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METASTABILITY OF THE CONTACT PROCESS ON FAST EVOLVING SCALE-FREE NETWORKS

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We study the contact process in the regime of small infection rates on finite scale-free networks with stationary dynamics based on simultaneous updating of all connections of a vertex. We allow the update rates of individual vertices to increase with the strength of a vertex, leading to a fast evolution of the network. We first develop an approach for inhomogeneous networks with general kernel and then focus on two canonical cases, the factor kernel and the preferential attachment kernel. For these specific networks, we identify and analyse four possible strategies how the infection can survive for a long time. We show that there is fast extinction of the infection when neither of the strategies is successful, otherwise there is slow extinction and the most successful strategy determines the asymptotics of the metastable density as the infection rate goes to zero. We identify the domains in which these strategies dominate in terms of phase diagrams for the exponent describing the decay of the metastable density.

1. Introduction. The spread of disease, information or opinion on networks has been one of the most studied problems in mathematical network science over the past decade. There has been tremendous progress related to a variety of spreading processes and underlying network models. For the vast majority of these studies, the network has been assumed to be fixed—at least on the time scales of the processes running on the network. Real networks however undergo change and this change is often on a similar time scale as the spreading processes running on the networks. The problem of temporal variability of the networks, and how this variability can interfere with processes on the network has received little attention so far in the mathematical literature. The aim of this paper is therefore to investigate the possible effects of stationary dynamics of a network on the spread of an infection by offering an extensive case study based on the following basic assumptions:

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- **Scale-free network model.** We look at the class of sparse *inhomogeneous random graphs*. For this class, vertices are labelled by indices from $\{1, \dots, N\}$ and edges exist independently with the probability of an edge $\{i, j\}$ given as $\frac{1}{N} p(i/N, j/N) \wedge 1$ for a suitable kernel $p\colon (0, 1] \times (0, 1] \to (0, \infty)$. We focus on two universal types of kernel, which produce scale-free networks. The factor kernel reproduces the asymptotic connection probabilities of most standard scale-free network models without significant correlations when the vertices are ordered by decreasing strength. This includes the Chung–Lu [4], Norros–Reittu [11] and configuration models [9]. The preferential attachment kernel reproduces the connection probabilities of various preferential attachment models [1, 5]; see [12] for a recent survey of static network models.
- **Fast network evolution.** We focus on *fast dynamics*, which arise, for example, as a rough approximation of migration effects in networks where links correspond to physical proximity. In our network dynamics, all edges adjacent to a vertex are updated simultaneously at a rate that may depend increasingly on the vertex strength, so that the most relevant vertices can update relatively quickly. Upon updating a vertex loses all its connections, and new connections are built independently. The connection probabilities of a vertex remain the same before and after the update, so that the network evolution is stationary.
- **SIS type epidemic process on the network.** We investigate the *contact process*, or SIS infection. The key feature which makes this process interesting from our point of view is that in order to survive the infection travels many times along individual edges, so that temporal changes in the status of edges become relevant for the behaviour of the infection. For the contact process on scale-free networks with the factor kernel, this feature leads to different qualitative behaviour of the static model and its classical mean-field approximation, as explained in [3]; see also [2, 6]. It is therefore a natural question to ask how the contact process behaves for dynamic models that interpolate between the static and the mean-field case.

An *evolving network* is a (random) family $(\mathscr{G}_t^{(N)} : t \geq 0, N \in \mathbb{N})$ of graphs, where $\mathscr{G}_t^{(N)}$ has vertex set $\{1, \dots, N\}$. Conditionally on this network evolution, the *contact process* on $(\mathscr{G}_t^{(N)} : t \geq 0)$ is a time-inhomogeneous Markov process that can be defined as follows: Every vertex v may be healthy or infected; if infected, every adjacent healthy vertex gets infected with rate λ up until the recovery of v, which happens at rate one. When a vertex recovers, it is again susceptible to infection. We write $X_t(v) = 1$ if the vertex $v \in \{1, \dots, N\}$ is infected at time t and $X_t(v) = 0$ otherwise.

The state when every vertex is healthy is absorbing and can be reached at any time from every other state in finite time with positive probability. Hence there exists a finite time T_{ext}, called the *extinction time*, which is the infimum over all times where the contact process is in the absorbing state. If the evolving network is itself a (time-homogeneous) Markov process, then $(\mathscr{G}_t^{(N)}, X_t)_{t\geq 0}$ is also a (time-homogeneous) Markov process, and we will work within this context. More precisely, the evolving network we consider is a stationary Markov process, and unless otherwise specified, we start the process with the network distributed according to the stationary measure, and the contact process with every vertex infected. Our interest is in the size of the extinction time in that case.

We say that the system experiences *fast extinction* if, for some sufficiently small infection rate $\lambda > 0$, the expected extinction time is bounded by a power of the network size. We say that we have *slow extinction* if, for every infection rate $\lambda > 0$, the extinction time is at least exponential in the network size with high probability, more precisely there exists a positive constant c such that, uniformly in $N > 0$, we have

$$\mathbb{P}(T_{\text{ext}} \leq e^{cN}) \leq e^{-cN}.$$

Slow extinction is a phenomenon of *metastability*, a physical system reaching its equilibrium very slowly because it spends a lot of time in states which are local energy minima, the so-called metastable states. Metastability in our model suggests, informally, that starting from all vertices infected the density of infected vertices is likely to decrease rapidly to a *metastable density*, and stay close to this density up to the exponential survival time of the infection. Metastable densities for the contact process have been studied in the case of static networks by Mountford et al. in [10]. Our interest in metastable densities stems from the fact that, when seen as a function of small λ, they reflect which is the optimal survival strategy for the infection. As we shall see, the optimal survival strategies changes as we vary the network parameters, defining phase transitions.

To understand the mechanisms behind slow extinction, we follow [2] and first look at a *star graph*, that is, a single central vertex connected to k neighbouring vertices of degree one. If only the centre is initially infected, at the time of its first recovery it has on average $\frac{\lambda k}{\lambda+1} \sim \lambda k$ infected neighbours. The probability that none of these neighbours reinfects the centre is therefore approximately

$$\left(\frac{1}{1+\lambda}\right)^{\lambda k} \sim e^{-\lambda^2 k}.$$

Hence the infection survives for a long time on the star graph if $k \gg \lambda^{-2}$ and in this case the survival time is exponential in $\lambda^2 k$.

If now the central vertex in the star graph updates at fixed rate κ, and upon updating is connected to k uninfected vertices, at the time r of the first recovery we have on average order λk infected neighbours. The probability that none of them reinfects the centre before it updates is

$$\frac{\kappa}{\lambda^2 k + \kappa}$$

and in this case we call r a *true recovery* (as opposed to *simple recovery*) since then (and only then) the infection becomes extinct. Again the infection survives for a long time on the star graph if $k \gg \kappa\lambda^{-2}$ but now the survival time in this case is of order $\lambda^2 k$, that is, linear in the degree of the central vertex as opposed to exponential as in the case of the star graph without updating.

To understand the survival of infections on a (static or evolving) inhomogeneous random graph, we classify vertices as *stars* and *connectors* where stars have large degree and connectors do not. Assuming that the kernel p is decreasing in every component, we use a function $a(\lambda) \downarrow 0$ to perform such a classification, where the set of stars $\mathscr{S}$ is

$$\mathscr{S} := \{1, \dots, \lfloor a(\lambda)N \rfloor\},$$

and its elements have degree asymptotically bounded from below by $\int_0^1 p(a(\lambda), x)\, dx$. We think of stars acting locally like the centres in a star graph (hence the name) with most of their neighbours in the complementary set $\mathscr{C}$ of connectors. In particular, an individual star can hold the infection for a long time if

$$\int_0^1 p\big(a(\lambda), x\big)\, dx \gg \lambda^{-2}.$$

Slow extinction of the infection is based on a collective strategy such that, given that a positive proportion of vertices in the set $\mathscr{S}$ is infected, up to an exponentially small error probability a positive proportion of vertices in $\mathscr{S}$ will again be infected after a time span given by the recovery cycle of the stars. The existence of such a strategy ensures that the infection is kept alive on $\mathscr{S}$ for an exponentially long time, making this set the *skeleton* of the infection. To obtain the metastable density associated to any given *survival strategy*, we find first a maximal function $a(\lambda) \downarrow 0$ (which defines $\mathscr{S}$) such that the strategy holds, and obtain the density as the number of infected direct neighbours of $\mathscr{S}$ divided by the total number N of vertices.

We have identified *four relevant survival strategies* for the infection:

(i) *Quick direct spreading*
Stars directly infect sufficiently many other stars before simple recoveries, so that the infection can be kept alive for a long time on the subgraph of stars alone. The connectors play no role for the survival of the infection.

(ii) *Delayed direct spreading*
As described for the star graph above, in this mechanism a star can retain an infection on a longer time scale if the lower bound on its degree is of larger order than λ^{-2}. Operating on this longer time-scale stars spread the infection directly to other stars and keep the infection alive.

(iii) *Quick indirect spreading*
Stars infect a large number of their neighbours before a simple recovery, and these neighbours then pass on the infection to other stars. In this way, stars indirectly infect sufficiently many other stars keeping the infection alive.

(iv) *Delayed indirect spreading*
As described for the delayed direct mechanism, a star retains the infection on a longer time scale if the lower bound on its degree is of larger order than λ^{-2}. On this time-scale, stars pass the infection to other stars via their infected neighbours, as in the quick indirect mechanism.

Assume now that $\mathscr{G}_0^{(N)}$ is an inhomogeneous random graph with a kernel p and suppose that in the evolving network $(\mathscr{G}_t^{(N)} : t \geq 0)$ every vertex updates with rate κ and upon updating it receives a new set of adjacent edges with the same probability as before, given by the kernel p. We now formulate and explain heuristically our results for the case of updating with constant rate κ in the case of the factor and preferential attachment kernel. Results for more general kernel and update rules will be formulated in the next section when we present our main results.

Define

$$I_N(t) := \frac{1}{N}\mathbb{E}\left[\sum_{v=1}^{N} X_t(v)\right],$$

to be the expected density of infected vertices at time t. Using the self-duality of the contact process [8], Chapter VI, we get

$$I_N(t) = \frac{1}{N}\sum_{v=1}^{N} \mathbb{P}_v(T_{\text{ext}} > t), \tag{1}$$

where $\mathbb{P}_v$ refers to the contact process started with only vertex v infected. We say that the contact process has *lower metastable density* $\rho^-(\lambda)$ and *upper metastable density* $\rho^+(\lambda)$ if, whenever t_N is going to infinity slower than exponentially, we have[4]

$$0 < \rho^-(\lambda) = \liminf_{N\to\infty} I_N(t_N) \leq \limsup_{N\to\infty} I_N(t_N) = \rho^+(\lambda).$$

We say that ξ is the *metastability exponent* of the process if the lower and upper metastability densities exist for sufficiently small $\lambda > 0$ and satisfy

$$\xi := \lim_{\lambda\downarrow 0} \frac{\log \rho^-(\lambda)}{\log \lambda} = \lim_{\lambda\downarrow 0} \frac{\log \rho^+(\lambda)}{\log \lambda}.$$

Loosely speaking, the metastability exponent measures the rate of decay of the metastable density as the infection rate λ approaches the critical value zero.

We first look at the *factor kernel*

$$p(x, y) = \beta x^{-\gamma} y^{-\gamma} \quad \text{for some } \beta > 0 \text{ and } 0 < \gamma < 1.$$

It is easy to see that the inhomogeneous networks with kernel p are scale-free with *power-law exponent* $\tau = 1 + \frac{1}{\gamma}$. Our first result shows that in the case of factor

[4] Actually, a more precise and slightly stronger metastability definition is given in next section.

kernels there are two phase transitions in the behaviour of the contact process with small infection rates.

PROPOSITION 1. *Suppose p is a factor kernel with parameter $0 < \gamma < 1$.*

(a) *If $0 < \gamma < \frac{1}{3}$ we have fast extinction, and if $\frac{1}{3} < \gamma < 1$ slow extinction.*
(b) *If $\frac{1}{3} < \gamma < 1$ the metastability exponent exists and equals*

$$\xi = \begin{cases} \dfrac{2}{3\gamma - 1} & \text{if } \dfrac{1}{3} < \gamma < \dfrac{2}{3}, \\ \dfrac{\gamma}{2\gamma - 1} & \text{if } \dfrac{2}{3} < \gamma < 1. \end{cases}$$

(a) is the main result of Jacob and Mörters [7].

We argue now informally that in the regime $1/3 < \gamma < 2/3$ the strategy of delayed direct spreading prevails, whereas for $\gamma > 2/3$ it is quick direct spreading that is most successful. For $\gamma < 1/3$, none of the strategies succeed.

Under quick direct spreading, the infection can be sustained on $\mathscr{S}$ if $a(\lambda)$ satisfies

$$\int_0^{a(\lambda)} \int_0^{a(\lambda)} \lambda p(x, y)\, dx\, dy \approx a(\lambda),$$

which arises from equating the initial amount of infected stars with the vertices in $\mathscr{S}$ infected by those stars before one unit time, which is the average time it takes to have simple recoveries. This equation yields $a(\lambda) \approx \lambda^{1/(2\gamma-1)}$, which is admissible if $\gamma > \frac{1}{2}$. We hence get a lower bound for the lower metastable density

$$\rho^-(\lambda) \approx \int_0^{a(\lambda)} \int_0^1 \lambda p(x, y)\, dx\, dy \approx \lambda a(\lambda)^{1-\gamma} \approx \lambda^{\frac{\gamma}{2\gamma-1}}.$$

For the delayed mechanism, on the other hand, we note that the lower bound on the expected degree of a star is $a(\lambda)^{-\gamma}$, and hence the infection can be held at a star on a time scale of

$$T(\lambda) = \lambda^2 a(\lambda)^{-\gamma},$$

which the average time until a true recovery. Now by the same principle as in the quick mechanism $a(\lambda)$ has to satisfy

$$T(\lambda) \int_0^{a(\lambda)} \int_0^{a(\lambda)} \lambda p(x, y)\, dx\, dy \approx a(\lambda),$$

hence $a(\lambda) \approx \lambda^{\frac{3}{3\gamma-1}}$ which is admissible if $\gamma > \frac{1}{3}$. This yields a lower bound of the form

$$\rho^-(\lambda) \approx \lambda a(\lambda)^{1-\gamma} \approx \lambda^{\frac{2}{3\gamma-1}}.$$

Comparing both densities, the delayed strategy therefore wins if $\frac{1}{3} < \gamma < \frac{2}{3}$, but if $\gamma > \frac{2}{3}$ the quick strategy wins. The other two strategies we have identified turn out to be inferior in any case. If $\gamma < \frac{1}{3}$ none of the strategies succeeds, that is, gives an admissible value of $a(\lambda)$.

The situation is quite different for *preferential attachment kernels* given by

$$p(x, y) = \beta(x \wedge y)^{-\gamma}(x \vee y)^{\gamma-1} \quad \text{for some } \beta > 0 \text{ and } 0 < \gamma < 1.$$

As before, the networks with kernel p are easily seen to be scale-free with the same power-law exponent $\tau = 1 + \frac{1}{\gamma}$.

PROPOSITION 2. *Suppose p is a preferential attachment kernel with parameter $0 < \gamma < 1$.*

(a) *For all $0 < \gamma < 1$ there is slow extinction.*
(b) *The metastability exponent exists and equals*

$$\xi = \begin{cases} \dfrac{3-2\gamma}{\gamma} & \text{if } \gamma < \dfrac{3}{5}, \\ \dfrac{3-\gamma}{3\gamma-1} & \text{if } \gamma > \dfrac{3}{5}. \end{cases}$$

We now explain heuristically that in the regime $\gamma < 3/5$ the strategy of delayed direct spreading prevails, whereas for $\gamma > 3/5$ it is delayed indirect spreading that is most successful.

For delayed direct spreading $a(\lambda)$ again has to satisfy

$$T(\lambda) \int_0^{a(\lambda)} \int_0^{a(\lambda)} \lambda p(x, y)\, dx\, dy \approx a(\lambda),$$

for the time scale $T(\lambda) = \lambda^2 a(\lambda)^{-\gamma}$. For the preferential attachment kernel, this gives $a(\lambda) \approx \lambda^{3/\gamma}$, which is always admissible. This mechanism then yields

$$\rho^-(\lambda) \approx \lambda a(\lambda)^{1-\gamma} \approx \lambda^{\frac{3-2\gamma}{\gamma}}.$$

For the indirect mechanism, the equation for $a(\lambda)$ changes to

$$T(\lambda) \int_0^{a(\lambda)} \int_{a(\lambda)}^1 \int_0^{a(\lambda)} \lambda^2 p(x, y) p(y, z)\, dx\, dy\, dz \approx a(\lambda),$$

where the term on the left represents the amount of stars infected by connectors that where in turn infected by the initially infected stars in a time-scale of order $T(\lambda)$. This gives $a(\lambda) \approx \lambda^{4/(3\gamma-1)}$ for $\gamma > 1/2$, which is admissible and yields

$$\rho^-(\lambda) \approx \lambda a(\lambda)^{1-\gamma} \approx \lambda^{\frac{3-\gamma}{3\gamma-1}}.$$

Comparing once again the resulting densities, the indirect strategy therefore wins if $\gamma > 3/5$, otherwise the direct strategy wins.

To better understand the metastability phenomenon and explore the full range of possible optimal strategies, we move in the next section to a model where update rates can depend on the vertex strength. A rich and beautiful picture emerges from this.

2. Statement of the main results. Recall that for $N \in \mathbb{N}$ the inhomogeneous random graph $\mathscr{G}^{(N)}$ has vertex set $\{1, \dots, N\}$ and every edge $\{i, j\}$ exists independently with probability

$$p_{i,j} := \frac{1}{N} p\left(\frac{i}{N}, \frac{j}{N}\right) \wedge 1,$$

where $p\colon (0, 1] \times (0, 1] \to (0, \infty)$ is a kernel for which we make the following assumptions:

(1) p is symmetric, continuous and decreasing in both parameters,
(2) there is $\gamma \in (0, 1)$ and constants $0 < c_1 < c_2$ such that for all $a \in (0, 1)$,

$$c_1 a^{-\gamma} \le p(a, 1) \le \int_0^1 p(a, s)\, ds < c_2 a^{-\gamma}. \tag{2}$$

Observe that for every $f\colon (0, 1] \to (0, \infty)$ decreasing, continuous and integrable, the kernel $p(s, t) = (s \wedge t)^{-\gamma} f(s \vee t)$ satisfies conditions (1) and (2). The choices $f(x) = \beta x^{-\gamma}$ and $f(x) = \beta x^{\gamma-1}$ give the factor and preferential attachment kernels, respectively.

We take $\mathscr{G}_0^{(N)} = \mathscr{G}^{(N)}$ and obtain the evolving network $(\mathscr{G}_t^{(N)} : t \ge 0)$ using the following dynamics: Each vertex i updates independently with rate

$$\kappa_i = \kappa_0 \left(\frac{N}{i}\right)^{\gamma\eta} \quad \text{for } i \in \{1, \dots, N\},$$

where $\eta \in \mathbb{R}$ and $\kappa_0 > 0$ are fixed constants. When vertex i updates, every unordered pair $\{i, j\}$, for $j \ne i$ forms an edge with probability $p_{i,j}$, independently of its previous state and of all other edges. The remaining edges $\{k, l\}$ with $k, l \ne i$ remain unchanged.

Observe that this evolution is stationary. The expected degree of vertex i does not depend on time and is of order $(N/i)^\gamma$ so κ_i is proportional to its degree raised to the power η. If $\eta > 0$, powerful vertices update more quickly and as η passes from zero to ∞ we interpolate between the evolving networks with fixed update rates and the mean field model in which no memory of edges present is retained. We call this a *fast evolving dynamics*. Conversely, if $\eta < 0$ powerful vertices update slowly and we can consider the connection between them as fixed during long periods of time. As η passes to $-\infty$, we interpolate between evolving networks with fixed update rates and the static model. In this work, we only consider the fast evolving case $\eta \ge 0$ as the slowly evolving case requires additional techniques.

We now consider the contact process with infection rate $\lambda \in (0, 1)$ on the evolving graphs $(\mathscr{G}_t^{(N)})$. The assumption $\lambda < 1$ is unessential but simplifies a bit the presentation of our results. We say the contact process exhibits:

- *metastability* if there there exists $\varepsilon > 0$ such that:
  – whenever t_N is going to infinity slower than $e^{\varepsilon N}$, we have

$$\liminf_{N\to\infty} I_N(t_N) > 0;$$

  – whenever s_N and t_N are going to infinity slower than $e^{\varepsilon N}$, we have

$$I_N(s_N) - I_N(t_N) \underset{N\to\infty}{\longrightarrow} 0.$$

  In that case, we can unambiguously define the *lower metastable density* $\rho^-(\lambda) = \liminf I_N(t_N) > 0$ and the *upper metastable density* $\rho^+(\lambda) = \limsup I_N(t_N)$.
- *a metastable density* $\rho(\lambda)$ if there is metastability and $\rho^-(\lambda) = \rho^+(\lambda) = \rho(\lambda)$. Equivalently, whenever t_N is going to infinity slower than $e^{\varepsilon N}$, we have

$$\lim I_N(t_N) = \rho(\lambda) > 0.$$

In the following theorem, we identify conditions on the kernel p for the four survival strategies identified in the first section to successfully sustain the infection. We deduce slow extinction and metastability and derive lower bounds on the lower metastable densities in each case. We also believe that there is a metastable density as soon as there is slow extinction, but we do not prove this.

THEOREM 1. *Define* $\theta = \exp(-2(1+\kappa_0 2^{\gamma\eta}))$. *For* $a, \lambda > 0$ *define* $T = T(a,\lambda)$ *by*

$$T\log^2(T) = \frac{c_1\theta}{400\kappa_0(3\kappa_0+1)}\lambda^2 a^{-\gamma(1-2\eta)}, \tag{3}$$

where c_1 *is as in* (2). *There exist positive and finite constants* $M_{(i)}$, $M_{(ii)}$, $M_{(iii)}$, $M_{(iv)}$ (*that might depend on* γ *and* η), *such that, for every* $\lambda \in (0,1)$, *slow extinction and metastability are guaranteed as soon as one can find* $a = a(\lambda) \in (0, 1/2)$ *satisfying at least one of the following conditions*:

(i) **(Quick Direct Spreading)**

$$\lambda a p(a,a) > M_{(i)}.$$

(ii) **(Quick Indirect Spreading)**

$$\lambda^2 a p(a,1)^2 > M_{(ii)}.$$

(iii) **(Delayed Direct Spreading)**

$$T(a,\lambda) > M_{(iii)} \quad \textit{and} \quad \lambda a T(a,\lambda) p(a,a) > M_{(iii)}.$$

(iv) **(Delayed Indirect Spreading)**

$$T(a,\lambda) > M_{(iv)} \quad \textit{and} \quad \lambda^2 a T(a,\lambda) p(a,1)^2 > M_{(iv)}.$$

Moreover, in each of these cases we have

$$\rho^{-}(\lambda) \geq c'\big(\lambda a p(a,1) \wedge 1\big), \tag{4}$$

where $c' > 0$ is a universal constant (*independent of* λ).

While the lower bounds above can be verified by investigating each of the four explicit survival strategies separately, upper bounds require a general, more implicit, method that yields information independent of any chosen strategy. Our approach is a supermartingale technique which gives upper bounds based on the choice of a scoring function. By carefully selecting a proper scoring function the technique will produce upper bounds which match the lower bounds in each of the cases investigated here.

THEOREM 2. *Let the positive constant D be defined by*

$$D = \min\left\{\frac{\kappa_0}{4}, \frac{\kappa_0^2}{64c_2}, \frac{1}{16}\right\}, \tag{5}$$

where c_2 is as in (2). *For $\lambda \in (0,1)$, define the time-scale function $T_\lambda\colon (0,1) \to (0,\infty)$ as*

$$T_\lambda(x) = \max\{\lambda^2 x^{-\gamma(1-2\eta)}, 1\}. \tag{6}$$

(a) *For $\lambda \in (0,1)$, suppose there is some nonincreasing function $S\colon (0,1] \to (0,\infty)$ with $\int_0^1 S(x)\,dx < \infty$ and*

$$\lambda T_\lambda(x) \int_0^1 p(x,y) S(y)\,dy \leq D S(x) \quad \forall x \in (0,1]. \tag{7}$$

Then the expected extinction time is at most linear in N and in particular there is fast extinction. More precisely, writing $(H)_\delta$ the hypothesis that $T_\lambda S^{-\delta}$ is a bounded function, we have

(i) $(H)_1$ *is a consequence of* (7).

(ii) *If $(H)_\delta$ is satisfied for some $\delta \in (0,1]$, then there exists $\omega = \omega(\lambda,\delta)$ such that, for every N, we have*

$$\mathbb{E}[T_{\text{ext}}] \leq \omega N^\delta.$$

(b) *For $\lambda \in (0,1)$, suppose there exists some $a = a(\lambda) > 0$ and some nonincreasing function $S : [a,1] \to (0,\infty)$ such that*

$$\lambda T_\lambda(x) \int_0^1 p(x,y) S(y \vee a)\,dy \leq D S(x) \quad \forall x \in (0,1]. \tag{8}$$

Then there exists $\omega = \omega(\lambda) > 0$ and a function $\varepsilon = \varepsilon(N)$ converging to 0 as $N \uparrow \infty$ such that, for all N and all $t \geq 0$, we have

$$I_N(t) \leq a + \frac{1}{S(a)} \int_a^1 S(y)\,dy + \frac{\omega}{t} + \varepsilon(N). \tag{9}$$

In particular, if there is metastability, then the upper metastable density satisfies

$$\rho^+(\lambda) \le a(\lambda) + \frac{1}{S(a(\lambda))} \int_{a(\lambda)}^1 S(y)\, dy. \tag{10}$$

Applying these two theorems to the kernels considered yields our main result.

THEOREM 3.

(a) *Suppose p is the factor kernel.*

(i) *If $\eta < \frac{1}{2}$ and $\gamma < \frac{1}{3-2\eta}$, or if $\eta \ge \frac{1}{2}$ and $\gamma < \frac{1}{2}$, there is fast extinction.*

(ii) *if $\eta < \frac{1}{2}$ and $\gamma > \frac{1}{3-2\eta}$, or if $\eta \ge \frac{1}{2}$ and $\gamma > \frac{1}{2}$, there is slow extinction and metastability, and the metastability exponent satisfies*

$$\xi = \begin{cases} \dfrac{2-2\gamma\eta}{3\gamma - 2\gamma\eta - 1} & \text{if } \gamma < \dfrac{2}{3+2\eta}, \\ \dfrac{\gamma}{2\gamma - 1} & \text{if } \gamma > \dfrac{2}{3+2\eta}. \end{cases} \tag{11}$$

(b) *Suppose p is the preferential attachment kernel.*

(i) *If $\eta \ge \frac{1}{2}$ and $\gamma < \frac{1}{2}$, there is fast extinction.*

(ii) *If $\eta < \frac{1}{2}$, or if $\eta \ge \frac{1}{2}$ and $\gamma > \frac{1}{2}$, there is slow extinction and metastability, and the metastability exponent satisfies*

$$\xi = \begin{cases} \dfrac{3-2\gamma-2\gamma\eta}{\gamma - 2\gamma\eta} & \text{if } \eta < \dfrac{1}{2} \text{ and } 0 < \gamma < \dfrac{3}{5+2\eta}, \\ \dfrac{3-\gamma-2\gamma\eta}{3\gamma - 2\gamma\eta - 1} & \text{if } \eta < \dfrac{1}{2} \text{ and } \dfrac{3}{5+2\eta} < \gamma < \dfrac{1}{1+2\eta}, \\ \dfrac{1}{2\gamma - 1} & \text{if } \dfrac{1}{1+2\eta} < \gamma. \end{cases} \tag{12}$$

REMARK 1. For both kernels, when $\eta > \frac{1}{2}$ and $\gamma < \frac{1}{2}$ we have $T_\lambda(x) = 1$, and hence $(H)_\delta$ holds for any $\delta > 0$. By Theorem 2(1), we then get that $\mathbb{E}[T_{\mathrm{ext}}]$ is even subpolynomial in N.

REMARK 2. The different exponents for the metastable densities in Theorem 3 are indicative of different survival strategies for the infection, as indicated in Figure 1. Propositions 1 and 2 follow from Theorem 3 by letting $\eta = 0$.

REMARK 3. In the cases of slow extinction, our results are actually slightly more precise than stated in Theorem 3. In particular, the upper metastable density always satisfies $\rho^+(\lambda) \le c\lambda^\xi$ for some constant c. In the phases when quick direct/indirect spreading prevails, the lower metastable density also satisfies $\rho^-(\lambda) \ge$

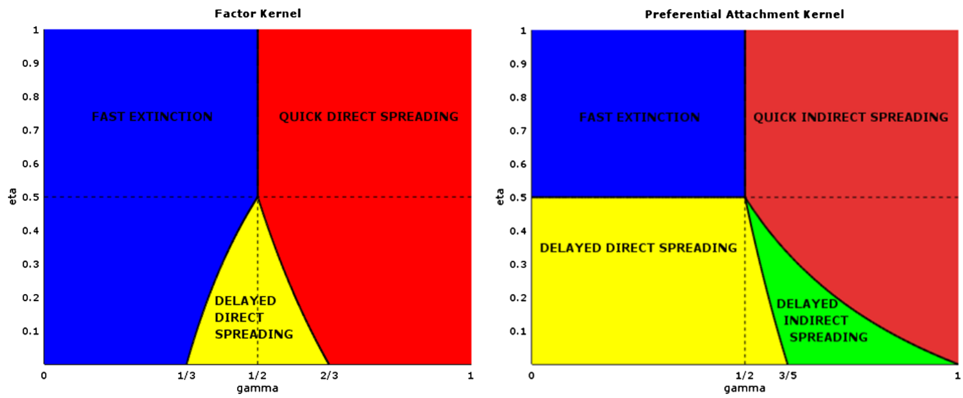

FIG. 1. *The figures summarise Theorem 3 in the form of phase diagrams for the factor kernel (left) and the preferential attachment kernel (right).*

$c\lambda^{\xi}$ for some $c > 0$, so in these phases we obtain the metastable densities up to a bounded multiplicative factor.

The rest of this paper is organized as follows; In Section 3, we introduce a graphical representation of the evolving network and contact process which allows us to define the process rigorously, yielding at the same time useful properties such as self-duality and monotonicity. In Sections 4 and 5, we give the proofs of Theorems 1 and 2, respectively, and finally in Sections 6 and 7 we apply those theorems to deduce Theorem 3.

3. Graphical representation. The evolving network model $(\mathscr{G}_t^{(N)} : t \geq 0, N \in \mathbb{N})$ is represented with the help of the following independent random variables;

(1) For each $x \in \mathbb{N}$, a Poisson point process $\mathcal{U}^x = (U_n^x)_{n\geq 1}$ of intensity κ_x, describing the updating times of the vertex x. Given $x \neq y$ we also write $\mathcal{U}^{x,y} = (U_n^{x,y})_{n\geq 1}$ for the union $\mathcal{U}^x \cup \mathcal{U}^y$, which is a Poisson point process of intensity $\kappa_x + \kappa_y$, describing the updating times of the potential edge $\{x, y\}$.

(2) For each $\{x, y\}$ with $x \neq y$ and $x, y \leq N$, a sequence of independent random variables $(C_n^{x,y})_{n\geq 0}$, all Bernoulli with parameter $p_{x,y}$, describing the presence/absence of the edge in the network after the successive updating times of the potential edge $\{x, y\}$. More precisely, if $t \geq 0$ then $\{x, y\}$ is an edge in $\mathscr{G}_t^{(N)}$ if and only if $C_n^{x,y} = 1$ for $n = |[0, t] \cap \mathcal{U}^{x,y}|$. We denote $\mathcal{C}^x := (C_n^{x,y} : y \leq N, n \in \mathbb{N})$.

Given the network we represent the infection by means of the following set of independent random variables;

(3) For each $x \in \mathbb{N}$, a Poisson point process $\mathcal{R}^x = (R_n^x)_{n\geq 1}$ of intensity one describing the recovery times of x.

(4) For each $\{x, y\}$ with $x \neq y$, a Poisson point process $\mathcal{I}_0^{x,y}$ with intensity λ describing the infection times along the edge $\{x, y\}$. Only the trace $\mathcal{I}^{x,y}$ of this process on the set

$$\bigcup_{n=0}^{\infty} \{[U_n^{x,y}, U_{n+1}^{x,y}) \colon C_n^{x,y} = 1\} \subset [0, \infty)$$

can actually cause infections. Write $(I_n^{x,y})_{n \geq 1}$ for the ordered points of $\mathcal{I}^{x,y}$. If just before time $I_n^{x,y}$ vertex x is infected and y is healthy, then x infects y at time $I_n^{x,y}$. If y is infected and x healthy, then y infects x; otherwise, nothing happens.

The infection is now described by a process $(X_t(x), x \in \{1, \dots, N\} \colon t \geq 0)$ with values in $\{0, 1\}^N$, such that $X_t(x) = 1$ if x is infected at time t, and $X_t(x) = 0$ if x is healthy at time t. More formally, the infection process associated to this graphical representation and to a starting set A_0 of infected vertices, is the càdlàg process with $X_0(x) = \mathbf{1}_{A_0}(x)$ evolving only at times $t \in \mathcal{R}^x \cup \bigcup_{n=1}^{\infty} I_n^{x,y}$, according to the following rules:

- If $t \in \mathcal{R}^x$, then $X_t(x) = 0$ (whatever $X_{t-}(x)$).
- If $t \in \mathcal{I}^{x,y}$, then

$$\big(X_t(x), X_t(y)\big) = \begin{cases} (0, 0) & \text{if } \big(X_{t-}(x), X_{t-}(y)\big) = (0, 0), \\ (1, 1) & \text{otherwise.} \end{cases}$$

The process $(\mathscr{G}_t^{(N)}, X_t \colon t \geq 0)$ is a Markov process describing the simultaneous evolution of the network and of the infection. We call $(\mathfrak{F}_t \colon t \geq 0)$ its canonical filtration.

Using the graphical representation, we obtain monotonicity and duality properties of the contact process on the evolving graph. The proof of the following proposition is standard within the context of the contact process (see [8]) and, therefore, omitted here.

PROPOSITION 3.

(1) **Monotonicity.** *If* $(X_t^1 \colon t \geq 0)$, $(X_t^2 \colon t \geq 0)$ *are processes constructed as above with* $X_0^1 \leq X_0^2$ *and infection rates* $\lambda_1 \leq \lambda_2$, *then* $X_t^1 \leq X_t^2$ *stochastically.*

(2) **Self-duality.** *If* X^A, X^B *correspond to the process with initial condition* $X_0 = \mathbf{1}_A$ *and* $X_0 = \mathbf{1}_B$, *respectively, then for all* $t > 0$,

$$\mathbb{P}\big(\exists x \in A, X_t^B(x) = 1\big) = \mathbb{P}\big(\exists x \in B, X_t^A(x) = 1\big).$$

The only added subtlety in the proof of the proposition above when compared to [8] is that the duality property combines both the duality property of the contact process, and that of the network dynamics.

4. Slow extinction and lower bounds. In this section, we prove Theorem 1 by showing four different *survival strategies* which can sustain the infection exponentially long. All these strategies are based on a division between powerful and weak vertices given by a parameter $a = a(\lambda) \in (0, 1/2)$ as

$$\mathscr{S} := \{1, 2, \dots, \lfloor aN \rfloor\}, \qquad \mathscr{C} := \{\lfloor aN \rfloor + 1, \dots, N\}.$$

The elements of $\mathscr{S}$ are called *stars* and the elements of $\mathscr{C}$ are called *connectors*. Notice that when λ decreases, any vertex with fixed degree has a lower chance of infecting its neighbours. Our definition of a star changes accordingly, that is, $a(\lambda) \downarrow 0$ as $\lambda \downarrow 0$. We denote by $\mathscr{S}_0 = \{x \in \mathscr{S}, X_0(x) = 1\}$ the set of initially infected stars, and by $S_0 = |\mathscr{S}_0|$ its cardinality. We start the proof with a relatively simple lemma, which already contains the flavour of the kind of inequalities we will use throughout the proof.

LEMMA 1. *Fix $r > 0$ and suppose one is given an initial condition $(X_0, \mathscr{G}_0)$ such that $S_0 \geq raN$. Then there exists a constant $C > 0$ (independent of λ, a, N) such that, for all $t \in [2, 3]$,*

$$\mathbb{E}\left[\sum_{v=1}^{N} X_t(v) \middle| X_0, \mathscr{G}_0\right] \geq C\big(\lambda a p(a, 1) \wedge 1\big)N. \tag{13}$$

REMARK 4. Lemma 1 remains true if t is in an arbitrary compact set bounded away from zero, changing only the value of C. For our purposes, the above formulation suffices.

PROOF. We introduce a terminology specific to this proof as follows:

$$\mathscr{S}' := \{x \in \mathscr{S}_0 : \mathcal{R}^x \cap [0, 2] = \varnothing, \mathcal{U}^x \cap [0, 1] \neq \varnothing\},$$
$$\mathscr{C}' := \{y \in \mathscr{C} : \mathcal{R}^y \cap [1, 3] = \varnothing, \exists x \in \mathscr{S}', \mathcal{I}^{x,y} \cap [1, 2] \neq \varnothing\}.$$

Each $x \in \mathscr{S}_0$ belongs to $\mathscr{S}'$ independently with probability $e^{-2}(1 - e^{-\kappa_x}) \geq e^{-2}(1 - e^{-\kappa_0})$. Therefore, the cardinality S' of $\mathscr{S}'$ dominates a binomial random variable with parameters S_0 and $e^{-2}(1 - e^{-\kappa_0})$.

For $y \in \mathscr{C}$, the event $\mathcal{R}^y \cap [1, 3] = \varnothing$ has probability e^{-2} and is independent of the event $E^y := \{\exists x \in \mathscr{S}', \mathcal{I}^{x,y} \cap [1, 2] \neq \varnothing\}$, whose probability we want to estimate. Conditionally on $x \in \mathscr{S}'$, a sufficient condition for the event E^y to be satisfied is that

(1) $\mathcal{I}_0^{x,y} \cap [1, 2] \neq \varnothing$, which happens with probability $1 - e^{-\lambda}$.

(2) At time $t = \min(\mathcal{I}_0^{x,y} \cap [1, 2])$, the edge $\{x, y\}$ belongs to the network. This happens with probability $p_{x,y}$, independently of the configuration of the network at time 0, as vertex x has updated on the time interval $[0, 1]$.

The probabilities here obtained are independent from each other and from the realization chosen for the $\mathcal{U}^x$, $\mathcal{R}^x$, hence x infects y on time interval $[1,2]$ (namely $\mathcal{I}^{x,y} \cap [1,2] \neq \varnothing$) with probability at least $p_{x,y}(1-e^{-\lambda})$, which we can bound from below by $(1-e^{-\lambda})p(a,1)/N$, by the definition of $p_{x,y}$ and the monotonicity of $p(\cdot,\cdot)$. If we now condition on $\mathscr{S}'$, the events $\mathcal{I}^{x,y} \cap [1,2] \neq \varnothing$ are independent and we get

$$\begin{aligned}\mathbb{P}(E^y|\mathscr{S}') &\geq 1 - \left(1 - (1-e^{-\lambda})\frac{p(a,1)}{N}\right)^{S'} \\ &\geq 1 - \exp\left(-(1-e^{-\lambda})\frac{p(a,1)S'}{N}\right) \geq \frac{\lambda p(a,1)S'}{4N} \wedge \frac{1}{2},\end{aligned}$$

where in the last inequality we used twice the inequality $1 - e^{-x} \geq (x \wedge 1)/2$ for $x \geq 0$ (we also used $\lambda < 1$). Finally we obtain that, given the initial condition of the network and the infection and conditionally on $\mathscr{S}'$, the cardinality of $\mathscr{C}'$ dominates a binomial random variable with parameters $|\mathscr{C}| \geq N/2$ and

$$\rho = \frac{e^{-2}\lambda p(a,1)S'}{4N} \wedge \frac{e^{-2}}{2}.$$

To conclude, we first consider the case $\lambda a p(a,1) < 1$. Then we always have $\rho = \frac{e^{-2}\lambda p(a,1)S'}{4N}$, and we easily get that the expectation of $|\mathscr{C}'|$ is bounded from below by

$$\frac{e^{-4}(1-e^{-\kappa_0})}{8}\lambda a p(a,1) r N.$$

In the case $\lambda a p(a,1) > 1$, we have $\rho \geq \frac{e^{-2}S'}{4aN}$, and we get a bound of

$$\frac{e^{-4}(1-e^{-\kappa_0})}{8} r N.$$

This altogether proves (13) with $C = re^{-4}(1-e^{-\kappa_0})/8$. $\square$

Denoting by $\mathscr{S}_k$ the set of infected stars at time $k \in \mathbb{N}$, and by $S_k = |\mathscr{S}_k|$ its cardinality, we aim to prove that for some $r > 0$ the events $\mathscr{E}_k^r := \{S_k > raN\}$ hold for a sufficiently long time. With this in mind, we say that a family of events $\mathscr{E}_k$ depending on $k \in \mathbb{N}$ holds *exponentially long* if there exists $c > 0$ such that, for all N,

$$\mathbb{P}\left(\bigcap_{k \leq e^{cN}} \mathscr{E}_k\right) \geq 1 - e^{-cN}.$$

As $\mathscr{E}_k^r \subset \{T_{\text{ext}} > k\}$, if the events $\mathscr{E}_k^r$ hold exponentially long, we have slow extinction. Moreover, from Lemma 1 we have

$$C(\lambda a p(a,1) \wedge 1)\mathbb{P}\left(\left|\{x \in \mathscr{S} : X_{\lfloor t \rfloor - 2}(x) = 1\}\right| > raN\right) \leq I_N(t)$$

for any $t > 2$, so if $\mathscr{E}_k^r$ holds exponentially long, the left-hand side above is bounded from below by $C(\lambda ap(a, 1) \wedge 1)(1 - e^{-cN})$ for some $c > 0$, and hence we deduce the lower bound (4) on the metastable density. Our aim is therefore not only to prove slow extinction, but the stronger result that under any of the conditions given in Theorem 1, the events $\mathscr{E}_k^r$ hold exponentially long. Actually, we will also allow a conditioning on any initial configuration included in the event $\mathscr{E}_0^r$, and still show that the events $\mathscr{E}_k^r$ hold exponentially long.

The proof of metastability, on the other hand, will also follow from this result. First, note that from (1) it is easy to see that $I_N(\cdot)$ is decreasing, and hence it suffices to show that, whenever $t_N \le e^{\varepsilon N}$,

$$\limsup_{N\to\infty} \left| I_N(t) - I_N(t_N)\right| \underset{t\to\infty}{\longrightarrow} 0.$$

Using self-duality, we can write

$$I_N(t) - I_N(t_N) = \frac{1}{N}\sum_{v=1}^{N} \mathbb{P}_v(t < T_{\text{ext}} < t_N),$$

where we recall that $\mathbb{P}_v$ stands for the probability measure corresponding to the infection starting with only vertex v infected. So metastability follows if we can prove that $\mathbb{P}_v(t < T_{\text{ext}} < e^{\varepsilon N})$ converges to 0 as $t \to \infty$, uniformly in N and $v \in \{1, \dots, N\}$. We separate this proof into three steps:

(1) For all $n \ge 0$, uniformly in N and v, $\mathbb{P}_v(\max S_k < n, T_{\text{ext}} > t) \longrightarrow_{t\to\infty} 0$.
(2) Uniformly in N and v, $\mathbb{P}_v(\max S_k \ge raN \,|\, \max S_k \ge n) \longrightarrow_{n\to\infty} 1$.
(3) Uniformly in v, $\mathbb{P}_v(T_{\text{ext}} \ge e^{\varepsilon N} \,|\, \max S_k \ge raN) \longrightarrow_{N\to\infty} 1$.

These three steps easily give the result. Indeed, for any given $\varepsilon > 0$, choosing N_0 as in (3), then n as in (2), then t as in (1), we obtain that for any $N \ge N_0$ and $v \in \{1, \dots, N\}$,

$$\mathbb{P}_v\bigl(t < T_{\text{ext}} < e^{\varepsilon N}\bigr) \le 3\varepsilon.$$

The *first step* is fairly easy and relies on the observation that there is some constant $c > 0$ depending only on n such that for all $k \ge 0$,

$$\mathbb{P}_v(S_{k+1} \ge n|\mathfrak{F}_k) \ge c\mathbf{1}_{\{S_k>0\}}.$$

We deduce $\mathbb{P}_v(\max S_k < n, S_t > 0) \le (1 - c)^{\lfloor t\rfloor}$, which proves the first step.

For the *second step*, we introduce the stopping time $K := \inf\{k \ge 0, S_k \ge n\}$ and prove $\mathbb{P}_v(\max S_k \ge raN|K < +\infty, \mathfrak{F}_K) \to \infty$ uniformly on N, v and on the σ-field $\mathfrak{F}_K$. In other words, we provide a uniform bound in all the possible configurations $(\mathscr{G}_K^{(N)}, X_K)$ for which $S_K \ge n$. This bound follows from the analysis of the process (S_k) below (see in particular Lemma 3(1) and Lemma 4(1)), and concludes the second step.

Finally, the *third step* also follows from the analysis of the process (S_k). If the stopping time $\tilde{K} := \inf\{k \geq 0, S_k \geq raN\}$ is finite, then the event $\mathscr{E}^r_{\tilde{K}}$ holds, and further the events $\mathscr{E}^r_{k+\tilde{K}}$ hold exponentially long, proving the third step.

We now provide the detailed analysis of the process (S_k). A useful tool, which we will use repeatedly, is the following large deviation estimate, which can be directly derived from Chernoff's inequality.

LEMMA 2. *Let X be a binomial random variable with parameters n and q. Then*

$$\mathbb{P}(X < snq) \leq e^{-Dn} \quad \textit{where } D = sq\log(s) + (1-sq)\log\left(\frac{1-sq}{1-q}\right) \textit{ and } s < 1.$$

4.1. *Quick direct spreading.* Let us start with quick direct spreading, which is arguably the simplest mechanism, that makes no use of connectors and is only based on stars infecting directly other stars before recovery. This strategy can only succeed when the subgraph $\mathscr{S}$ is sufficiently connected. In this case, for $x, y \in \mathscr{S}$ and small $\lambda > 0$, our choice $\eta \geq 0$ implies that there is typically an updating event $\mathcal{U}^{x,y}$ between two infections in $\mathcal{I}_0^{x,y}$ so the times $\mathcal{I}^{x,y}$ when infections pass the edge $\{x, y\}$ can therefore be approximated by a Poisson point process with rate $\lambda p_{x,y}$. If λ is small and N large, these rates tend to zero, and hence, during a brief interval of time, the infection starting from a single vertex is unlikely to infect twice the same vertex, resulting in the infection spreading like a Galton–Watson process.

We use, specifically in this quick direct spreading subsection, the terminology

$$\mathscr{S}'_k := \{x \in \mathscr{S}_k : \mathcal{R}^x \cap [k, k+1] = \varnothing\},$$

and $S'_k = |\mathscr{S}'_k|$. Clearly, $\mathscr{S}'_k \subset \mathscr{S}_{k+1}$ and thus $S'_k \leq S_{k+1}$, as infected stars that do not recover on $[k, k+1]$ are still infected at time $k+1$. Further, we let

$$\begin{aligned}\mathscr{S}''_k := \{&x \in \mathscr{S} \backslash \mathscr{S}_k : \mathcal{U}^x \cap [k, k+1/2] \neq \varnothing, \mathcal{R}^x \cap [k+1/2, k+1] = \varnothing,\\ &\exists y \in \mathscr{S}'_k, \mathcal{I}^{x,y} \cap [k+1/2, k+1] \neq \varnothing\},\end{aligned}$$

and $S''_k = |\mathscr{S}''_k|$. Clearly, we also have $\mathscr{S}''_k \subset \mathscr{S}_{k+1}$, as the stars in $\mathscr{S}''_k$ have been infected on $[k+1/2, k+1]$ and did not recover on that time interval.

An advantageous property of $\mathscr{S}'_k$ and $\mathscr{S}''_k$, compared to $\mathscr{S}_{k+1}$, is that their conditional laws knowing $(\mathscr{G}_k^{(N)}, X_k)$, depend only on $\mathscr{S}_k$, and not on the network structure $\mathscr{G}_k^{(N)}$. So, the cardinality S'_k of $\mathscr{S}'_k$ is (conditionally) a binomial random variable with parameters S_k and e^{-1}.

Now, if x is in $\mathscr{S} \backslash \mathscr{S}_k$, then it satisfies $\mathcal{U}^x \cap [k, k+1/2] \neq \varnothing$ and $\mathcal{R}^x \cap [k+1/2, k+1] = \varnothing$ with probability $e^{-1/2}(1 - e^{-\kappa_x/2}) \geq e^{-1/2}(1 - e^{-\kappa_0/2})$. Conditioning on this event and on $\mathscr{S}'_k$, a similar argument to the one used in the proof of

Lemma 1 gives that x belongs to S_k'' with probability at least

$$\left[1-\left(1-\frac{\lambda p(a,a)}{4N}\right)^{S_k'}\right] \geq \left[1-\exp\left(-\frac{\lambda p(a,a)S_k'}{4N}\right)\right] \\ \geq \frac{\lambda p(a,a)S_k'}{8N} \wedge \frac{1}{2}.$$

As a consequence, S_k'' dominates a binomial random variable with parameters $S - S_k$ and

$$e^{-1/2}(1-e^{-\kappa_0/2})\left(\frac{\lambda p(a,a)S_k'}{8N} \wedge \frac{1}{2}\right).$$

Gathering these results with $S_{k+1} \geq S_k' + S_k''$, we obtain a stochastic lower bound for the conditional distribution of S_{k+1} given $\mathfrak{F}_k$, which we exploit in the following lemma.

LEMMA 3. *Suppose ρ, ρ' and c are three positive constants such that $\rho\rho' > 1$. Suppose $M_0, M_0', M_1, M_1', \dots$ is a process on $\{0, 1, \dots, n\}$, adapted to a filtration $(\mathfrak{F}_0, \mathfrak{F}_0', \mathfrak{F}_1, \mathfrak{F}_1', \dots)$, such that*

- *given $\mathfrak{F}_k$ the random variable M_k' is binomially distributed with parameters M_k and ρ;*
- *given $\mathfrak{F}_k'$ the random variable $M_{k+1} - M_k'$ dominates a binomially distributed random variable with parameters $n - M_k$ and $\rho' \frac{M_k'}{n+1} \wedge c$.*

Then there exist positive constants $r, l, \varepsilon > 0$ such that for large n:

(1) *For every initial condition $M_0 = m_0$, the probability that the process M_k goes above value rn is at least $1 - e^{-lm_0}$.*

(2) *For every initial condition $M_0 = m_0 \geq rn$, with probability at least $1 - e^{-\varepsilon n}$, the process (M_k) stays above value rn at all times $k \leq e^{\varepsilon n}$.*

Under the hypothesis $\lambda ap(a,a) > 8e/e^{-1/2}(1-e^{-\kappa_0/2})$, we can apply Lemma 3 with the choice $M_k = S_k$, $M_k' = S_k'$, $\mathfrak{F}_k' = \sigma(\mathfrak{F}_k, S_k')$, $n = \lfloor aN \rfloor$, $\rho = e^{-1}$,

$$\rho' = \frac{e^{-1/2}(1-e^{-\kappa_0/2})}{8}\lambda ap(a,a) \quad \text{and} \quad c = \frac{e^{-1/2}(1-e^{-\kappa_0/2})}{2}.$$

Item (2) then completes the proof of slow extinction, while items (1) and (2) complete the proof of metastability. Thus, using Lemma 3, we have proven the quick direct spreading part of Theorem 1, with $M_{(i)} = 8e/e^{-1/2}(1-e^{-\kappa_0/2})$.

PROOF OF LEMMA 3. We first prove the second item. Choose $r < c \wedge \frac{1}{\rho'}$, which, together with $\rho\rho' > 1$, implies $r < \rho$. If $M_k \geq r'n$ with $r' > r/\rho$, then Lemma 2 implies $M_{k+1} \geq M_k' \geq rn$ with probability at least $1 - e^{-cn}$, for some constant $c > 0$. If $rn \le M_k \le r'n$, choosing ι in the nonempty interval $(\frac{1}{\rho'}, \rho \wedge \frac{c}{r\rho'})$, a further application of Lemma 2 yields $M_k' \ge \iota M_k$, with error probability bounded by some e^{-cn} (possibly with a new value of $c > 0$).

Further, we can bound $\rho' \frac{M_k'}{n+1}$ from below by $\iota\rho' \frac{n}{n+1} r$, which is in (r, c) for large n. A last application of Lemma 2 yields $M_{k+1} - M_k' \ge r(n - M_k)$ and then $M_{k+1} \ge rn$, with error probability bounded by $e^{-\varepsilon n}$. This easily gives item (2) of the theorem (dividing ε by 2 if needed).

For the first item we let $K := \inf\{k \ge 0, M_k = 0 \text{ or } M_k \ge r'n\}$, where r' is a constant to be determined later. We possibly have $r' < r$, but (2) still holds if we replace[5] r by $r \wedge r'$. Under the hypothesis $r' < \eta/\rho'$ and $k < K$, we can bound below the law of M_{k+1} knowing M_k' by a binomial random variable with parameters $(1 - r')n$ and $\rho' M_k'/(n+1)$. We now prove that for some well-chosen $l > 0$, the process $(e^{-lM_{k\wedge K}})_{k\ge 0}$ is a positive supermartingale. Note that the result then follows from a standard stopping theorem.

It suffices to prove the inequality

$$\mathbb{E}[e^{-lM_1} | \mathfrak{F}_0] \le e^{-lM_0}$$

on the event $M_0 < r'n$. But on this event, the Laplace transform of Binomial random variables easily gives the following:

$$\mathbb{E}[e^{-lM_1} | \mathfrak{F}_0'] \le \left(1 - \frac{\rho' M_0'}{n+1}(1 - e^{-l})\right)^{(1-r')n} \le \exp\left(-\frac{(1-r')n}{n+1}\rho' M_0'(1 - e^{-l})\right).$$

Thus, a further Laplace transform gives

$$\begin{aligned}\mathbb{E}[e^{-lM_1} | \mathfrak{F}_0] &\le \left(1 - \rho\left(1 - e^{-\frac{(1-r')n}{n+1}\rho' M_0'(1-e^{-l})}\right)\right)^{M_0} \\ &\le \exp\left(-\rho M_0\left(1 - e^{-\frac{(1-r')n}{n+1}\rho'(1-e^{-l})}\right)\right).\end{aligned}$$

When l goes to 0, the last expression is $\exp(-l \frac{(1-r')n}{n+1} \rho\rho' M_0(1 + o(1)))$. Choosing $r' > 0$ small and n large so that $\frac{(1-r')n}{n+1}\rho\rho' > 1$, and then $l > 0$ small, we can guarantee $\mathbb{E}[e^{-lM_1} | \mathfrak{F}_0] \le e^{-lM_0}$, and this completes the proof. □

4.2. *Quick indirect spreading*. Quick indirect spreading is a mechanism similar to quick direct spreading in the sense that stars spread the infection before seeing a simple recovery event, but in this case the infection spreads to connectors which in turn infect stars again. Being a two stage mechanism, quick indirect spreading can be less efficient than its direct version as we see in the case of the factor kernel. However, as it relies on the connectedness of the whole network rather than the connectedness among stars it can be advantageous when the latter

[5] Also, the reader can check that once the process has gone above level $r'n$, it is actually likely to go above level rn, too.

is scarce. As with the direct version, the quick update of stars allows us to think of valid infections as Poisson point processes with parameter $\lambda p_{x,y}$ and then approximate the behaviour of X_t as a Galton–Watson process.

We keep from the quick direct spreading subsection the notation $S_k = |\mathscr{S}_k|$ as well as $S'_k = |\mathscr{S}'_k| = |\{x \in \mathscr{S}_k : \mathcal{R}^x \cap [k, k+1] = \varnothing\}|$. Again, conditionally on the network and infection evolution up to time k, the cardinality S'_k of $\mathscr{S}'_k$ is binomial with parameters S_k and e^{-1}. In order to consider indirect spreading, we now introduce the following terminology, specific to this subsection:

$$\begin{aligned}\mathscr{C}_k &:= \{y \in \mathscr{C} : \mathcal{U}^y \cap [k, k+1/3] \neq \varnothing, \mathcal{R}^y \cap [k+1/3, k+1] = \varnothing,\\ &\qquad \exists x \in \mathscr{S}'_k, \mathcal{I}^{x,y} \cap [k+1/3, k+2/3] \neq \varnothing\},\\ \mathscr{S}''_k &:= \{x \in \mathscr{S} \backslash \mathscr{S}_k : \mathcal{R}^x \cap [k+2/3, k+1] = \varnothing,\\ &\qquad \exists y \in \mathscr{C}_k, \mathcal{I}^{x,y} \cap [k+2/3, k+1] \neq \varnothing\}.\end{aligned}$$

We also denote the cardinality of these sets by $C_k = |\mathscr{C}_k|$, and $S''_k = |\mathscr{S}''_k|$, respectively. It should be clear that $S_{k+1} \geq S'_k + S''_k$. Indeed, for each star $x \in \mathscr{S}''_k$, we can find a star $z \in \mathscr{S}'_k$ that infects a connector $y \in \mathscr{C}_k$ on time interval $[k+1/3, k+2/3]$, which stays infected until it infects x on time interval $[k+2/3, k+1]$. Note that the condition that connectors in $\mathscr{C}_k$ should update on $[k, k+1/3]$, is useful for the law of C_k, conditionally on $\mathscr{S}'_k$ and on the network and infection evolution up to time k, to actually only depend on $\mathscr{S}'_k$. More precisely, each $y \in \mathscr{C}$ belongs to $\mathscr{C}_k$ with probability at least

$$\begin{aligned}&e^{-2/3}(1-e^{-\kappa_0/3})\left(1-(1-e^{-\lambda/3})\frac{p(a,1)}{N}\right)^{S'_k}\\ &\quad \geq e^{-2/3}(1-e^{-\kappa_0/3})\left(\frac{\lambda p(a,1)S'_k}{6N} \wedge \frac{1}{2}\right).\end{aligned}$$

Hence C_k dominates a binomial random variable with parameters $(1-a)N$ and $e^{-2/3}(1-e^{-\kappa_0/3})(\lambda p(a,1)S'_k/(6N) \wedge \frac{1}{2})$. Similarly, conditionally on S_k, S'_k and C_k, we can bound S''_k from below by a binomial random variable with parameters $aN - S_k$ and $e^{-1/3}(\lambda p(a,1)C_k/(6N) \wedge \frac{1}{2})$. Lemma 3 has to be replaced by the following lemma.

LEMMA 4. *Suppose ρ, ρ', ρ'', ρ''' and c, c' are positive constants such that $\rho\rho'\rho''\rho''' > 1$, and $M_0, M'_0, M''_0, M_1, M'_1, M''_1, \ldots$ is a process on $\{0, 1, \ldots, n\}$ adapted to the filtration $(\mathfrak{F}_0, \mathfrak{F}'_0, \mathfrak{F}''_0, \mathfrak{F}_1, \mathfrak{F}'_1, \mathfrak{F}''_1, \ldots)$ such that*:

- *given $\mathfrak{F}_k$ the random variable M'_k is binomially distributed with parameters M_k and ρ;*
- *given $\mathfrak{F}'_k$ the random variable M''_k dominates a binomially distributed random variable with parameters $\lceil \rho''' n \rceil$ and $\frac{\rho'}{n+1} M'_k \wedge c$;*

- *given $\mathfrak{F}''_k$ the random variable $M_{k+1} - M'_k$ dominates a binomially distributed random variable with parameters $n - M_k$, and $\frac{\rho''}{n+1} M''_k \wedge c'$.*

Then there exist positive constants $r, l, \varepsilon > 0$ such that for large n:

(1) *For every initial condition $M_0 = m_0$, the probability that the process M_k goes above value rn is at least $1 - e^{-lm_0}$.*

(2) *For every initial condition $M_0 = m_0 \geq rn$, with probability at least $1 - e^{-\varepsilon n}$, the process (M_k) stays above value rn at all times $k \leq e^{\varepsilon n}$.*

The proof is similar to that of Lemma 3. It just involves more calculation, which is not so informative, so we omit it. We can now apply this lemma with the parameters

$$\begin{aligned} &M_k = S_k, \qquad M'_k = S'_k, \qquad M''_k = C_k, \\ &\mathfrak{F}'_k = \sigma(\mathfrak{F}_k, S'_k), \qquad \mathfrak{F}''_k = \sigma(\mathfrak{F}'_k, C_k), \qquad n = \lfloor aN \rfloor, \\ &\rho = e^{-1}, \qquad \rho' = e^{-2/3}(1 - e^{-\kappa_0/3})\lambda a p(a,1)/12, \\ &\rho'' = e^{-1/3}\lambda a p(a,1)/12, \end{aligned}$$

and $\rho''' = (1-a)/a$, under the condition $\rho\rho'\rho''\rho''' > 1$, which is satisfied if $a < 1/2$ and $\lambda^2 a p(a,1)^2 > 288e^2/(1 - e^{-\kappa_0/3})$. We now conclude the quick indirect spreading part of Theorem 1 just like the quick direct spreading part, with $M_{(ii)} = 288e^2/(1 - e^{-\kappa_0/3})$.

4.3. *Delayed direct spreading.* Delayed direct spreading is a mechanism similar to quick direct spreading in the sense that the infection spreads directly from star to star. The main difference is that the infection is kept alive at a star on a longer time scale with the aid of connectors. A single vertex, if powerful enough, can survive a recovery event by infecting a connector which in turn infects it back before an updating event (where the connection is lost with a high probability) thus prolonging the recovery cycle of stars. In contrast with the stars studied at [2] which survive for an amount of time exponential in their degree, the survival time here is roughly linear in this parameter, which is explained by the cost of maintaining the right conditions on the network for this effect to take place.

To begin our proof, for each $k \in \mathbb{N}$, $k \geq 1$ define $\bar{\mathscr{C}_k}$ as

$$\bar{\mathscr{C}_k} := \{y \in \mathscr{C} : [\mathcal{U}^y \cup \mathcal{R}^y] \cap [k, k+2] = \varnothing\}$$

that is, $\bar{\mathscr{C}_k}$ is the set of all **stable connectors** in the interval $[k, k+2]$. As each $y \in \mathscr{C}$, $y > N/2$ belongs to $\bar{\mathscr{C}_k}$ independently with probability at least $\theta = \exp(-2(\kappa_0 2^{\gamma\eta} + 1))$, Lemma 2 shows that $\mathbb{P}(|\bar{\mathscr{C}_k}| > \theta N/4) > 1 - e^{-cN}$ for some fixed $c > 0$, and hence these events hold exponentially long. *For the entire remainder of this section, we therefore fix a realization of $\mathcal{U}^y, \mathcal{R}^y$, $y \in \mathscr{C}$ such that*

$\{|\bar{\mathscr{C}}_k| > \theta N/4\}$ *holds exponentially long and all probabilities will be taken to be conditional on such a realization.*

Next, we denote by $\mathscr{S}_k$ the set of infected stars at time kT, and $S_k = |\mathscr{S}_k|$. Note that we use the same notation as before, though the length of a recovery cycle has been modified, from one to $T = T(a, \lambda)$ as defined in Theorem 1. Our hope is that this will not confuse the reader, but rather stress the unity of the approach.

4.3.1. *Properties of stars.* As the probability $p_{x,y}$ of having a connection between a star x and a connector y is bounded from below by $\frac{1}{N}p(a, 1) \geq \frac{1}{N}c_1 a^{-\gamma}$, we deduce that for given t, the number of connectors $y \in \bar{\mathscr{C}}_{\lfloor t \rfloor}$ connected to x dominates a binomial random variable with parameters $\lceil \theta N/4 \rceil$ and $c_1 a^{-\gamma}/N$. By Lemma 2, one can deduce that

$$\mathbb{P}\big(\big|\{y \in \bar{\mathscr{C}}_{\lfloor t \rfloor} : \{x, y\} \in \mathscr{G}_t^{(N)}\}\big| > c_1 \theta a^{-\gamma}/5\big) > 1 - e^{-ca^{-\gamma}}, \tag{14}$$

for some $c > 0$, uniformly in $a \leq 1/2$ and $N \geq c_1 a^{-\gamma}$.

DEFINITION 1. For any $T > 0$, let

$$\bar{\mathcal{U}}^x := \{0\} \cup \bigcup_{U_m^x \in \mathcal{U}^x} \{U_m^x + n : n \in \mathbb{N}_0 \cap [0, U_{m+1}^x - U_m^x]\}.$$

A star $x \in \mathscr{S}$ is T-**stable** if:

(i) $|\mathcal{U}^x \cap [0, T]| < 3\kappa_x T$;
(ii) at times $t \in \bar{\mathcal{U}}^x \cap [0, T]$, the vertex x has at least $\frac{c_1\theta}{5} a^{-\gamma}$ neighbours in $\bar{\mathscr{C}}_{\lfloor t \rfloor}$.

The set $\bar{\mathcal{U}}^x$ arises by adding points to $\mathcal{U}^x$ between consecutive updating events when these are further than one unit of time apart. Loosely speaking, T-stability means that x does not update too much and that at every time in this enlarged updating set it has sufficiently many neighbouring stable connectors. The next result follows from (14) and the fact that $T(a, \lambda) \leq C\lambda^2 a^{-\gamma}$. We omit its easy proof.

LEMMA 5. $\lim_{T\to\infty} \liminf_{N\to\infty} \inf_{x\in\mathscr{S}} \mathbb{P}(x \textit{ is } T\textit{-stable}) = 1$.

Since the events $\{x \text{ is } T\text{-stable}\}_{x\in\mathscr{S}}$ are independent, we deduce that for large T and N, most stars will exhibit this property. We define next the concept of $[L, T]$-**susceptibility** of x, depending only on $\mathcal{U}^x$, $\mathcal{R}^x$, which loosely speaking means that recoveries of x are not too frequent, and not too close to its updating events.

DEFINITION 2. For any $L \in [0, T)$, we say that a star $x \in \mathscr{S}$ is $[L, T]$-**susceptible** if:

(i) there are no recovery events in $[L, L + 1]$ or $[T - 1, T]$,
(ii) $|\mathcal{R}^x \cap [L, T]| < 2T$,

(iii) for every pair of consecutive times $t_1, t_2 \in \bar{\mathcal{U}}^x \cap [L, T]$, such that $\mathcal{R}^x \cap [t_1, t_2] \neq \varnothing$ we have

$$(r_1 - t_1)(t_2 - r_2) > \kappa_0 \big[20(3\kappa_0 + 1)\kappa_x^2 T \log(T)\big]^{-1}, \tag{15}$$

where r_1 and r_2 are the first and last recoveries in $[t_1, t_2]$, respectively.

There is no reason most stars should be $[L, T]$-susceptible, however the probability of having this property is bounded from zero uniformly for all stars, if T and N are large.

LEMMA 6. *There exists $q_1 > 0$ such that for all T large and $L \in [0, T)$,*

$$\liminf_{N\to\infty} \inf_{x\in\mathscr{S}} \mathbb{P}\big(x \textit{ is } [L, T]\textit{-susceptible and } T\textit{-stable}\big) > q_1.$$

PROOF. Fix a realization of $\mathcal{U}^x$, $\mathcal{C}^x$ making x T-stable and notice that in this case $[L, T]$-susceptibility depends on the process $\mathcal{R}^x$ alone. In fact, all three conditions in Definition 2 are decreasing with $\mathcal{R}^x$. By Harris' inequality, the three conditions are therefore positively correlated, and hence $\mathbb{P}(x \text{ is } [L, T]\text{-susceptible}|\mathcal{U}^x, \mathcal{C}^x)$ is larger than the product of the individual conditional probabilities, which we now calculate. It is easy to see that the probability of conditions (i) and (ii) are independent of $\mathcal{U}^x$, $\mathcal{C}^x$ and their probability is bounded from below by e^{-2} and $1/2$, respectively (the latter value is obtained from a rough Markov inequality). For the bound on (iii) call $t_1, t_2, \dots$ the elements of $\bar{\mathcal{U}}^x \cap [L, \infty)$ in increasing order, and observe that for (iii) to fail, it must do so in some interval $[t_i, t_{i+1}]$. We take two consecutive t_i, t_{i+1} with $t_i \le T$ and find an upper bound for the probability P_i of the event where (15) fails at $[t_i, t_{i+1}]$. To do so, call $l_i = t_{i+1} - t_i$ and $D = \kappa_0[20(3\kappa_0 + 1)\kappa_x^2 T \log(T)]^{-1}$, and split this event in two scenarios:

First suppose $l_i < 2\sqrt{D}$ and observe that in this case for any point $r \in [t_i, t_{i+1}]$ we have $(r - t_i)(t_{i+1} - r) < D$, so (15) fails if and only if $\mathcal{R}^x \cap [t_i, t_{i+1}] \neq \varnothing$. We bound this probability by $1 - e^{-l_i} \le l_i$.

Second suppose $2\sqrt{D} \le l_i \le 1$ and observe that we still need $\mathcal{R}^x \cap [t_i, t_{i+1}] \neq \varnothing$, which we divide into the events Υ_{10}, Υ_{01} and Υ_{11} where there is a recovery event only at the first half, the second half or both halves of the interval, respectively. In Υ_{10}, we have $(r_1 - t_i)(t_{i+1} - r_2) \ge (r_1 - t_i)\frac{l_i}{2}$ so we necessarily have $r_1 - t_i \le \frac{2D}{l_i}$, and hence the probability of failure is bounded in this scenario by $e^{-l_i/2}(1 - e^{-2D/l_i}) \le \frac{2D}{l_i}$. By symmetry, the same bound holds in the event Υ_{01}. In the event Υ_{11}, the random variables $r_1 - t_i$ and $t_{i+1} - r_2$ are dominated by independent exponential random variables. We deduce that the probability of failure in this scenario is equal to

$$\int_0^{\frac{l_i}{2}} \int_0^{\frac{l_i}{2}} e^{-y-z} \mathbf{1}_{\{z<\min\{D/y, l_i/2\}\}}\, dz\, dy.$$

Dividing the integral as to which expression is smaller in the indicator function yields

$$\int_0^{\frac{2D}{l_i}} \int_0^{\frac{l_i}{2}} e^{-y-z}\,dz\,dy + \int_{\frac{2D}{l_i}}^{\frac{l_i}{2}} \int_0^{\frac{D}{y}} e^{-y-z}\,dz\,dy,$$

and bounding all the exponentials by 1, the expression above is less than $D(1+\log(\frac{l_i^2}{4D}))$. Putting together the bounds for (i), (ii) and (iii), we finally obtain

$$\begin{aligned}&\mathbb{P}\big(x \text{ is } [L,T]\text{-susceptible and } T\text{-stable}|\mathcal{U}^x,\mathcal{C}^x\big)\\&\quad\geq \frac{1}{2}e^{-2}\Bigg(1-\sum_{\substack{i\in\mathbb{N}\\ t_i\in\bar{\mathcal{U}}^x\cap[0,T]}} P_i\Bigg)\mathbf{1}_{\{x \text{ is } T\text{-stable}\}},\end{aligned}$$

and hence, to find the lower bound for the probability, we take expectation with respect to the realization of $\mathcal{U}^x, \mathcal{C}^x$ in this expression. From our previous argument, we have

$$P_i \leq l_i \mathbf{1}_{\{l_i<2\sqrt{D}\}} + D\bigg(\frac{4}{l_i}+1+\log\bigg(\frac{l_i^2}{4D}\bigg)\bigg)\mathbf{1}_{\{2\sqrt{D}\leq l_i\leq 1\}}$$

for each P_i, which depends on l_i alone. Notice that from the definition of $\bar{\mathcal{U}}^x$, each l_i is exponentially distributed with rate κ_k in [0, 1] and has mass $e^{-\kappa_x}$ at 1, so that

$$\begin{aligned}\mathbb{E}P_i \leq{}& \int_0^{2\sqrt{D}} l_i\kappa_x e^{-\kappa_x l_i}\,dl_i + \int_{2\sqrt{D}}^1 D\bigg(\frac{4}{l_i}+1+\log\bigg(\frac{l_i^2}{4D}\bigg)\bigg)\kappa_x e^{-\kappa_x l_i}\,dl_i\\&+e^{-\kappa_x}D\big(5-\log(4D)\big).\end{aligned}$$

For the first integral, we bound the exponential term by 1 to deduce that it is less than $2D\kappa_x$, while for the remaining expression observe that it can be written as

$$\int_{2\sqrt{D}}^{\infty} D\bigg(\frac{4}{l_i\wedge 1}+1+\log\bigg(\frac{(l_i\wedge 1)^2}{4D}\bigg)\bigg)\kappa_x e^{-\kappa_x l_i}\,dl_i \leq \int_{2\sqrt{D}}^{\infty} D\bigg(\frac{4}{l_i}+5\bigg)\kappa_x e^{-\kappa_x l_i}\,dl_i,$$

since the logarithmic term is at most zero. Changing variables we obtain that this expression is equal to

$$5De^{-2\kappa_x\sqrt{D}} + 4D\kappa_x \int_{2\kappa_x\sqrt{D}}^{\infty} \frac{e^{-y}}{y}\,dy \leq 5D + 4D\kappa_x \log\bigg(1+\frac{1}{2\kappa_x\sqrt{D}}\bigg),$$

where the inequality follows from an elementary bound on the exponential integral $\mathrm{Ei}(x)$. We conclude that, since T is large,

$$\begin{aligned}\mathbb{E}P_i &\leq 11D\kappa_x\log\bigg(1+\frac{1}{2\kappa_x\sqrt{D}}\bigg) = \frac{11\kappa_0}{20(3\kappa_0+1)\kappa_x T\log(T)}\log\big(1+\sqrt{T\log(T)}\big)\\&\leq \frac{11\kappa_0}{20(3\kappa_0+1)\kappa_x T}.\end{aligned}$$

Noticing that each l_i is independent of t_i, we finally deduce

$$\mathbb{E}\Bigl(\sum_{\substack{i\in\mathbb{N}\\ t_i\in\bar{\mathcal{U}}^x\cap[0,T]}} P_i\mathbf{1}_{\{x\text{ is }T\text{-stable}\}}\Bigr)\le\frac{11\kappa_0}{20(3\kappa_0+1)\kappa_x T}\bigl(3\kappa_x T+\lceil T\rceil\bigr)\le\frac{3}{5},$$

where the first inequality follows from T-stability and from observing that to build $\bar{\mathcal{U}}^x$ we add at most $\lceil T\rceil$ additional points to $\mathcal{U}^x$ in $[0,T]$. It follows that

$$\mathbb{P}\bigl(x\text{ is }[L,T]\text{-susceptible and }T\text{-stable}\bigr)\ge\frac{e^{-2}}{2}\Bigl(\mathbb{P}(x\text{ is }T\text{-stable})-\frac{3}{5}\Bigr),$$

and we conclude the result by taking T large such that $\mathbb{P}(x\text{ is }T\text{-stable})>\frac{4}{5}$. □

As it will be seen later, $[L,T]$-susceptibility gives stars enough time after each update to infect stable connectors that, in turn, have enough time to reinfect them back. This amount of time, however, is not enough to infect other stars. For this purpose, we introduce the concept of **T-infectiousness** as follows.

DEFINITION 3. A star $x\in\mathscr{S}$ is **T-infectious** if

$$\bigl|\bigl\{t\in\mathcal{U}^x\cap[0,T]:\ (\mathcal{U}^x\cup\mathcal{R}^x)\cap[t,t+\kappa_x^{-1})=\{t\}\bigr\}\bigr|>\frac{1}{2}e^{-1-\kappa_0^{-1}}\kappa_x T.$$

A large deviation argument yields $\lim_{T\to\infty}\liminf_{N\to\infty}\inf_{x\in\mathscr{S}}\mathbb{P}(x\text{ is }T\text{-infectious})=1$, so most of the stars will have this property if T is sufficiently large. Gathering all the results obtained here, we obtain the following lemma.

LEMMA 7. *There exists $q_1>0$ as in Lemma 6 such that, for all large T and $L\in[0,T)$,*

$$\liminf_{N\to\infty}\inf_{x\in\mathscr{S}}\mathbb{P}\bigl(x\textit{ is }T\textit{-infectious},T\textit{-stable and }[L,T]\textit{-susceptible}\bigr)>q_1.$$

4.3.2. *Survival and spreading*. The star properties mentioned in the last lemma are useful to bound from below the probability that a star maintains the infection on $[0,T]$ and infects another star.

DEFINITION 4. A star $x\in\mathscr{S}$ is **$[L,T]$-infected** if $X_L(x)=X_T(x)=1$ and, for all $t\in\bar{\mathcal{U}}^x\cap[L,T]$, we have $X_t(x)=1$.

If x is $[0,T]$-infected, we say the infection is maintained at x on $[0,T]$ (although the star may of course have recovered several times on this time interval).

LEMMA 8. *There exists $q_2>0$ independent of N, such that for all λ, a such that $T=T(a,\lambda)$ is sufficiently large, and for all $x\in\mathscr{S}$ and $L\in[0,T)$, we have*

$$\begin{aligned}&\mathbb{P}\bigl(x\textit{ is }[L,T]\textit{-infected}\,|\,X_L(x)=1,\\ &\qquad x\textit{ is }T\textit{-stable and }[L,T]\textit{-susceptible}\bigr)>q_2.\end{aligned}\tag{16}$$

PROOF. Fix a realization of $\mathcal{U}^x$, $\mathcal{R}^x$ and $(C_n^{x,y})_{y\in\mathscr{C}}$ making x a T-stable and $[L,T]$-susceptible star, and assume that $X_L(x)=1$. Call $t_0=L$ and $\{t_1,\dots,t_n\}$ the elements of $\bar{\mathcal{U}}^x\cap[L,T]$ in increasing order. By definition of $[L,T]$-susceptibility, there are no recovery events in $[L,L+1]$ which gives $X_{t_1}=1$. From the same argument, we have $X_T=1$ as soon as $X_{t_n}=1$, so along with conditional probabilities and the Markov property, this allows us to bound from below the probability in the statement by

$$\prod_{i=1}^{n-1}\mathbb{P}\big(X_{t_{i+1}}(x)=1|X_{t_i}(x)=1\big). \tag{17}$$

To control this product observe first that for all terms with $[t_i,t_{i+1}]\cap\mathcal{R}^x=\varnothing$, we trivially have $\mathbb{P}(X_{t_{i+1}}(x)=1|X_{t_i}(x)=1)=1$. Fix now some t_i such that $\mathcal{R}^x\cap[t_i,t_{i+1}]\neq\varnothing$ and define r_1 and r_2 as the first and last element in that intersection, respectively. A sufficient scenario for $X_{t_{i+1}}(x)=1$ is that:

- x infects some neighbour $y\in\bar{\mathscr{C}}_{\lfloor t_i\rfloor}$ during $[t_i,r_1]$,
- since $y\in\bar{\mathscr{C}}_{\lfloor t_i\rfloor}$ and $t_{i+1}-t_1\leq 1$, it remains infected (and also a neighbour of x) up until time t_{i+1},
- y infects x back during $[r_2,t_{i+1}]$.

The scenario above follows from the event

$$\bigcup_{\substack{x\sim y\text{ at time }t_i\\ y\in\bar{\mathscr{C}}_{\lfloor t_i\rfloor}}}\big\{\mathcal{I}_0^{x,y}\cap[t_i,r_1]\neq\varnothing\text{ and }\mathcal{I}_0^{x,y}\cap[r_2,t_{i+1}]\neq\varnothing\big\}, \tag{18}$$

and using independence we can calculate its probability as

$$1-\prod_{\substack{x\sim y\text{ at time }t_i\\ y\in\bar{\mathscr{C}}_{\lfloor t_i\rfloor}}}\big[1-\big(1-e^{-\lambda(r_1-t_i)}\big)\big(1-e^{-\lambda(t_{i+1}-r_2)}\big)\big].$$

Since $t_{i+1}-t_i\leq 1$ and $\lambda>0$ small, we can use the bound

$$\begin{aligned}\big(1-e^{-\lambda(r_1-t_i)}\big)\big(1-e^{-\lambda(t_{i+1}-r_2)}\big)&\geq\frac{\lambda^2(r_1-t_i)(t_{i+1}-r_2)}{4}\\&\geq\frac{\kappa_0\lambda^2}{80(3\kappa_0+1)\kappa_x^2T\log T},\end{aligned}$$

which follows from the third condition of $[L,T]$-susceptibility, to obtain that (18) has probability at least

$$1-\prod_{\substack{x\sim y\text{ at time }t_i\\ y\in\bar{\mathscr{C}}_{\lfloor t_i\rfloor}}}\bigg[1-\frac{\kappa_0\lambda^2}{20(3\kappa_0+1)\kappa_x^2T\log(T)}\bigg]$$

$$\begin{aligned}&\geq 1-\exp\left(-\frac{\lambda^2 c_1\theta a^{-\gamma}}{400a^{-2\gamma\eta}\kappa_0(3\kappa_0+1)T\log(T)}\right)\\&=1-\frac{1}{T},\end{aligned}$$

where the inequality follows from the T-stability of x and the monotonicity of κ and the equality from the definition of T. This allows us to bound the argument in (17) by $(1-1/T)$, and from $[L,T]$-stability there are at most $2T$ recoveries, giving

$$\left|\left\{t_i\in\mathcal{U}^x\cap[L,T]\colon \mathcal{R}^x\cap[t_i,t_{i+1}]\neq\varnothing\right\}\right|\leq 2T,$$

so we finally obtain

$$\prod_{i=0}^{n-1}\mathbb{P}\big(X_{t_{i+1}}(x)=1|X_{t_i}(x)=1\big)\geq\left(1-\frac{1}{T}\right)^{2T}\geq e^{-4},$$

which holds if T is large, giving the result. $\square$

Denote by $\mathscr{S}_0'\subset\mathscr{S}_0$ the set of initially infected stars, that are also T-infectious, T-stable, $[0,T]$-susceptible, and $[0,T]$-infected. From Lemmas 7 and 8, when T and N are large, conditionally on S_0, the random variable $S_0'=|\mathscr{S}_0'|$ dominates a binomial random variable with parameters S_0 and q_1q_2. Further, we bound from below the number of initially uninfected stars, that get infected by some star in $\mathscr{S}_0'$, and are still infected at time T.

LEMMA 9. *We have, uniformly in* $\lambda<1$ *and* $a<1/2$,

$$\begin{aligned}&\liminf_{N\to\infty}\inf_{x\in\mathscr{S}\setminus\mathscr{S}_0}\mathbb{P}\big(\exists y\in\mathscr{S}_0',t\in\mathcal{I}^{x,y}\cap[0,T]\colon X_t(y)=1|\mathscr{S}_0'\big)\\&\quad>\frac{\lambda Tp(a,a)S_0'}{8e^{1+1/\kappa_0}N}\wedge\frac{1}{2}.\end{aligned}\tag{19}$$

PROOF. Fix $y\in\mathscr{S}_0'$ and consider $t\in\mathcal{U}^y\cap[0,T]$ such that $[t,t+\kappa_x^{-1}]\cap(\mathcal{R}^y\cup\mathcal{U}^y)=\varnothing$, which exists since y is T-infectious. As $y\in\mathscr{S}_0'$, it satisfies the condition in Lemma 8, and hence it is infected throughout the interval $[t,t+\kappa_x^{-1}]$, which gives a small time interval for y to infect x. To find a lower bound for the event $\{\mathcal{I}^{x,y}\cap[t,t+\kappa_x^{-1}]\neq\varnothing\}$, it is enough that $\mathcal{I}_0^{x,y}\cap[t,t+\kappa_x^{-1}]\neq\varnothing$ and that at the first infection event the edge $\{x,y\}$ belongs to the graph, but this happens with probability

$$p_{x,y}\big[1-e^{-\lambda\kappa_x^{-1}}\big]\geq 1-e^{-\frac{\lambda p_{x,y}}{2\kappa_x}}.$$

Using independence, we deduce

$$\mathbb{P}\big(\exists t \in \mathcal{I}^{x,y} \cap [0,T]\colon X_t(y)=1\big) \geq 1 - \prod_{\substack{t \in \mathcal{U}^y \cap [0,T] \\ [t, t+\kappa_x^{-1}] \cap \mathcal{R}^y = \varnothing}} \exp\left(-\frac{\lambda p_{x,y}}{2\kappa_x}\right)$$
$$\geq 1 - \exp\left(-\frac{\lambda p_{x,y} T}{4e^{1+1/\kappa_0}}\right),$$

where the last inequality is due to the fact that $y \in \mathscr{S}_0'$, and hence it is T-infectious. Finally, to deduce (19), we use independence one last time to deduce

$$\mathbb{P}\big(\exists y \in \mathscr{S}_0', t \in \mathcal{I}^{x,y} \cap [0,T], X_t(y)=1\big)$$
$$\geq 1 - \exp\left(-\frac{\lambda T}{4e^{1+1/\kappa_0}} \sum_{y \in \mathscr{S}'_0} p_{x,y}\right)$$
$$\geq 1 - \exp\left(-\frac{\lambda T p(a,a) S_0'}{4e^{1+1/\kappa_0} N}\right) > \frac{\lambda T p(a,a) S_0'}{8e^{1+1/\kappa_0} N} \wedge \frac{1}{2}. \qquad \square$$

Using Lemma 9, we get that, when T and N are large, each $x \in \mathscr{S} \backslash \mathscr{S}_0$ has probability at least $\frac{\lambda T p(a,a) S_0'}{8e^{1+1/\kappa_0} N} \wedge \frac{1}{2}$ to receive an infection in $[0,T]$. Calling $T_x \in [0,T]$ the first time when this occurs, we can use Lemma 6 to deduce that with probability at least q_1 the star x is T-stable and $[T_x, T]$-susceptible. Now, being infected at time T_x, Lemma 8 gives that with probability at least q_2, the star x will be infected at time T. Since all these events are independent for different values of x, we deduce that $S_1 - S_0'$ dominates a binomial random variable with parameters $\lfloor aN \rfloor - S_0$ and

$$q_1 q_2 \left(\frac{\lambda T p(a,a) S_0'}{8e^{1+1/\kappa_0} N} \wedge \frac{1}{2}\right).$$

Finally, the same reasoning applies for the whole process S_k, which can now be studied similarly as for quick direct spreading. More precisely, if T is large and under the hypothesis

$$\frac{q_1^2 q_2^2 \lambda T a p(a,a)}{8e^{1+1/\kappa_0}} > 1,$$

we can apply Lemma 3 with $n = \lfloor aN \rfloor$, $\rho = q_1 q_2$, $\rho' = \frac{q_1 q_2 \lambda T a p(a,a)}{8e^{1+1/\kappa_0}}$. In that case, slow extinction and metastability follow just as before. To actually deduce the lower bound for the lower metastable density, inequality (4), we need not only that the events $S_k \geq raN$ hold exponentially long for some $r > 0$ (as we get from Lemma 3), but that the events $|\{x \in \mathscr{S} : X_k(x) = 1\}| \geq r'aN$ with $k \in \mathbb{N}$, hold exponentially long, for some $r' > 0$. However, it is clear from our proofs that we do get this, for some $r' < r$. This altogether proves the case of delayed direct spreading of Theorem 1.

4.4. *Delayed indirect spreading*. In the delayed indirect spreading strategy stars survive during long periods of time, and the spreading between stars does not occur directly but using connectors as intermediaries. The proof in this case takes most of the work already done for the other mechanisms, with the sole exception being that infections on connectors have a very limited lifespan, which forces us to be a little more careful.

In particular, we introduce a more restrictive notion of stability for connectors. For every $k \geq 1$, we introduce

$$\bar{\bar{\mathscr{C}}}_k := \{y \in \bar{\mathscr{C}}_k : \mathcal{U}^y \cap [k-1, k] \neq \varnothing\},$$

and we use these connectors in the spreading mechanism. Similarly, as for stable connectors, the events $\{|\bar{\bar{\mathscr{C}}}_k| > \theta' N/4\}$ hold exponentially long, where $\theta' = \theta(1 - e^{-\kappa_0})$. We now *work conditionally on a realization of* $\mathcal{U}^y, \mathcal{R}^y$, $y \in \mathscr{C}$ *such that both* $\{|\bar{\mathscr{C}}_k| > \theta N/4\}$ *and* $\{|\bar{\bar{\mathscr{C}}}_k| > \theta' N/4\}$ *hold exponentially long*. Recall the concepts of T-stability and $[L, T]$-susceptibility as given in Definitions 1 and 2. We replace the concept of T-infectiousness used for spreading among stars, by the following definition.

DEFINITION 5. We say that $x \in \mathscr{S}$ is T–$\mathscr{C}$-infectious if

$$\big|\{k \in \mathbb{N} \cap [0, T-1] : \mathcal{R}^x \cap [k, k+1] = \varnothing\}\big| > \frac{T}{3}.$$

Using Lemma 2, we know that each $x \in \mathscr{S}$ is T–$\mathscr{C}$-infectious with probability $1 - e^{-cT}$ for some $c > 0$, and hence $\liminf_{T\to\infty} \liminf_{N\to\infty} \inf_{x\in\mathscr{S}} \mathbb{P}(x$ is T–$\mathscr{C}$-infectious$) = 1$, which is why we can obtain the following result, in analogy to Lemma 7.

LEMMA 10. *There exists $q_1 > 0$ as in Lemma 6 such that for all large T and $L \in [0, T)$,*

$$\liminf_{N\to\infty} \inf_{x\in\mathscr{S}} \mathbb{P}\big(x \textit{ is } T\text{–}\mathscr{C}\textit{-infectious}, T\textit{-stable and } [L, T]\textit{-susceptible}\big) > q_1,$$

and these events are independent for different $x \in \mathscr{S}$.

We set $\mathscr{S}_0' \subset \mathscr{S}_0$ to be the subset of the initially infected stars that are also T-stable, T–$\mathscr{C}$-infectious, $[0, T]$-susceptible and $[0, T]$-infected. If a is small and N large, its cardinality S_0' dominates a binomial random variable with parameters S_0 and $q_1 q_2$. Moreover, for $x \in \mathscr{S}_0$, the event $x \in \mathscr{S}_0'$ is increasing in the processes $\mathcal{I}^{x,y}$, $y \neq x$, and thus by Harris' inequality, it is positively correlated with every event which is increasing in the processes $\mathcal{I}^{x,y}$, $y \neq x$. Further, let

$$K := \{k \in \mathbb{N} \cap [0, T] : \big|\{x \in \mathscr{S}_0', \mathcal{R}^x \cap [k, k+1] = \varnothing\}\big| \geq S_0'/6\}.$$

We necessarily have

$$\frac{S_0'T}{3} \le \sum_{k\in\mathbb{N}\cap[0,T]} \sum_{x\in\mathscr{S}_0'} 1_{\{\mathcal{R}^x\cap[k,k+1]=\varnothing\}} \le |K|S_0' + \big(T-|K|\big)\frac{S_0'}{6},$$

where the left inequality follows from the definition of T–$\mathscr{C}$-infectiousness, and the right inequality from the definition of K. It follows $|K| \ge T/5$. We use this set to search for times in which stars can infect sufficiently many stable connectors. More precisely, we let

$$\mathscr{P}_0 := \big\{(k,y), k\in K, y\in \overline{\overline{\mathscr{C}}}_k, \exists x\in\mathscr{S}_0', \mathcal{I}^{x,y}\cap[k,k+1]\neq\varnothing\big\}.$$

Conditionally, on $k\in K$, $y\in\overline{\overline{\mathscr{C}}}_k$ and $x\in\mathscr{S}_0'$, we have $\mathcal{I}_0^{x,y}\cap[k,k+1]\neq\varnothing$ with probability at least $1-e^{-\lambda}$, thanks to the positive correlation with the event $x\in\mathscr{S}_0'$, and at time $t=\min(I_0^{x,y}\cap[k,k+1])$ we have $\{x,y\}\in\mathscr{G}_t^{(N)}$ with probability at least $p_{x,y}\ge p(a,1)/N$, thanks to the update of y on time interval $[k-1,k]$.

Proceeding as before, we obtain that for large T and large N and conditionally on S_0', on K and on $(C_k)_{k\in K}$, the cardinality P_0 of $\mathscr{P}_0$ dominates a binomial random variable with parameters

$$\left\lceil\frac{\theta' TN}{20}\right\rceil \quad\text{and}\quad \frac{\lambda p(a,1)}{24N}S_0'\wedge\frac{1}{2}.$$

Finally, we define $\mathscr{S}_0'' := \{x\in\mathscr{S}\backslash\mathscr{S}_0 \colon \exists(k,y)\in\mathscr{P}_0, \mathcal{I}^{x,y}\cap[k+1,k+2]\neq\varnothing, X_T(x)=1\}$ and $S_0''=|\mathscr{S}_0''|$, and observe that $S_1\ge S_0'+S_0''$. Conditionally on $\mathscr{S}_0$, $\mathscr{S}_0'$ and $\mathscr{P}_0$, we have, independently for each $(k,y)\in\mathscr{P}_0$,

$$\mathbb{P}\big(\mathcal{I}^{x,y}\cap[k+1,k+2]\neq\varnothing\big)\ge\frac{\lambda p(a,1)}{2N},$$

whence the probability that there exists $(k,y)\in\mathscr{P}_0$ such that $\mathcal{I}^{x,y}\cap[k+1,k+2]\neq\varnothing$ is at least $\frac{\lambda P_0 p(a,1)}{4N}\wedge\frac{1}{2}$. Now, the probability that x gets infected on $[0,T]$ is at least $\frac{\lambda P_0 p(a,1)}{4N}\wedge\frac{1}{2}$. Conditionally on this, writing T_x the first time when it gets infected, we have that with probability at least q_1q_2 (when T and N large), the star x is T-stable, $[T_x,T]$-susceptible and $[T_x,T]$-infected, whence $X_T(x)=1$. In other words, we can bound S_0'' from below by a binomial random variable with parameters

$$\lfloor aN\rfloor - S_0 \quad\text{and}\quad q_1q_2\bigg(\frac{\lambda p(a,1)}{4N}P_0\wedge\frac{1}{2}\bigg).$$

Gathering the results, when T is large we can use Lemma 4 with $n=\lfloor aN\rfloor$, $\rho=q_1q_2$, $\rho'=\frac{\lambda}{24}ap(a,1)$, $\rho''=\lambda q_1q_2ap(a,1)/4$, and $\rho'''=\theta'T/20a$, under the condition $\rho\rho'\rho''\rho'''>1$. This condition is satisfied if $\lambda^2T(a,\lambda)ap(a,1)^2$ is large enough, which concludes the proof of the delayed indirect spreading case of Theorem 1.

5. Fast extinction and upper bounds. To obtain upper bounds we need to show that no mechanism can outperform the ones examined in the lower bounds. This cannot be done explicitly, but requires a more abstract supermartingale argument, which we now introduce. We start by coupling our process to a simpler process which is a stochastic upper bound.

5.1. *A coupling*. We construct a coupling between the contact process on the dynamic network, described by the pair of processes $(X, \mathscr{G}^{(N)})$, and a process Y, which we call the "wait-and-see" process. The process $(Y_t : t > 0)$ takes values in $\{0, 1\}^N \times \{0, 1\}^{N \otimes N}$, where $N \otimes N$ is the set of potential edges, that is, unordered pairs of distinct vertices in $\{1, \dots, N\}$. We say a vertex x is infected at time t (for the wait-and-see process) if $Y_t(x) = 1$, and we say a potential edge $\{x, y\}$ is *revealed* at time t if $Y_t(x, y) = 1$. Informally, a potential edge is unrevealed at time t if we have no information about its presence in the dynamic network $\mathscr{G}_t^{(N)}$. The wait-and-see model evolves according to the following rules:

- Every infected vertex x recovers at rate 1.
- If x is infected then it infects every uninfected vertex y,
  - with rate λ if $\{x, y\}$ is revealed (i.e., if $Y_t(x, y) = 1$) and
  - with rate $\lambda p_{x,y}$ if it is unrevealed (i.e., if $Y_t(x, y) = 0$).

  In the latter case, when x infects y, the value of $Y_t(x, y)$ immediately turns to 1.
- If x, y are both infected and $\{x, y\}$ is unrevealed, it gets revealed at rate $\lambda p_{x,y}$.
- Finally, each vertex updates at rate κ_x. Updating of x means that all its adjacent potential edges turn to unrevealed.

LEMMA 11. *Fix deterministic initial conditions $X_0(v) \le Y_0(v)$ for all $v \in \{1, \dots, N\}$. There exists a coupling of*:

- *the dynamic random network $(\mathscr{G}_t^{(N)} : t \ge 0)$,*
- *the original infection process on this network $(X_t : t \ge 0)$, and*
- *the wait-and-see process $(Y_t : t \ge 0)$, started from vertices in Y_0 infected and all its edges unrevealed,*

such that, at all times $t \ge 0$, we have $X_t(v) \le Y_t(v)$ for all $v \in \{1, \dots, N\}$, and every revealed edge is an edge in $\mathscr{G}_t^{(N)}$.

The proof of this lemma is similar to the proof of Proposition 6.1 in [7], so we omit it.

5.2. *Proof of Theorem* 2. In this section, we prove Theorem 2 for the vertex component of Y, and hence for X since we have that $X \le Y$ stochastically. We use the function S, given in the assumptions of the theorem, to define a function m_t which, based on the state of Y, attaches a score to every vertex, in such a way that the accumulated score of the vertices in the network is a supermartingale. Raising it to the power δ and adding a small drift still yields a supermartingale if we stop it at the extinction time. We then exploit optional stopping to get upper bounds and prove both statements of the theorem.

We now suppose $S:(0,1]\to(0,+\infty)$ satisfies (7) and $(H)_\delta$ for some given $\delta\in(0,1]$, and for vertices $x\in\{1,\dots,N\}$, we introduce the notation $s(x)=S(x/N)$ and $t(x)=\frac{4D}{T_\lambda(x/N)\kappa_x}s(x)$, where we recall $D=\min\{\frac{\kappa_0}{4},\frac{\kappa_0^2}{64c_2},\frac{1}{16}\}$. Observe that the monotonicity properties of p and S easily imply

$$\lambda\sum_y p_{x,y}s(y)\le\frac{\lambda}{N}\sum_y p\Big(\frac{x}{N},\frac{y}{N}\Big)S\Big(\frac{y}{N}\Big)\le\lambda\int_0^1 p\Big(\frac{x}{N},y\Big)S(y)\,dy,$$

so inequality (7) implies

$$\lambda\sum_y p_{x,y}s(y)\le\frac{D}{T_\lambda(\frac{x}{N})}s(x). \tag{20}$$

We now define the score of a configuration as

$$M_t:=\sum_{x=1}^N m_t(x),$$

where

$$m_t(x)=\begin{cases} s(x)+\Big(2\kappa_x^{-1}\lambda N_t(x)\wedge\dfrac{1}{2}\Big)\big(2t(x)\big) & \text{if } Y_t(x)=1,\\ \big(2\kappa_x^{-1}\lambda N_t(x)\wedge 1\big)\big(s(x)+t(x)\big) & \text{if } Y_t(x)=0,\end{cases}$$

and $N_t(x)=\sum_{y\neq x}Y_t(x,y)$ is the number of revealed neighbours of x at time t.

Even though it may seem a bit obscure at first glance, the score is actually natural; every vertex has a base score of $s(x)$ or 0 (depending on whether it is infected or not) and $m_t(x)$ increases linearly on the amount of its revealed neighbours, which reflects the fact that revealed neighbours make the propagation of the infection easier. In both cases, the score grows linearly up until some maximal cap at which m_t is the same for infected and noninfected vertices; a natural choice since from a certain amount of revealed neighbours on, we can think of vertices as permanently infected. Observe also that:

- From the inequalities $T_\lambda(x/N)\ge 1$ and $\kappa_x\ge\kappa_0$ and the definition of D, we always have $t(x)\le s(x)$.
- The score of a vertex is monotone with respect to the value of $Y_t(x)$ and of $N_t(x)$.
- The maximal value $m_t(x)=s(x)+t(x)$ is obtained if either $Y_t(x)=1$ and $N_t(x)\ge\lambda^{-1}\kappa_x/4$, or $Y_t(x)=0$ and $N_t(x)\ge\lambda^{-1}\kappa_x/2$.

We now aim at proving that M_t is a supermartingale, as well as $Z_{t\wedge T_{\mathrm{ext}}}$, where T_{ext} is the extinction time of the infection and $Z_t:=M_t^\delta+\delta\varepsilon t$, for some suitable

$\varepsilon > 0$. We do this by showing that the expected infinitesimal change of M_t is less than $-\varepsilon M_t^{1-\delta}$ when $M_t > 0$. To begin, we bound the expected infinitesimal change of $m_t(x)$ according to the values of $Y_t(x)$ and $N_t(x)$.

(i) $\boldsymbol{Y_t(x) = 1}$, $\boldsymbol{N_t(x) \geq \lambda^{-1}\kappa_x/4}$. In this case, the score of x can only decrease (or remain unchanged) with each possible change, so we obtain the bound considering only an update event at x, which yields

$$\frac{1}{dt}\mathbb{E}\big[m_{t+dt}(x) - m_t(x)|\mathfrak{F}_t\big] \leq -\kappa_x t(x).$$

(ii) $\boldsymbol{Y_t(x) = 1}$, $\boldsymbol{N_t(x) < \lambda^{-1}\kappa_x/4}$. In this case, we can bound the infinitesimal change by the expression

$$\big(2\kappa_x^{-1}\lambda N_t(x)\big[s(x) - t(x)\big] - s(x)\big) + 4\lambda^2\kappa_x^{-1}t(x) \sum_{y:\, Y_t(x,y)=0} p_{x,y},$$

where the first term comes from the recovery at x and the second one from the possible revealing of a neighbouring edge. As $t(x) \leq s(x)$ and $N_t(x) < \lambda^{-1}\kappa_x/4$ the first term is bounded by

$$-\frac{s(x) + t(x)}{2} \leq -\frac{s(x)}{2} = -\frac{1}{8D}\kappa_x t(x) T_\lambda\left(\frac{x}{N}\right).$$

On the other hand, since $\sum_y p_{x,y}$ can be bounded by $\int p(\frac{x}{N}, t)\, dt \leq c_2(\frac{x}{N})^{-\gamma}$, we can bound the second term by

$$\begin{aligned} 4c_2\kappa_0^{-2}\big(\kappa_x t(x)\big)\left[\lambda^2\left(\frac{x}{N}\right)^{-\gamma+2\gamma\eta}\right] &\leq 4c_2\kappa_0^{-2}\kappa_x t(x) T_\lambda\left(\frac{x}{N}\right) \\ &\leq \frac{1}{16D}\kappa_x t(x) T_\lambda\left(\frac{x}{N}\right). \end{aligned}$$

Adding the two terms, we obtain

$$\frac{1}{dt}\mathbb{E}\big[m_{t+dt}(x) - m_t(x)|\mathfrak{F}_t\big] \leq -\frac{\kappa_x t(x) T_\lambda(\frac{x}{N})}{16D} \leq -\kappa_x t(x).$$

(iii) $\boldsymbol{Y_t(x) = 0}$**,** $\boldsymbol{N_t(x) \geq \lambda^{-1}\kappa_x/2}$. As in the first scenario, the score is maximal in this case. We obtain the bound again considering only an update event at x, which yields

$$\frac{1}{dt}\mathbb{E}\big[m_{t+dt}(x) - m_t(x)|\mathfrak{F}_t\big] \leq -\kappa_x m_t(x).$$

(iv) $\boldsymbol{Y_t(x) = 0}$, $\boldsymbol{N_t(x) \leq \lambda^{-1}\kappa_x/2}$. In this case, we can bound every positive increment of $m(x)$ by the maximal score $s(x) + t(x)$, hence we can bound the infinitesimal change by

$$-\kappa_x m_t(x) + \lambda N_t(x)\big[s(x) + t(x)\big] + \sum_{y:\, Y_t(y)=1} \lambda p_{x,y}\big[s(x) + t(x)\big],$$

where the first term comes from possible updates of x, the second from infections coming through neighbouring revealed edges and the third from infections coming through unrevealed edges. Since $N_t(x) \le \lambda^{-1}\kappa_x/2$, the second term is exactly $\frac{\kappa_x}{2} m_t(x)$, and hence, since $t(x) \le s(x)$, we obtain

$$\frac{1}{dt}\mathbb{E}\big[m_{t+dt}(x) - m_t(x)|\mathfrak{F}_t\big] \le -\frac{\kappa_x}{2} m_t(x) + 2\lambda \sum_{y\colon Y_t(y)=1} p_{x,y} s(x).$$

Now, we can consider the whole score, and write

$$\begin{aligned}
&\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)|\mathfrak{F}_t\big] \\
&\quad = \sum_x \frac{1}{dt}\mathbb{E}\big[m_{t+dt}(x) - m_t(x)|\mathfrak{F}_t\big] \\
&\quad \le \sum_{\substack{x\colon Y_t(x)=0 \\ N_t(x)\ge\lambda^{-1}\kappa_x/2}} -\kappa_x m_t(x) + \sum_{x\colon Y_t(x)=1} -\kappa_x t(x) \\
&\qquad + \sum_{\substack{x\colon Y_t(x)=0 \\ N_t(x)<\lambda^{-1}\kappa_x/2}} -\frac{\kappa_x}{2} m_t(x) + \sum_{\substack{x\colon Y_t(x)=0 \\ N_t(x)<\lambda^{-1}\kappa_x/2}} 2\lambda \sum_{y\colon Y_t(y)=1} p_{x,y} s(x).
\end{aligned}$$

For the last term, we can reverse the role of x and y obtaining the expression

$$\sum_{x\colon Y_t(x)=1} 2\lambda \sum_y p_{x,y} s(y) \le 2D \sum_{x\colon Y_t(x)=1} \frac{s(x)}{T_\lambda(x/N)} \le \frac{1}{2} \sum_{x\colon Y_t(x)=1} \kappa_x t(x),$$

where the first inequality comes from (20). We thus arrive at

$$\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)|\mathfrak{F}_t\big] \le -\frac{1}{2} \sum_{x\colon Y_t(x)=0} \kappa_x m_t(x) - \frac{1}{2} \sum_{x\colon Y_t(x)=1} \kappa_x t(x), \tag{21}$$

which is clearly negative, and hence $(M(t)\colon t \ge 0)$ is a supermartingale. To show that $(Z_{t\wedge T_{\text{ext}}}\colon t \ge 0)$ is a supermartingale, recall that $\mathfrak{c} := \|T_\lambda S^{-\delta}\|_\infty$ is finite by $(H)_\delta$, and observe the following inequality is satisfied:

$$\frac{2\lambda}{\kappa_x} s(x) \mathbf{1}_{m_t(x)>0} \le m_t(x) \le 2s(x),$$

by the definition of $m_t(x)$ and the property $t(x) \le s(x)$. Further,

$$\begin{aligned}
\kappa_x t(x) &= \frac{4Ds(x)}{T_\lambda(x/N)} \ge \frac{4D}{\mathfrak{c}} s(x)^{1-\delta} \ge \frac{2^{1+\delta}D}{\mathfrak{c}} m_t(x)^{1-\delta}, \\
\kappa_x m_t(x) &\ge \big(2\lambda S(1)\big)^\delta \kappa_0^{1-\delta} m_t(x)^{1-\delta},
\end{aligned}$$

with the understanding (in the last inequality) $m_t(x)^{1-\delta} = 0$ if $m_t(x) = 0$ and $\delta = 1$. This together with (21) yields the existence of $\varepsilon > 0$ depending on λ but not on N such that

$$\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)|\mathfrak{F}_t\big] \le -\varepsilon \sum_x m_t(x)^{1-\delta} \le -\varepsilon M(t)^{1-\delta}, \tag{22}$$

where the last inequality is due to $0 < \delta \le 1$, and hence $Z_{t\wedge T_{\text{ext}}}$ defines a positive supermartingale which converges almost surely to $Z_{T_{\text{ext}}}$. Since T_{ext} is increasing in the initial condition of Y it is enough to take $Y_0 = 1$ to prove the theorem. We infer from the optional stopping theorem that

$$\delta\varepsilon\mathbb{E}[T_{\text{ext}}] = \mathbb{E}[Z_{T_{\text{ext}}}] \le \mathbb{E}[Z_0] = \mathbb{E}\big[M_0^\delta\big] = N^\delta\left[\frac{1}{N}\sum_{x=1}^N s(x)\right]^\delta, \tag{23}$$

but the expression inside the brackets is bounded by $\int_0^1 S(x)\,dx$ which is a fixed constant and hence the first statement of Theorem 2 is proved.

In order to prove the second statement, we use the duality described in Proposition 3 to deduce $I_N(t) = \frac{1}{N}\sum_{x=1}^N \mathbb{P}_x(X_t \ne 0)$ where under $\mathbb{P}_x$ the process X is started from initial condition $X_0 = \delta_x$. Since Y_t stochastically bounds X_t from above, we get

$$I_N(t) \le \frac{1}{N}\sum_{x=1}^N \mathbb{P}_x(t < T_{\text{ext}}), \tag{24}$$

where in this context T_{ext} is the extinction time of Y, started from $Y_0 = \delta_x$. Defining T_{hit} as the first time that $Y_t(x) = 1$ for some $x \le \lceil aN \rceil$, and $T := T_{\text{ext}} \wedge T_{\text{hit}}$, we obtain

$$\mathbb{P}(t < T_{\text{ext}}) \le \mathbb{P}(T_{\text{hit}} < T_{\text{ext}}) + \mathbb{P}(t < T), \tag{25}$$

which leads us to use the stopped process $Y_{t\wedge T}$ instead of Y_t.

We suppose a and $S\colon [a,1] \to (0,\infty)$ nonincreasing are such that (8) is satisfied. We extend the function S to the whole interval $[0,1]$ by setting $S(x) = S(a)$ for $x < a$. A close look at the latter proof shows that if we define $s(x)$, $t(x)$, $m_t(x)$, and M and Z as before, with $\delta = 1$ and $\varepsilon > 0$ small enough, then the stopped process $Z_{t\wedge T}$ is a positive supermartingale. Furthermore, it converges to

$$Z_T = M_T + \varepsilon T = M_{T_{\text{hit}}}\mathbf{1}_{\{T_{\text{hit}} < T_{\text{ext}}\}} + \varepsilon T \ge S(a)\mathbf{1}_{\{T_{\text{hit}} < T_{\text{ext}}\}} + \varepsilon T.$$

By the optional stopping theorem, we get $\mathbb{E}_x[Z_T] \le \mathbb{E}_x[Z_0] = s(x)$, and thus in particular

$$\mathbb{P}_x(T_{\text{hit}} < T_{\text{ext}}) \le \frac{s(x)}{S(a)}, \qquad \mathbb{P}_x(t < T) \le \frac{1}{t}\mathbb{E}_x T \le \frac{s(x)}{\varepsilon t}.$$

Gathering this with (24) and (25), and bounding the probability of survival by 1 whenever $x \le \lceil aN \rceil$, we obtain

$$I_N(t) \le \frac{\lceil aN \rceil}{N} + \frac{1}{NS(a)} \sum_{x=\lceil aN \rceil+1}^{N} s(x) + \frac{1}{\varepsilon t N} \sum_{x=\lceil aN \rceil+1}^{N} s(x).$$

Noticing that all these terms converge when $N \to \infty$, we have that for all N and $t \ge 0$,

$$I_N(t) \le a + \frac{1}{S(a)} \int_a^1 S(x)\,dx + \frac{1}{\varepsilon t} \int_a^1 S(x)\,dx + \varepsilon'(N),$$

with $\varepsilon'(N) \to 0$, which concludes the result when taking t large enough (note both integrals are finite since S is bounded by $S(a)$).

6. Application to the factor kernel. In this section, we prove the first half of Theorem 3 applying both Theorems 1 and 2 to the factor kernel $p(x, y) = \beta x^{-\gamma} y^{-\gamma}$. Our proof is structured as follows:

(1) For each of the four strategies in Theorem 1 we find the function $a(\lambda)$ of maximal order satisfying the respective condition given in the theorem.

(2) We define $a_0(\lambda)$ as the maximum of these functions over the four strategies. This function gives for each λ the definition of the set of stars. Using Theorem 1, we derive from it a lower bound of order $\lambda a_0 p(a_0, 1) \wedge 1$, or more simply $\lambda a_0(\lambda)^{1-\gamma}$, for the lower metastable density.

(3) We search for a nonincreasing function S and for $a_1 = a_1(\lambda) \in [0, 1]$ as small as possible such that inequality (8) is satisfied for small λ and $a = a_1(\lambda)$.

If we can take $a_1 = 0$, then we can apply Theorem 2, (1) and deduce fast extinction for small λ. In the other cases, we will always have proven already metastability, and Theorem 2, (2) then gives us, for small λ, an upper bound on the upper metastable density as

$$\rho^+(\lambda) \le a_1(\lambda) + \frac{1}{S(a_1(\lambda))} \int_{a_1(\lambda)}^1 S(y)\,dy.$$

Note we have not discussed how to choose the function S. This will be further discussed in the examples below.

To avoid cluttered notation we henceforth assume $\beta = 1$, which does not affect the results.

The function $a_0(\lambda)$. Our aim is to find for each strategy the maximal function satisfying the respective condition in Theorem 1.

- **Quick direct spreading**: We study the expression $\lambda a p(a, a) = \lambda a^{1-2\gamma}$ and check whether it is bounded away from zero. If $\gamma \le 1/2$, this is never satisfied. If $\gamma > 1/2$, we impose that the expression be constant and obtain

$$a(\lambda) = r\lambda^{\frac{1}{2\gamma-1}} \quad \text{for some } r > 0.$$

- **Delayed direct spreading**: We study the expression $\lambda T a p(a, a) = \lambda T a^{1-2\gamma}$ with T as in Theorem 1. To facilitate our study, we impose $T \to \infty$ instead of just being large, but this translates into $T \log^2(T) = C\lambda^2 a^{-\gamma(1-2\eta)} \to \infty$ which can only occur if $\eta < 1/2$. To ensure boundedness of the expression from zero, we observe that $\lambda T a^{1-2\gamma} \le \lambda^3 a^{1-3\gamma+2\gamma\eta}$, and hence it suffices that $1 - 3\gamma + 2\gamma\eta < 0$. Assuming this and $\eta < 1/2$, we find $a(\lambda)$ by imposing that $\lambda T a^{1-2\gamma}$ be constant, say equal to c, and deduce from the definition of T that

$$[\log(c) - \log(\lambda a^{1-2\gamma})]^2 = \frac{C}{c}\lambda^3 a^{1-3\gamma+2\gamma\eta}.$$

  Since $T \to \infty$, the expression on the left goes to infinity, and hence a is of the form

$$a(\lambda) = \lambda^{\frac{3}{3\gamma-2\gamma\eta-1}} f(\lambda),$$

  for some function f going to zero as $\lambda \to 0$. Replacing this new expression for a, we obtain $[\log(c) - c_1 \log(\lambda) - c_2 \log(f(\lambda))]^2 = \frac{C}{c} f(\lambda)^{1-3\gamma+2\gamma\eta}$. The expression on the right tends to infinity polynomially in f, so the equality holds only if $\log\lambda$ dominates $\log f(\lambda)$ giving $f(\lambda)$ of order $[-\log\lambda]^{\frac{2}{1-3\gamma+2\gamma\eta}}$. We finally get the maximal a of the form

$$a(\lambda) = r\left[\frac{\lambda^3}{(\log\lambda)^2}\right]^{\frac{1}{3\gamma-2\gamma\eta-1}} \quad \text{for some } r > 0.$$

- **Quick indirect spreading**: We study the expression $\lambda^2 a^{1-\gamma} p(a, 1) = \lambda^2 a^{1-2\gamma}$. If $\gamma \le 1/2$, this inevitably tends to zero and otherwise we obtain that the maximal a is of the form

$$a(\lambda) = r\lambda^{\frac{2}{2\gamma-1}} \quad \text{for some } r > 0.$$

- **Delayed indirect spreading**: In this case, we need to consider the expression $\lambda^2 T a^{1-\gamma} p(a, 1) = \lambda^2 T a^{1-2\gamma}$ whose study is analogous to what was done in the delayed direct spreading case, obtaining that condition (iv) can only hold when $\eta < 1/2$ and $1 - 3\gamma + 2\gamma\eta < 0$ and in this case a must be of the form

$$a(\lambda) = r\left[\frac{\lambda^4}{(-\log\lambda)^2}\right]^{\frac{1}{3\gamma-2\gamma\eta-1}} \quad \text{for some } r > 0.$$

For the final form of a_0, we notice first that on the set

$$\left\{(\gamma, \eta) \colon \gamma \le \frac{1}{3-2\eta}, \eta \le \frac{1}{2} \text{ or } \gamma \le \frac{1}{2}, \eta \ge \frac{1}{2}\right\} \tag{26}$$

none of the conditions of Theorem 1 hold, so we expect that fast extinction occurs for parameters inside this region. For the construction of a_0 on the complement of this set, we note that all of the functions a have the form $\lambda^{\mathfrak{e}'+o(1)}$, and since $\lambda < 1$, for each λ small the dominant survival strategy (i.e., the one which gives the largest lower bound for the density) corresponds to the expression with the smallest exponent. If $\eta < 1/2$, this gives

$$a_0(\lambda) = \begin{cases} r\left[\dfrac{\lambda^3}{(-\log\lambda)^2}\right]^{\frac{1}{3\gamma-2\gamma\eta-1}} & \text{if } \dfrac{1}{3-2\eta} < \gamma < \dfrac{2}{3+2\eta}, \\ r\lambda^{\frac{1}{2\gamma-1}} & \text{if } \dfrac{2}{3+2\eta} < \gamma < 1, \end{cases}$$

while in the case $\eta > 1/2$ we obtain

$$a_0(\lambda) = r\lambda^{\frac{1}{2\gamma-1}} \quad \text{if } \gamma > \frac{1}{2}.$$

Computing the lower bound for the density as $\lambda a_0^{1-\gamma}$ for the values of the parameter where a_0 is defined gives the lower bound in (11).

The function S. In the case of the factor kernel, inequality (8) takes the simple form

$$\lambda T_\lambda(x)x^{-\gamma} \int_0^1 y^{-\gamma} S(y \vee a)\, dy \le DS(x).$$

Since the integral does not depend on x, a natural choice is to consider the function

$$S(x) = T_\lambda(x)x^{-\gamma}.$$

This scoring function is also somehow natural, as we now explain. The average degree of x is of order $x^{-\gamma}$ and if x is infected, we should wait on average at most time $T_\lambda(x)$ before x turns to healthy and surrounded by unrevealed edges. More precisely, this should happen roughly at time $\max\{T(x,\lambda), 1\}$, where $T(x,\lambda)$ is as in (3), but this is bounded by $T_\lambda(x)$, and even of the same order, up to logarithmic terms.[6] Thus $\lambda S(x)$ should be a reasonable upper bound for the average number of infections sent by vertex x before the first time when it is healthy and surrounded by unrevealed edges (namely the first time t for which $m_t(x) = 0$), and thus S seems a reasonable scoring function.

Using this choice of S, inequality (8) becomes

$$\lambda \int_0^1 y^{-\gamma} S(y \vee a)\, dy \le D.$$

[6]The discrepancy between the time-scale function $T_\lambda(x)$ used in Theorem 2 and that in Theorem 1 explains why, as we will see, the lower bounds for $\rho^-(\lambda)$ and the upper bounds for $\rho^+(\lambda)$ that we get match only up to logarithmic terms. They match indeed up to a constant multiplicative term when $T_\lambda(x) = 1$.

Using that $T_\lambda(x) \le 1 + \lambda^2 x^{-\gamma(1-2\eta)}$, the left-hand side can be bounded as follows:

$$\begin{aligned}\lambda \int_0^1 y^{-\gamma} S(y \vee a)\, dy &= \lambda a^{1-2\gamma} T_\lambda(a) + \lambda \int_a^1 S(y) y^{-\gamma}\, dy \\ &\le \lambda a^{1-2\gamma} T_\lambda(a) + \lambda^3 \int_a^1 y^{-\gamma(3-2\eta)}\, dy + \lambda \int_a^1 y^{-2\gamma}\, dy \\ &\le \rho\big(\lambda^3 a^{1-\gamma(3-2\eta)} + \lambda a^{1-2\gamma} + \lambda\big),\end{aligned}$$

for some constant $\rho > 0$ depending only on γ, η, under the hypothesis $\gamma \notin \{\frac{1}{2}, \frac{1}{3-2\eta}\}$. Note that in the last inequality we have used that $\int_a^1 y^\iota\, dy \le \frac{1}{|1+\iota|}(1 + a^{1+\iota})$ for $\iota \neq -1$, and the upper bound is sharp up to a multiplicative constant.

We now consider three different cases.

(1) **The case $\eta < \frac{1}{2}$ and $\gamma < \frac{1}{3-2\eta}$, or $\eta \ge \frac{1}{2}$ and $\gamma < \frac{1}{2}$.**
Here, we can take $a_1 = 0$. Indeed, we have $\lambda \int_0^1 y^{-\gamma} S(y)\, dy \le \rho\lambda$, therefore, inequality (8) is satisfied if $\lambda \le D/\rho$. Moreover, the function S satisfies $T_\lambda(x) \le S(x)^\delta$ for:

- $\delta \ge \frac{1-2\eta}{2-2\eta}$ whenever $0 \le \eta < 1/2$,
- $\delta > 0$ in the case $\eta > 1/2$.

Using Theorem 2, (1), we deduce fast extinction for small λ. More precisely, when $\eta < 1/2$, we get $\mathbb{E}[T_{\text{ext}}] \le \omega' N^{(1-2\eta)/(2-2\eta)}$ for some $\omega' < +\infty$. In the case $\eta \ge 1/2$, for every $\delta > 0$, there is some $\omega' < +\infty$ such that $\mathbb{E}[T_{\text{ext}}] \le \omega' N^\delta$. In particular, the extinction time grows even slower than polynomially.

(2) **The case $\eta < \frac{1}{2}$ and $\frac{1}{3-2\eta} < \gamma < \frac{2}{3+2\eta}$.**
A sufficient condition for Inequality (8) to be satisfied is

$$\max\big(\lambda^3 a^{1-\gamma(3-2\eta)}, \lambda a^{1-2\gamma}, \lambda\big) \le D/3\rho.$$

This requires, in particular, that

$$a \ge a_1(\lambda) := r\lambda^{\frac{3}{3\gamma-2\eta-1}},$$

where $r = (D/(3\rho))^{\frac{1}{3\gamma-2\eta-1}} < \infty$. One can check this is also a sufficient condition for small λ when $\gamma < \frac{2}{3+2\eta}$. Applying Theorem 2, (2), we deduce that

$$\rho^+(\lambda) \le a_1(\lambda) + \frac{1}{S(a_1(\lambda))} \int_{a_1(\lambda)}^1 S(y)\, dy.$$

From now on, we write $f(\lambda) \lesssim g(\lambda)$ if the function $g(\lambda)/f(\lambda)$ is bounded from below by a positive constant, and similarly for $\gtrsim$. One can check the following:

$$\begin{aligned}\int_{a_1(\lambda)}^1 S(y)\, dy &\lesssim \int_{a_1(\lambda)}^1 \lambda^2 y^{-\gamma(2-2\eta)} + y^{-\gamma}\, dy \\ &\lesssim \lambda^2 a_1(\lambda)^{1-2\gamma+2\gamma\eta} + 1 \lesssim \lambda^{\frac{1+2\gamma\eta}{3\gamma-2\gamma\eta-1}} + 1 \lesssim 1.\end{aligned}$$

This, together with $S(a_1(\lambda)) \geq \lambda^2 a_1^{-\gamma(2-2\eta)} \gtrsim \lambda^{-\frac{2-2\gamma\eta}{3\gamma-2\gamma\eta-1}}$, gives

$$\rho^+(\lambda) \leq c\lambda^{\frac{2-2\gamma\eta}{3\gamma-2\gamma\eta-1}},$$

for some constant $c < \infty$. Thus we obtain the upper bound (11) in this region.

(3) **The case $\eta < \frac{1}{2}$ and $\gamma > \frac{2}{3+2\eta}$, or $\eta \geq 1/2$ and $\gamma > 1/2$.** Similarly, as in the previous case, it suffices for small λ to require

$$a \geq a_1(\lambda) := r\lambda^{\frac{1}{2\gamma-1}},$$

with $r = (D/(3\rho))^{\frac{1}{2\gamma-1}}$. We have $S(x) = x^{-\gamma}$ for $x \geq a_1(\lambda)$, and Theorem 2, (2), yields

$$\rho^+(\lambda) \leq a_1^\gamma \int_0^1 y^{-\gamma}\, dy \leq c\lambda^{\frac{\gamma}{2\gamma-1}},$$

for some constant $c < \infty$. This gives again the upper bound (11) and concludes our study of the metastable densities for the factor kernel.

7. Application to the preferential attachment kernel. In this section, we derive results for the preferential attachment kernel given by $p(x, y) = \beta \min\{x, y\}^{-\gamma} \max\{x, y\}^{\gamma-1}$ following the programme set out at the beginning of Section 6. Again, we assume $\beta = 1$ for simplicity. Taking into account that $p(a, a) = a^{-1}$ and $p(a, 1) = a^{-\gamma}$, straightforward calculations allow us to deduce the values of the maximal order functions $a(\lambda)$ summarised in Table 1. The third column gives the conditions needed to define the maximal function $a(\lambda)$.

TABLE 1
Maximal order function $a(\lambda)$ for given spreading strategy, case of preferential attachment kernel

Strategy	$a(\lambda)$	Condition
Quick Direct Spreading	–	–
Delayed Direct Spreading	$\left[\frac{\lambda^3}{(-\log\lambda)^2}\right]^{\frac{1}{\gamma(1-2\eta)}}$	$\eta < 1/2$
Quick Indirect Spreading	$\lambda^{\frac{2}{2\gamma-1}}$	$\gamma > 1/2$
Delayed Indirect Spreading	$\left[\frac{\lambda^4}{(-\log\lambda)^2}\right]^{\frac{1}{3\gamma-2\gamma\eta-1}}$	$\eta < \frac{1}{2}$ and $\gamma > \frac{1}{3-2\eta}$

Taking the maximum over permissible strategies we deduce $a_0(\lambda)$. If $\eta < 1/2$, we get

$$a_0(\lambda) = \begin{cases} r\Big[\dfrac{\lambda^3}{(-\log\lambda)^2}\Big]^{\frac{1}{\gamma(1-2\eta)}} & \text{if } 0 < \gamma < \dfrac{3}{5+2\eta}, \\ r\Big[\dfrac{\lambda^4}{(-\log\lambda)^2}\Big]^{\frac{1}{3\gamma-2\gamma\eta-1}} & \text{if } \dfrac{3}{5+2\eta} < \gamma < \dfrac{1}{1+2\eta}, \\ r\lambda^{\frac{2}{2\gamma-1}} & \text{if } \dfrac{1}{1+2\eta} < \gamma < 1. \end{cases}$$

If $\eta > 1/2$ and $\gamma > 1/2$, we get $a_0(\lambda) = r\lambda^{\frac{2}{2\gamma-1}}$, but if $\eta > 1/2$ and $\gamma < 1/2$ none of the strategies succeed. Calculating $\lambda a_0(\lambda)^{1-\gamma}$ gives the lower bounds as in (12) and there is slow extinction for all parameters except for the case $\eta > 1/2$ and $\gamma < 1/2$, as expected.

To get upper bounds in the case of the preferential attachment kernel, the choice of a scoring function S is much more delicate. Our initial approach has been to search for a function S giving equality in (8), or in the related Fredholm equation of the second kind

$$\int_{a_1(\lambda)}^{1} T_\lambda(x)p(x,y)S(y)\,dy = \frac{D}{\lambda}S(x), \tag{27}$$

as such an S is a plausible candidate to give the best possible bounds in Theorem 2. To carry out this programme requires extensive calculations with Bessel functions and modified Bessel functions. However, it turns out that relatively crude approximations to these functions also suffice and this is the approach we now follow.

The upper bound for $\gamma > 1/2$. We start with the case $\gamma > 1/2$, and we define S by

$$S(x) = T_\lambda(x)\big(x^{\gamma-1} + \rho\lambda x^{-\gamma}\big),$$

where $\rho > 0$ is a constant to be chosen later. To argue why S may be a "reasonable scoring function", it is useful to note that the cardinality of the sets $\{y \le x : y \sim x\}$ and $\{y \le x : \exists z \ge x, y \sim z \sim x\}$ are of order $x^{\gamma-1}$ and $x^{-\gamma}$, respectively. Thus, $\lambda S(x)$ might be a reasonable upper bound for the number of other strong vertices a strong vertex x can typically infect, either directly or indirectly, before it totally recovers (namely $m_t(x) = 0$).

Using this function, inequality (8) becomes

$$\frac{\lambda}{x^{\gamma-1} + \rho\lambda x^{-\gamma}} \int_0^1 p(x,y)S(y \vee a)\,dy \le D.$$

We denote the left-hand side by $I(x,a)$, and observe that the hypothesis $\gamma > 1/2$ implies

$$p(x,y) \le x^{\gamma-1}y^{-\gamma} + x^{-\gamma}y^{\gamma-1} \le 2p(x,y).$$

If $x \geq a$, we can bound I using the notation $\alpha = \gamma(1-2\eta)$ as

$$\begin{aligned}
I(x,a) &= \frac{\lambda}{x^{\gamma-1}+\rho\lambda x^{-\gamma}}\biggl[\int_0^a p(x,y)S(a)\,dy \\
&\quad + \int_a^1 p(x,y)T_\lambda(y)\bigl(y^{\gamma-1}+\rho\lambda y^{-\gamma}\bigr)\,dy\biggr] \\
&\leq \frac{\lambda}{x^{\gamma-1}+\rho\lambda x^{-\gamma}}\biggl[x^{\gamma-1}S(a)\int_0^a y^{-\gamma}\,dy \\
&\quad + \int_a^1 p(x,y)\bigl(1+\lambda^2 y^{-\alpha}\bigr)\bigl(y^{\gamma-1}+\rho\lambda y^{-\gamma}\bigr)\,dy\biggr] \\
&\leq \frac{\lambda}{x^{\gamma-1}+\rho\lambda x^{-\gamma}}\biggl[x^{\gamma-1}\biggl(\frac{a^{1-\gamma}S(a)}{1-\gamma} \\
&\quad + \int_a^1 \bigl(y^{-1}+\rho\lambda y^{-2\gamma}+\lambda^2 y^{-1-\alpha}+\rho\lambda^3 y^{-\alpha-2\gamma}\bigr)\,dy\biggr) \\
&\quad + x^{-\gamma}\int_a^1 \bigl(y^{2\gamma-2}+\rho\lambda y^{-1}+\lambda^2 y^{2\gamma-2-\alpha}+\rho\lambda^3 y^{-1-\alpha}\bigr)\,dy\biggr] \\
&\leq \lambda\biggl(\frac{a^{1-\gamma}S(a)}{1-\gamma}+\int_a^1 \bigl(y^{-1}+\rho\lambda y^{-2\gamma}+\lambda^2 y^{-1-\alpha}+\rho\lambda^3 y^{-\alpha-2\gamma}\bigr)\,dy\biggr) \\
&\quad + \frac{1}{\rho}\int_a^1 \bigl(y^{2\gamma-2}+\rho\lambda y^{-1}+\lambda^2 y^{2\gamma-2-\alpha}+\rho\lambda^3 y^{-1-\alpha}\bigr)\,dy.
\end{aligned}$$

At this point, we observe that the bounds we used are tight up to a multiplicative constant. Indeed, replacing a "max" by a sum can multiply the result by 2 at worst. The last inequality is tight because taking $x = 1$ gives at least $1/(1+\rho\lambda) \geq 1/2$ times the first term (for $\lambda < 1/\rho$), while taking $x = a$ gives at least $1/2$ times the second term, if we further suppose $\rho\lambda a^{-\gamma} > a^{\gamma-1}$ or $a \leq (\rho\lambda)^{1/(2\gamma-1)}$ (one can check a posteriori that a_1 below will always satisfy this property).

For simplicity,[7] we now suppose that the exponents in the integrals are different from -1, and use again the inequality $\int_a^1 y^\iota\,dy \leq \frac{1}{|1+\iota|}(a^{1+\iota}+1)$ for $\iota \neq -1$. This allows to give a relatively simple upper bound for $I(x,a)$ (again tight up to a multiplicative constant) as

$$\begin{aligned}
I(x,a) &\lesssim \frac{1}{\rho}+\rho\lambda^2+\lambda|\log a|+\rho\lambda^2 a^{1-2\gamma}+\lambda^3 a^{-\alpha}+\rho\lambda^4 a^{1-\alpha-2\gamma} \\
&\quad + \frac{1}{\rho}a^{2\gamma-1}+\frac{1}{\rho}\lambda^2 a^{2\gamma-1-\alpha}.
\end{aligned}$$

[7] We get an additional factor $\log a$ when the exponent is -1. However, the reader can check this actually never concerns the leading term, so our results also hold when one of the exponents equals -1.

We want this to be smaller than D. To this end, we now fix $\rho = 2/D$, so the first term is smaller than $D/2$. Now, we can ensure $I(x,a)$ is smaller than D by requesting:

- $a > r\lambda^{3/\alpha}$ with r large,
- $a > r\lambda^{2/(2\gamma-1)}$ with r large,
- $a > r\lambda^{4/(2\gamma+\alpha-1)}$ with r large,
- $a > r\lambda^{2/(\alpha+1-2\gamma)}$ with r large (only in the case $\alpha + 1 - 2\gamma > 0$).

We then choose $a_1 = a_1(\lambda)$ the smallest value making all these requests satisfied. After some more computations, these give:

- $a_1(\lambda) = r\lambda^{3/\alpha}$ in the case $1/2 < \gamma < 3/(5+2\eta)$,
- $a_1(\lambda) = r\lambda^{4/(2\gamma+\alpha-1)}$ in the case $3/(5+2\eta) < \gamma < 1/(1+2\eta)$,
- $a_1(\lambda) = r\lambda^{2/(2\gamma-1)}$ in the case $\gamma > 1/(1+2\eta)$.

Theorem 2, (2), now gives the upper bound for the upper metastable density

$$\rho^+(\lambda) \le a_1(\lambda) + \frac{1}{S(a_1(\lambda))} \int_{a_1(\lambda)}^1 S(y)\,dy.$$

The reader can check that in all three cases, S is integrable and the integral gives a constant term, while $S(a_1(\lambda))^{-1}$ is of same order as $\lambda a_1^{1-\gamma}$. Actually, the expression $\lambda a^{1-\gamma} S(a)$ appears in the upper bound of $I(1,a)$, and the choice of a_1 made it small, but of constant order. Finally, we get an upper bound of order $\lambda a_1^{1-\gamma}$, which matches (12).

The upper bound for $\gamma < 1/2$ and $\eta < 1/2$. Here, we define the scoring function S by

$$S(x) = \left(x^{-\gamma} + \lambda x^{\gamma-1}\right) T_\lambda(x)$$

and avoid to use the inequality $p(x,y) \le x^{\gamma-1}y^{-\gamma} + x^{-\gamma}y^{\gamma-1}$, as it is not sharp anymore. Now inequality (8) is equivalent to the inequality $I(x,a) \le D$ for $x \ge a$, where

$$\begin{aligned}
I(x,a) &:= \frac{\lambda}{x^{-\gamma} + \lambda x^{\gamma-1}} \left(\int_0^x y^{-\gamma} x^{\gamma-1} S(y \vee a)\,dy + \int_x^1 x^{-\gamma} y^{\gamma-1} S(y)\,dy \right) \\
&\le \frac{\lambda x^{\gamma-1}}{x^{-\gamma} + \lambda x^{\gamma-1}} \\
&\quad \times \left(\frac{a^{1-\gamma} S(a)}{1-\gamma} + \int_a^x \left(y^{-2\gamma} + \lambda y^{-1} + \lambda^2 y^{-2\gamma-\alpha} + \lambda^3 y^{-1-\alpha}\right) dy \right) \\
&\quad + \frac{\lambda x^{-\gamma}}{x^{-\gamma} + \lambda x^{\gamma-1}} \int_x^1 \left(y^{-1} + \lambda y^{2\gamma-2} + \lambda^2 y^{-1-\alpha} + \lambda^3 y^{2\gamma-2-\alpha}\right) dy \\
&\le I_1(x,a) + I_2(x).
\end{aligned}$$

We again suppose for simplicity that the exponents do not equal -1, and after tedious but straightforward calculations we obtain the following simple upper bounds for $x \in [a, 1]$:

$$I_1(x, a) \lesssim \lambda|\log a| + \lambda^3 a^{-\alpha} + \lambda^2 a^{1-2\gamma-\alpha},$$
$$I_2(x)) \lesssim \lambda|\log x| + \lambda^3 x^{-\alpha}.$$

For example, one of the terms of $I_2(x)$ is

$$\begin{aligned}\frac{\lambda x^{-\gamma}}{x^{-\gamma} + \lambda x^{\gamma-1}} \int_x^1 \lambda^3 y^{2\gamma-2-\alpha}\, dy &\lesssim \frac{\lambda x^{-\gamma}}{x^{-\gamma} + \lambda x^{\gamma-1}} (\lambda^3 + \lambda^3 x^{2\gamma-1-\alpha}) \\ &\lesssim \lambda^4 \frac{x^{-\gamma}}{x^{-\gamma} + \lambda x^{\gamma-1}} + \lambda^3 x^{-\alpha} \frac{\lambda x^{\gamma-1}}{x^{-\gamma} + \lambda x^{\gamma-1}} \\ &\lesssim \lambda^4 + \lambda^3 x^{-\alpha} \lesssim \lambda|\log x| + \lambda^3 x^{-\alpha},\end{aligned}$$

and we bound the other terms similarly. As in the case $\gamma > 1/2$, we search for the minimal value making I_1 and I_2 small, and find that in the region $\gamma < 1/2$, $\eta < 1/2$, we can always take $a_1(\lambda) = r\lambda^{3/\alpha}$. Finally, as in the case $\gamma > 1/2$, we obtain the upper bound

$$\rho^+(\lambda) \le a_1(\lambda) + \frac{1}{S(a_1(\lambda))} \int_{a_1(\lambda)}^1 S(y)\, dy.$$

Again, the integral is of constant order, while $S(a_1(\lambda))^{-1}$ is of same order as $\lambda a_1(\lambda)^{1-\gamma}$, yielding an upper bound matching (12).

The upper bound for $\gamma < 1/2$ and $\eta > 1/2$. In this case, none of the scoring functions introduced before enable us to prove slow extinction. Besides, the time-scale function is simply $T_\lambda(x) = 1$. We define the scoring function $S(x) = x^{-\gamma'}$, with $\gamma' \in (\gamma, 1-\gamma)$. Inequality (8) for $a_1 = 0$ becomes

$$\lambda \int_0^1 p(x, y) y^{-\gamma'}\, dy \le D x^{-\gamma'}.$$

But simple calculations give

$$\begin{aligned}\lambda \int_0^1 p(x, y) y^{-\gamma'}\, dy &= \lambda x^{\gamma-1} \int_0^x y^{-\gamma-\gamma'}\, dy + \lambda x^{-\gamma} \int_x^1 y^{\gamma-1-\gamma'}\, dy \\ &\le \lambda x^{-\gamma'} \left(\frac{1}{1-\gamma-\gamma'} + \frac{1}{\gamma'-\gamma} \right),\end{aligned}$$

thus inequality (8) is satisfied for small λ. Moreover, $T_\lambda(x) = 1$ satisfies Hypothesis $(H)_\delta$ for every $\delta > 0$, thus we have fast extinction, and the expected extinction time grows subpolynomially. This completes our analysis for the preferential attachment kernel.

8. Conclusions. We have investigated the effect of fast network dynamics on the behaviour of the contact process on scale-free networks modelled as inhomogeneous random network with suitable connection kernels. The stationary network dynamics consists of vertices updating their neighbourhoods independently of the contact process. Variation of a parameter η, which controls the rate at which the most powerful vertices update, allows an interpolation between a scenario where vertices update on the time-scale of the contact process ($\eta = 0$) and a mean-field model where updates occur on a time scale of much faster order ($\eta \uparrow \infty$). We develop general techniques to study the behaviour of the extinction time and metastable densities at small infection rates for this class of models. Lower bounds are based on the identification of four core survival strategies for the contact process, and upper bounds are proved using coupling and supermartingale techniques.

Our focus is on two paradigmatic connection kernels, the factor kernel and the preferential attachment kernel, which exhibit very different behaviour. For the factor kernel, we identify a phase transition between fast and slow extinction, and, in case $\eta < \frac{1}{2}$, a further transition within the slow extinction phase between two types of metastable densities. For the preferential attachment kernel, a phase transition between fast and slow extinction only occurs when $\eta > \frac{1}{2}$. For $\eta < \frac{1}{2}$, we always have slow extinction and two phase transitions in the behaviour of the metastable densities.

In a future paper, we will discuss slowly evolving networks. This will include updating edges individually as well as vertex updating in the case where the most powerful vertices update very slowly ($\eta < 0$) and will allow us to interpolate between a scenario where vertices update on the time-scale of the contact process ($\eta = 0$) and the case of static networks ($\eta \downarrow -\infty$). The mathematical problems emerging in this work will require, in part, significantly different methods from those explained in this paper.

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