect that the spectrum is discrete thus it has eigenvectors, which we will require to be polynomials in the variables (x^i). Moreover, we will require that those polynomial eigenvectors to be ranked according to their degrees, i.e., if we denote by $\mathcal{H}_n$ the space of polynomials with total degree at most n, then for each n we need that there exists an orthonormal basis of $\mathcal{H}_n$ which is made of eigenvectors for L. Equivalently, we require that L maps $\mathcal{H}_n$ into itself. This situation is quite rare, and imposes some strong restriction on the domain Ω that we will describe below.

When we have such a Markov triple (Ω, Γ, μ), where Ω has a piecewise smooth (at least $\mathcal{C}^1$) boundary, μ has a smooth density with respect to the Lebesgue measure on Ω. Then we call Ω a polynomial domain and (Ω, Γ, μ) a polynomial model.

In dimension 1 for example, up to affine transformations, there are only 3 cases of polynomial models, corresponding to the Jacobi, Laguerre and Hermite polynomials, see for example [3].

1. The Hermite case corresponds to the case where $\Omega = \mathbb{R}$, μ is the Gaussian measure $\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$ on $\mathbb{R}$ and L is the Ornstein–Uhlenbeck operator
$$\mathrm{L}_{\mathrm{OU}} = \frac{d^2}{dx^2} - x\frac{d}{dx}. \tag{2.8}$$
The Hermite polynomial H_n of degree n satisfy $\mathrm{L}_{\mathrm{OU}} P_n = -nP_n$.
2. The Laguerre polynomials correspond to the case where $\Omega = (0, \infty)$, the measure μ depends on a parameter $a > 0$ and is $\mu_a(dx) = C_a x^{a-1} e^{-x}\,dx$ on $(0, \infty)$, and L is the Laguerre operator
$$\mathrm{L}_a = x\frac{d^2}{dx^2} + (a - x)\frac{d}{dx}. \tag{2.9}$$
The Laguerre polynomial $L_n^{(a)}$ with degree n satisfies $\mathrm{L}_a L_n^{(a)} = -nL_n^{(a)}$.
3. The Jacobi polynomials correspond to the case where $\Omega = (-1, 1)$, the measure μ depends on two parameters a and b, $a, b > 0$ and is is $\mu_{a,b}(dx) = C_{a,b}(1 - x)^{a-1}(1 + x)^{b-1}\,dx$ on $(-1, 1)$, and L is the Jacobi operator
$$\mathrm{L}_{a,b} = \bigl(1 - x^2\bigr)\frac{d^2}{dx^2} - \bigl(a - b + (a + b)x\bigr)\frac{d}{dx}. \tag{2.10}$$
The Jacobi polynomial $(J_n^{(a,b)})_n$ with degree n satisfy
$$\mathrm{L}_{a,b} J_n^{(a,b)} = -n(n + a + b - 1)J_n^{(a,b)}.$$

In higher dimensions, when Ω is bounded, we recall from [4] the following results.

Proposition 2.1. *Let (Ω, Γ, μ) be a polynomial model in $\mathbb{R}^d$. Then, with* L *described by equation* (2.1), *we have*

1. *For $i = 1, \dots, d$, b^i is a polynomial with* $\deg(b^i) \le 1$.
2. *For $i, j = 1, \dots, d$, g^{ij} is a polynomial with* $\deg(g^{ij}) \le 2$.
3. *The boundary $\partial\Omega$ is included in the algebraic set* $\{\det(g^{ij}) = 0\}$.
4. *If $\{P_1 \cdots P_k = 0\}$ is the reduced equation of the boundary $\partial\Omega$ (see Remark* 2.3 *below), then, for each $q = 1, \dots, k$, each $i = 1, \dots, d$, one has*
$$\Gamma(\log P_q, x_i) = L_{i,q}, \tag{2.11}$$
where $L_{i,q}$ is a polynomial with $\deg(L_{i,q}) \le 1$;
5. *All the measures $\mu_{\alpha_1,\dots,\alpha_k}$ with densities $C_{\alpha_1,\dots,\alpha_k}|P_1|^{\alpha_1} \cdots |P_k|^{\alpha_k}$ on Ω, where the α_i are such that the density is is integrable on Ω, are such that $(\Omega, \Gamma, \mu_{\alpha_1,\dots,\alpha_k})$ is a polynomial model.*
6. *When the degree of $P_1 \cdots P_k$ is equal to the degree of* $\det(g^{ij})$, *there are no other measures.*

Conversely, assume that some bounded domain Ω is such that the boundary $\partial\Omega$ is included in an algebraic surface and has reduced equation $\{P_1 \cdots P_k = 0\}$. Assume moreover that there exists a matrix $(g^{ij}(x))$ which is positive definite in Ω and such that each component $g^{ij}(x)$ is a polynomial with degree at most 2. *Let Γ denote the associated carré du champ operator. Assume moreover that equation* (2.11) *is satisfied for any $i = 1, \dots, d$ and any $q = 1, \dots, k$, with $L_{i,q}$ a polynomial with degree at most* 1.

Let $(\alpha_1, \dots, \alpha_k)$ be such that the $|P_1|^{\alpha_1} \cdots |P_k|^{\alpha_k}$ is integrable on Ω with respect to the Lebesgue measure, and denote

$$\mu_{\alpha_1,\dots,\alpha_k}(dx) = C_{\alpha_1,\dots,\alpha_k}|P_1|^{\alpha_1} \cdots |P_k|^{\alpha_k}\, dx,$$

where $C_{\alpha_1,\dots,\alpha_k}$ is the normalizing constant such that $\mu_{\alpha_1,\dots,\alpha_k}$ is a probability measure.

Then $(\Omega, \Gamma, \mu_{\alpha_1,\dots,\alpha_k})$ is a polynomial model.

Remark 2.2. The determination of the polynomial domains therefore amounts to the determination of the domains Ω with an algebraic boundary, with the property that the reduced equation of $\partial\Omega$ is such that the set of equations (2.11) has a non trivial solution, for g^{ij} and $L_{i,q}$. Given the reduced equation of $\partial\Omega$, equations (2.11) are linear homogeneous ones in the coefficients of the polynomials g^{ij} and of the polynomials $L_{i,k}$. Unfortunately, in general we need much more equations to determine the unknowns uniquely, and this requires very strong constraints on the polynomials appearing in the reduced equation of the boundary. We will see that both the simplex and the matrix simplex (in the complex and real case) are such domains where the choice of the co-metric Γ is not unique.

Remark 2.3. The set of equations (2.11), which are central in the study of polynomial models, may be reduced to less equations, when $k > 1$. Indeed, if we set $P = P_1 \cdots P_k$, it reduces to

$$\Gamma(x_i, \log P) = L_i, \quad \deg(L_i) \le 1. \tag{2.12}$$

In fact assume that this last equation holds with some polynomial L_i, then on the regular part of the boundary described by $\{P_q(x) = 0\}$, we have $\Gamma(x_i, P_q) = 0$ since

$$\Gamma(x_i, P_q) = P_q\left(L_i - \sum_{l \ne q} \frac{\Gamma(x_i, P_l)}{P_l}\right).$$

Therefore, $\Gamma(x_i, P_q)$ vanishes on the regular part, and P_q is irreducible, dividing $\Gamma(x_i, P_q)$, which leads to $\Gamma(x_i, P_q) = L_{i,q} P_q$, where $\deg(L_{i,q}) \le 1$. Thus we obtain the equation (2.11).

Remark 2.4. A bounded polynomial domain is therefore any bounded domain Ω with algebraic boundary, on which there exists a symmetric matrix (g^{ij}) with entries which are polynomials with degree at most 2, that is positive definite on Ω and defines on Ω an operator Γ satisfying equation (2.12).

In [4], a complete description of all polynomial domains and models in dimension 2 is provided: 11 different cases are given up to affine transformations. This description only relies on algebraic considerations on those algebraic curves in the plane where the boundary condition (2.11) has a non trivial solution. This reflects the fundamental role played by the boundary equation.

Among the 11 bounded domains provided by the classification in [4], the triangle appears to be one of the few ones (with the unit ball and a particular case of the parabolic bi-angle) where the metric is not unique, which may be generalized in higher dimension as the simplex. Here we extend it furthermore to the matrix simplex.

3. Dirichlet measure on the simplex

In this section, we recall a few facts about the simplex and provide some diffusion operators on it which shows that the simplex is a polynomial domain in the sense of Section 2.

Definition 3.1. The n dimensional simplex $\Delta_n \subset \mathbb{R}^n$ is the set

$$\Delta_n = \left\{ 0 \le x_i \le 1, i = 1, \dots, n, \sum_{i=1}^n x_i \le 1 \right\}.$$

Given $\mathbf{a} = (a_1, a_2, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$, where $a_i > 0$, $i = 1, \dots, n+1$, the Dirichlet distribution $D_{\mathbf{a}}$ is the probability measure given by

$$\frac{1}{B_{\mathbf{a}}} x_1^{a_1-1} \cdots x_n^{a_n-1} (1 - x_1 - \cdots - x_n)^{a_{n+1}-1} \mathbf{1}_{\Delta_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n,$$

where $B_{\mathbf{a}} = \frac{\Gamma(a_1)\cdots\Gamma(a_{n+1})}{\Gamma(a_1+\cdots+a_{n+1})}$ is the normalizing constant.

The Dirichlet measure can be considered as a n-dimensional generalization of the beta distribution on the real line,

$$\beta(a_1, a_2) = \frac{1}{B(a_1, a_2)} x^{a_1-1}(1-x)^{a_2-1} \mathbf{1}_{(0,1)}(x)\, dx,$$

which is indeed $D_{(a_1,a_2)}$.

It turns out that the simplex Δ_n is a polynomial diffusion domain in $\mathbb{R}^n$ in the sense of [4], as described in Section 2. More precisely, there exist many different polynomial models on the simplex which admit the Dirichlet measure as their reversible measures.

Theorem 3.2. *Let $\mathbf{A} = (A_{ij})_{i,j=1,\dots,n+1}$ be a symmetric $(n+1) \times (n+1)$ matrix where all the coefficients A_{ij} are non negative, and $A_{ii} = 0$. Let $\mathbf{a} = (a_1, \dots, a_{n+1})$ be a $(n+1)$-tuple of positive real numbers. Let $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ be the symmetric diffusion operator defined on the simplex Δ_n defined by*

$$\Gamma_{\mathbf{A}}(x_i, x_j) = -A_{ij} x_i x_j + \delta_{ij} \sum_{k=1}^{n+1} A_{ik} x_k x_i, \tag{3.1}$$

$$\mathrm{L}_{\mathbf{A},\mathbf{a}}(x_i) = -x_i \sum_{j=1}^{n+1} A_{ij} a_j + a_i \sum_{j=1}^{d+1} A_{ij} x_j, \tag{3.2}$$

where $x_{n+1} = 1 - \sum_{j=1}^n x_j$.

Then, as soon as $A_{ij} > 0$ for all $i, j = 1, \dots, n+1$, $i \neq j$, $(\Delta_n, D_{\mathbf{a}}, \Gamma_{\mathbf{A}})$ is a polynomial model, and the operator $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ is elliptic on Δ_n. Moreover, any polynomial diffusion model on Δ_n having $D_{\mathbf{a}}$ as its reversible measure has this form (however without the requirement that the coefficients A_{ij} are positive).

Observe that the condition $A_{ii}=0$ is irrelevant in formulas (3.1), since the coefficient A_{ii} vanishes in the formulas.

Proof of Theorem 3.2. Let us prove first that any polynomial model with the usual degree has this form. According to Proposition 2.1, to be a polynomial model, $\Gamma(x_i,x_j)$ must be a polynomial no more than degree 2 and satisfy the boundary equation (2.11): for $1\le i,j\le n$,

$$\Gamma_{\mathbf{A}}(x_i,\log x_j)=L^{i,j}, \tag{3.3}$$

$$\Gamma_{\mathbf{A}}(x_i,\log x_{n+1})=L^{i,n+1}, \tag{3.4}$$

where $\{L^{i,j}, 1\le j\le n+1\}$ are polynomials with degree at most 1.

From equation (3.3), we get

$$\Gamma_{\mathbf{A}}(x_i,x_j)=x_jx_jL^{ij},$$

so that $\Gamma_{\mathbf{A}}(x_i,x_j)$ is divisible by x_i, and similarly by x_j. Therefore when $i\neq j$, there exists a constant A_{ij}, with $A_{ij}=A_{ij}$, such that

$$\Gamma_{\mathbf{A}}(x_i,x_j)=-A_{ij}x_jx_i.$$

When $i=j$, $\Gamma_{\mathbf{A}}(x_i,x_i)$ is divisible by x_i and from equation (3.4), we obtain

$$\sum_{j=1}^{n}\Gamma_{\mathbf{A}}(x_i,x_j)=-x_{n+1}L^{i,n+1},$$

which writes

$$\Gamma_{\mathbf{A}}(x_i,x_i)=x_i\left(\sum_{j=1,j\neq i}^{n}A_{ij}x_j\right)-x_{n+1}L^{i,n+1}.$$

This implies that x_i divides $L^{i,n+1}$, and therefore that $L^{i,n+1}=-A_{i(n+1)}x_i$, so that

$$\Gamma_{\mathbf{A}}(x_i,x_i)=x_i\sum_{i=1,j\neq i}^{n+1}A_{ij}x_j.$$

Conversely, it is quite immediate every operator $\Gamma_{\mathbf{A}}$ defined by equation (3.3) satisfies the boundary condition on the simplex.

The formulas for $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ are just a direct result from the reversible measure equation (2.3).

The ellipticity of $\Gamma_{\mathbf{A}}$ when all the coefficients $A_{ij}, i\neq j$ are positive is a particular case of the real version of Theorem 4.1, which will be proved in Section 4. □

When all the coefficients A_{ij} are equal to 1, the operator (3.1) is an image of the spherical Laplacian when all the parameters a_i are half integers. Indeed, consider the unit sphere $\sum_{i=1}^{N}y_i^2=1$ in $\mathbb{R}^N$, and split the set $\{1,\dots,N\}$ in a partition of $n+1$ disjoint subsets $I_1,\dots,I_{n+1}$ with size $p_1,\dots,p_{n+1}$. For $i=1,\dots,n+1$, set $x_i=\sum_{j\in I_i}y_j^2$, so that $x_{n+1}=1-\sum_{i=1}^{n}x_i$. Then it is easy to check that, for the spherical Laplace operator $\Delta_{\mathbb{S}^{N-1}}$ in $\mathbb{R}^N$ (see [2] for details), $\Delta_{\mathbb{S}^{N-1}}(x_i)$ and $\Gamma_{\mathbb{S}^{N-1}}(x_i,x_j)$ coincide with those given in equations (3.1) whenever $a_i=\frac{p_i}{2}$.

The next proposition generalizes this geometric interpretation for the general choice of the parameters A_{ij}.

Proposition 3.3. *Let $I_1,\dots,I_{n+1}$ be a partition of $\{1,\dots,N\}$ into disjoint sets with size $p_1,\dots,p_{n+1}$. For $i,j\in\{1,\dots,n+1\}$, with $i\neq j$, let $\mathrm{L}_{i,j}$ be the operator acting on the unit sphere $\mathbb{S}^{N-1}\subset\mathbb{R}^N$ as*

$$\mathrm{L}_{i,j}=\sum_{p\in I_i,q\in I_j}(y_p\partial_{y_q}-y_q\partial_{y_p})^2.$$

Then, setting $x_i = \sum_{p\in I_i} y_i^2$, $\{x_1,\dots,x_{n+1}\}$ *are a closed system for any* $\mathrm{L}_{i,j}$, *and the image of*

$$\frac{1}{2}\sum_{i<j} A_{ij}\mathrm{L}_{i,j}$$

is the operator $\mathrm{L}_{\mathbf{A},\mathbf{a}}$, *where* $a_i = \frac{p_i}{2}$.

The proof follows from a direct application of the change of variable formulas (2.6) and (2.7) applied to the functions $\{x_p\}$. For any operator $\mathrm{L}_{i,j}$, the associated carré du champ operator is $\Gamma_{i,j}(f,g) = \sum_{p\in I_i, q\in I_j}(y_p\partial_{y_q} - y_q\partial_{y_p})(f)(y_p\partial_{y_q} - y_q\partial_{y_p})(g)$. We just have to identify $\mathrm{L}_{i,j}(x_p)$ and $\Gamma_{i,j}(x_p,x_q)$ through a direct and easy computation, and observe that it fits with the coefficients of A_{ij} in the same expression in the definition of $\mathrm{L}_{\mathbf{A},\mathbf{a}}$.

It is worth to observe that the spherical Laplace operator is nothing else than $\sum_{i,j}\mathrm{L}_{i,j}$. The disappearance of the operators $\mathrm{L}_{i,i}$ in the general form for $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ comes from from the fact that the action of $\mathrm{L}_{i,i}$ on any of the variables x_p vanishes.

To come back to the case where all the coefficients A_{ij} are equal to 1, and as a consequence of the previous observation, the Dirichlet measure on the simplex is an image of the uniform measure on the sphere when the parameters are half integers, through the map that we just described $(y_1,\dots,y_N)\mapsto(x_1,\dots,x_d)$ where $x_i = \sum_{j\in I_i} y_j^2$. In the same way that the uniform measure on a sphere is an image of a Gaussian measure through the map $y\in\mathbb{R}^N \mapsto \frac{y}{\|y\|}\in\mathbb{S}^{N-1}$, the Dirichlet measure may be seen as the law of the random variables $\sum_{j\in I_i} y_j^2$ where y_j are independent standard Gaussian variables. More precisely, if $(y_1,\dots,y_N)$ are independent real valued Gaussian variables, setting $S_i = \sum_{j\in I_i} y_j^2$ and $S = \sum_{i=1}^{d+1} S_i$, we see that $(\frac{S_1}{S},\dots,\frac{S_d}{S})$ follows the Dirichlet law $D_{\mathbf{a}}$ whenever $a_i = \frac{p_i}{2}$.

Since the variables S_i have γ distribution $\frac{1}{\Gamma(\frac{p_i}{2})2^{\frac{p_i}{2}}}x^{\frac{p_i}{2}-1}e^{-x/2}\,dx$, it is not surprising that this representation of Dirichlet laws may be extended for the general case, that is when the parameters a_i are no longer half integers, by replacing norms of Gaussian vectors by independent variables having γ distribution. We quote the following proposition from [14], which gives a construction of Dirichlet random variable through gamma distributions $\gamma_{\alpha,\beta}$ on $\mathbb{R}^+$ given by

$$\gamma_{\alpha,\beta}(dx) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}\mathbf{1}_{(0,\infty)}(x)\,dx,$$

where $\alpha > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$.

Proposition 3.4. *Consider independent random variables* $x_1,\dots,x_n,x_{n+1}$ *such that each* x_k *has gamma distribution* $\gamma_{\alpha_k,2}$. *Define* $S = \sum_{k=1}^{n+1} x_k$. *Then* S *is independent of* $\frac{1}{S}(x_1,\dots,x_n)$, *and the distribution of* $\frac{1}{S}(x_1,\dots,x_n)$ *has the density* $D_{(\alpha_1,\dots,\alpha_n)}$.

When $(y_1,\dots,y_p)$ are independent standard $\mathcal{N}(0,1)$ Gaussian variables, the random variable $x = \sum_{i=1}^p y_i^2$ follows a $\gamma_{p/2,2}$ distribution. Then Proposition 3.4 generalizes the previous discription of the Dirichlet law from standard Gaussian variables in $\mathbb{R}^N$.

Indeed, Proposition 3.4 is still valid at the level of the processes. Start first from an N-dimensional Ornstein–Uhlenbeck process, which admits the standard Gaussian measure in $\mathbb{R}^N$ as reversible measure. One should be careful here, since the spherical Brownian motion is not directly the image of an Ornstein–Uhlenbeck process through $x\mapsto \frac{x}{\|x\|}$. Indeed, writing the Ornstein–Uhlenbeck operator $\mathrm{L}_{\mathrm{OU}} = \Delta - x\cdot\nabla$ in $\mathbb{R}^N\setminus\{0\}$ in polar coordinates ($r = \|x\|$, $\phi = \frac{x}{\|x\|}$) $\in (0,\infty)\times\mathbb{S}^{N-1}$, we derive

$$\mathrm{L}_{\mathrm{OU}} = \partial_r^2 + \left(\frac{N-1}{r} - r\right)\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}},$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the spherical Brownian Laplace operator acting on the variable $\phi\in\mathbb{S}^{N-1}$. The structure of L_{OU} shows that it appears as a warped product of a one dimensional operator and the spherical Laplace operator.

Starting again with some Ornstein–Uhlenbeck process in $\xi_t = (\xi_t^{(1)}, \dots, \xi_t^{(N)})$ in $\mathbb{R}^N$, and cutting the set of indices in $n+1$ parts I_i as above, with $|I_i| = p_i$, we may now consider the variables $\sigma_t^{(i)} = S_i(\xi_t)$ and denote $\sigma_t = S(\xi_t)$, where $S_i(x) = \frac{1}{2}\sum_{j\in I_i} x_j^2$ and $S(x) = \sum_i S_i(x)$. Then $\sigma_t^{(i)}$ are independent Laguerre processes with generator $\mathrm{L}_{a_i}(f) = 2(xf'' + (a_i - x)f')$, where $a_i = \frac{p_i}{2}$. One sees that σ_t is again a Laguerre process with generator $\mathrm{L}_a(f) = 2(xf'' + (a-x)f')$, with $a = N/2$. Then, setting $\zeta_t^{(i)} = \frac{\sigma_t^{(i)}}{\sigma_t}$ and $\zeta_t = (\zeta_t^{(1)}, \dots, \zeta_t^{(n)})$, ζ_t takes values in the simplex Δ_n, and a simple computation shows that (σ_t, ζ_t) is a Markov process with generator $\mathrm{L}_a + \frac{1}{S}\mathrm{L}_{\mathbf{A},\mathbf{a}}$, where $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ is the operator defined in Theorem 3.2, with $A_{ij} = 2$ and $a_i = \frac{p_i}{2}$.

This may be extended to the general case where the parameters a_i are no longer half integers. Starting from the standard Laguerre operator, we have

Proposition 3.5. *Let $\bar{a} = (a_1, \dots, a_{n+1})$ be positive integers and $\Sigma_t = (\sigma_t^{(1)}, \dots, \sigma_t^{(n+1)})$ be independent Laguerre processes on $\mathbb{R}_+$ with generator $\mathrm{L}_{a_i}(f) = xf'' + (a_i - x)f'$. Let $\sigma_t = \sum_{i=1}^{n+1}\sigma_t^{(i)}$ and $\zeta_t = (\frac{\sigma_t^{(1)}}{\sigma_t}, \dots, \frac{\sigma_t^{(d)}}{\sigma_t})$. Then, the pair $(\sigma_t, \zeta_t) \in \mathbb{R}_+ \times \Delta_n$ is a diffusion process with generator*

$$S\partial_S^2 + (a - S)\partial_S + \frac{1}{S}\mathrm{L}_{\mathbf{A},\mathbf{a}},$$

where $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ is defined in Theorem 3.2 with $A_{ij} = 1$ for $i \neq j$, $\mathbf{a} = \bar{a}$ and $a = \sum_{i=1}^{n+1} a_i$.

Proof. Following the notations of Section 2, we consider he generator of the process Σ_t, which is

$$\mathrm{L}_{\bar{a}} = \sum_{i=1}^{n+1} y_i\partial_{y_i}^2 f + \sum_{i=1}^{n+1}(a_i - y_i)\partial_{y_i} f,$$

where $y = (y_1, \dots, y_{n+1}) \in \mathbb{R}_+^{(n+1)}$. Let $\Gamma_{\bar{a}}$ be its associated carré du champ operator. With $S = \sum_{i=1}^{n+1} y_i$ and $z_i = \frac{y_i}{S}$, $i = 1, \dots, n$, we just have to check that

1. $\mathrm{L}_{\bar{a}}(S) = a - S$;
2. $\mathrm{L}_{\bar{a}}(z_i) = \frac{1}{S}(a_i - az_i)$;
3. $\Gamma_{\bar{a}}(S, S) = S$;
4. $\Gamma_{\bar{a}}(S, z_i) = 0$;
5. $\Gamma_{\bar{a}}(z_i, z_j) = \frac{1}{S}(\delta_{ij}z_i - z_iz_j)$.

This follows directly from a straightforward computation. □

Remark 3.6. As pointed out by the referee, a special case of Proposition 3.5 has appeared in [17], where Warren and Yor showed that the ratio of two squared Bessel process is a real time-changed Jacobi process.

Remark 3.7. Proposition 3.5 provides a construction of the process on the simplex with generator $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ for the general $\mathbf{a}$ but only when $A_{ij} = 1$ for $i \neq j$. To obtain a construction in the general case, we may use the same generalization that we did on the sphere with the operators $\mathrm{L}_{i,j} = \sum_{k\in I_i, l\in I_j}(y_k\partial_l - y_l\partial_k)^2$, where $y = (y_i) \in \mathbb{S}^n$, i.e., consider the operator

$$\mathrm{L}_{\mathrm{OU}}^{\mathbf{A}} = \partial_r^2 + \left(\frac{N-1}{r} - r\right)\partial_r + \frac{1}{2r^2}\sum_{i<j} A_{ij}\mathrm{L}_{i,j},$$

where we use $\frac{1}{2}\sum_{i<j} A_{ij}\mathrm{L}_{i,j}$ instead of $\Delta_{\mathbb{S}^{N-1}}$ in the Ornstein–Uhlenbeck operator.

Its reversible measure is still $e^{-\frac{1}{2}\|x\|^2}\,dx_1\cdots dx_{N+1}$ on $\mathbb{R}^{N+1}$. Now let $\mathrm{L}_{\mathrm{OU}}^{\mathbf{A}}$ act on the variables $x^i = \sum_{p\in I_i} y_p^2$, and $S = \sum_i x_i$ $z_i = \frac{x_i}{S}$, $i = 1, \dots, n$, then it is easy to check that the image of $\mathrm{L}_{\mathrm{OU}}^{\mathbf{A}}$ is $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ with $a_i = \frac{p_i}{2}$.

4. Matrix Dirichlet processes

4.1. *The complex matrix simplex and their associated Dirichlet measures*

We now introduce complex matrix simplex, the complex matrix Dirichlet processes and their corresponding diffusion operators. In particular, we give two families of models. The first ones come from extracted matrices from the Brownian motion on $SU(N)$, and are analogues to the construction of the scalar Dirichlet process from spherical Brownian motion. Similarly, Proposition 3.5 may be extended to a similar construction from Wishart matrix processes, which are a natural matrix extension of Laguerre processes.

As the complex matrix simplex has its real version, it is not hard to derive the real counterparts of our results.

First we define the complex matrix generalization $\Delta_{n,d}$ of the n-dimensional simplex as the set of n-tuple of Hermitian non negative matrices $\{\mathbf{Z}^{(1)},\dots,\mathbf{Z}^{(n)}\}$ such that

$$\sum_{k=1}^{n}\mathbf{Z}^{(k)}\le \mathrm{Id}. \tag{4.1}$$

Accordingly, the complex matrix Dirichlet measure $D_{\mathbf{a}}$ on $\Delta_{n,d}$ is given by

$$C_{d;\mathbf{a}}\prod_{k=1}^{n}\det\bigl(\mathbf{Z}^{(k)}\bigr)^{a_k-1}\det\left(\mathrm{Id}-\sum_{k=1}^{n}\mathbf{Z}^{(k)}\right)^{a_{n+1}-1}\prod_{k=1}^{n}d\mathbf{Z}^{(k)}, \tag{4.2}$$

where $\mathbf{a}=(a_1,\dots,a_{n+1})$, $\{a_i\}_{i=1}^{n+1}$ are all positive constants, and $d\mathbf{Z}^{(k)}$ is the Lebesgue measure on the entries of $\mathbf{Z}^{(k)}$, i.e., if $\mathbf{Z}^{(k)}$ has complex entries $Z_{pq}^{(k)}=X_{pq}^{(k)}+iY_{pq}^{(k)}$, with $Z_{qp}^{(k)}=\bar{Z}_{pq}^{(k)}$,

$$d\mathbf{Z}^{(k)}=\prod_{1\le p\le q\le d}dX_{pq}^{(k)}dY_{pq}^{(k)}.$$

This measure is finite exactly when the constants a_i are positive, and the normalizing constant $C_{d;\mathbf{a}}$ makes $D_{\mathbf{a}}$ a probability measure on $\Delta_{n,d}$.

The normalization constant $C_{d;\mathbf{a}}$ may be explicitly computed with the help of the matrix gamma function

$$\Gamma_d(a)=\int_{A>0}e^{-\operatorname{trace}(A)}\det(A)^{a-1}\,dA=\pi^{\frac{1}{2}d(d-1)}\prod_{i=1}^{d}\Gamma(a+d-i),$$

where $\{A>0\}$ denotes the domain of $d\times d$ positive-definite, Hermitian matrices. Then, the normalization constant may be written as

$$C_{d;\mathbf{a}}=\frac{\prod_{i=1}^{n+1}\Gamma_d(a_i)}{\Gamma_d(\sum_{i=1}^{n+1}a_i)}.$$

It turns out that $\Delta_{n,d}$ is polynomial domain as described in Remark 2.4, with boundary described by the equation

$$\det\bigl(\mathbf{Z}^{(1)}\bigr)\cdots\det\bigl(\mathbf{Z}^{(n)}\bigr)\det\bigl(\mathrm{Id}-\mathbf{Z}^{(1)}-\cdots-\mathbf{Z}^{(n)}\bigr)=0, \tag{4.3}$$

which is a polynomial domain. For convenience, and as described in Section 2, we will use complex coordinates $(Z_{pq}^{(k)},\bar{Z}_{pq}^{(k)},1\le k\le n,1\le p<q\le d)$, and real ones $Z_{pp}^{(k)}$, to describe the various diffusion operators acting on $\Delta_{n,d}$. Moreover, instead of using $(Z_{ij}^{(k)},\bar{Z}_{ij}^{(k)})$, $1\le i<j\le d$ as coordinates, in our case it is simpler to use $(Z_{ij}^{(k)}$, $i,j=1,\dots,d)$, due to the fact that $\bar{Z}_{ij}^{(k)}=Z_{ji}^{(k)}$. As in the scalar case studied in Section 3, it turns out that there are many possible operators Γ acting on $\Delta_{n,d}$ such that $(\Delta_{n,d},\Gamma,D_{a_1,\dots,a_{n+1}})$ is a polynomial model.

To simplify the notations, we shall always set $\mathbf{Z}^{(n+1)}=\mathrm{Id}-\sum_{i=1}^{n}\mathbf{Z}^{(i)}$, with entries $Z_{ij}^{(n+1)}$, $i,j=1,\dots,d$.

According to the boundary equation (2.12), the carré du champ operator Γ of the matrix Dirichlet process is such that each entry must be a polynomial of degree at most 2 in the variables $(Z_{ij}^{(k)}, \bar{Z}_{ij}^{(k)})$ and must satisfy

$$\Gamma\big(Z_{ij}^{(p)}, \log\det\big(\mathbf{Z}^{(k)}\big)\big) = L^{p,k}$$

for every $p = 1, \dots, n$, $k = 1, \dots, n+1$, $i, j = 1, \dots d$, where $L^{p,k}$ is an affine function of the entries of $\{\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)}\}$. However, since there are many variables in the diffusion operator Γ, we are not in the position to describe all the possible solutions for Γ as we did in the scalar case. Therefore, we will restrict ourselves to a simpler form. Namely, we assume that for any $1 \le p, q \le n+1$,

$$\Gamma\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) = \sum_{abcdrs} \big(A_{ij,kl}^{p,q}\big)_{r,s}^{ab,cd} Z_{ab}^{(r)} Z_{cd}^{(s)},$$

with some constant coefficients $(A_{ij,kl}^{p,q})_{r,s}^{ab,cd}$. Then the tensor $(\mathbf{A}_{ij,kl}^{p,q})$, whose entries are denoted by $\{(A_{ij,kl}^{p,q})_{r,s}^{ab,cd}\}$, should satisfy the following restrictions:

- Since Γ is symmetric, we have $(A_{ij,kl}^{p,q})_{ab,cd} = (A_{kl,ij}^{q,p})_{ab,cd}$, and the choice of $(\mathbf{A}_{ij,kl}^{p,q})$ should ensure that Γ is elliptic on the matrix simplex.
- The fact that the diffusions live in the matrix simplex gives rise to

$$\sum_{p=1}^{n+1} \sum_{abcd} \big(A_{ij,kl}^{p,q}\big)_{ab,cd} Z_{ab}^{(p)} Z_{cd}^{(q)} = 0.$$

- The boundary equation (4.4) leads to

$$\Gamma\big(Z_{ij}^{(p)}, \log\det\mathbf{Z}^{(q)}\big) = \sum_{abcd} \big(Z^{(q)}\big)_{lk}^{-1} \big(A_{ij,kl}^{p,q}\big)_{ab,cd} Z_{ab}^{(p)} Z_{cd}^{(q)} = L^{p,q} \tag{4.4}$$

  for every $p = 1, \dots, n$, $q = 1, \dots, n+1$, where $L^{p,q}$ is an affine function in the entries of $\{\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)}\}$, and $(Z^{(q)})_{ij}^{-1}$ are the entries of the inverse matrix $(\mathbf{Z}^{(q)})^{-1}$.

  This last equation comes from the diffusion property for the operator Γ (equation (2.7)), together with the fact that, for any matrix $\mathbf{Z}$ with entries Z_{ij},

$$\partial_{Z_{ij}} \log\det(\mathbf{Z}) = Z_{ji}^{-1},$$

  where Z_{ij}^{-1} are the entries of the inverse matrix $\mathbf{Z}^{-1}$.

Even under above restrictions, it is still hard to give any complete description of such tensors $\{(\mathbf{A}_{ij,kl}^{p,q})\}$. In the following we give two models of matrix Dirichlet process, which appear quite naturally as projections from the Brownian motion on $SU(N)$ and from complex Wishart processes, as we have mentioned before.

4.2. *Two polynomial diffusion models on $\Delta_{n,d}$*

Our first model is defined by the following Theorem 4.1. As we will see in next section, it appears naturally in some projections of diffusion models on $SU(N)$.

Theorem 4.1. *Let the matrix $\mathbf{A} = (A_{pq})$, $1 \le p, q \le n+1$ be a symmetric matrix. Then, consider the diffusion $\Gamma_{\mathbf{A}}$ operator given by*

$$\Gamma_{\mathbf{A}}\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) = \delta_{pq} \sum_{s=1}^{n+1} A_{sp}\big(Z_{il}^{(s)} Z_{kj}^{(p)} + Z_{kj}^{(s)} Z_{il}^{(p)}\big) - A_{pq}\big(Z_{il}^{(q)} Z_{kj}^{(p)} + Z_{il}^{(p)} Z_{kj}^{(q)}\big) \tag{4.5}$$

for $1 \le p, q \le n$, $1 \le i, j, k, l \le d$. Following the notations of Section 4.1, in this model we have

$$\big(\mathbf{A}^{p,q}_{ij,kl}\big)^{ab,cd}_{r,s} = A_{rs}(\delta_{pq}\delta_{pr} - \delta_{pr}\delta_{qs})(\delta_{ak}\delta_{bj}\delta_{ci}\delta_{dl} + \delta_{ai}\delta_{bl}\delta_{ck}\delta_{dj}).$$

Then $\Gamma_{\mathbf{A}}$ is elliptic if and only if the matrix A has non negative entries and is irreducible. In this case, $(\Delta_{n,d}, \Gamma_{\mathbf{A}}, D_{\mathbf{a}})$ is a polynomial model and

$$\mathrm{L}_{\mathbf{A},\mathbf{a}}\big(Z^{(p)}_{ij}\big) = \sum_{q}^{n+1} 2(a_p + d - 1)A_{pq}Z^{(q)}_{ij} - \sum_{q}^{n+1} 2(a_q + d - 1)A_{pq}Z^{(p)}_{ij}. \tag{4.6}$$

We recall that a matrix A with non negative entries is irreducible if and only if for any $p \neq q$, there exits a path $p = p_0, p_1, \dots, p_k = q$ such that for any $i = 0, \dots, k-1$, $A_{p_i p_{i+1}} \neq 0$.

Proof of Theorem 4.5. Recall that our coordinates are the entries $Z^{(k)}_{ij}$ of the Hermitian matrices $\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)}$ and that $\mathbf{Z}^{(n+1)} = \mathrm{Id} - \sum_{k=1}^{n} \mathbf{Z}^{(k)}$.

The only requirement here (apart from the ellipticity property that we willl deal with below) is that equation (4.4) is satisfied for this particular choice of the tensor $(\mathbf{A}^{p,q}_{ij,kl})$. Thus for $1 \le p, q \le n$, we have

$$\Gamma_{\mathbf{A}}\big(\log\det\big(\mathbf{Z}^{(q)}\big), Z^{(p)}_{ij}\big) = \delta_{pq}\sum_{s}^{n+1} 2A_{sp}Z^{(s)}_{ij} - 2A_{pq}Z^{(p)}_{ij},$$

while

$$\Gamma_{\mathbf{A}}\big(\log\det\big(\mathbf{Z}^{(n+1)}\big), Z^{(p)}_{ij}\big) = -2A_{(n+1)p}Z^{(p)}_{ij},$$

which shows that the boundary equation is satisfied for this model.

We now prove that on the matrix simplex $\Delta_{n,d}$, $\Gamma_{\mathbf{A}}$ given by (4.5) is elliptic if and only if for $p, q = 1, \dots, n+1$, $A_{pq} > 0$.

For fixed (p, q), we consider $\Gamma_{\mathbf{A}}(Z^{(p)}_{ij}, Z^{(q)}_{kl})$ as the (ij, kl) element in a $d^2 \times d^2$ matrix; and $(\Gamma_{\mathbf{A}}(Z^{(p)}_{ij}, Z^{(q)}_{kl}))$ is the (p, q) element in a $n \times n$ matrix of $d^2 \times d^2$ matrices. Notice that we may write

$$(\Gamma_{\mathbf{A}}) = \sum_{1 \le p < q \le n} A_{pq}\Gamma^{(pq)} + \mathbf{A}_{n+1}\Gamma^{(n+1)},$$

where

1. (Γ^{pq}) is a $nd^2 \times nd^2$ block matrix with

$$\big(\Gamma^{(pq)}\big)_{pq,(ij,kl)} = -\big(Z^{(q)}_{il}Z^{(p)}_{kj} + Z^{(p)}_{il}Z^{(q)}_{kj}\big),$$
$$\big(\Gamma^{(pq)}\big)_{qp,(ij,kl)} = -\big(Z^{(q)}_{il}Z^{(p)}_{kj} + Z^{(p)}_{il}Z^{(q)}_{kj}\big),$$
$$\big(\Gamma^{(pq)}\big)_{pp,(ij,kl)} = Z^{(q)}_{il}Z^{(p)}_{kj} + Z^{(p)}_{il}Z^{(q)}_{kj},$$
$$\big(\Gamma^{(pq)}\big)_{qq,(ij,kl)} = Z^{(q)}_{il}Z^{(p)}_{kj} + Z^{(p)}_{il}Z^{(q)}_{kj},$$

   and other entries are 0;
2. $\Gamma^{(n+1)}$ is a diagonal block matrix with $\Gamma^{(n+1)}_{pp,(ij,kl)} = Z^{(n+1)}_{il}Z^{(p)}_{kj} + Z^{(p)}_{il}Z^{(n+1)}_{kj}$ and other entries are 0;
3. $\mathbf{A}_{n+1}$ is a diagonal block matrix of size $nd^2 \times nd^2$ satisfying $(\mathbf{A}_{n+1})_{pp} = A_{p(n+1)}\mathrm{Id}_{d^2\times d^2}$.

We first observe that each operator associated with $\Gamma^{(pq)}$ is non negative. To see this, we have to check that for any sequence $(\Lambda) = (\Lambda^{(1)}, \ldots, \Lambda^{(n)})$ of Hermitian matrices with entries $\lambda_{ij}^{(k)}$,

$$\sum_{rs,ij,kl} \Gamma^{(pq)}_{rs,(ij,\bar{k}l)} \lambda_{ij}^{(r)} \bar{\lambda}_{kl}^{(s)} \geq 0.$$

In fact,

$$\begin{aligned}\sum_{rs,ij,kl} \Gamma^{(pq)}_{rs,(ij,\bar{k}l)} \lambda_{ij}^{(r)} \bar{\lambda}_{kl}^{(s)} &= \bigl(\Lambda^{(p)} - \Lambda^{(q)}\bigr)\Gamma^{(pq)}_{pp}\bigl(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)}\bigr) \\ &= \operatorname{trace}\bigl(\mathbf{Z}^{(q)}\bigl(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)}\bigr)\mathbf{Z}^{(p)}\bigl(\Lambda^{(p)} - \Lambda^{(q)}\bigr)^t\bigr) \\ &\quad + \operatorname{trace}\bigl(\mathbf{Z}^{(p)}\bigl(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)}\bigr)\mathbf{Z}^{(q)}\bigl(\Lambda^{(p)} - \Lambda^{(q)}\bigr)^t\bigr).\end{aligned}$$

For any p, $\mathbf{Z}^{(p)}$ is Hermitian and non negative-definite, so are $(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)})\mathbf{Z}^{(p)}(\Lambda^{(p)} - \Lambda^{(q)})^t$, $(\Lambda^{(p)} - \Lambda^{(q)})^t\mathbf{Z}^{(p)}(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)})$, then we have

$$\begin{aligned}&\operatorname{trace}\bigl(\mathbf{Z}^{(q)}\bigl(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)}\bigr)\mathbf{Z}^{(p)}\bigl(\Lambda^{(p)} - \Lambda^{(q)}\bigr)^t\bigr) \geq 0, \\ &\operatorname{trace}\bigl(\mathbf{Z}^{(p)}\bigl(\bar{\Lambda}^{(p)} - \bar{\Lambda}^{(q)}\bigr)\mathbf{Z}^{(q)}\bigl(\Lambda^{(p)} - \Lambda^{(q)}\bigr)^t\bigr) \geq 0.\end{aligned}$$

Therefore $\Gamma^{(pq)}$ are all non negative-definite matrices.

Similarly, we have

$$\begin{aligned}&\sum_{p,ij,kl} A_{p(n+1)} \Gamma^{(n+1)}_{pp,(ij,\bar{k}l)} \lambda_{ij}^{(p)} \bar{\lambda}_{kl}^{(p)} \\ &\quad = \sum_p A_{p(n+1)} \bigl(\operatorname{trace}\bigl(\mathbf{Z}^{(n+1)}\bar{\Lambda}^{(p)}\mathbf{Z}^{(p)}\bigl(\Lambda^{(p)}\bigr)^t\bigr) + \operatorname{trace}\bigl(\mathbf{Z}^{(n+1)}\bigl(\Lambda^{(p)}\bigr)^t\mathbf{Z}^{(p)}\bar{\Lambda}^{(p)}\bigr)\bigr).\end{aligned}$$

Thus we know $\Gamma^{(n+1)} \geq 0$.

We then prove the following lemma,

Lemma 4.2. *Let A, B be $d \times d$ Hermitian positive definite matrices. Then for a given $d \times d$ matrix U, if* $\operatorname{trace}(AUBU^*) = 0$, *then* $U = 0$.

Proof. Suppose A has the spectral decomposition $A = P^*DP$, where P is unitary and $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_d\}$ with all λ_i positive. Then

$$\operatorname{trace}\bigl(AUBU^*\bigr) = \operatorname{trace}\bigl(P^*DPUBU^*\bigr) = \operatorname{trace}\bigl(DPUBU^*P^*\bigr).$$

Notice that since B is positive definite, if $PU \neq 0$, then $PUBU^*P^*$ is also positive definite, whose elements on the diagonal are all positive, implying $\operatorname{trace}(DPUBU^*P^*) > 0$. Therefore, $\operatorname{trace}(AUBU^*) = 0$ holds only when $PU = 0$. Since P is unitary, this happens only when $U = 0$. □

Recall the boundary equation of $\Delta_{n,d}$ (4.3), then inside $\Delta_{n,d}$ we know all $\mathbf{Z}^{(i)}$ are positive definite. Now we claim that $\Gamma_{\mathbf{A}}$ is elliptic inside $\Delta_{n,d}$ if and only if $\mathbf{A}$ has non negative entries and is irreducible.

Previous computations show that $\Gamma_{\mathbf{A}} \geq 0$ and if there exists Λ_0 such that $\Lambda_0\Gamma_{\mathbf{A}}\Lambda_0^* = 0$, we have

$$\begin{aligned}&\sum_{p<q}^{n} A_{pq}\bigl(\operatorname{trace}\bigl(\mathbf{Z}^{(q)}\bigl(\bar{\Lambda}_0^{(p)} - \bar{\Lambda}_0^{(q)}\bigr)\mathbf{Z}^{(p)}\bigl(\Lambda_0^{(p)} - \Lambda_0^{(q)}\bigr)^t\bigr) \\ &\qquad + \operatorname{trace}\bigl(\mathbf{Z}^{(p)}\bigl(\bar{\Lambda}_0^{(p)} - \bar{\Lambda}_0^{(q)}\bigr)\mathbf{Z}^{(q)}\bigl(\Lambda_0^{(p)} - \Lambda^{(q)}\bigr)_0^t\bigr)\bigr)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p}^{n} A_{p(n+1)}\big(\operatorname{trace}\big(\mathbf{Z}^{(n+1)}\bar{\Lambda}^{(p)}\mathbf{Z}^{(p)}\big(\Lambda^{(p)}\big)^{t}\big) \\
&+\operatorname{trace}\big(\mathbf{Z}^{(n+1)}\big(\Lambda^{(p)}\big)^{t}\mathbf{Z}^{(p)}\bar{\Lambda}^{(p)}\big)\big) \\
=&\ 0.
\end{aligned}
\tag{4.7}
$$

Since each term is non-negative, they should all vanish. By the fact that $\mathbf{A}$ is irreducible, we know that for any $1 \le p \le n$, there exists a path connecting p and $n+1$, noted by $p, p_1, p_2, \dots, p_r, (n+1)$, which leads to $A_{pp_1}, A_{p_1p_2}, \dots, A_{p_r(n+1)} > 0$. Then by Lemma 4.2, we have

$$
\Lambda_0^{(p)} = \Lambda_0^{(p_1)} = \cdots = \Lambda_0^{(p_r)} = 0,
$$

i.e. $\Lambda_0 = 0$, which implies that $\Gamma_{\mathbf{A}}$ is elliptic.

Conversely, when $\Gamma_{\mathbf{A}}$ is elliptic, then if $\Lambda_0\Gamma_{\mathbf{A}}\Lambda_0^* = 0$, we must have $\Lambda_0 = 0$. First we prove that $\mathbf{A}$ has non negative entries. For given $1 \le p \le (n+1)$, assume that some $A_{pq} < 0$ for some $q \neq p$. Then, choose the sequence $\Lambda_1 = (\Lambda_1^{(1)}, \dots, \Lambda_1^{(n)})$ to be such that $\Lambda^{(p)} = ((\mathbf{Z}^{(p)})^{-\frac{1}{2}})^t$ and all others are 0, then from $\Lambda_1\Gamma_{\mathbf{A}}\Lambda_1^* > 0$, we conclude that for $(\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)}) \in \Delta_{n,d}$,

$$
\sum_{r\neq p}^{n+1} 2A_{pr}\operatorname{trace}\big(\mathbf{Z}^{(r)}\big) > 0.
$$

Now, choose $Z^{(q)} = (1-\varepsilon)\mathrm{Id}$ and $Z^{(r)} = \frac{\varepsilon}{n}\mathrm{Id}$ for $r \neq q$, $0 < \varepsilon < 1$. Then when $\varepsilon < \frac{|A_{pq}|}{\sum_{s\neq p}^{n+1}|A_{ps}|}$ we obtain a contradiction.

Now suppose $\mathbf{A}$ is not irreducible, then it has at least two strongly connected components $\mathcal{A}_1, \mathcal{A}_2$, such that $A_{pq} = 0$ for $p \in \mathcal{A}_1$, $q \in \mathcal{A}_2$. Suppose $(n+1) \in \mathcal{A}_1$, then for all $p \in \mathcal{A}_1$, choose $\Lambda_0^{(p)} = 0$, while for $q \in \mathcal{A}_2$, $\Lambda_0^{(q)} \neq 0$, such that we have $\Lambda_0 \neq 0$ which satisfies $\Lambda_0\Gamma_{\mathbf{A}}\Lambda_0^* = 0$. Thus there is a contradiction, so we must have $\mathbf{A}$ to be irreducible.

Finally, by a direct computation, we have

$$
\begin{aligned}
\mathrm{L}_{\mathbf{A},\mathbf{a}}\big(Z_{ij}^{(p)}\big) &= \sum_{q=1}^{n+1}(a_q - 1)\Gamma\big(\log\det\big(\mathbf{Z}^{(q)}\big), Z_{ij}^{(p)}\big) + \sum_{q=1}^{n}\sum_{kl}\partial_{Z_{kl}^{(q)}}\Gamma\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) \\
&= \sum_{q}^{n+1} 2(a_p + d - 1)A_{pq}Z_{ij}^{(q)} - \sum_{q}^{n+1} 2(a_q + d - 1)A_{pq}Z_{ij}^{(p)}.
\end{aligned}
$$

□

The second model is given by the following Theorem 4.3, which can be naturally derived from the Ornstein–Uhlenbeck process on the complex matrices.

Theorem 4.3. *Let $\mathbf{A}$ be a $d \times d$ positive-definite Hermitian matrix and $\mathbf{B}$ be a $d^2 \times d^2$ positive-definite Hermitian matrix. Consider the diffusion $\Gamma_{\mathbf{A},\mathbf{B}}$ operator given by*

$$
\begin{aligned}
\Gamma_{\mathbf{A},\mathbf{B}}\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) &= \delta_{pq}\big(A_{il}Z_{kj}^{(p)} + A_{kj}Z_{il}^{(p)}\big) - A_{kj}\big(Z^{(p)}Z^{(q)}\big)_{il} - A_{il}\big(Z^{(q)}Z^{(p)}\big)_{kj} \\
&\quad + \sum_{ab}\big(B_{ia,lb}Z_{aj}^{(p)}Z_{kb}^{(q)} + B_{aj,bk}Z_{ia}^{(p)}Z_{bl}^{(q)} - B_{aj,lb}Z_{ia}^{(p)}Z_{kb}^{(q)} - B_{ia,bk}Z_{aj}^{(p)}Z_{bl}^{(q)}\big)
\end{aligned}
\tag{4.8}
$$

for $1 \le p, q \le n$, $1 \le i, j, k, l \le d$. Following the notations of Section 4.1, in this model we have

$$
\begin{aligned}
\big(\mathbf{A}_{ij,kl}^{p,q}\big)_{r,s}^{ab,cd} &= \delta_{pq}\delta_{pr}A_{cd}(\delta_{ak}\delta_{bj}\delta_{ci}\delta_{dl} + \delta_{ai}\delta_{bl}\delta_{ck}\delta_{dj}) - \delta_{pr}\delta_{qs}(A_{kj}\delta_{ai}\delta_{bc}\delta_{dl} + A_{il}\delta_{ck}\delta_{ad}\delta_{bj}) \\
&\quad + \delta_{pr}\delta_{qs}(B_{ia,ld}\delta_{bj}\delta_{ck} + B_{bj,ck}\delta_{ai}\delta_{dl} - B_{bj,ld}\delta_{ai}\delta_{ck} - B_{ia,ck}\delta_{bj}\delta_{dl}).
\end{aligned}
$$

Then $(\Delta_{n,d}, \Gamma_{\mathbf{A},\mathbf{B}}, D_{a_1,\dots,a_{n+1}})$ *is a polynomial model. Moreover, in this case,*

$$\begin{aligned}\mathrm{L}_{\mathbf{A},\mathbf{B},\mathbf{a}}\bigl(Z_{ij}^{(p)}\bigr) &= 2(a_p-1+d)A_{ij} - \sum_{q=1}^{n}(a_q-1+d)\bigl(\bigl(AZ^{(p)}\bigr)_{ij} + \bigl(Z^{(p)}A\bigr)_{ij}\bigr) \\ &\quad - (a_{n+1}-1)\bigl(\bigl(AZ^{(p)}\bigr)_{ij} + \bigl(Z^{(p)}A\bigr)_{ij}\bigr) - 2A_{ij}\operatorname{trace}\bigl(Z^{(p)}\bigr) \\ &\quad + \sum_{ab}\bigl(B_{ia,jb}Z_{ab}^{(p)} + B_{bj,ai}Z_{ab}^{(p)} - B_{ia,ba}Z_{bj}^{(p)} - B_{bj,ba}Z_{ia}^{(p)}\bigr),\end{aligned} \tag{4.9}$$

where $\mathbf{Z}^{(n+1)} = \mathrm{Id} - \sum_{p=1}^{n}\mathbf{Z}^{(p)}$.

Proof. First, let us show that equations (4.8) and (4.9) define a polynomial model. In fact, for $1 \le p, q \le n$,

$$\begin{aligned}\Gamma\bigl(Z_{ij}^{(p)}, \log\det\mathbf{Z}^{(q)}\bigr) &= 2A_{ij}\delta_{pq} - \bigl(AZ^{(p)}\bigr)_{ij} - \bigl(Z^{(p)}A\bigr)_{ij}, \\ \Gamma\bigl(Z_{ij}^{(p)}, \log\det\mathbf{Z}^{(n+1)}\bigr) &= -\bigl(AZ^{(p)}\bigr)_{ij} - \bigl(Z^{(p)}A\bigr)_{ij},\end{aligned}$$

which is a polynomial model by Proposition 2.1.

Direct computations yield

$$\begin{aligned}\mathrm{L}\bigl(Z_{ij}^{(p)}\bigr) &= 2(a_p-1+d)A_{ij} - \sum_{q=1}^{n}(a_q-1+d)\bigl(\bigl(AZ^{(p)}\bigr)_{ij} + \bigl(Z^{(p)}A\bigr)_{ij}\bigr) \\ &\quad - (a_{n+1}-1)\bigl(\bigl(AZ^{(p)}\bigr)_{ij} + \bigl(Z^{(p)}A\bigr)_{ij}\bigr) - 2A_{ij}\operatorname{trace}\bigl(Z^{(p)}\bigr) \\ &\quad + \sum_{ab}\bigl(B_{ia,jb}Z_{ab}^{(p)} + B_{bj,ai}Z_{ab}^{(p)} - B_{ia,ba}Z_{bj}^{(p)} - B_{bj,ba}Z_{ia}^{(p)}\bigr).\end{aligned}$$

Now we prove that if $\mathbf{A}$ is a $d \times d$ Hermitian and positive-definite matrix and $\mathbf{B}$ is a $d^2 \times d^2$ Hermitian and positive-definite matrix, then $\Gamma_{\mathbf{A},\mathbf{B}}$ is elliptic inside the matrix simplex $\Delta_{n,d}$. In fact, consider $\Gamma_{\mathbf{A},\mathbf{B}}$ as a $n \times n$ block matrix, and each block is of size $d^2 \times d^2$, then we may write

$$\Gamma_{\mathbf{A},\mathbf{B}} = \Gamma_{\mathbf{A}} + \Gamma_{\mathbf{B}},$$

where $\Gamma_{\mathbf{A}}$ is the block matrix containing $\mathbf{A}$ and $\Gamma_{\mathbf{B}}$ is the block matrix containing $\mathbf{B}$.

Let $(\Lambda^1, \dots, \Lambda^n)$ be any sequence of $d \times d$ Hermitian matrices. Then,

$$\begin{aligned}&\sum_{p,q=1}^{n}\sum_{ijkl}\lambda_{ij}^{p}\bar{\lambda}_{kl}^{q}\bigl(\Gamma_{\mathbf{A}}^{p,q}\bigr)_{ij,lk} \\ &\quad = \operatorname{trace}\left(A\left(\sum_{p=1}^{n}\bar{\Lambda}^{p}\mathbf{Z}^{(p)}\bigl(\Lambda^{p}\bigr)^{t} - \sum_{p,q=1}^{n}\bar{\Lambda}^{p}\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bigl(\Lambda^{q}\bigr)^{t}\right)\right) \\ &\qquad + \operatorname{trace}\left(A\left(\sum_{p=1}^{n}\bigl(\Lambda^{p}\bigr)^{t}\mathbf{Z}^{(p)}\bar{\Lambda}^{p} - \sum_{p,q=1}^{n}\bigl(\Lambda^{p}\bigr)^{t}\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bar{\Lambda}^{q}\right)\right).\end{aligned}$$

Then since A is positive definite, we just need to prove that

$$\sum_{p=1}^{n}\bar{\Lambda}^{p}\mathbf{Z}^{(p)}\bigl(\Lambda^{p}\bigr)^{t} - \sum_{p,q=1}^{n}\bar{\Lambda}^{p}\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bigl(\Lambda^{q}\bigr)^{t}, \qquad \sum_{p=1}^{n}\bigl(\Lambda^{p}\bigr)^{t}\mathbf{Z}^{(p)}\bar{\Lambda}^{p} - \sum_{p,q=1}^{n}\bigl(\Lambda^{p}\bigr)^{t}\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bar{\Lambda}^{q},$$

are non negative definite. For the first one, given a vector $X=(X_1,\dots,X_d)$, we have

$$\begin{aligned}
&X\left(\sum_{p=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\big(\Lambda^p\big)^t-\sum_{p,q=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\big(\Lambda^q\big)^t\right)X^*\\
&\quad=X\bar{\Lambda}\big(\mathbf{Z}-YY^*\big)\Lambda^t X^*\\
&\quad=X\bar{\Lambda}\mathbf{Z}^{\frac12}\big(\mathrm{Id}_{nd}-\big(\mathbf{Z}^{-\frac12}Y\big)\big(\mathbf{Z}^{-\frac12}Y\big)^*\big)\big(X\bar{\Lambda}\mathbf{Z}^{\frac12}\big)^*,
\end{aligned}$$

where Λ is a vector of matrices $\Lambda=(\Lambda^1,\dots,\Lambda^n)$, $\mathbf{Z}=\mathrm{diag}(\mathbf{Z}^{(1)},\dots,\mathbf{Z}^{(n)})$ and Y is a vector of matrices such that $Y=(\mathbf{Z}^{(1)},\dots,\mathbf{Z}^{(n)})^*$. Then by Sylvester determinant theorem, we are able to compute the eigenvalues of $\mathrm{Id}-(\mathbf{Z}^{-\frac12}Y)(\mathbf{Z}^{-\frac12}Y)^*$,

$$\begin{aligned}
&\det\big(\lambda\mathrm{Id}_{nd}-\big(\mathrm{Id}_{nd}-\big(\mathbf{Z}^{-\frac12}Y\big)\big(\mathbf{Z}^{-\frac12}Y\big)^*\big)\big)\\
&\quad=\det\big((\lambda-1)\mathrm{Id}_{nd}+\big(\mathbf{Z}^{-\frac12}Y\big)\big(\mathbf{Z}^{-\frac12}Y\big)^*\big)\\
&\quad=(\lambda-1)^{(n-1)d}\det\big(\lambda\mathrm{Id}-\mathbf{Z}^{(n+1)}\big).
\end{aligned}\tag{4.10}$$

Since $\mathbf{Z}^{(n+1)}=\mathrm{Id}-\sum_{p=1}^n\mathbf{Z}^{(p)}$ is also a non negative-definite Hermitian matrix, the above equation means that the eigenvalues of $\mathrm{Id}-(\mathbf{Z}^{-\frac12}Y)(\mathbf{Z}^{-\frac12}Y)^*$ are all non negative, indicating that it is a non negative definite matrix, such that

$$X\left(\sum_{p=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\big(\Lambda^p\big)^t-\sum_{p,q=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\big(\Lambda^q\big)^t\right)X^*\geq 0,$$

thus $\sum_{p=1}^n\bar{\Lambda}^p\mathbf{Z}^{(p)}(\Lambda^p)^t-\sum_{p,q=1}^n\bar{\Lambda}^p\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}(\Lambda^q)^t$ is non negative definite, so is $\sum_{p=1}^n(\Lambda^p)^t\mathbf{Z}^{(p)}\bar{\Lambda}^p-\sum_{p,q=1}^n(\Lambda^p)^t\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bar{\Lambda}^q$. Therefore by the fact that A is a positive-definite, Hermitian matrix, we have

$$\begin{aligned}
&\mathrm{trace}\left(A\left(\sum_{p=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\big(\Lambda^p\big)^t-\sum_{p,q=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\big(\Lambda^q\big)^t\right)\right)\geq 0,\\
&\mathrm{trace}\left(A\left(\sum_{p=1}^n \big(\Lambda^p\big)^t \mathbf{Z}^{(p)}\bar{\Lambda}^p-\sum_{p,q=1}^n \big(\Lambda^p\big)^t \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\bar{\Lambda}^q\right)\right)\geq 0.
\end{aligned}$$

Since

$$\mathrm{trace}\left(A\left(\sum_{p=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\big(\Lambda^p\big)^t-\sum_{p,q=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\big(\Lambda^q\big)^t\right)\right)=\mathrm{trace}\big(A\bar{\Lambda}\big(\mathbf{Z}-YY^*\big)\Lambda^t\big),$$

and by (4.10) we know that inside $\Delta_{n,d}$, $\mathbf{Z}-YY^*$ is positive definite. Thus following the proof of Lemma 4.2, we may conclude that if

$$\mathrm{trace}\left(A\left(\sum_{p=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\big(\Lambda^p\big)^t-\sum_{p,q=1}^n \bar{\Lambda}^p \mathbf{Z}^{(p)}\mathbf{Z}^{(q)}\big(\Lambda^q\big)^t\right)\right)=0,$$

then $\Lambda=0$. This implies that $\Gamma_{\mathbf{A}}$ is elliptic inside $\Delta_{n,d}$.

As for $\Gamma_{\mathbf{B}}$, notice that

$$\sum_{i,j,k,l=1}^d \lambda_{ij}^p\bar{\lambda}_{kl}^q\big(\Gamma_{\mathbf{B}}^{p,q}\big)_{ij,\bar{k}l}=\big(\Lambda^p\bar{\mathbf{Z}}^{(p)}-\bar{\mathbf{Z}}^{(p)}\Lambda^p\big)B\big(\bar{\mathbf{Z}}^{(q)}\big(\Lambda^q\big)^*-\big(\Lambda^q\big)^*\bar{\mathbf{Z}}^{(q)}\big).$$

Let $H^{(p)} = \Lambda^p \bar{\mathbf{Z}}^{(p)} - \bar{\mathbf{Z}}^{(p)} \Lambda^p$, since $\mathbf{B}$ is positive-definite, Hermitian in the sense that $B_{ij,kl} = \bar{B}_{kl,ij}$, then

$$\sum_{p,q=1}^{n} \sum_{i,j,k,l=1}^{d} \lambda_{ij}^{p} \bar{\lambda}_{kl}^{q} \left(\Gamma_B^{p,q}\right)_{ij,lk} = \left(\sum_{p=1}^{n} H^{(p)}\right) B \left(\sum_{q=1}^{n} \left(H^{(q)}\right)^*\right) \geq 0,$$

which means $\Gamma_{\mathbf{B}}$ is non-negative definite. Since $\Gamma_{\mathbf{A},\mathbf{B}} = \Gamma_{\mathbf{A}} + \Gamma_{\mathbf{B}}$, we know that $\Gamma_{\mathbf{A},\mathbf{B}}$ is elliptic inside $\Delta_{n,d}$. Then we finish the proof. □

5. Examples

5.1. *The construction from $SU(N)$*

In this section, we show that our first model of matrix Dirichlet processes can be realized by the projections from Brownian motion on $SU(N)$. The construction relies on the matrix-extracting procedure, extending the construction of matrix Jacobi processes described in [10]. In both this case and the special case where all the parameters A_{ij} are equal to d, the associated generator may be considered as an image of the Casimir operator on $SU(N)$, whenever the coefficients a_i in the measure are integers. Moreover, for the general case, similar to the construction in the scalar case in Proposition 3.3, we provide a construction from more general Brownian motions on $SU(N)$, where the generator is no longer the Casimir operator on $SU(N)$.

The so-called "matrix-extracting procedure" is the following: consider a matrix u on $SU(N)$, then take the first d lines, and split the set of all N columns into $(n+1)$ disjoint sets $I_1, \dots, I_{n+1}$. For $1 \leq i \leq (n+1)$, define $d_i = |I_i|$ such that $N = d_1 + \cdots + d_{n+1}$. Then we get $(n+1)$ extracted matrices $\{\mathbf{W}^{(i)}\}$, respectively of size $d \times d_1, d \times d_2, \dots, d \times d_{n+1}$. The matrix Jacobi process is obtained by considering the Hermitian matrix $\mathbf{W}^{(1)}\mathbf{W}^{(1)*}$. Here, we extend this procedure, defining $\mathbf{Z}^{(i)} = \mathbf{W}^{(i)}(\mathbf{W}^{(i)})^*$ for $i = 1, \dots, n+1$. Then, $\sum_{i=1}^{n+1} \mathbf{Z}^{(i)} = \mathrm{Id}$, and the process $(\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)})$ lives in the complex matrix Dirichlet simplex $\Delta_{d,n}$.

The compact Lie group $SU(N)$ is semi-simple compact. There is, up to a scaling constant, a unique elliptic diffusion operator on it which commutes both with the right and the left multiplication. This operator is called the Casimir operator, see [4,18] for more details. The Brownian motion on $SU(N)$ is the diffusion process which has the Casimir operator as its generator. It may be described by the vector fields $\mathcal{V}_{R_{ij}}$, $\mathcal{V}_{S_{ij}}$ and $\mathcal{V}_{D_{ij}}$ which are given on the entries $\{u_{ij}\}$ of $\mathbf{u} \in SU(N)$ matrix as

$$\mathcal{V}_{R_{ij}} = \sum_k (u_{kj}\partial_{u_{ki}} - u_{ki}\partial_{u_{kj}} + \bar{u}_{kj}\partial_{\bar{u}_{ki}} - \bar{u}_{ki}\partial_{\bar{u}_{kj}}), \tag{5.1}$$

$$\mathcal{V}_{S_{ij}} = i\sum_k (u_{kj}\partial_{u_{ki}} + u_{ki}\partial_{u_{kj}} - \bar{u}_{kj}\partial_{\bar{u}_{ki}} - \bar{u}_{ki}\partial_{\bar{u}_{kj}}), \tag{5.2}$$

$$\mathcal{V}_{D_{ij}} = i\sum_k (u_{ki}\partial_{u_{ki}} - u_{kj}\partial_{u_{kj}} - \bar{u}_{ki}\partial_{\bar{z}_{ki}} + \bar{u}_{kj}\partial_{\bar{u}_{kj}}). \tag{5.3}$$

Then for the Casimir operator $\Delta_{SU(N)}$ on $SU(N)$, we have

$$\Delta_{SU(N)} = \frac{1}{4N}\sum_{i<j}\left(\mathcal{V}_{R_{ij}}^2 + \mathcal{V}_{S_{ij}}^2 + \frac{2}{N}\mathcal{V}_{D_{ij}}^2\right),$$

and

$$\Gamma_{SU(N)}(f,g) = \frac{1}{4N}\left(\sum_{i<j}\mathcal{V}_{R_{ij}}(f)\mathcal{V}_{R_{ij}}(g) + \mathcal{V}_{S_{ij}}(f)\mathcal{V}_{S_{ij}}(g) + \frac{2}{N}\mathcal{V}_{D_{ij}}(f)\mathcal{V}_{D_{ij}}(g)\right).$$

From this, a simple computation provides

$$\Gamma_{SU(N)}(u_{ij}, u_{kl}) = -\frac{1}{2N}u_{il}u_{kj} + \frac{1}{2N^2}u_{ij}u_{kl},$$

$$\Gamma_{SU(N)}(u_{ij}, \bar{u}_{kl}) = \frac{1}{2N}\delta_{ik}\delta_{jl} - \frac{1}{2N^2}u_{ij}\bar{u}_{kl},$$

$$\Delta_{SU(N)}(u_{ij}) = -\frac{N^2-1}{2N^2}u_{ij}, \qquad \Delta_{SU(N)}(\bar{u}_{ij}) = -\frac{N^2-1}{2N^2}\bar{u}_{ij},$$

and these formulas describe entirely the Brownian motion on $SU(N)$.

A simple application of the diffusion property (equations (2.6) and (2.7)) yields, for $1 \le p, q \le n$, for the entries $Z_{ij}^{(p)}$ of the matrices $\mathbf{Z}^{(p)}$,

$$\Gamma_{SU(N)}\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) = \frac{1}{2N}\delta_{pq}\big(\delta_{il}Z_{kj}^{(p)} + \delta_{kj}Z_{il}^{(p)}\big) - \frac{1}{2N}\big(Z_{kj}^{(p)}Z_{il}^{(q)} + Z_{il}^{(p)}Z_{kj}^{(q)}\big), \tag{5.4}$$

$$\Delta_{SU(N)}\big(Z_{ij}^{(p)}\big) = -Z_{ij}^{(p)} + \frac{1}{N}d_p\delta_{ij}. \tag{5.5}$$

Comparing (5.4), (5.5) with (4.5) and (4.6), we obtain a matrix Dirichlet operator described in Theorem 4.1, with

$$A_{pq} = \frac{1}{2N}, \qquad a_i = d_i - d + 1$$

for any $1 \le p, q \le n$ and $1 \le i \le (n+1)$. Therefore, the density of the reversible measure is

$$C\prod_{p=1}^{n}\det\big(\mathbf{Z}^{(p)}\big)^{d_p-d}\det\left(\mathrm{Id} - \sum_{p=1}^{n}\mathbf{Z}^{(p)}\right)^{d_{n+1}-d}. \tag{5.6}$$

To have this measure finite, we need $d_i > d-1$ for all $1 \le i \le (n+1)$, i.e., $d_i \ge d$ since these parameters are integers. It is worth to observe that this restriction is necessary for the matrices $\mathbf{Z}^{(i)}$ to be non degenerate. If it is not satisfied, the matrices $\mathbf{Z}^{(i)}$ live on an algebraic submanifold and their law may not have any density with respect to the Lebesgue measure.

To summarize, we have

Proposition 5.1. *The image of the Brownian motion on $SU(N)$ under the matrix-extracting procedure is a diffusion process on the complex matrix simplex $\Delta_{d,n}$ with carré du champ $\Gamma_{\mathbf{A}}$ and reversible measure $D_{a_1,\ldots,a_{n+1}}$ where $A_{pq} = \frac{1}{2N}$, $p, q = 1, \ldots, n+1$, $p \ne q$, $a_i = d_i - d + 1$.*

As a corollary, we get

Corollary 5.2. *Whenever $d_i \ge d$, $i = 1, \ldots, n+1$, the image of the Haar measure on $SU(N)$ through the matrix-extracting procedure is the matrix Dirichlet measure $D_{d_1-d,\ldots,d_{n+1}-d}$.*

For the general case where the parameters A_{ij} are not equal, we may follow Proposition 3.3. For $p \ne q$, we define the following $\mathrm{L}^{(pq)}$ acting on the matrix simplex $\{\mathbf{Z}^{(i)}, \sum_{i=1}^{n}\mathbf{Z}^{(i)} \le \mathrm{Id}\}$,

$$\mathrm{L}^{(pq)} = \sum_{i\in I_p, j\in I_q}\mathcal{V}_{R_{ij}}^2 + \mathcal{V}_{S_{ij}}^2 + \frac{2}{N}\mathcal{V}_{D_{ij}}^2,$$

with its corresponding carré du champ operator $\Gamma^{(pq)}$,

$$\Gamma^{(pq)}(f,g) = \sum_{i\in I_p, j\in I_q}\mathcal{V}_{R_{ij}}(f)\mathcal{V}_{R_{ij}}(g) + \mathcal{V}_{S_{ij}}(f)\mathcal{V}_{S_{ij}}(g) + \frac{2}{N}\mathcal{V}_{D_{ij}}(f)\mathcal{V}_{D_{ij}}(g).$$

Lemma 5.3. *For the entries u_{ij} of an $SU(N)$ matrix, and denoting by $Z_{ij}^{(p)}$ the entries of the extracted matrix $\mathbf{Z}^{(p)}$, we have*

$$\mathcal{V}_{R_{ij}} Z_{ab}^{(p)} = \mathbf{1}_{i\in I_p}(u_{aj}\bar{u}_{bi} + u_{ai}\bar{u}_{bj}) - \mathbf{1}_{j\in I_p}(u_{ai}\bar{u}_{bj} + u_{aj}\bar{u}_{bi}), \tag{5.7}$$

$$\mathcal{V}_{S_{ij}} Z_{ab}^{(p)} = \sqrt{-1}\big(\mathbf{1}_{i\in I_p}(u_{aj}\bar{u}_{bi} - u_{ai}\bar{u}_{bj}) + \mathbf{1}_{j\in I_p}(u_{ai}\bar{u}_{bj} - u_{aj}\bar{u}_{bi})\big), \tag{5.8}$$

$$\mathcal{V}_{D_{ij}} Z_{ab}^{(p)} = 0, \tag{5.9}$$

and

$$\mathcal{V}_{R_{ij}}^2 Z_{ab}^{(p)} = \mathbf{1}_{i\in I_p}(2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi}) - \mathbf{1}_{j\in I_p}(2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi}), \tag{5.10}$$

$$\mathcal{V}_{S_{ij}}^2 Z_{ab}^{(p)} = \mathbf{1}_{i\in I_p}(2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi}) - \mathbf{1}_{j\in I_p}(2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi}). \tag{5.11}$$

Then,

$$\Gamma^{(pq)}\big(Z_{ab}^{(p)}, Z_{cd}^{(q)}\big) = -2Z_{cb}^{(p)}Z_{ad}^{(q)} - 2Z_{ad}^{(p)}Z_{cb}^{(q)}, \tag{5.12}$$

$$\Gamma^{(pq)}\big(Z_{ab}^{(p)}, Z_{cd}^{(p)}\big) = 2Z_{cb}^{(p)}Z_{ad}^{(q)} + 2Z_{ad}^{(p)}Z_{cb}^{(q)}, \tag{5.13}$$

$$\Gamma^{(pq)}\big(Z_{ab}^{(q)}, Z_{cd}^{(q)}\big) = 2Z_{cb}^{(p)}Z_{ad}^{(q)} + 2Z_{ad}^{(p)}Z_{cb}^{(q)}. \tag{5.14}$$

For a pair $(r,s) \neq (p,q)$, we have

$$\Gamma^{(pq)}\big(Z_{ab}^{(r)}, Z_{cd}^{(s)}\big) = 0. \tag{5.15}$$

Moreover,

$$\mathrm{L}^{(pq)}\big(Z_{ij}^{(r)}\big) = \mathbf{1}_{r=p}4\big(d_p Z_{ij}^{(q)} - d_q Z_{ij}^{(p)}\big) - \mathbf{1}_{r=q}4\big(d_p Z_{ij}^{(q)} - d_q Z_{ij}^{(p)}\big). \tag{5.16}$$

Remark 5.4. The action of $\mathcal{V}_{D_{ij}}$ vanishes on the variables $Z_{ab}^{(p)}$ such that indeed we could replace $\mathrm{L}^{(pq)}$ by

$$\mathrm{L}^{(pq)} = \sum_{i\in I_p, j\in I_q} \mathcal{V}_{R_{ij}}^2 + \mathcal{V}_{S_{ij}}^2,$$

and the dimension N disappears from the definition.

Proof of Lemma 5.3. Recall (5.1). Then letting $\mathcal{V}_{R_{ij}}$ act on $\mathbf{Z}^{(p)} = \mathbf{W}^{(p)}(\mathbf{W}^{(p)})^*$ and writing $(u_{ij})_{1\le i\le d, j\in I_p}$ for the entries of $\mathbf{W}^{(p)}$, which are indeed entries of an $SU(N)$ matrix, we have

$$\begin{aligned}\mathcal{V}_{R_{ij}} Z_{ab}^{(p)} &= \sum_{r\in I_p} \mathcal{V}_{R_{ij}}(u_{ar}\bar{u}_{br}) \\ &= \mathbf{1}_{i\in I_p}(u_{aj}\bar{u}_{bi} + u_{ai}\bar{u}_{bj}) - \mathbf{1}_{j\in I_p}(u_{ai}\bar{u}_{bj} + u_{aj}\bar{u}_{bi}).\end{aligned}$$

Similarly we can prove (5.8), (5.9).
By (5.7), (5.8), we obtain

$$\mathcal{V}_{R_{ij}}(u_{aj}\bar{u}_{bi} + u_{ai}\bar{u}_{bj}) = 2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi},$$

$$\mathcal{V}_{S_{ij}}(u_{aj}\bar{u}_{bi} - u_{ai}\bar{u}_{bj}) = -\sqrt{-1}(2u_{aj}\bar{u}_{bj} - 2u_{ai}\bar{u}_{bi}),$$

then (5.15), (5.10) follow.

By (5.7), (5.8) and (5.9) we have

$$\begin{aligned}
&\Gamma^{(pq)}\big(Z_{ab}^{(p)}, Z_{cd}^{(q)}\big)\\
&\quad= \sum_{i\in I_p, j\in I_q} \mathcal{V}_{R_{ij}}\big(Z_{ab}^{(p)}\big)\mathcal{V}_{R_{ij}}\big(Z_{cd}^{(q)}\big) + \mathcal{V}_{S_{ij}}\big(Z_{ab}^{(p)}\big)\mathcal{V}_{S_{ij}}\big(Z_{cd}^{(q)}\big)\\
&\quad= -2Z_{cb}^{(p)} Z_{ad}^{(q)} - 2Z_{ad}^{(p)} Z_{cb}^{(q)},
\end{aligned}$$

which proves (5.12).

In the same way, we can deduce (5.13), (5.14) and (5.15).

(5.16) is proved by (5.10), (5.11) and

$$\begin{aligned}
\mathrm{L}^{(pq)}\big(Z_{ij}^{(r)}\big) &= \sum_{k\in I_p, l\in I_q} \mathcal{V}_{R_{kl}}^2 Z_{ij}^{(r)} + \mathcal{V}_{S_{kl}}^2 Z_{ij}^{(r)}\\
&= \mathbf{1}_{r=p}4\big(d_p Z_{ij}^{(q)} - d_q Z_{ij}^{(p)}\big) - \mathbf{1}_{r=q}4\big(d_p Z_{ij}^{(q)} - d_q Z_{ij}^{(p)}\big).
\end{aligned}$$

□

Now by Lemma 5.3 we may derive the following conclusion.

Proposition 5.5. $(\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)})$ *is a closed system for any* $\mathrm{L}^{(pq)}$ *and the image of*

$$\frac{1}{2}\sum_{p<q}^{n+1} A_{pq}\mathrm{L}^{(pq)}$$

is the operator $\mathrm{L}_{\mathbf{A},\mathbf{a}}$ *in Theorem* 4.1 *with* $a_i = d_i - d + 1$.

5.2. *The construction from Wishart processes*

In what follows, we show that our second model may be constructed as a projection from complex Wishart processes, which are matrix generalizations of Laguerre processes. We first recall the definition of the complex Wishart distribution.

Definition 5.6. A $d \times d$ Hermitian positive definite matrix W is said to have a Wishart distribution with parameters $d, r \geq d$, if its distribution has a density given by

$$C_{r,d} \det(W)^{r-d} e^{-\frac{1}{2}\operatorname{trace}(W)}, \tag{5.17}$$

where $C_{r,d} = (2^{rd}\pi^{\frac{1}{2}d(d-1)}\Gamma(r)\Gamma(r-1)\cdots\Gamma(r-d+1))^{-1}$ is the normalization constant.

When $r \geq d$ is an integer, this distribution can be derived from the Gaussian distributed complex matrix. Indeed, if we consider a $d \times r$ complex matrix X with its elements being independent Gaussian centered random variables, then $W = XX^*$ has the complex Wishart distribution with parameters d, r.

A $d \times d$ complex Wishart process $\{W_t, t \geq 0\}$ is usually defined as a solution to the following stochastic differential equation,

$$dW_t = \sqrt{W_t}\, dB_t + dB_t^*\sqrt{W_t} + (\alpha W_t + \beta \mathrm{Id}_d)\, dt, \quad W_t = W_0,$$

where $\{B_t, t \geq 0\}$ is a $d \times d$ complex Brownian motion, W_0 is a $d \times d$ Hermitian matrix. Wishart processes have been deeply studied, see [6,9] etc. There exists more general form of Wishart processes that we will not consider here. In what follows, we extend the construction of Wishart laws from matrix Gaussian ones at the level of processes, exactly as in the scalar case where Laguerre processes may be constructed (with suitable parameters) from Ornstein–Uhlenbeck ones. We will apply the matrix extracting procedure again.

The generator of an Ornstein–Uhlenbeck process on $N \times N$ complex matrices is given, on the entries $\{z_{ij}\}$ of a complex matrix $\mathbf{z}$, by

$$\Gamma(z_{ij}, z_{kl}) = 0,$$
$$\Gamma(z_{ij}, \bar{z}_{kl}) = 2\delta_{ik}\delta_{jl},$$
$$\mathrm{L}(z_{ij}) = -z_{ij}.$$

This describes a process on matrices where the entries are independent complex Ornstein–Uhlenbeck processes. Now we start the "matrix-extracting" procedure on $\mathbf{z}$, as we did before on $SU(N)$ in Section 5.1, i.e., take the first d lines and split the N columns into $(n+1)$ parts $I_1, \dots, I_{n+1}$, such that $d_i = |I_i|$, for $1 \le i \le (n+1)$ and $N = d_1 + \cdots + d_{n+1}$, then we have $(n+1)$ extracted matrices $\mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(n+1)}$. For $1 \le p \le (n+1)$, define $\mathbf{W}^{(p)} = \mathbf{Y}^{(p)}(\mathbf{Y}^{(p)})^*$ with entries $W_{ij}^{(p)}$.

Proposition 5.7. $\{\mathbf{W}^{(p)}, 1 \le p \le (n+1)\}$ *form a family of independent complex Wishart processes, whose reversible measure respectively given by* (5.17) *with* $r_p = d_p$ *for* $1 \le p \le (n+1)$. *Moreover, the image of complex Gaussian measure through the "matrix-extracting" procedure is a product of complex Wishart distributions.*

Proof. One may check

$$\Gamma\big(W_{ij}^{(p)}, W_{kl}^{(q)}\big) = 2\delta_{pq}\big(\delta_{jk} W_{il}^{(p)} + \delta_{il} W_{kj}^{(p)}\big), \tag{5.18}$$

$$\mathrm{L}\big(W_{ij}^{(p)}\big) = 4d_p\delta_{ij} - 2W_{ij}^{(p)}. \tag{5.19}$$

Let ρ be the density of the reversible measure of $\{\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(n+1)}\}$. Then,

$$\Gamma\big(\log\rho, W_{ij}^{p}\big) = 4(d_p - d)\delta_{ij} - 2W_{ij}^{(p)},$$

and we also have

$$\Gamma\big(\log\det\big(W^{(p)}\big), W_{ij}^{(p)}\big) = 4\delta_{ij},$$
$$\Gamma\big(\operatorname{trace} W^{(p)}, W_{ij}^{(p)}\big) = 4W_{ij}^{(p)},$$

therefore,

$$\rho = C\prod_{p=1}^{n+1}\det\big(W^{(p)}\big)^{d_p - d} e^{-\frac{1}{2}\operatorname{trace}(\sum_{p=1}^{n+1} W^{(p)})},$$

which shows that, under the reversible measure, we have a family of $d \times d$ independent Wishart matrices $\{\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(n+1)}\}$. □

We now construct a process on the complex matrix simplex from independent Wishart processes $(\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(n+1)})$. As in the scalar case, we obtain a kind of warped product on the set $\mathbf{D} \times \Delta_{n,d}$, where $\mathbf{D}$ denotes the set of real diagonal matrices with positive diagonal entries.

Let $\mathbf{S} = \sum_{p=1}^{n+1} \mathbf{W}^{(p)}$. Since $\mathbf{S}$ is a positive-definite Hermitian matrix, we may assume that it has a spectral decomposition $\mathbf{S} = \mathbf{U}\mathbf{D}^2\mathbf{U}^*$, where $\mathbf{U}$ is unitary and $\mathbf{D} = \operatorname{diag}\{\lambda_1, \dots, \lambda_d\}$. Observe that $\mathbf{U}$ is not uniquely determined, since we may change $\mathbf{U}$ into $\mathbf{UP}$ where $\mathbf{P} = \operatorname{diag}\{e^{i\phi_1}, \dots, e^{i\phi_d}\}$, $0 \le \phi_1, \dots, \phi_d \le 2\pi$, and this amounts to the choice of a phase for the eigenvectors. In this paper, we choose $\mathbf{U}$ to be the one that has real elements on its diagonal, such that $\mathbf{U}$ is an analytic function of $\mathbf{S}$ in the Weyl chamber $\{\lambda_1 < \cdots < \lambda_d\}$. This choice will be irrelevant to the construction of the process. Moreover, we introduce $(\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(d)})$, whose elements are given by

$$V_{ij}^{(p)} = U_{ip}U_{pj}^* \tag{5.20}$$

for $1 \le i, j, p \le d$, and we see that the choice of phase in $\mathbf{U}$ has no influence on $\{\mathbf{V}^{(p)}\}$.

Then for $1 \le i \le (n+1)$, write

$$\mathbf{M}^{(i)} = \mathbf{S}^{-\frac{1}{2}}\mathbf{W}^{(i)}\mathbf{S}^{-\frac{1}{2}}, \qquad \mathbf{Z}^{(i)} = \mathbf{U}^*\mathbf{M}^{(i)}\mathbf{U} \tag{5.21}$$

for which we have the following result.

Theorem 5.8. *Let $(\mathbf{W}^1, \dots, \mathbf{W}^{(n+1)})$ be $(n+1)$ independent Wishart processes. Then with $\mathbf{D} = \mathrm{diag}\{\lambda_1, \dots, \lambda_d\}$, $0 \le \lambda_1 \le \cdots \le \lambda_d$ and $\mathbf{Z}^{(i)}$ defined as in equation (5.21), $(\mathbf{D}, \mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)})$ is a Markov diffusion process, where $(\mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)})$ lives in the matrix simplex $\Delta_{d,n}$. The generator of the process is*

$$\sum_{i=1}^{d}\left(\partial_{\lambda_i}^2 + \left(\frac{2(N-d)+1}{\lambda_i} - \lambda_i + \sum_{j\neq i}\frac{4\lambda_i}{\lambda_i^2 - \lambda_j^2}\right)\partial_{\lambda_i}\right) + \mathrm{L}_{\mathbf{A},\mathbf{B},\mathbf{a}}, \tag{5.22}$$

where $\mathrm{L}_{\mathbf{A},\mathbf{B},\mathbf{a}}$ is defined in Theorem 4.3 with

$$A_{ij} = 2\lambda_i^{-2}\delta_{ij},$$

$$B_{ij,kl} = \begin{cases} 2\frac{\lambda_i^2+\lambda_j^2}{(\lambda_i^2-\lambda_j^2)^2}\delta_{ik}\delta_{jl}, & i \neq j \text{ and } k \neq l, \\ \frac{1}{\lambda_i^2}\delta_{ij}\delta_{ik}\delta_{jl}, & i = j \text{ or } k = l \end{cases}$$

for $1 \le i, j, k, l \le d$ and $a_p = d_p - d + 1$ for $1 \le p \le (n+1)$.

It is known that starting from random matrices $(W^{(1)}, \dots, W^{(n+1)})$ distributed as independent Wishart distributions, one could get the matrix Dirichlet distribution through $M^i = S^{-\frac{1}{2}}W^{(i)}S^{-\frac{1}{2}}$, where $S = \sum_{i=1}^{n+1} W^i$, see [11]. Also from our results in the scalar case (Section 3), it is natural to guess that $(\mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$ may be a matrix Dirichlet process. As we will see in the following proposition, given $\mathbf{S}$, $(\mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$ is indeed a matrix Dirichlet process; However, the operator of $(\mathbf{S}, \mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$ is much more complicated than the one of $(\mathbf{D}, \mathbf{Z}^{(1)}, \dots, \mathbf{Z}^{(n)})$, since $\Gamma(\mathbf{S}, \mathbf{M}^{(i)}) \neq 0$, and therefore does not have the structure of a (generalized) warped product.

Proposition 5.9. *$(\mathbf{S}, \mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$ is a Markov diffusion process where $(\mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$ lives on the matrix simplex $\Delta_{n,d}$, and the generator of the process is*

$$\begin{aligned}&\sum_{ijkl} 2(\delta_{jk}S_{il} + \delta_{il}S_{kj})\partial_{S_{ij}}\partial_{S_{kl}} + \sum_{ij}(4N\delta_{ij} - 2S_{ij})\partial_{S_{ij}} + \mathrm{L}_{\mathbf{A},\mathbf{B},\mathbf{a}} \\ &\quad + \sum_{ijkl,p}\sum_{a,b=1}^{d} 2\frac{\lambda_a - \lambda_b}{\lambda_a + \lambda_b}\Big(\big(M^{(p)}V^{(a)}\big)_{il}V_{kj}^{(b)} - V_{il}^{(a)}\big(V^{(b)}M^{(p)}\big)_{kj}\Big)\partial_{S_{ij}}\partial_{M_{kl}^{(p)}},\end{aligned} \tag{5.23}$$

where $V^{(i)}$ defined by equation (5.20), $\mathrm{L}_{\mathbf{A},\mathbf{B},\mathbf{a}}$ is defined in Theorem 3.2 with

$$A = 2S^{-1}, \qquad B_{ij,kl} = \sum_{r,s=1}^{d}\frac{4}{(\lambda_r + \lambda_s)^2}V_{ik}^{(r)}V_{lj}^{(s)},$$

and $a_p - 1 + d = d_p$.

Before proving Theorem 5.8 and Proposition 5.9, we first give the following lemmas regarding the action of the diffusion operators of $\mathbf{N} = \mathbf{S}^{\frac{1}{2}}$, $\{\mathbf{M}^{(i)}, 1 \le i \le n\}$ and $\{\mathbf{Z}^{(i)}, 1 \le i \le n\}$.

Lemma 5.10. *For the generator of independent Wishart matrices* $(\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(n+1)})$ *and* $\mathbf{S} = \sum_{p=1}^{n+1} \mathbf{W}^{(p)}$, *we have*

$$\Gamma(S_{ij}, S_{kl}) = 2(\delta_{jk} S_{il} + \delta_{il} S_{kj}), \tag{5.24}$$

$$\mathrm{L}(S_{ij}) = 4N\delta_{ij} - 2S_{ij}, \tag{5.25}$$

$$\Gamma\big(S_{ij}, W_{kl}^{(p)}\big) = 2\big(\delta_{jk} W_{il}^{(p)} + \delta_{il} W_{kj}^{(p)}\big). \tag{5.26}$$

Moreover, suppose $\mathbf{S}$ *has a spectral decomposition* $\mathbf{S} = \mathbf{U}\mathbf{D}^2\mathbf{U}^*$, *where* $\mathbf{U}$ *is unitary and* $\mathbf{D} = \mathrm{diag}\{\lambda_1, \dots, \lambda_d\}$, *we have*

$$\Gamma(\lambda_i, \lambda_j) = \delta_{ij}, \tag{5.27}$$

$$\mathrm{L}(\lambda_i) = \frac{2(N-d)+1}{\lambda_i} - \lambda_i + 4\lambda_i \sum_{j \neq i} \frac{1}{\lambda_i^2 - \lambda_j^2}. \tag{5.28}$$

Proof. Formulas (5.24), (5.26) and (5.25) are straight-forward from (5.18) and (5.19).

By (5.24), (5.25), we are able to compute the diffusion operators of $D = \{\lambda_1, \dots, \lambda_d\}$. Let $P(X) = \det(S - X\mathrm{Id})$, $P(Y) = \det(S - Y\mathrm{Id})$. Notice that

$$\Gamma\big(\log P(X), \log P(Y)\big) = \frac{4}{Y-X}\left(\frac{XP'(X)}{P(X)} - \frac{YP'(Y)}{P(Y)}\right),$$

and compare it with

$$\Gamma\big(\log P(X), \log P(Y)\big) = \sum_{i,j} \frac{4\lambda_i\lambda_j\Gamma(\lambda_i, \lambda_j)}{(\lambda_i^2 - X)(\lambda_j^2 - Y)},$$

implying that

$$\Gamma(\lambda_i, \lambda_j) = \delta_{ij}.$$

Now we compute $\mathrm{L}(\lambda_i)$. By (5.24), (5.25) we derive

$$\mathrm{L}\big(\log P(X)\big) = 4(d-N)\frac{P'(X)}{P(X)} - 4\frac{X(P'(X))^2}{P^2(X)} - 2d + 2\frac{XP'(X)}{P(X)}. \tag{5.29}$$

On the other hand, let $\eta_i = \lambda_i^2$ be the eigenvalues of $\mathbf{S}$, we have

$$\mathrm{L}\big(\log P(X)\big) = \sum_{i=1}^{d}\left(-\frac{4\eta_i}{(\eta_i - X)^2} + \frac{\mathrm{L}(\eta_i)}{\eta_i - X}\right).$$

Comparing it with (5.29) leads to

$$\mathrm{L}(\eta_i) = 8\eta_i \sum_{j \neq i} \frac{1}{\eta_i - \eta_j} + 4(N-d+1) - 2\eta_i,$$

such that

$$\mathrm{L}(\lambda_i) = \frac{2(N-d)+1}{\lambda_i} - \lambda_i + 4\lambda_i \sum_{j \neq i} \frac{1}{\lambda_i^2 - \lambda_j^2}.$$

□

Lemma 5.11. *Let S be a positive definite, Hermitian matrix and N be its positive definite square root. Suppose S has the spectral decomposition $S = UD^2U^*$, where U is the unitary part and $D = \mathrm{diag}\{\lambda_1, \dots, \lambda_d\}$, with λ_i all positive. With $N = UDU^*$, and $V_{ij}^{(p)} = U_{ip}U_{pj}^*$ for $1 \le i, j, p \le d$, we have for $1 \le i, j, k, l \le d$*

$$\partial_{S_{kl}} N_{ij} = \sum_{r,s=1}^{d} \frac{1}{\lambda_r + \lambda_s} V_{ik}^{(r)} V_{lj}^{(s)}. \tag{5.30}$$

Proof. For fixed k, l, write $\mathcal{D}_{ij}^{kl} = \partial_{S_{kl}} N_{ij}$. Then from $N_{ij}^2 = S_{ij}$, we have

$$\mathcal{D}^{kl}\mathbf{N} + \mathbf{N}\mathcal{D}^{kl} = E^{kl},$$

where E^{kl} is the matrix satisfying $E_{ab}^{kl} = \delta_{ak}\delta_{bl}$. Since $\mathbf{S} = \mathbf{U}\mathbf{D}^2\mathbf{U}^*$, we may write $\mathbf{N} = \mathbf{U}\mathbf{D}\mathbf{U}^*$, such that

$$\big(\mathbf{U}^*\mathcal{D}^{kl}\mathbf{U}\big)\mathbf{D} + \mathbf{D}\big(\mathbf{U}^*\mathcal{D}^{kl}\mathbf{U}\big) = \mathbf{U}^* E^{kl}\mathbf{U}.$$

Therefore, we deduce

$$\big(U^*\mathcal{D}^{kl}U\big)_{ij} = \frac{U_{ik}^* U_{lj}}{\lambda_i + \lambda_j},$$

so that

$$\mathcal{D}_{ij}^{kl} = \sum_{r,s} \frac{1}{\lambda_r + \lambda_s} V_{ik}^{(r)} V_{lj}^{(s)},$$

which finishes the proof. □

Lemma 5.12. *The following formulas hold for $\mathbf{N} = \mathbf{S}^{\frac{1}{2}}$,*

$$\Gamma\big(N_{ij}, W_{kl}^{(p)}\big) = \sum_{r,s=1}^{d} \frac{2}{\lambda_r + \lambda_s} V_{il}^{(r)} \big(W^{(p)}V^{(s)}\big)_{kj} + \sum_{r,s=1}^{d} \frac{2}{\lambda_r + \lambda_s} V_{kj}^{(s)} \big(V^{(r)}W^{(p)}\big)_{il}, \tag{5.31}$$

$$\Gamma(N_{ij}, N_{kl}) = \sum_{r,s=1}^{d} 2\frac{\lambda_r^2 + \lambda_s^2}{(\lambda_r + \lambda_s)^2} V_{il}^{(r)} V_{kj}^{(s)}, \tag{5.32}$$

$$\mathrm{L}(N_{ij}) = 4\sum_{r,s=1}^{d} \frac{\lambda_s}{(\lambda_r + \lambda_s)^2} V_{ij}^{(r)} - N_{ij} + 2(N-d)N_{ij}^{-1}. \tag{5.33}$$

Furthermore,

$$\Gamma\big(N_{ij}^{-1}, N_{kl}\big) = -\sum_{r,s=1}^{d} 2\frac{\lambda_r^2 + \lambda_s^2}{\lambda_r\lambda_s(\lambda_r + \lambda_s)^2} V_{il}^{(r)} V_{kj}^{(s)}, \tag{5.34}$$

$$\Gamma\big(N_{ij}^{-1}, W_{kl}^{(p)}\big) = -2\sum_{r,s=1}^{d} \frac{1}{\lambda_r\lambda_s(\lambda_r + \lambda_s)} \Big(V_{il}^{(r)}\big(W^{(p)}V^{(s)}\big)_{kj} + V_{kj}^{(s)}\big(V^{(r)}W^{(p)}\big)_{il}\Big), \tag{5.35}$$

$$\Gamma\big(N_{ij}^{-1}, N_{kl}^{-1}\big) = 2\sum_{r,s=1}^{d} \frac{\lambda_r^2 + \lambda_s^2}{\lambda_r^2\lambda_s^2(\lambda_r + \lambda_s)^2} V_{il}^{(r)} V_{kj}^{(s)}, \tag{5.36}$$

$$\mathrm{L}\big(N_{ij}^{-1}\big) = 4\sum_{r,s=1}^{d} \frac{1}{\lambda_s(\lambda_r+\lambda_s)^2} V_{ij}^{(r)} + N_{ij}^{-1} - 2(N-d)\big(S^{-1}N^{-1}\big)_{ij}. \tag{5.37}$$

Proof. By (5.30), we are able to compute

$$\begin{aligned}\Gamma\big(N_{ij}, W_{kl}^{(p)}\big) &= \sum_{ab} \mathcal{D}_{ij}^{ab}\Gamma\big(S_{ab}, W_{kl}^{(p)}\big)\\ &= 2\sum_{r,s=1}^{d}\frac{1}{\lambda_r+\lambda_s}V_{il}^{(r)}\big(W^{(p)}V^{(s)}\big)_{kj} + 2\sum_{r,s=1}^{d}\frac{1}{\lambda_r+\lambda_s}V_{kj}^{(s)}\big(V^{(r)}W^{(p)}\big)_{il},\end{aligned}$$

and

$$\begin{aligned}\Gamma(N_{ij}, N_{kl}) &= 2\sum_{r,s=1}^{d}\frac{\lambda_r^2+\lambda_s^2}{(\lambda_r+\lambda_s)^2}V_{il}^{(r)}V_{kj}^{(s)},\\ \mathrm{L}(N_{ij}) &= 4\sum\frac{\lambda_s}{(\lambda_r+\lambda_s)^2}V_{ij}^{(r)} - N_{ij} + 2(N-d)N_{ij}^{-1}.\end{aligned}$$

Moreover, due to the fact that

$$\partial_{N_{ab}}N_{ij}^{-1} = -N_{ia}^{-1}N_{bj}^{-1},$$

we have

$$\begin{aligned}\Gamma\big(N_{ij}^{-1}, W_{kl}^{(p)}\big) &= \sum_{a,b=1}^{d}\partial_{N_{ab}}N_{ij}^{-1}\Gamma\big(N_{ab}, W_{kl}^{(p)}\big)\\ &= -2\sum_{r,s=1}^{d}\frac{1}{\lambda_r\lambda_s(\lambda_r+\lambda_s)}V_{il}^{(r)}\big(W^{(p)}V^{(s)}\big)_{kj} - 2\sum_{r,s=1}^{d}\frac{1}{\lambda_r\lambda_s(\lambda_r+\lambda_s)}V_{kj}^{(s)}\big(V^{(r)}W^{(p)}\big)_{il}.\end{aligned}$$

Similar computations yield

$$\begin{aligned}\Gamma\big(N_{ij}^{-1}, N_{kl}\big) &= -2\sum_{r,s=1}^{d}\frac{\lambda_r^2+\lambda_s^2}{\lambda_r\lambda_s(\lambda_r+\lambda_s)^2}V_{il}^{(r)}V_{kj}^{(s)},\\ \Gamma\big(N_{ij}^{-1}, N_{kl}^{-1}\big) &= 2\sum_{r,s=1}^{d}\frac{\lambda_r^2+\lambda_s^2}{\lambda_r^2\lambda_s^2(\lambda_r+\lambda_s)^2}V_{il}^{(r)}V_{kj}^{(s)},\\ \mathrm{L}\big(N_{ij}^{-1}\big) &= \sum_{k,l=1}^{d}\partial_{N_{kl}}N_{ij}^{-1}\mathrm{L}(N_{kl}) + \sum_{k,l,a,b=1}^{d}\Gamma(N_{kl}, N_{ab})\partial_{N_{ab}}\partial_{N_{kl}}N_{ij}^{-1}\\ &= 4\sum_{r,s=1}^{d}\frac{1}{\lambda_s(\lambda_r+\lambda_s)^2}V_{ij}^{(r)} + N_{ij}^{-1} - 2(N-d)\big(S^{-1}N^{-1}\big)_{ij}.\end{aligned}$$

□

Lemma 5.13. *With $V^{(i)}$ defined in equation (5.20), we have*

$$\begin{aligned}&\Gamma\big(M_{ij}^{(p)}, M_{kl}^{(q)}\big)\\ &\quad = 2\delta_{pq}\big(S_{il}^{-1}M_{kj}^{(p)} + S_{kj}^{-1}M_{il}^{(p)}\big) - 2S_{kj}^{-1}\big(M^{(p)}M^{(q)}\big)_{il} - 2S_{il}^{-1}\big(M^{(q)}M^{(p)}\big)_{kj}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{a,b=1}^{d} \frac{4}{(\lambda_a+\lambda_b)^2}\big(V^{(b)}M^{(p)}\big)_{kj}\big(V^{(a)}M^{(q)}\big)_{il} - \sum_{a,b=1}^{d} \frac{4}{(\lambda_a+\lambda_b)^2}\big(M^{(p)}V^{(a)}\big)_{il}\big(M^{(q)}V^{(b)}\big)_{kj} \\
&+ \sum_{a,b=1}^{d} \frac{4}{(\lambda_a+\lambda_b)^2} V^{(a)}_{kj}\big(M^{(p)}V^{(b)}M^{(q)}\big)_{il} + \sum_{a,b=1}^{d} \frac{4}{(\lambda_a+\lambda_b)^2} V^{(a)}_{il}\big(M^{(q)}V^{(b)}M^{(p)}\big)_{kj},
\end{aligned}
\tag{5.38}
$$

and

$$
\begin{aligned}
\mathrm{L}\big(M^{(p)}_{ij}\big) = {}& 4d_p S^{-1}_{ij} - 2(N-d)\big(S^{-1}M^{(p)}\big)_{ij} - 2(N-d)\big(M^{(p)}S^{-1}\big)_{ij} - 4S^{-1}_{ij}\operatorname{trace}\big(M^{(p)}\big) \\
& - 4\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2}\big(V^{(a)}M^{(p)}\big)_{ij} - 4\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2}\big(M^{(p)}V^{(b)}\big)_{ij} \\
& + 8\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2} V^{(a)}_{ij}\operatorname{trace}\big(V^{(b)}M^{(p)}\big).
\end{aligned}
\tag{5.39}
$$

Moreover, we have

$$
\Gamma\big(M^{(p)}_{ij}, S_{kl}\big) = \sum_{a,b=1}^{d} 2\frac{\lambda_a-\lambda_b}{\lambda_a+\lambda_b}\Big(\big(M^{(p)}V^{(a)}\big)_{il}V^{(b)}_{kj} - V^{(a)}_{il}\big(V^{(b)}M^{(p)}\big)_{kj}\Big).
\tag{5.40}
$$

Proof. Since

$$
\Gamma\big(M^{(p)}_{ij}, M^{(q)}_{kl}\big) = \Gamma\Big(\big(N^{-1}W^{(p)}N^{-1}\big)_{ij}, \big(N^{-1}W^{(q)}N^{-1}\big)_{kl}\Big),
$$

then by (5.18), (5.35) and (5.36) we are able to prove (5.38).

As for (5.39), direct computations yield

$$
\begin{aligned}
\mathrm{L}\big(M^{(p)}_{ij}\big) &= \sum_{r,s=1}^{d} \mathrm{L}\big(N^{-1}_{ir}W^{(p)}_{rs}N^{-1}_{sj}\big) \\
&= 8\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2} V^{(a)}_{ij}\operatorname{trace}\big(V^{(b)}M^{(p)}\big) - 4S^{-1}_{ij}\operatorname{trace}\big(M^{p}\big) \\
&\quad - 4\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2}\big(V^{(a)}M^{(p)}\big)_{ij} - 4\sum_{a,b=1}^{d} \frac{1}{(\lambda_a+\lambda_b)^2}\big(M^{(p)}V^{(b)}\big)_{ij} \\
&\quad + 4d_p S^{-1}_{ij} - 2(N-d)\big(S^{-1}M^{(p)}\big)_{ij} - 2(N-d)\big(M^{(p)}S^{-1}\big)_{ij},
\end{aligned}
$$

where the last equality is due to (5.19), (5.35), (5.36) and (5.37).

By (5.31), (5.34), we get

$$
\begin{aligned}
&\Gamma\big(M^{(p)}_{ij}, N_{kl}\big) \\
&= \sum_{a,b=1}^{d} 2\frac{\lambda_a-\lambda_b}{(\lambda_a+\lambda_b)^2}\big(M^{(p)}V^{(a)}\big)_{il}V^{(b)}_{kj} - 2\sum_{a,b=1}^{d} \frac{\lambda_a-\lambda_b}{(\lambda_a+\lambda_b)^2} V^{(a)}_{il}\big(V^{(b)}M^{(p)}\big)_{kj},
\end{aligned}
\tag{5.41}
$$

which leads to (5.40), showing that in fact $\mathbf{M}^{(p)}$ and $\mathbf{S}$ are not independent. □

Now we may obtain Proposition 5.9 directly from Lemma 5.13.

Proof of Proposition 5.9. Combining (5.24), (5.25) and (5.38), (5.39) and (5.40), we derive (5.23) as the operator of $(\mathbf{S}, \mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)})$. □

To examine the relation of $\mathbf{N}$ and $\mathbf{M}^{(p)}$ more precisely, we decompose $\mathbf{N}$ into $\mathbf{D}$ and $\mathbf{U}$, and explore their relations with $\mathbf{M}^{(p)}$ separately. The method is adapted from [5], i.e., to obtain the action of the Γ operator of the spectrum of $\mathbf{N}$ and $\mathbf{M}^{(p)}$ by computing its action on the characteristic polynomial of $\mathbf{N}$ and $\mathbf{M}^{(p)}$.

Lemma 5.14.

$$\Gamma\big(M_{ij}^{(p)}, \lambda_k\big) = 0, \tag{5.42}$$

and

$$\Gamma\big(M_{ij}^{(p)}, U_{kl}\big) = \sum_{a=1}^{d} g_{al}\big(M^{(p)}U\big)_{il} U_{aj}^{*} U_{ka} - \sum_{a=1}^{d} g_{al} U_{il}\big(U^{*}M^{(p)}\big)_{aj} U_{ka}, \tag{5.43}$$

$$\Gamma\big(U_{kl}^{*}, M_{ij}^{(p)}\big) = -\sum_{a=1}^{d} g_{ka}\big(M^{(p)}U\big)_{ia} U_{al}^{*} U_{kj}^{*} + \sum_{a=1}^{d} g_{ka} U_{ia} U_{al}^{*}\big(U^{*}M^{(p)}\big)_{kj}, \tag{5.44}$$

where

$$g_{ij} = \begin{cases} \frac{2}{(\lambda_i+\lambda_j)^2}, & i \neq j, \\ 0, & i = j \end{cases}$$

for $1 \le i, j \le d$.

Proof. Let $P_{\mathbf{N}}(X) = \det(\mathbf{N} - X\mathrm{Id}) = \prod_{i=1}^{d}(\lambda_i - X)$ be the characteristic polynomial of $\mathbf{N}$. Notice that

$$\begin{aligned}
\Gamma\big(\log P_N(X), M_{ij}^{(p)}\big) &= \sum_{k,l=1}^{d} N^{-1}(X)_{lk}\Gamma\big(N_{kl}, M_{ij}^{(p)}\big) \\
&= \sum_{r,s} 2\frac{\lambda_r - \lambda_s}{(\lambda_r + \lambda_s)^2}\frac{1}{\lambda_r - X}\delta_{rs} U_{sj}^{*}\big(M^{(p)}U\big)_{ir} \\
&\quad - 2\sum_{r,s}\frac{\lambda_r - \lambda_s}{(\lambda_r + \lambda_s)^2}\frac{1}{\lambda_r - X}\delta_{rs} U_{ir}\big(U^{*}M^{(p)}\big)_{sj} \\
&= 0,
\end{aligned}$$

from which we deduce that

$$\Gamma\big(M_{ij}^{(p)}, \lambda_k\big) = 0,$$

since

$$\Gamma\big(\log P_N(X), M_{ij}^{(p)}\big) = \sum_{k=1}^{d}\frac{1}{\lambda_k - X}\Gamma\big(\lambda_k, M_{ij}^{(p)}\big) = 0.$$

Also notice that $\Gamma(N_{kl}, M_{ij}^{(p)})$ is invariant under the unitary transformation

$$\big(\mathbf{N}, \mathbf{M}^{(p)}\big) \to \big(\mathbf{U}^0\mathbf{N}\big(\mathbf{U}^0\big)^{*}, \mathbf{U}^0\mathbf{M}^{(p)}\big(\mathbf{U}^0\big)^{*}\big)$$

for any unitary matrix U^0, we may compute $\Gamma(M_{ij}^{(p)}, U_{kl})$ at $U = \mathrm{Id}$, then obtain it at any U. More precisely, by (5.41), we obtain at $U = \mathrm{Id}$,

$$\Gamma\big(N_{kl}, M_{ij}^{(p)}\big) = 2\frac{\lambda_l - \lambda_k}{(\lambda_l + \lambda_k)^2}\delta_{kj} M_{il}^{(p)} - 2\frac{\lambda_l - \lambda_k}{(\lambda_l + \lambda_k)^2}\delta_{il} M_{kj}^{(p)},$$

and on the other hand at $U = \mathrm{Id}$,

$$\Gamma\big(N_{kl}, M_{ij}^{(p)}\big) = \Gamma\big(U_{kl}, M_{ij}^{(p)}\big)(\lambda_l - \lambda_k).$$

Combining the two equalities together, we have for $k \neq l$

$$\Gamma\big(U_{kl}, M_{ij}^{(p)}\big)(U = \mathrm{Id}) = 2\frac{1}{(\lambda_l + \lambda_k)^2}\big(\delta_{kj} M_{il}^{(p)} - \delta_{il} M_{kj}^{(p)}\big). \tag{5.45}$$

Moreover, at $U = \mathrm{Id}$, for $1 \le k \le d$

$$\Gamma(U_{kk}, \cdot) = -\Gamma\big(U_{kk}^*, \cdot\big).$$

Then by the fact that U_{kk} is real, we have at $U = \mathrm{Id}$,

$$\Gamma\big(U_{kk}, M_{ij}^{(p)}\big) = 0.$$

Now write

$$g_{ij} = \begin{cases} \frac{2}{(\lambda_i+\lambda_j)^2}, & i \neq j, \\ 0, & i = j, \end{cases}$$

then at any U, we derive (5.43) and (5.44) by the invariance of unitary transformation. □

Lemma 5.15. *The diffusion operators of $\{\mathbf{Z}^i\}$ are given by*

$$\Gamma\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) = 2\delta_{pq}\left(\delta_{il}\frac{1}{\lambda_i^2} Z_{kj}^{(p)} + \delta_{kj}\frac{1}{\lambda_j^2} Z_{il}^{(p)}\right) - 2\left(\delta_{kj}\frac{1}{\lambda_j^2}\big(Z^{(p)} Z^{(q)}\big)_{il} + \delta_{il}\frac{1}{\lambda_i^2}\big(Z^{(q)} Z^{(p)}\big)_{kj}\right)$$
$$+ \sum_{a=1}^{d}\big(y_{ia}\delta_{il} Z_{aj}^{(p)} Z_{ka}^{(q)} + y_{ka}\delta_{kj} Z_{ia}^{(p)} Z_{al}^{(q)}\big) - y_{ik} Z_{kj}^{(p)} Z_{il}^{(q)} - y_{jl} Z_{il}^{(p)} Z_{kj}^{(q)}, \tag{5.46}$$

where $y_{ij} = 2\frac{\lambda_i^2+\lambda_j^2}{(\lambda_i^2-\lambda_j^2)^2}$ when $i \neq j$ and $y_{ii} = \frac{1}{\lambda_i^2}$ for $1 \le i \le d$, and

$$\mathrm{L}\big(Z_{ij}^{(p)}\big) = 4d_p\frac{1}{\lambda_i^2}\delta_{ij} - 2(N-d)\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2}\right) Z_{ij}^{(p)} - 4\frac{1}{\lambda_i^2}\delta_{ij}\,\mathrm{trace}\big(Z^{(p)}\big)$$
$$+ 2\sum_{a=1}^{d} y_{ia}\delta_{ij} Z_{aa}^{(p)} - \sum_{a=1}^{d} y_{ja} Z_{ij}^{(p)} - \sum_{a=1}^{d} y_{ia} Z_{ij}^{(p)}. \tag{5.47}$$

Moreover,

$$\Gamma\big(Z_{ij}^{(p)}, \lambda_k\big) = 0, \tag{5.48}$$

$$\Gamma\big(Z_{ij}^{(p)}, U_{kl}\big) = \delta_{il}\sum_{a \neq l}^{d} d_{al} U_{ka} Z_{aj}^{(p)} - d_{jl} U_{kj} Z_{il}^{(p)}, \tag{5.49}$$

where $d_{ij} = 4\frac{\lambda_i\lambda_j}{(\lambda_i^2-\lambda_j^2)^2}$ when $i \neq j$ and $d_{ii} = 0$.

Proof. Notice the diffusion operator of $(\mathbf{U},\mathbf{M}^{(1)},\dots,\mathbf{M}^{(n)})$ are all invariant through the map

$$\big(\mathbf{U},\mathbf{M}^{(1)},\dots,\mathbf{M}^{(n)}\big)\to\big(\mathbf{U}^0\mathbf{U},\mathbf{U}^0\mathbf{M}^{(1)}\big(\mathbf{U}^0\big)^*,\dots,\mathbf{U}^0\mathbf{M}^{(n)}\big(\mathbf{U}^0\big)^*\big),$$

hence the diffusion operator of $\mathbf{Z}=(\mathbf{Z}^{(1)},\dots,\mathbf{Z}^{(n)})$ are also invariant. Therefore to compute $\Gamma(\mathbf{Z}^{(p)},\mathbf{Z}^{(q)})$, we may first consider the case at $\mathbf{U}=\mathrm{Id}$. By direct computations, we have

$$\begin{aligned}\Gamma\big(Z_{ij}^{(p)},Z_{kl}^{(q)}\big)={}&\sum_{r,s=1}^{d}\big(\Gamma\big(M_{ij}^{(p)},M_{kl}^{(q)}\big)+\Gamma\big(M_{ij}^{(p)},U_{kr}^*U_{sl}\big)M_{rs}^{(q)}+M_{rs}^{(p)}\Gamma\big(U_{ir}^*U_{sj},M_{kl}^{(q)}\big)\big)\\&+\sum_{u,v,r,s=1}^{d}\Gamma\big(U_{iu}^*U_{vj},U_{kr}^*U_{sl}\big)M_{uv}^{(p)}M_{rs}^{(q)}.\end{aligned}$$

The first term $\Gamma(M_{ij}^{(p)},M_{kl}^{(q)})$ at $U=\mathrm{Id}$ is straightforward from (5.38).
By (5.45), we get

$$\begin{aligned}&\sum_{r,s=1}^{d}\Gamma\big(M_{ij}^{(p)},U_{kr}^*U_{sl}\big)M_{rs}^{(q)}\\&\quad=\sum_{a=1}^{d}\big(-g_{ka}\delta_{kj}Z_{ia}^{(p)}Z_{al}^{(q)}-g_{al}\delta_{il}Z_{ka}^{(q)}Z_{aj}^{(p)}\big)+g_{jl}Z_{il}^{(p)}Z_{kj}^{(q)}+g_{ik}Z_{kj}^{(p)}Z_{il}^{(q)},\\&\sum_{r,s=1}^{d}\Gamma\big(M_{kl}^{(q)},U_{ir}^*U_{sj}\big)M_{rs}^{(p)}\\&\quad=\sum_{a=1}^{d}\big(-g_{ia}\delta_{il}Z_{ka}^{(q)}Z_{aj}^{(p)}-g_{aj}\delta_{kj}Z_{ia}^{(p)}Z_{al}^{(q)}\big)+g_{jl}Z_{il}^{(p)}Z_{kj}^{(q)}+g_{ik}Z_{kj}^{(p)}Z_{il}^{(q)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{u,v,r,s=1}^{d}\Gamma\big(U_{iu}^*U_{vj},U_{kr}^*U_{sl}\big)M_{uv}^{(p)}M_{rs}^{(q)}\\&\quad=\sum_{a=1}^{d}\big(r_{ia}\delta_{il}M_{aj}^{(p)}M_{ka}^{(q)}+r_{ka}\delta_{kj}M_{ia}^{(p)}M_{al}^{(q)}\big)-r_{ik}M_{kj}^{(p)}M_{il}^{(q)}-r_{jl}M_{il}^{(p)}M_{kj}^{(q)}.\end{aligned}$$

Therefore, putting all the four terms together we obtain at $U=\mathrm{Id}$,

$$\begin{aligned}&\Gamma\big(Z_{ij}^{(p)},Z_{kl}^{(q)}\big)\\&\quad=2\delta_{pq}\left(\delta_{il}\frac{1}{\lambda_i^2}Z_{kj}^{(p)}+\delta_{kj}\frac{1}{\lambda_j^2}Z_{il}^{(p)}\right)-2\left(\delta_{kj}\frac{1}{\lambda_j^2}\big(Z^{(p)}Z^{(q)}\big)_{il}+\delta_{il}\frac{1}{\lambda_i^2}\big(Z^{(q)}Z^{(p)}\big)_{kj}\right)\\&\qquad-\frac{1}{\lambda_i^2}\delta_{ik}Z_{ij}^{(p)}Z_{il}^{(q)}-\frac{1}{\lambda_j^2}\delta_{jl}Z_{ij}^{(p)}Z_{kj}^{(q)}+\frac{1}{\lambda_j^2}\delta_{kj}Z_{ij}^{(p)}Z_{jl}^{(q)}+\frac{1}{\lambda_i^2}\delta_{il}Z_{ij}^{(p)}Z_{ki}^{(q)}\\&\qquad+\sum_{a=1}^{d}\big(r_{ia}\delta_{il}Z_{aj}^{(p)}Z_{ka}^{(q)}+r_{ka}\delta_{kj}Z_{ia}^{(p)}Z_{al}^{(q)}\big)-r_{ik}Z_{kj}^{(p)}Z_{il}^{(q)}-r_{jl}Z_{il}^{(p)}Z_{kj}^{(q)}\end{aligned}$$

and defining

$$y_{ij} = \begin{cases} r_{ij}, & i \neq j, \\ \frac{1}{\lambda_i^2}, & i = j \end{cases}$$

for $1 \leq i, j \leq d$, we obtain

$$\begin{aligned}
&\Gamma\big(Z_{ij}^{(p)}, Z_{kl}^{(q)}\big) \\
&\quad = 2\delta_{pq}\left(\delta_{il}\frac{1}{\lambda_i^2}Z_{kj}^{(p)} + \delta_{kj}\frac{1}{\lambda_j^2}Z_{il}^{(p)}\right) - 2\left(\delta_{kj}\frac{1}{\lambda_j^2}\big(Z^{(p)}Z^{(q)}\big)_{il} + \delta_{il}\frac{1}{\lambda_i^2}\big(Z^{(q)}Z^{(p)}\big)_{kj}\right) \\
&\qquad + \sum_{a=1}^{d}\big(y_{ia}\delta_{il}Z_{aj}^{(p)}Z_{ka}^{(q)} + y_{ka}\delta_{kj}Z_{ia}^{(p)}Z_{al}^{(q)}\big) - y_{ik}Z_{kj}^{(p)}Z_{il}^{(q)} - y_{jl}Z_{il}^{(p)}Z_{kj}^{(q)}.
\end{aligned}$$

This formula is also valid at any U, because under the map $(U, M^p) \to (U^0U, U^0M^p(U^0)^*)$, $Z^{(p)}$ does not change. Thus we derive (5.46).

Let $P(X) = \det(S - X\mathrm{Id})$. Then

$$\Gamma\big(S_{ij}, \log P(X)\big) = \sum_{r=1}^{d}\frac{1}{\lambda_r^2 - X}\Gamma\big(S_{ij}, \lambda_r^2\big) = \sum_{p,q=1}^{d}(S - X)_{qp}^{-1}\Gamma(S_{ij}, S_{pq}).$$

Setting $V_{ijp,k} = \Gamma(U_{ip}\bar{U}_{jp}, \lambda_k^2)$, we may derive from the above formula that

$$4U\frac{D}{D - X}U^* + \sum_{kp}\frac{\lambda_p^2}{\lambda_k^2 - X}V_{ijp,k} = 4\frac{H}{H - X},$$

which leads to

$$\sum_p \lambda_p^2 V_{ijp,k} = 0.$$

On the other hand, since $\mathbf{U}$ is unitary, we know that $\Gamma(U_{ij}, \cdot) = -\Gamma(\bar{U}_{ji}, \cdot)$ at $\mathbf{U} = \mathrm{Id}$. Then taking $\sum_p X_p V_{ijp,k} = 0$ at $U = \mathrm{Id}$, we derive

$$\Gamma(U_{ij}, \lambda_k) = 0 \tag{5.50}$$

for any $i \neq j$ and k.

Computing $\Gamma(S_{ij}, S_{kl})$ at $\mathbf{U} = \mathrm{Id}$ leads to

$$2\delta_{il}\delta_{kj}\big(\lambda_i^2 + \lambda_j^2\big) = 4\delta_{i=j=k=l}\lambda_i^2 + \big(\lambda_i^2 - \lambda_j^2\big)\big(\lambda_k^2 - \lambda_l^2\big)\Gamma(U_{ij}, U_{kl}),$$

from which we may deduce that for $i \neq j$ or $k \neq l$

$$\Gamma(U_{ij}, U_{kl})(\mathrm{Id}) = -r_{ij}\delta_{il}\delta_{kj}, \tag{5.51}$$

where $r_{ij} = 2\frac{(\lambda_i^2 + \lambda_j^2)}{(\lambda_i^2 - \lambda_j^2)^2}$. Also since U_{ii} is real, we have $r_{ii} = 0$.

As for $\mathrm{L}(U_{ij})$, by the fact that $\mathbf{U}$ is unitary we obtain at $\mathbf{U} = \mathrm{Id}$,

$$\mathrm{L}(\bar{U}_{ji}) + \mathrm{L}(U_{ij}) + 2\sum_r \Gamma(U_{ir}, \bar{U}_{jr}) = 0,$$

thus

$$\mathrm{L}(U_{ij}) = \mathrm{L}(\bar{U}_{ji}) = -\sum_{r \neq i} r_{ir}\delta_{ij}. \tag{5.52}$$

By (5.45), (5.51) and (5.52), we have (5.47).
(5.45) and (5.50) lead to

$$\Gamma\bigl(Z_{ij}^{(p)}, \lambda_k\bigr) = 0.$$

Also by (5.51), we have at $U = \mathrm{Id}$ and at any U,

$$\Gamma\bigl(Z_{ij}^{(p)}, U_{kl}\bigr) = 4\frac{\lambda_k\lambda_l}{(\lambda_k^2 - \lambda_l^2)^2}\bigl(\delta_{il} Z_{kj}^{(p)} - \delta_{kj} Z_{il}^{(p)}\bigr).$$

Let $d_{ij} = 4\frac{\lambda_i\lambda_j}{(\lambda_i^2-\lambda_j^2)^2}$ when $i \neq j$ and $d_{ii} = 0$, then at any U, we obtain (5.49). □

Finally we are in the position to prove Theorem 5.8.

Proof of Theorem 5.8. By (5.27), (5.28), we obtain the generator of **D**.
Together with (5.48), and comparing (5.46), (5.47) with Theorem 4.3, we obtain the operator (5.22) with

$$A = 2D^{-2}, \qquad B_{ij,kl} = y_{ij}\delta_{ik}\delta_{jl},$$

and $a_p = d_p - d + 1$. □

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