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A PROBABILISTIC APPROACH TO DIRAC CONCENTRATION IN NONLOCAL MODELS OF ADAPTATION WITH SEVERAL RESOURCES[1]

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This work is devoted to the study of scaling limits in small mutations and large time of the solutions u^ε of two deterministic models of phenotypic adaptation, where the parameter $\varepsilon > 0$ scales the size or frequency of mutations. The second model is the so-called Lotka–Volterra parabolic PDE in $\mathbb{R}^d$ with an arbitrary number of resources and the first one is a version of the second model with finite phenotype space. The solutions of such systems typically concentrate as Dirac masses in the limit $\varepsilon \to 0$. Our main results are, in both cases, the representation of the limits of $\varepsilon \log u^\varepsilon$ as solutions of variational problems and regularity results for these limits. The method mainly relies on Feynman–Kac-type representations of u^ε and Varadhan's lemma. Our probabilistic approach applies to multiresources situations not covered by standard analytical methods and makes the link between variational limit problems and Hamilton–Jacobi equations with irregular Hamiltonians that arise naturally from analytical methods. The finite case presents substantial difficulties since the rate function of the associated large deviation principle (LDP) has noncompact level sets. In that case, we are also able to obtain uniqueness of the solution of the variational problem and of the associated differential problem which can be interpreted as a Hamilton–Jacobi equation in finite state space.

1. Introduction. We are interested in the dynamics of a population subject to mutations and selection driven by competition for resources. Each individual in the population is characterized by a phenotypic trait, or simply *trait* (e.g., the size of individuals, their mean size at division for bacteria, their rate of nutrients intake or their efficiency in nutrients assimilation). The article deals with models with discrete and continuous trait space. In the case of finite trait space E, we consider

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the system of ordinary differential equations

$$\dot{u}^\varepsilon(t,i) = \sum_{j\in E} \exp\left(-\frac{\mathfrak{T}(i,j)}{\varepsilon}\right)\left(u^\varepsilon(t,j) - u^\varepsilon(t,i)\right) + \frac{1}{\varepsilon}u^\varepsilon(t,i)R\left(i, v_t^\varepsilon\right) \tag{1.1}$$
$$\forall t \in \mathbb{R}_+, \forall i \in E$$

with $u^\varepsilon(0,i) = \exp(-\frac{h(i)}{\varepsilon})$, with $h : E \to \mathbb{R}_+$, where $u^\varepsilon(t,i)$ is the density of population with trait $i \in E$ at time $t \geq 0$, $\mathfrak{T}(i,j) > 0$ for all $i \neq j \in E$ and $v_t^\varepsilon = (v_t^{1,\varepsilon}, \dots, v_t^{r,\varepsilon})$ is given by

$$v_t^{p,\varepsilon} = \sum_{j\in E} u^\varepsilon(t,j)\Psi_p(j) \qquad \forall 1 \leq p \leq r,$$

for some functions $\Psi_p : E \to (0, +\infty)$. The first term of the right-hand side of (1.1) models the effect of mutations on the population density: $\exp(-\frac{\mathfrak{T}(i,j)}{\varepsilon})$ can be interpreted as the mutation rate from trait j to trait i. More precisely, we can recover the more classical form of mutation as

$$\dot{u}^\varepsilon(t,i) = \sum_{j\in E}\left[e^{-\mathfrak{T}(i,j)/\varepsilon}u^\varepsilon(t,j) - e^{-\mathfrak{T}(j,i)/\varepsilon}u^\varepsilon(t,i)\right] + \frac{1}{\varepsilon}u^\varepsilon(t,i)R_\varepsilon\left(i, v_t^\varepsilon\right)$$
$$\forall t \in \mathbb{R}_+, \forall i \in E,$$

by modifying R as

$$R^\varepsilon(i,v) = R(i,v) + \varepsilon\left(e^{-\mathfrak{T}(j,i)/\varepsilon} - e^{-\mathfrak{T}(i,j)/\varepsilon}\right) \qquad \forall i \in E, v \in \mathbb{R}^r. \tag{1.2}$$

This modification has no impact on our results, but our analysis is simpler to present from (1.1).

Growth and competition are modeled through the function $R : E \times \mathbb{R}^r \to \mathbb{R}$: the quantity $R(i, v_t^\varepsilon)$ represents the growth rate of the population with trait i at time t and competition occurs through the functions $v_t^{p,\varepsilon}$. A typical example of function R is given by

$$R(i,v) = \sum_{p=1}^{r} \frac{c_p\Psi_p(i)}{1+v_p} - d(i) \qquad \forall i \in E, v = (v_1, \dots, v_r) \in \mathbb{R}_+^r, \tag{1.3}$$

where the first term models births which occur through the consumption of r resources, whose concentrations at time t are given by $c_p/(1+v_p)$ and with a trait-dependent consumption efficiency given by the function $\Psi_p(i)$. The second term corresponds to deaths at trait-dependent rate $d(i)$. This form of function R is relevant for populations of microorganisms in a chemostat, and has been studied for related models in lots of works [10–12, 19].

The parameter $\varepsilon > 0$ in (1.1) introduces a scaling (in the limit $\varepsilon \to 0$) of exponentially rare mutations at rate $e^{-\mathfrak{T}(i,j)/\varepsilon}$ and of strong selection in the coefficient $R(i, v_t^\varepsilon)/\varepsilon$. Similar scalings have been applied to various mutation-competition models in continuous trait space, for example, reaction-diffusion equations with nonlocal density-dependence such as

$$(1.4)\qquad \partial_t u^\varepsilon(t,x) = \frac{\varepsilon}{2}\Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon}u^\varepsilon(t,x)R\big(x, v_t^\varepsilon\big) \qquad \forall t>0, x\in\mathbb{R}^d,$$

where

$$v_t^{p,\varepsilon} = \int_{\mathbb{R}^d} \Psi_p(x)u^\varepsilon(t,x)\,dx, \qquad 1\le p\le r,$$

for some functions $\Psi_p : \mathbb{R}^d \to \mathbb{R}_+$.

Here, the parameter $\varepsilon > 0$ introduces a scaling of small or rare mutations and can be interpreted either as a large time scaling or a strong selection scaling. This parameter scaling has been used to study front propagation in standard (local) reaction-diffusion problems [2, 23, 24, 26] and was later introduced in models of adaptive dynamics (with nonlocal competition) in [19]. In this context, the qualitative outcome of this scaling is that solutions to (1.4) concentrate as Dirac masses, and this concentration is studied using the WKB ansatz

$$u^\varepsilon(t,x) = \exp\left(\frac{V^\varepsilon(t,x)}{\varepsilon}\right)$$

in [3, 4, 10, 19, 31, 34, 39] for different particular cases of (1.4) and also in [17, 27, 31, 40, 41] for models with competitive Lotka–Volterra competition.

Several of these works prove the convergence along a subsequence $(\varepsilon_k)_{k\ge1}$ converging to 0 of V^{ε_k} to a solution V of the Hamitlon–Jacobi problem

$$(1.5)\qquad \partial_t V(t,x) = R(x, v_t) + \frac{1}{2}\big|\nabla V(t,x)\big|^2,$$

where v_t is expected to take the form

$$v_t^p = \int_{\mathbb{R}^d} \Psi_p(x)\mu_t(dx),$$

where μ_t is some (measure, weak) limit of $u^{\varepsilon_k}(t,x)$. Due to the fact that, under general assumptions, the total mass of the population $\int_{\mathbb{R}^d} u^\varepsilon(t,x)\,dx$ is uniformly bounded and bounded away from 0, the function V satisfies the constraint $\sup_{x\in\mathbb{R}^d} V(t,x) = 0$ for all $t\ge 0$, and the measure μ_t is expected to have support in $\{V(t,\cdot)=0\}$. In addition, the measure μ_t is expected to be *metastable* in the sense that $R(x,v_t)\le 0$ for all x such that $V(t,x)=0$ and $R(x,v_t)=0$ for all x in the support of μ_t (to preserve the condition $\sup_{x\in\mathbb{R}^d} V(t,x)=0$). However, this is rigorously proved only in the case of a single resource ($r=1$) [31, 34, 39] or for a very specific model when $r\ge 2$ [10]. Recently, new works [28] study metastable behaviors for discrete systems similar to (1.1).

The study of the Hamilton–Jacobi problem (1.5) with (some or all of) the previous constraints is difficult. For example, uniqueness is only known in general in the case $r=1$ [35, 36] (see also [7, 39]). Yet, the case $r\ge 2$ is of particular biological interest since it is the only case where a phenomenon of diversification known as *evolutionary branching* [18, 33] can occur (see [11], Proposition 3.1).

In this work, our goal is to prove, using a probabilistic approach, the convergence of $V_\varepsilon(t,\cdot)$ to a limit $V(t,\cdot)$ for models (1.1) and (1.4), which is solution to a variational problem related to the Hamilton–Jacobi equation (1.5) or a discrete version of this equation.

For the model (1.1), our results take the following form. Under monotonicity assumptions on R (see Section 2.1) and under the assumption that, for all $A \subset E$, the dynamical system

$$\dot{u_i} = u_i R\Big(i; \sum_{j\in A} \Psi_p(j) u_j, 1 \le p \le r\Big), \qquad i \in A$$

admits a unique metastable steady state u^*_A (see Hypothesis (H) in Section 4.4), we prove (see Corollaries 4.14 and 4.12 and Theorem 4.13) that the family $(u^\varepsilon(t,i), t \ge 0, i \in E)_{\varepsilon>0}$ converges locally weakly to $(u_i(t), t \ge 0, i \in E)$ defined for all $i \in E$ and $t \ge 0$ by

$$u_i(t) = \begin{cases} u^*_{\{V(t,\cdot)=0\},i} & \text{if } V(t,i) = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

where $V(t,i)$ is the unique solution to

$$\begin{aligned} V(t,i) = \sup_{\varphi:[0,t]\to E \text{ càdlàg}, \varphi(0)=i} \Big\{ &-h(\varphi(t)) \\ &+ \int_0^t R\big(\varphi(u), F\big(\{V(t-u,\cdot)=0\}\big)\big)\, du - I_t(\varphi)\Big\} \end{aligned} \tag{1.7}$$

with

$$F(A) = \Big(\sum_{j\in E} \Psi_p(j) u^*_{A,j}\Big)_{1\le p\le r} \qquad \forall A \subset E,$$

such that $V(0,i) = -h(i)$ for all $i \in E$ and $t \mapsto F(\{V(t,\cdot)=0\})$ is right continuous, and where

$$I_t(\varphi) = \sum_{0<s\le t} \mathfrak{T}(\varphi_{s-}, \varphi_s),$$

with the convention $\mathfrak{T}(i,i) = 0$. This function $V(t,i)$ is also the unique solution such that $t \mapsto F(\{V(t,\cdot)=0\})$ is right continuous to the problem

$$\begin{aligned} &\dot V(t,i) = \sup\big\{R_j\big(F\big(\{V(t,\cdot)=0\}\big)\big) \mid j \in E, V(t,j) - \mathfrak{T}(j,i) = V(t,i)\big\}, \\ &V(0,i) = h(i) \qquad \forall i \in E. \end{aligned} \tag{1.8}$$

Our approach relies on a Feynman–Kac representation of the solution to (1.1), on a large deviations principle with rate function I_t when $\varepsilon \to 0$ for the Markov process whose generator is given by the mutation term in (1.1), and on the application of Varadhan's lemma to characterize the limit V of V^ε as the solution of the optimization problem (1.7). Uniqueness is obtained from the key property that one can characterize any accumulation point of v^ε_{t+s} for small $s>0$ only from the zeroes of $V(t,\cdot)$, which implies that $t\mapsto F(\{V(t,\cdot)=0\})$ is right continuous, and from a careful study of the problems (1.7) and (1.8). An important difficulty comes from the fact that the rate function I_t does not have compact level sets, and hence is *not a good* rate function, so that the proofs of the large deviations principle and of Varadhan's lemma are nonstandard and need some care.

The first part of this proof (Feynman–Kac representation, large deviations principle and Varadhan's lemma) does not rely on the specific structure of the mutation operator, so it applies to other models, including the partial-differential equation (1.4). Since it is (surprisingly) substantially simpler in this case, we first present our method for this model in Section 2. We are able to obtain in Theorem 2.7 the convergence of $\varepsilon\log u^\varepsilon$ along a subsequence to the solution of the variational problem associated to the Hamilton–Jacobi equation. In the case of classical reaction-diffusion equations, this stochastic approach is actually not new since it goes back to works of Freidlin [24, 26]. In [25], Freidlin also studied similar questions for models close to (1.4) (with different initial conditions), but only for a single ressource ($r=1$). This condition is also assumed in the more recent works [3, 31, 34, 39], but is not needed in our study.

We discuss in Section 3 consequences of the last results on the Hamilton–Jacobi problem (1.5) and possible extensions. In cases where the convergence to the Hamilton–Jacobi problem is known, we deduce as a side result the equality between the solution to the Hamilton–Jacobi problem and its variational formulation (see Section 3.1). Interestingly, this result does not seem to be covered by existing general results on this topic because of the possible discontinuities of the coefficients of the Hamilton–Jacobi problem. We also take care to avoid the use of precise properties of the heat semigroup, so that it is easy to extend our results to other mutation operators than the Laplace operator, as discussed in Section 3.2.

The extension to (1.1) is studied in details in Section 4. The nonstandard large deviations principle and Varadhan's lemma are proved in Section 4.3. The characterization of V as the unique solution to (1.7) such that $t\mapsto F(\{V(t,\cdot)=0\})$ is right continuous, its equality with the unique solution to the discrete version of the Hamilton–Jacobi problem (1.8) and the convergence of u^ε to (1.6) are proved in Section 4.4.

2. The Lotka–Volterra parabolic PDE with several resources. We first study in this section the reaction-diffusion model (1.4). We first state our assumptions and gather basic preliminary results in Section 2.1. We give the Feynman–Kac representation in Section 2.2. We finally study the limit of small ε and prove the main result of the section, Theorem 2.7, giving a variational characterization of the limit of $\varepsilon\log u^\varepsilon$ along appropriate subsequences, in Section 2.3.

2.1. *Problem statement and preliminary results.* We consider the following partial differential equation in $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \dfrac{\varepsilon}{2}\Delta u^\varepsilon(t,x) + \dfrac{1}{\varepsilon}u^\varepsilon(t,x)R^\varepsilon\big(x, v_t^\varepsilon\big) & \forall t > 0, x \in \mathbb{R}^d \\ u^\varepsilon(0,x) = \exp\Big(-\dfrac{h_\varepsilon(x)}{\varepsilon}\Big) & \forall x \in \mathbb{R}^d \end{cases} \tag{2.1}$$

with

$$v_t^\varepsilon = \big(v_t^{1,\varepsilon}, \dots, v_t^{r,\varepsilon}\big) \qquad \text{where } v_t^{p,\varepsilon} = \int_{\mathbb{R}^d} \Psi_p(x) u^\varepsilon(t,x)\,dx, 1 \le p \le r, \tag{2.2}$$

R^ε is a map from $\mathbb{R}^d \times \mathbb{R}^r$ to $\mathbb{R}$ and Ψ_p and h_ε are maps from $\mathbb{R}^d$ to $\mathbb{R}$.

Let us state our assumptions of R^ε, Ψ_p and h_ε.

1. *Assumptions on* Ψ_p

- There exist $\Psi_{\min}$ and $\Psi_{\max}$, two positive real numbers such that

$$\begin{gathered} \Psi_{\min} \le \Psi_p(x) \le \Psi_{\max} \qquad \forall x \in \mathbb{R}^d \quad \text{and} \\ \Psi_p \in W^{2,\infty}\big(\mathbb{R}^d\big) \qquad \forall 1 \le p \le r. \end{gathered} \tag{2.3}$$

2. *Assumptions on* R^ε

(a) When $\varepsilon \to 0$, R^ε converges in $L^\infty(\mathbb{R}^d \times \mathbb{R}^r)$ to a continuous function R from $\mathbb{R}^d \times \mathbb{R}^r$ to $\mathbb{R}$.

(b) There exists A a positive real number such that, for all $\varepsilon > 0$,

$$-A \le \partial_{v_p} R^\varepsilon(x, v_1, \dots, v_r) \le -A^{-1}$$
$$\forall p \in \{1, \dots, r\}, x \in \mathbb{R}^d, v_1, \dots, v_r \in \mathbb{R}.$$

(c) There exist two positive constants $v_{\min} < v_{\max}$ such that, for all $\varepsilon > 0$,

- if $\min_{1 \le p \le r} v_i > v_{\max}$, then $\max_{x \in \mathbb{R}^d} R^\varepsilon(x, v) < 0$,
- if $\max_{1 \le p \le r} v_i < v_{\min}$, then $\min_{x \in \mathbb{R}^d} R^\varepsilon(x, v) > 0$.

(d) Let $\mathcal{H}$ denotes the annulus defined by

$$\mathcal{H} := \left\{ v \in \mathbb{R}_+^r : \frac{\Psi_{\min} v_{\min}}{2\Psi_{\max}} \le \|v\|_1 \le \frac{2\Psi_{\max} v_{\max}}{\Psi_{\min}} \right\}.$$

There exists a positive constant M such that, for all $\varepsilon > 0$,

$$\sup_{v \in \mathcal{H}} \big\| R^\varepsilon(\cdot, v) \big\|_{W^{1,\infty}} < M.$$

Note that the constant 2 in the definition of $\mathcal{H}$ could be replaced by any constant strictly larger than 1.

3. *Assumptions on* h_ε

(a) h_ε is Lipschitz-continuous on $\mathbb{R}^d$, uniformly with respect to $\varepsilon > 0$.

(b) h_ε converges in $L^\infty(\mathbb{R}^d)$ as $\varepsilon \to 0$ to a function h.
(c) For all $\varepsilon > 0$ and all $1 \le p \le r$,

$$v_{\min} \le \int_{\mathbb{R}^d} \Psi_p(x) \exp\left(-\frac{h_\varepsilon(x)}{\varepsilon}\right) dx \le v_{\max}.$$

In particular, $u^\varepsilon(0, x)$ is bounded in $L^1(\mathbb{R}^d)$.

Note that the limit h of h_ε is continuous, and hence, in order to satisfy Assumption (3c), it must satisfy $h(x) \ge 0$ for all $x \in \mathbb{R}^d$.

This type of assumptions is standard in this domain [3, 31, 36, 39], but the previous references only studied the case $r = 1$. These references also assume bounds on second-order derivatives of R^ε and h_ε and linear or quadratic bounds at infinity for these functions (see Corollary 3.1 in Section 3.1). The case $r \ge 2$ was only studied in [10], but for a very specific form of the function R, and in [39] but without the convergence to the Hamilton–Jacobi problem.

Now, we present some preliminary results which are needed to study the asymptotic behavior of the solution of (2.1). The first result, Proposition 2.1, gives preliminary estimates on the solution of (2.1). The second one, Theorem 2.2, provides the existence and uniqueness of the solution of the equation. These two results are direct adaptations of the results of [3, 31, 39] and [3], so we omit their proofs.

PROPOSITION 2.1 (A priori estimates). *Suppose that there exists a nonnegative weak solution u^ε in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$. We have, for all positive time t and all $p \in \{1, \dots, r\}$,*

$$\frac{\Psi_{\min}}{\Psi_{\max}} v_{\min} - \frac{A\varepsilon^2 \Psi_{\min}}{\|\Psi_p\|_{W^{2,\infty}}} \le v_t^{p,\varepsilon} \le \frac{\Psi_{\max}}{\Psi_{\min}} v_{\max} + \frac{A\varepsilon^2 \Psi_{\min}}{\|\Psi_p\|_{W^{2,\infty}}},$$

$$\begin{aligned} &\frac{\Psi_{\min}}{\Psi_{\max}} v_{\min} - \frac{A\varepsilon^2 \Psi_{\min}}{\inf_{1\le p\le r} \|\Psi_p\|_{W^{2,\infty}}} \\ &\quad \le \|v_t^\varepsilon\|_1 \le \frac{\Psi_{\max}}{\Psi_{\min}} v_{\max} + \frac{A\varepsilon^2 \Psi_{\min}}{\inf_{1\le p\le r} \|\Psi_p\|_{W^{2,\infty}}} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\Psi_{\max}^{-1}\left(\frac{\Psi_{\min}}{\Psi_{\max}} v_{\min} - \frac{A\varepsilon^2 \Psi_{\min}}{\inf_{1\le p\le r} \|\Psi_p\|_{W^{2,\infty}}}\right) \\ &\quad \le \int_{\mathbb{R}^d} u^\varepsilon(t, x)\, dx \\ &\quad \le \left(\frac{\Psi_{\max}}{\Psi_{\min}} v_{\max} + \frac{A\varepsilon^2 \Psi_{\min}}{\inf_{1\le p\le r} \|\Psi_p\|_{W^{2,\infty}}}\right) \Psi_{\min}^{-1}. \end{aligned} \tag{2.5}$$

THEOREM 2.2. *For $\varepsilon > 0$ small enough, there exists a unique nonnegative solution u^ε in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ of (2.1).*

Since $R^\varepsilon(x, v_t^\varepsilon)$ is a Lipschitz function, one can actually get higher regularity from the regularizing effect of the Laplace operator. However, since we plan to extend our method to more general mutation operators, we shall only make use in the sequel of the fact that $u^\varepsilon \in C(\mathbb{R}_+, L^1(\mathbb{R}^d))$.

2.2. *Feynman–Kac representation of the solution*. The purpose of this section is to prove the following integral representation of the solution of (2.1).

THEOREM 2.3 (Feynman–Kac representation of the solution of (2.1)). *Let u^ε be the unique weak solution of* (2.1), *then*

$$u^\varepsilon(t,x) = \mathbb{E}_x\left[\exp\left(-\frac{h_\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\int_0^t R^\varepsilon\big(X_t^\varepsilon, v_{t-s}^\varepsilon\big)\,ds\right)\right]$$
$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{2.6}$$

where for all $x \in \mathbb{R}^d$, $\mathbb{E}_x$ is the expectation associated to the probability measure $\mathbb{P}_x$, under which $X_0^\varepsilon = x$ almost surely and the process $B_t = (X_t^\varepsilon - x)/\sqrt{\varepsilon}$ is a standard Brownian motion in $\mathbb{R}^d$.

Such results are classical but, for generalization purposes, we want to give a proof making only use of the existence of a solution $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ to (2.1). Let us first recall usual notions of weak solutions to (2.1) (cf., e.g., [20, 32, 37, 38]). We say that a function u in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ is a mild solution of problem (2.1) if it satisfies the following integral equation:

$$u(t,x) = P_t^\varepsilon g_\varepsilon(x) + \frac{1}{\varepsilon}\int_0^t P_{t-s}^\varepsilon\big(u(s,x)R^\varepsilon\big(x, v_s^\varepsilon\big)\big)\,ds, \tag{2.7}$$

where $g_\varepsilon(x) = \exp(-h_\varepsilon(x)/\varepsilon)$ and $(P_t^\varepsilon)_{t\in\mathbb{R}_+}$ is the standard heat semigroup defined on $L^1(\mathbb{R}^d)$ by

$$P_t^\varepsilon f(x) = \int_{\mathbb{R}^d} \frac{1}{(2\pi t\varepsilon)^{n/2}} e^{\frac{-\|x-y\|^2}{2t\varepsilon}} f(y)\,dy \qquad \forall f \in L^1\big(\mathbb{R}^d\big).$$

We also say that a function u in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ is a weak solution of problem (2.1) if for any compactly supported test function φ of $C^\infty([0,\infty) \times \mathbb{R}^d)$, we have

$$\begin{aligned}&\int_{\mathbb{R}_+\times\mathbb{R}^d} u(t,x)\left(-\partial_t\varphi(t,x) - \frac{\varepsilon}{2}\Delta\varphi(t,x)\right)dx\,dt \\ &\quad = \frac{1}{\varepsilon}\int_{\mathbb{R}_+\times\mathbb{R}^d} u(t,x)R^\varepsilon\big(x, v_t^\varepsilon\big)\varphi(t,x)\,dx\,dt + \int_{\mathbb{R}^d} g_\varepsilon(x)\varphi(0,x)\,dx.\end{aligned}$$

We point out that this notion of weak solution is the one for which existence and uniqueness hold in Theorem 2.2.

LEMMA 2.4. *Let $\bar{u}^\varepsilon$ be defined as the right-hand side of* (2.6), *where v^ε is defined by* (2.2) *and u^ε is the solution given in Theorem* 2.2. *The function $\bar{u}^\varepsilon$ belongs to $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ and is a mild solution of problem* (2.1).

PROOF. Let t and h be two positive real numbers:

$$\begin{aligned}
&\|\bar{u}^\varepsilon(t+h,\cdot)-\bar{u}^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d)}\\
&\quad\le \int_{\mathbb{R}^d}\mathbb{E}_x\biggl[\exp\biggl(\frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t+h-s}^\varepsilon)\,ds\biggr)\\
&\qquad\times\biggl|\exp\biggl(\frac{1}{\varepsilon}\int_0^h R^\varepsilon(X_{t+s}^\varepsilon, v_{h-s}^\varepsilon)\,ds\biggr)g_\varepsilon(X_{t+h}^\varepsilon)-g_\varepsilon(X_t^\varepsilon)\biggr|\biggr]\,dx\\
&\qquad+\int_{\mathbb{R}^d}\mathbb{E}_x\biggl[g_\varepsilon(X_t^\varepsilon)\biggl|\exp\biggl(\frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t+h-s}^\varepsilon)\,ds\biggr)\\
&\qquad-\exp\biggl(\frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\biggr)\biggr|\biggr]\,dx.
\end{aligned}$$

Using Hypothesis (2d), we get

$$\begin{aligned}
&\|\bar{u}^\varepsilon(t+h,\cdot)-\bar{u}^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d)}\\
&\quad\le\int_{\mathbb{R}^d}e^{\frac{Mt}{\varepsilon}}\{\mathbb{E}_x[g_\varepsilon(X_{t+h}^\varepsilon)(e^{\frac{Mh}{\varepsilon}}-1)]+\mathbb{E}_x[|g_\varepsilon(X_{t+h}^\varepsilon)-g_\varepsilon(X_t^\varepsilon)|]\}\,dx\\
&\qquad+\frac{A}{\varepsilon}e^{\frac{Mt}{\varepsilon}}\int_{\mathbb{R}^d}\mathbb{E}_x[g_\varepsilon(X_t^\varepsilon)]\int_0^t|v_{t+h-s}^\varepsilon-v_{t-s}^\varepsilon|\,ds\,dx.
\end{aligned}$$

Since $(P_t)_{t\in\mathbb{R}_+}$ preserves the $L^1(\mathbb{R}^d)$ norm, we obtain

$$\begin{aligned}
\|\bar{u}^\varepsilon(t+h,\cdot)-\bar{u}^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d)}&\le\|g_\varepsilon\|_{L^1(\mathbb{R}^d)}e^{\frac{Mt}{\varepsilon}}\biggl(e^{\frac{Mh}{\varepsilon}}-1+\frac{A}{\varepsilon}\int_0^t|v_{s+h}^\varepsilon-v_s^\varepsilon|\,ds\biggr)\\
&\quad+e^{\frac{Mt}{\varepsilon}}\mathbb{E}\biggl[\int_{\mathbb{R}^d}|P_h^\varepsilon g_\varepsilon(x+X_t^\varepsilon)-g_\varepsilon(x+X_t^\varepsilon)|\,dx\biggr].
\end{aligned}$$

Since v^ε is continuous, this finally leads to

$$\|\bar{u}^\varepsilon(t+h,\cdot)-\bar{u}^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d)}\le o_h(1)+e^{\frac{Mt}{\varepsilon}}\|P_h^\varepsilon g_\varepsilon-g_\varepsilon\|_{L^1(\mathbb{R}^d)}\xrightarrow[h\to 0]{}0,$$

which proves that $\bar{u}^\varepsilon$ is in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$.

We now prove that $\bar{u}^\varepsilon$ is a mild solution of problem (2.1). First, Markov property gives

$$\begin{aligned}
&\mathbb{E}_x\biggl[g_\varepsilon(X_t^\varepsilon)\exp\biggl(\frac{1}{\varepsilon}\int_{t-s}^t R^\varepsilon(X_\theta^\varepsilon, v_{t-\theta}^\varepsilon)\,d\theta\biggr)R^\varepsilon(X_{t-s}^\varepsilon, v_s^\varepsilon)\biggr]\\
&\quad=\mathbb{E}_x\biggl[\mathbb{E}_{X_{t-s}^\varepsilon}\biggl[g_\varepsilon(X_s^\varepsilon)\exp\biggl(\frac{1}{\varepsilon}\int_0^s R^\varepsilon(X_\theta^\varepsilon, v_{s-\theta}^\varepsilon)\,d\theta\biggr)\biggr]R^\varepsilon(X_{t-s}^\varepsilon, v_s^\varepsilon)\biggr]\\
&\quad=\mathbb{E}_x[\bar{u}(s, X_{t-s}^\varepsilon)R^\varepsilon(X_{t-s}^\varepsilon, v_s^\varepsilon)].
\end{aligned}$$

Using the fact that

$$\exp\left(\frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\right) - 1$$
$$= \frac{1}{\varepsilon}\int_0^t \exp\left(\frac{1}{\varepsilon}\int_{t-s}^t R^\varepsilon(X_u^\varepsilon, v_{t-u}^\varepsilon)\,du\right) R^\varepsilon(X_{t-s}^\varepsilon, v_s^\varepsilon)\,ds,$$

we deduce that

$$\bar{u}^\varepsilon(t,x) = \mathbb{E}_x\left[g_\varepsilon(X_t^\varepsilon)\right] + \frac{1}{\varepsilon}\int_0^t \mathbb{E}_x\left[\bar{u}^\varepsilon(s, X_{t-s}^\varepsilon) R^\varepsilon(X_{t-s}^\varepsilon, v_s^\varepsilon)\right] ds$$
$$= P_t^\varepsilon g_\varepsilon(x) + \int_0^t P_{t-s}^\varepsilon\left(\bar{u}^\varepsilon(s,x) R^\varepsilon(x, v_s^\varepsilon)\right) ds.$$

This completes the proof that $\bar{u}^\varepsilon$ is a mild solution of problem (2.1). □

LEMMA 2.5. *Let $u \in C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ be a mild solution of problem* (2.1), *then u is a weak solution of problem* (2.1).

PROOF. Assume that u is the weak solution of problem (2.1) given by Theorem 2.2, and consider the following Cauchy problem:

$$\begin{cases} \partial_t w = \dfrac{\varepsilon}{2}\Delta w + \dfrac{1}{\varepsilon} u(t,x) R^\varepsilon(x, v_t^\varepsilon), \\ w(0,x) = \exp\left(-\dfrac{h_\varepsilon(x)}{\varepsilon}\right) =: g_\varepsilon(x). \end{cases} \tag{2.8}$$

Since the inhomogeneous term belongs to $L^1(\mathbb{R}^d)$, it is well known that this problem admits a unique weak solution given by Duhamel's formula:

$$w(t,x) = P_t^\varepsilon g_\varepsilon(x) + \frac{1}{\varepsilon}\int_0^t P_{t-s}^\varepsilon\left[u(t,x) R^\varepsilon(x, v_s^\varepsilon)\right] ds.$$

Because of uniqueness for (2.8), $w = u$ so u is a mild solution to (2.1). Uniqueness of such a mild solution in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ follows easily from the fact that $(P_t^\varepsilon)_{t\in\mathbb{R}_+}$ is a contraction semigroup in $L^1(\mathbb{R}^d)$ and from Gronwall's lemma (see [32, 37]), so the lemma is proved. □

2.3. *Small diffusion asymptotic*. Recall from Theorem 2.3 that

$$u^\varepsilon(t,x) = \mathbb{E}_x\left[\exp\left(-\varepsilon^{-1} h(X_t^\varepsilon) + \frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\right)\right], \tag{2.9}$$

where $X_t^\varepsilon = x + \sqrt{\varepsilon} B_t$ under $\mathbb{P}_x$. This formula suggests to apply Varadhan's lemma to study the convergence of $\varepsilon \log u^\varepsilon(t,x)$ as $\varepsilon \to 0$. Let us fix $t > 0$.

LEMMA 2.6. *The function* $\Phi_\varepsilon : C([0,t]) \to \mathbb{R}$ *defined by*

$$\Phi_\varepsilon(\varphi) = \int_0^t R^\varepsilon\big(\varphi_s, v_s^\varepsilon\big)\, ds$$

is Lipschitz continuous on $C([0,t])$ *endowed with the* L^∞*-norm, uniformly w.r.t. to* ε *for* ε *small enough.*

Moreover, there exists a kernel $\mathcal{M}$ *on* $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^k)$ *such that, along a subsequence* $(\varepsilon_k)_{k\geq 1}$ *converging to* 0, *we have*

$$\Phi(\varphi) := \lim_{k\to\infty} \Phi_{\varepsilon_k}(\varphi) = \int_0^t \int_{\mathbb{R}^k} R(\varphi_s, y)\mathcal{M}_s(dy)\, ds \qquad \forall \varphi \in C\big([0,t]\big).$$

We recall that a kernel $\mathcal{M}$ *is a function from* $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^d)$ *into* $\mathbb{R}_+$ *such that, for all* $t \in \mathbb{R}_+$, $\mathcal{M}_s$ *is a measure on* $\mathcal{B}(\mathbb{R}^d)$ *and, for all* $A \in \mathcal{B}(\mathbb{R}^d)$, *the function* $s \to \mathcal{M}_s(A)$ *is measurable.*

PROOF. We begin by showing that $\Phi_\varepsilon(\varphi)$ is continuous for all ε, uniformly w.r.t. ε. Since $R^\varepsilon(\cdot, v)$ lies in $W^{1,\infty}$, it follows that it is Lipschitz continuous, uniformly for v is the annulus $\mathcal{H}$ (see Assumption (2d)). Hence, for $\psi, \varphi \in C([0,t])$, it follows from Morrey's inequality that

$$\begin{aligned}\int_0^t \big|R^\varepsilon\big(\varphi_s, v_s^\varepsilon\big) - R^\varepsilon\big(\psi_s, v_s^\varepsilon\big)\big|\, ds &\leq \int_0^t \big\|R^\varepsilon\big(\cdot, v_s^\varepsilon\big)\big\|_{W^{1,\infty}} \|\varphi_s - \psi_s\|\, ds \\ &\leq t \sup_{s\in[0,t]} \big\|R^\varepsilon\big(\cdot, v_s^\varepsilon\big)\big\|_{W^{1,\infty}} \|\varphi - \psi\|_{L^\infty([0,t])}.\end{aligned}$$

Hence, the result follows from (2.4).

Now, fix $T > 0$, and let Γ_T^ε, for all ε, be the measure defined on $([0,T] \times \mathcal{H}, \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathcal{H}))$ by

$$\Gamma_T^\varepsilon(A \times B) = \int_A \mathbb{1}_{v_s^\varepsilon \in B}\, ds \qquad \forall A \in \mathcal{B}\big([0,T]\big), B \in \mathcal{B}(\mathcal{H}).$$

Since $(\Gamma_T^\varepsilon)_{\varepsilon>0}$ is a family of finite measures defined on a compact metric space, it is weakly precompact. Hence, there exists a subsequence $(\varepsilon_k^T)_{k\geq 1}$ such that, the sequence of measures $(\Gamma_T^{\varepsilon_k^T})_{k\geq 1}$ converges weakly to some measure denoted by Γ_T. It follows from a diagonal argument that there exists a sequence $(\varepsilon_k)_{k\geq 1}$ such that, for any positive integer n, $\Gamma_n^{\varepsilon_k}$ converges weakly to a measure Γ_n. Now, if one wants to define a measure on $\mathbb{R}_+$ using the family $(\Gamma_n)_{n\geq 1}$, he needs that for $m < n$, the restriction of Γ_n on $[0,m]$ coincides with Γ_m.

To prove this we first remark that, for all $k \in \mathbb{N}$ and $\delta > 0$,

$$\Gamma_n^{\varepsilon_k}\big((m-\delta, m+\delta) \times \mathcal{H}\big) = \int_{m-\delta}^{m+\delta} \mathbb{1}_{v_s^{\varepsilon_k} \in \mathcal{H}}\, ds = 2\delta. \tag{2.10}$$

Let f be a bounded and continuous function on $[0,m] \times \mathcal{H}$ and $(f_\ell)_{\ell\geq 1}$ a sequence of uniformly bounded continuous function on $[0,n] \times \mathcal{H}$ such that

$$f_\ell(t,x) \xrightarrow[\ell\to\infty]{} f(t,x)\mathbb{1}_{t\in[0,m]} \qquad \forall (t,x) \in [0,n] \times \mathcal{H},$$

and $f_\ell = f$ on $[0, m-\delta) \times \mathcal{H}$ and $f_l = 0$ on $(m+\delta, n] \times \mathcal{H}$. Using (2.10) we have, for all $k, \ell \in \mathbb{N}$,

$$\left|\Gamma_m^{\varepsilon_k}(f) - \Gamma_n^{\varepsilon_k}(f_\ell)\right| \le c\delta,$$

where the constant c does not depend on k and ℓ and $\mu(f)$ is the integral of a function f w.r.t. a measure μ on the corresponding space. Since, in addition, $\Gamma_m^{\varepsilon_k}(f)$ converges to $\Gamma_m(f)$, $\Gamma_n^{\varepsilon_k}(f_\ell)$ converges to $\Gamma_n(f_\ell)$ when k goes to infinity and, by Lebesgue's theorem, $\Gamma_n(f_\ell)$ converges to $\Gamma_n(f)$ when ℓ goes to infinity, we deduce

$$\left|\Gamma_m(f) - \Gamma_n(f)\right| \le c\delta.$$

Since δ was arbitrary, we have proved that the restriction of Γ_n on $[0, m]$ coincides with Γ_m. As a consequence, we can define on $\mathbb{R}_+ \times \mathcal{H}$ a measure Γ whose restrictions on $[0, n] \times \mathcal{H}$ is Γ_n for all $n \in \mathbb{N}$.

In particular, for all t, $\Gamma_t^{\varepsilon_k}$ converges weakly to the restriction of Γ to $[0, t] \times \mathcal{H}$. In addition, for all function φ continuous on $[0, t]$, the map $(s, x) \mapsto R(\varphi_s, x)$ is continuous, and hence, for all $t \le T$,

$$\begin{aligned}\Phi(\varphi) &:= \lim_{k\to\infty} \int_0^t R^{\varepsilon_k}\big(\varphi_s, v_s^{\varepsilon_k}\big)\, ds = \lim_{k\to\infty} \int_0^t R\big(\varphi_s, v_s^{\varepsilon_k}\big)\, ds \\ &= \int_{[0,t]\times\mathbb{R}^d} R(\varphi_s, x)\Gamma(ds, dx).\end{aligned}$$

It remains to show that Γ can be disintegrated along $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda)$. Let A be a null set of $[0, T]$, for a fixed positive time T. It follows that there exists, for all $\eta > 0$, a denumerable family of open ball $(B_n^\eta)_{n\ge1}$ such that

$$A \subset \bigcup_{n=1}^{\infty} B_n^\eta \quad \text{and} \quad \lambda\left(\bigcup_{n=1}^{\infty} B_n^\eta\right) < \eta.$$

Let H be a measurable set of $\mathcal{H}$, then

$$\Gamma(A \times H) \le \Gamma_T\left(\bigcup_{n\ge1} B_n^\eta \times \mathcal{H}\right) \le \liminf_{k\to\infty} \Gamma_T^{\varepsilon_k}\left(\bigcup_{n=1}^{\infty} B_n^\eta \times \mathcal{H}\right) < C\eta$$

which implies that $\Gamma(A \times H) = 0$. According to Radon–Nikodym's theorem, there exists, for all H, an integrable function $s \to \mathcal{M}_s(H)$ such that

$$\Gamma(A \times H) = \int_{\mathbb{R}_+} \mathcal{M}_s(H)\, ds.$$

Usual theory (cf., e.g., [6]) ensures that there exists a modification of this map such that $H \in \mathcal{B}(\mathcal{H}) \to \mathcal{M}_s(H)$ is a measure for almost all $s \in \mathbb{R}_+$. $\square$

Note that (2.9) takes the form $u^\varepsilon(t, x) = \mathbb{E}_x[\exp \frac{1}{\varepsilon} F_\varepsilon(X^\varepsilon)]$ for functions F^ε on $C([0, t], \mathbb{R}^d)$ which are uniformly Lipschitz for the L^∞ norm and converge pointwise to a function F because of Lemma 2.6. Therefore, the following result is a classical extension of the proof of Varadhan's lemma (cf., e.g., [16]). We omit its proof.

THEOREM 2.7. *For all* (t,x) *in* $\mathbb{R}_+ \times \mathbb{R}^d$,

$$V(t,x) := \lim_{k\to\infty} \varepsilon_k \log u^{\varepsilon_k}(t,x) = \sup_{\varphi\in\mathcal{G}_{t,x}} \{-h(\varphi_0) + \Phi(\varphi) - I_t(\varphi)\} \tag{2.11}$$

with

$$I_t(\varphi) = \begin{cases} \int_0^t \|\varphi'(s)\|^2\, ds & \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise,} \end{cases}$$

$\mathcal{G}_{t,x}$ *denotes the set of continuous functions from* $[0,t]$ *to* $\mathbb{R}^d$ *such that* $\varphi_t = x$, *and* Φ *and* $(\varepsilon_k)_{k\geq 1}$ *are defined in Lemma* 2.6.

Thanks to the representation (2.9), the convergence given above can be enhanced.

THEOREM 2.8. *The convergence stated in Theorem* 2.7 *holds uniformly on compact sets and the limit* $V(t,x)$ *is Lipschitz w.r.t.* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

This result is an immediate consequence of the following lemma.

LEMMA 2.9. *The function*

$$\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R},$$
$$(t,x) \mapsto \varepsilon \log u^\varepsilon(t,x),$$

is Lipschitz w.r.t. $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *uniformly w.r.t.* ε (*at least for* ε *small enough*).

PROOF. Let (t,x) and (δ,y) in $\mathbb{R}_+ \times \mathbb{R}^d$. Using (2.9), and writing $X_t^\varepsilon = X_0^\varepsilon + \sqrt{\varepsilon} B_t$ for a Brownian motion B, we get

$$u^\varepsilon(t+\delta, x+y) = \mathbb{E}\biggl[\exp\biggl(-\frac{h_\varepsilon(x+y+\sqrt{\varepsilon}B_{t+\delta})}{\varepsilon} + \frac{1}{\varepsilon}\int_0^{t+\delta} R^\varepsilon\bigl(x+y+\sqrt{\varepsilon}B_s, v^\varepsilon_{t+\delta-s}\bigr)\, ds\biggr)\biggr].$$

Now, Markov's property entails

$$u^\varepsilon(t+\delta, x+y) = \mathbb{E}\biggl[\exp\biggl(\frac{1}{\varepsilon}\int_0^{\delta} R^\varepsilon\bigl(x+y+\sqrt{\varepsilon}B_s, v^\varepsilon_{t+\delta-s}\bigr)\, ds\biggr) \times u^\varepsilon(t, x+y+\sqrt{\varepsilon}B_\delta)\biggr].$$

Hence, using Assumption (2d), we get

$$
\begin{aligned}
e^{-\frac{M\delta}{\varepsilon}}\mathbb{E}\big[u^\varepsilon(t, x+y+\sqrt{\varepsilon}B_\delta)\big] &\le u^\varepsilon(t+\delta, x+y) \\
&\le e^{\frac{M\delta}{\varepsilon}}\mathbb{E}\big[u^\varepsilon(t, x+y+\sqrt{\varepsilon}B_\delta)\big]. \qquad (2.12)
\end{aligned}
$$

In addition, taking into account that $x \to R^\varepsilon(x, v)$ is Lipschitz uniformly w.r.t. $v \in \mathcal{H}$ with Lipstchiz norm bounded by M, we have

$$
\begin{aligned}
u^\varepsilon(t, x+y) \le \mathbb{E}\bigg[&\exp\bigg(-\frac{h_\varepsilon(x+\sqrt{\varepsilon}B_t)}{\varepsilon} + \frac{1}{\varepsilon}\int_0^t R^\varepsilon\big(x+\sqrt{\varepsilon}B_s, v^\varepsilon_{t-s}\big)\,ds\bigg) \\
&\times \exp\bigg(-\frac{h_\varepsilon(x+y+\sqrt{\varepsilon}B_t) - h_\varepsilon(x+\sqrt{\varepsilon}B_t)}{\varepsilon} + \frac{tM\|y\|}{\varepsilon}\bigg)\bigg].
\end{aligned}
$$

Since h_ε is Lipstchiz uniformly w.r.t. $\varepsilon > 0$,

$$
u^\varepsilon(t, x+y) \le u^\varepsilon(t, x)e^{\frac{tM+\sup_\varepsilon \|h_\varepsilon\|_{\mathrm{Lip}}}{\varepsilon}\|y\|}.
$$

Now, using (2.12), we get

$$
u^\varepsilon(t+\delta, x+y) \le u^\varepsilon(t, x)e^{\frac{M\delta}{\varepsilon}}\mathbb{E}\big[e^{\frac{tM+\sup_\varepsilon \|h_\varepsilon\|_{\mathrm{Lip}}}{\varepsilon}\|y+\sqrt{\varepsilon}B_\delta\|}\big].
$$

Then, using the inequality $\|y+\sqrt{\varepsilon}B_\delta\| \le \|y\| + C\sqrt{\varepsilon}\sum_i |B^i_\delta|$, we obtain

$$
u^\varepsilon(t+\delta, x+y) \le u^\varepsilon(t, x)e^{\frac{M\delta}{\varepsilon}}e^{\frac{tM+\sup_\varepsilon \|h_\varepsilon\|_{\mathrm{Lip}}}{\varepsilon}(\|y\|+dC\delta)}.
$$

Hence,

$$
\varepsilon \log u^\varepsilon(t+\delta, x+y) - \varepsilon \log u^\varepsilon(t, x) \le M\delta + \Big(tM + \sup_\varepsilon \|h_\varepsilon\|_{\mathrm{Lip}}\Big)\big(\|y\| + dC\delta\big).
$$

The lower bound is obtained similarly and completes the proof. $\square$

In the last results, one would like to be able to characterize the limit measure $\mathcal{M}_s$ in terms of the zeroes of $V(s, \cdot)$, in order to obtain a closed form of the optimization problem. Results on this question are known in models with a single resource ($r = 1$) [3, 31, 39], but the known results when $r \ge 2$ [9, 10] require stringent assumptions on the structure of the model, and indeed, it is possible to construct examples where there exist several measures $\mathcal{M}_s$ satisfying the metastability condition of [10]. Since our assumptions are more general, we cannot expect to obtain such results in full generality, so we will focus in Sections 4.3 and 4.4 on the finite case, where precise results can be obtained, granting uniqueness for the optimization problem (2.11).

3. Extensions and links between the variational and Hamiltom–Jacobi problems. In cases where the convergence to the Hamilton–Jacobi problem is known, Theorem 2.7 gives as a side result the equality between the solution to the Hamilton–Jacobi problem and its variational formulation. This is discussed in Section 3.1. We also study in Section 3.2 extensions of our results to other mutation operators than the Laplace operator.

3.1. *Links between the variational and Hamiltom–Jacobi problems*. The convergence of $\varepsilon \log u^\varepsilon$ has been studied in various works using PDE approaches, with a limit solving a Hamilton–Jacobi problem with constraints. With our approach, instead of a Hamilton–Jacobi equation, we obtain a variational characterization of the limit, under assumptions on R weaker than those of [3, 31, 39] and valid for any value of the parameter r, biologically interpreted as a number of resources. Therefore, we obtain naturally the identification between a solution to Hamilton–Jacobi problems and a variational problem. For example, the next result is a consequence of [31], equation (5.8).

COROLLARY 3.1. *In addition to the assumptions of Section 2.1, assume that* $r = 1$, $R^\varepsilon = R$ *does not depend on* ε, h_ε *is* C^2 *uniformly in* $\varepsilon > 0$ *and there exist constants* A, B, C, D *such that, for all* $x \in \mathbb{R}^d$, $v_{\min} \le \|v\|_1 \le v_{\max}$ *and* $\varepsilon > 0$,

$$-A|x|^2 \le R(x,v) \le B - A^{-1}|x|^2, \qquad -B + A^{-1}|x|^2 \le h_\varepsilon(x) \le B + A|x|^2,$$

$$-C\mathrm{Id} \le D^2 R(x,v) \le -C^{-1}\mathrm{Id}, \qquad C^{-1}\mathrm{Id} \le D^2 h_\varepsilon(x) \le C\mathrm{Id},$$

$$\Delta(\Psi_1 R) \ge -D,$$

where Id *is the* d*-dimensional identity matrix. Then,* v^{ε_k} *converges in* $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *to a nondecreasing limit* $\bar v$ *along the subsequence* ε_k *of Lemma 2.6, the kernel* $\mathcal{M}$ *satisfies*

$$\mathcal{M}_s(dy) = \delta_{\bar v(s)}(dy) \qquad \forall s \ge 0, \tag{3.1}$$

and the limit V *of Theorem 2.7 solves in the viscosity sense*

$$\begin{cases} \partial_t V(t,x) = R(x,\bar v_t) + \dfrac{1}{2}\left|\nabla V(t,x)\right| & \forall t \ge 0, x \in \mathbb{R}^d, \\ \max\limits_{x\in\mathbb{R}^d} V(t,x) = 0 & \forall t \ge 0. \end{cases} \tag{3.2}$$

REMARK 3.2. Note that Hamilton–Jacobi equations are known to be related to variational problems appearing in control theory. However, this link is only known in general in cases with continuous and time-independent coefficients in the Hamilton–Jacobi equation [22, 29]. In our case, $\bar v$ may be discontinuous. Several references study Hamilton–Jacobi or variational problems with discontinuous coefficients [1, 30] but none covers our case: irregular Lagrangian are studied, for example, in [14], but the link with Hamilton–Jacobi problems is not studied; measurable in time Hamiltonians are studied, for example, in [42], but without the integral term in the optimization problem.

Other mutation operators were also studied with the PDE approach. Up to our knowledge, the only work providing the Hamilton–Jacobi limit with several resources is [10]. It is also possible to deduce from our results a result similar to Corollary 3.1 for this model. Note that this model assumes an integral mutation operator, so we need to use the extension developed in the next subsection.

3.2. *Extensions of Theorem* 2.7. The proof of Theorem 2.7 only makes use of few properties of the Brownian motion and of the Laplace operator used to model mutations in (2.1). In particular, one expects that it may hold true for partial differential equations of the form

$$\partial_t u^\varepsilon(t,x) = L_\varepsilon u^\varepsilon(t,x) + \frac{1}{\varepsilon} u^\varepsilon(t,x) R\big(x, v_t^\varepsilon\big) \qquad \forall t > 0, x \in \mathbb{R}^d, \tag{3.3}$$

where L_ε is a linear operator describing mutations and where we assumed that R does not depend on ε for simplicity.

For our approach to work in this situation, the probabilistic interpretation of Theorem 2.3 must extend to this case, and Varadhan's lemma must be applied as in the proof of Theorem 2.7. For this, one needs that:

1. the operator L_ε is the infinitesimal generator of a Markov process $(X_t^\varepsilon, t \geq 0)$;
2. existence and uniqueness of a weak solution hold for the partial differential equation (3.3) in an appropriate functional space, for example, in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$, and any $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ mild solution to the PDE must be a weak solution;
3. the function

$$\bar{u}^\varepsilon(t,x) = \mathbb{E}_x\left[\exp\left(-\frac{h_\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\int_0^t R\big(X_s^\varepsilon, v_{t-s}^\varepsilon\big)\, ds\right)\right] \tag{3.4}$$

   can be shown to be $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ (note that the proof that $\bar{u}$ is a mild solution to the PDE only relies on the Markov property, so it is true in general);
4. the family of Markov processes $(X^\varepsilon)_{\varepsilon>0}$ must satisfy a large deviations principle with rate ε^{-1} and a good rate function.

Note that the compactness argument of Lemma 2.6 does not depend on the mutation operator. It only follows from our assumptions on R.

For example, all these points apply to the problem

$$\partial_t u^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \big[u^\varepsilon(t, x+\varepsilon z) - u^\varepsilon(t,x)\big] K(z)\, dz + \frac{1}{\varepsilon} u^\varepsilon(t,x) R\big(x, v_t^\varepsilon\big), \tag{3.5}$$

where $K : \mathbb{R}^d \to \mathbb{R}_+$ is such that

$$\int_{\mathbb{R}^d} zK(z)\, dz = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} e^{\langle a,z\rangle^2} K(z)\, dz < \infty \qquad \text{for all } a \in \mathbb{R}^d.$$

This form of mutation operator has already been studied in [3, 4, 10]. Similar equations are also considered in [17, 27, 41]. In this case, we can check all the points above as follows:

1. The Markov process X^ε is a continuous-time random walk, with jump rate $\frac{\|K\|_{L^1}}{\varepsilon}$ and i.i.d. jump steps distributed as εZ, where the random variable Z has law $\frac{K(z)}{\|K\|_{L^1}}\, dz$.

2. Existence and uniqueness of the solution to (3.5) in $C(\mathbb{R}_+, L^1(\mathbb{R}^d))$ follows from [3] (this is the point where the finiteness of quadratic exponential moments of K is needed).
3. Since random walks are shift-invariant as Brownian motion, the regularity of (3.4) can be proved exactly as in the proof of Theorem 2.3.
4. The family $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle (see [21], Section 10.3) with good rate function on $\mathbb{D}([0,t],\mathbb{R}^d)$, the set of càdlàg functions from $[0,t]$ to $\mathbb{R}^d$, given by

$$I_t(\varphi) = \begin{cases} \displaystyle\int_0^t \sup_{p\in\mathbb{R}^d}\left\{p\cdot\dot\varphi_s - \int_{\mathbb{R}^d}\big(e^{\langle z,p\rangle}-1\big)K(z)\,dz\right\}ds \\ \qquad\qquad \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty \qquad \text{otherwise,} \end{cases}$$

We consider in the next section another example of extension, where the trait space is finite. We obtain stronger results since we can fully characterize the limit of $\varepsilon \log u^\varepsilon$. This case will raise specific difficulties because the rate function of the associated large deviations principle has noncompact level sets.

4. The case of finite trait space. We consider here the case of a finite trait space with exponentially rare mutations. The model is given in Section 4.1. The large deviations principle is proved in Section 4.2 and we deduce from Varadhan's lemma the convergence of $\varepsilon \log u^\varepsilon$ in Section 4.3. Since in this case the equation satisfied by the limit V is simpler, we are able to study it in details in Section 4.4. In particular, we prove uniqueness of the solution to the variational problem and the associated discrete Hamilton–Jacobi problem.

4.1. *Problem statement and preliminary results.* We consider a finite set E and the system of ordinary differential equations

$$\begin{cases} \dot u^\varepsilon(t,i) = \displaystyle\sum_{j\in E\setminus\{i\}} \exp\left(-\frac{\mathfrak{T}(i,j)}{\varepsilon}\right)\big(u^\varepsilon(t,j)-u^\varepsilon(t,i)\big) \\ \qquad\qquad + \dfrac{1}{\varepsilon}u^\varepsilon(t,i)R^\varepsilon\big(i,v_t^\varepsilon\big) \qquad \forall t\in[0,T], \forall i\in E, \\ u^\varepsilon(0,i) = \exp\left(-\dfrac{h^\varepsilon(i)}{\varepsilon}\right), \end{cases} \tag{4.1}$$

where $\mathfrak{T}(i,j)\in(0,+\infty]$ for all $i\neq j\in E$, $R^\varepsilon(i,u): E\times\mathbb{R}^r\mapsto\mathbb{R}$, $h^\varepsilon: E\to\mathbb{R}$ and $v_t^\varepsilon = (v_t^{1,\varepsilon},\dots,v^{r,\varepsilon})$ is defined by

$$v_t^{p,\varepsilon} = \sum_{j\in E} u^\varepsilon(t,j)\Psi_p(j) \qquad \forall 1\le p\le r,$$

for some functions $\Psi_p: E\to(0,+\infty)$. The term $e^{-\mathfrak{T}(i,j)/\varepsilon}$ corresponds to the mutation rate from trait i to trait j. Its value is by convention 0 when $\mathfrak{T}(i,j) = +\infty$, which means that mutations are impossible from state i to state j.

The standing assumptions on Ψ, R^ε and h_ε given in Section 2.1 are still assumed to hold true here replacing $\mathbb{R}^d$ by E (except, of course, for the assumptions of regularity in the trait space). We also need the following assumption.

Assumption on $\mathfrak{T}$. For all distinct $i, j, k \in E$,

$$\mathfrak{T}(i,j) + \mathfrak{T}(j,k) > \mathfrak{T}(i,k) \tag{4.2}$$

(with the convention $\infty > \infty$) and we set

$$\eta := \inf_{i,j,k \in E \text{ distinct s.t. } \mathfrak{T}(i,k)<\infty} \mathfrak{T}(i,j) + \mathfrak{T}(j,k) - \mathfrak{T}(i,k) > 0. \tag{4.3}$$

Similarly, as for problem (1.4), the solution of problem (4.1) remains bounded. This is given in the following lemma, which can be proved similarly as in Proposition 2.1.

LEMMA 4.1. *We have, for all $t \geq 0$,*

$$\begin{aligned}\Big(\frac{\Psi_{\min}}{\Psi_{\max}} v_{\min} - A^{-1}\big(|E|-1\big)e^{-\beta/\varepsilon}\Big){\Psi_{\max}}^{-1} \\ \leq \sum_{i \in E} u^\varepsilon(t,i) \leq \Big(\frac{\Psi_{\max}}{\Psi_{\min}} v_{\max} + A\big(|E|-1\big)e^{-\gamma/\varepsilon}\Big){\Psi_{\min}}^{-1}\end{aligned}$$

holds true for any positive t as far as it holds for $t = 0$, where $|E|$ stands for the cardinality of E,

$$\gamma = \inf\{\mathfrak{T}(i,j) \mid i,j \in E, i \neq j\} > 0, \qquad \beta = \sup\{\mathfrak{T}(i,j) \mid i,j \in E, i \neq j\}$$

and the constants A, $v_{\min}$, $v_{\max}$, $\Psi_{\min}$ and $\Psi_{\max}$ are defined in Section 2.1.

Hence, we shall also assume in the sequel that

$$\begin{aligned}\Big(\frac{\Psi_{\min}}{\Psi_{\max}} v_{\min} - A^{-1}\big(|E|-1\big)e^{-\beta/\varepsilon}\Big){\Psi_{\max}}^{-1} \\ \leq \sum_{i \in E} e^{-h_\varepsilon(i)/\varepsilon} \leq \Big(\frac{\Psi_{\max}}{\Psi_{\min}} v_{\max} + A\big(|E|-1\big)e^{-\gamma/\varepsilon}\Big){\Psi_{\min}}^{-1}.\end{aligned}$$

We also define the compact set

$$\mathcal{S} := \Big\{u \in \mathbb{R}_+^E : \frac{\Psi_{\min}}{2\Psi_{\max}^2} v_{\min} \leq \|u\|_1 \leq \frac{2\Psi_{\max}}{\Psi_{\min}^2} v_{\max}\Big\}$$

which is invariant for the dynamics (4.1), for ε small enough.

Our first goal is to describe the solution u^ε of the system using an integral representation similar to (2.9). Let $(X_s^\varepsilon, s \in [0,T])$ be the Markov processes in E with infinitesimal generator

$$L^\varepsilon f(i) = \sum_{j \in E} \big(f(j) - f(i)\big)e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}},$$

that is, the Markov process which jumps from state $i \in E$ to $j \neq i$ with exponentially small rate $\exp(-\mathfrak{T}(i, j)/\varepsilon)$.

PROPOSITION 4.2 (Integral representation). *For any positive real number t and any element i of E, we have*

$$u^\varepsilon(t, i) = \mathbb{E}_i\left[\exp\left(-\frac{h^\varepsilon(X_t^\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\, ds\right)\right].$$

PROOF. First, note that the part of the proof of Lemma 2.4 showing that (2.9) is a mild solution of problem (2.1) do not rely in any manner on the Brownian nature of B^ε. As a consequence, one can directly deduce that $u^\varepsilon(t, i)$ satisfies

$$u^\varepsilon(t, i) = P_t^\varepsilon(e^{-h^\varepsilon/\varepsilon})(i) + \int_0^t P_{t-s}^\varepsilon\big(u^\varepsilon(s, i) R^\varepsilon(i, v_s^\varepsilon)\big)\, ds,$$

where $(P_t^\varepsilon, t \in \mathbb{R}_+)$ now stands for the semigroup generated by L^ε (i.e., a simple exponential of matrix since E is finite). This last expression is the Duhamel formulation of (4.1). $\square$

4.2. *Large deviations principle.* Large deviations properties of discrete-time Markov chains with exponentially small transition probabilities have been the object of numerous works, mainly with a perspective of estimates on exit times and metastability (see [8] or [13] and the references therein). However, we need to apply Varadhan's lemma to the continuous-time version of these processes. It actually appears that the process X^ε satisfies a large deviations principle with a rate function which is not good, which leads to substantial difficulties.

As a first step, the next result proves a weak large deviations principle (i.e., a large deviations principle with upper bounds only for compact sets of $\mathbb{D}([0, T], E)$) for the laws of X^ε.

PROPOSITION 4.3 (Weak LDP). *$(X^\varepsilon)_{\varepsilon \geq 0}$ satisfies a weak LDP with rate function*

$$I_T : \mathbb{D}\big([0, T], E\big) \mapsto \mathbb{R},$$

$$\varphi \to \sum_{l=1}^{N_\varphi} \mathfrak{T}(\varphi_{t_l^\varphi -}, \varphi_{t_l^\varphi}),$$

where $\mathbb{D}([0, T], E)$ is the space of càdlàg functions from $[0, T]$ to E and N_φ is the number of jumps of φ and $(t_l^\varphi)_{1 \leq l \leq N_\varphi}$ the increasing sequence of jump times of φ.

We shall also make use of the notation

$$I_T(\varphi) = \sum_{0 < s \leq T} \mathfrak{T}(\varphi_{s-}, \varphi_s)$$

with the implicit convention that $\mathfrak{T}(i, i) = 0$ for all $i \in E$.

Before proving this result, we focus on the following lemma which provides a convenient topological basis of the space $\mathbb{D}([0,T],E)$ equipped with the Skorohod topology.

LEMMA 4.4. *For all $\varphi \in \mathbb{D}([0,T],E)$ and $\delta < 1$, define*

$$B_{\mathrm{Sko}}(\varphi,\delta) = \big\{\psi \in \mathbb{D}\big([0,T],E\big) \mid N_\varphi = N_\psi, |t_l^\varphi - t_l^\psi| < \delta, \\ \varphi_0 = \psi_0, \textit{and } \varphi_{t_l^\varphi} = \psi_{t_l^\psi}\big\}.$$

Then the set

$$\big\{B_{\mathrm{Sko}}(\varphi,\varepsilon) \mid \varepsilon \in [0,1), \varphi \in \mathbb{D}\big([0,T],E\big)\big\}$$

is a topological basis of $\mathbb{D}([0,T],E)$.

PROOF. To prove the result, it is enough to show that, for all $\delta < 1$ and $\varphi \in \mathbb{D}([0,T],E)$ the set $B_{\mathrm{Sko}}(\varphi,\delta)$ is exactly the δ neighborhood of φ for a particular metric inducing the Skorokhod topology. We recall (see, e.g., [5]) that the Skorokhod topology can be defined through the metric d_S given by

$$d_S(\varphi,\psi) = \inf_{\lambda\in\Lambda}\Big\{\max\Big(\|\lambda - I\|_{L^\infty([0,T])}, \sup_{t\in[0,T]} d(\varphi_t, \psi\circ\lambda_t)\Big)\Big\},$$

where I is the identity function, Λ is the set of continuous increasing functions on $[0,T]$ with $\lambda_0 = 0$ and $\lambda_T = T$ and the distance d on E is defined as $d(i,j) = \mathbb{1}_{i\neq j}$. On one hand, let ψ and φ be such that $d_S(\varphi,\psi) < \delta$ for some $\delta < 1$, then there exists λ in Λ such that

$$\begin{cases} |\lambda_s - s| < \delta & \forall s \in [0,T], \\ d\big(\varphi_s, \psi(\lambda_s)\big) < \delta & \forall s \in [0,T]. \end{cases}$$

Since $\inf_{i,j\in E, i\neq j} d(i,j) = 1$, we have $\varphi_t = \psi(\lambda_t)$ for all $t \in [0,T]$. Hence, it follows that N_φ equals N_ψ and $t_l^\varphi = \lambda(t_l^\psi)$, for all l. Consequently, $|t_l^\varphi - t_l^\psi| < \delta$, for all l, and $\varphi_{t_l^\varphi} = \psi_{t_l^\psi}$.

On the other hand, assume that $\psi \in B_{\mathrm{Sko}}(\varphi,\delta)$, then one can consider the function λ, defined for any $t \in [t_n^\varphi, t_{n+1}^\varphi]$, by

$$\lambda_t = t_n^\psi + \frac{t_{n+1}^\psi - t_n^\psi}{t_{n+1}^\varphi - t_n^\varphi}\big(t - t_n^\varphi\big),$$

and obtain, using similar argument as above, that $\psi\circ\lambda = \varphi$. Moreover, on each interval $[t_n^\varphi, t_{n+1}^\varphi]$, since $\lambda_t - t$ is monotonous, it attains it maximal and minimal values on either at t_{n+1}^φ or t_n^φ. So, it is easy to see that $\|\lambda - I\|_\infty < \delta$, which gives $d_S(\varphi,\psi) < \delta$. $\square$

We can now prove the weak large deviations principle for X^ε.

PROOF OF PROPOSITION 4.3. Let $(S_l)_{l\geq 0}$ be the discrete time Markov chain associated to X^ε and let T_l be the lth inter-jump time, that is, $T_l = J_{l+1} - J_l$ for all $l \geq 0$, where J_l is the lth jump time of X^ε, for $l \geq 1$ and $J_0 = 0$. We recall that the Markov chain $(T_l, S_l)_{l\geq 0}$ has transition kernel given by

$$P\big((t,i), ds, dj\big) = \exp\bigg(-\frac{\mathfrak{T}(i,j)}{\varepsilon}\bigg)\exp\big(-c^\varepsilon(i)s\big)\,ds \otimes C(dj)$$
$$\forall (t,i) \in [0,T] \times E,$$

where

$$c^\varepsilon(i) = \sum_{j\in E, j\neq i} \exp\bigg(-\frac{\mathfrak{T}(i,j)}{\varepsilon}\bigg)$$

and $C(dj)$ is the counting measure on E and initial value $(T_0, S_0) = (0, i)$ $\mathbb{P}_i$-almost surely.

Let φ be an element of $\mathbb{D}([0,T], E)$ such that $\varphi(0) = i$ and $\delta < 1$. We set $t_0^\varphi = 0$. According to Lemma 4.4,

$$\begin{aligned}
&\mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi,\delta)\big)\\
&\quad\leq \mathbb{P}_i\Bigg(\bigcap_{\ell=0}^{N_\varphi-1}\big\{T_\ell \in \big[t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta, t_{\ell+1}^\varphi - t_\ell^\varphi + 2\delta\big], S_\ell = \varphi_{t_\ell^\varphi}\big\}\\
&\qquad\cap \big\{T_{N_\varphi} \geq T - t_{N_\varphi} - \delta, S_{N_\varphi} = \varphi_{t_{N_\varphi}^\varphi}\big\}\Bigg).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi,\delta)\big)\\
&\quad\leq \Bigg(\prod_{\ell=0}^{N_\varphi-1}\int_{(t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta)\vee 0}^{t_{\ell+1}^\varphi - t_\ell^\varphi + 2\delta} \exp\bigg(-\frac{\mathfrak{T}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi})}{\varepsilon}\bigg)\exp\big(-c^\varepsilon(\varphi_{t_\ell^\varphi})s_\ell\big)\,ds_\ell\Bigg)\\
&\qquad\times \int_{(T-t_{N_\varphi}^\varphi-\delta)\vee 0}^{\infty} c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi})\exp\big(-c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi})s\big)\,ds\\
&\quad= \prod_{\ell=0}^{N_\varphi-1}\exp\bigg(-\frac{\mathfrak{T}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi})}{\varepsilon}\bigg)e^{-c^\varepsilon(\varphi_{t_\ell^\varphi})(t_{\ell+1}^\varphi - t_\ell^\varphi - 2\delta)\vee 0}\\
&\qquad\times \frac{(1 - e^{-4\delta c^\varepsilon(\varphi_{t_\ell^\varphi})})}{c^\varepsilon(\varphi_{t_\ell^\varphi})} e^{-c^\varepsilon(\varphi_{t_{N_\varphi}^\varphi})(T-t_{N_\varphi}^\varphi-\delta)\vee 0}.
\end{aligned}$$

Now, using the facts that $c^\varepsilon(i) \le e^{-\frac{c}{\varepsilon}}$ with $c > 0$ and $(1 - \exp(-4\delta c^\varepsilon(\varphi_{t_\ell^\varphi})))/c^\varepsilon(\varphi_{t_\ell^\varphi}) \to 4\delta$ when $\varepsilon \to 0$, we get

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi, \delta)\big) \le -\sum_{\ell=0}^{N_\varphi - 1} \mathfrak{T}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

Similarly, for $\delta > 0$ small enough, using the bound

$$\mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi, \delta)\big) \ge \mathbb{P}_i\left(\bigcap_{\ell=0}^{N_\varphi}\left\{T_\ell \in \left(t_{\ell+1}^\varphi - t_\ell^\varphi - \frac{\delta}{N_\varphi}, t_{\ell+1}^\varphi - t_\ell^\varphi\right), S_\ell = \varphi_{t_\ell^\varphi}\right\}\right),$$

we obtain

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi, \delta)\big) \ge -\sum_{\ell=0}^{N_\varphi - 1} \mathfrak{T}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

Hence

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in B_{\mathrm{Sko}}(\varphi, \delta)\big) = -\sum_{\ell=0}^{N_\varphi - 1} \mathfrak{T}(\varphi_{t_\ell^\varphi}, \varphi_{t_{\ell+1}^\varphi}).$$

This classically entails (see, e.g., [16], Theorem 4.1.11) that X^ε satisfies a weak large deviations principle with rate function I_T. $\square$

Usually, a full large deviations principle is deduced from a weak one using exponential tightness of the laws of X^ε. However, in our case, exponential tightness does not hold. This is due to the fact that the function I_T is not a good rate function, as can be seen from the following example: let i and j be two elements of E and s a real number in $(0, T)$. Now, define for any positive integer n large enough,

$$\varphi_n(u) = \begin{cases} i & \text{if } u \in [0, s), \\ j & \text{if } u \in \left[s, s + \dfrac{1}{n}\right), \\ i & \text{if } u \in \left[s + \dfrac{1}{n}, T\right]. \end{cases}$$

Then the subset $\{\varphi_n \mid n \in \mathbb{N}\backslash\{0\}\}$ is clearly noncompact in $\mathbb{D}([0, T], E)$ while I_T is bounded on this set.

To prove the full large deviations principle, we need the following lemma.

LEMMA 4.5. *For all $N \ge 1$ and $t > 0$, we denote by N_t^ε the number of jumps of X^ε before t. There exists a constant $C_N \ge 0$ such that, for all $i \in E$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(N_t^\varepsilon \ge N\big) \le -C_N,$$

and for all $t > 0$, $\lim_{N\to\infty} C_N = +\infty$.

PROOF. Let us fix $x_0, x_1, \dots, x_N \in E$ such that $\mathfrak{T}(x_l, x_{l+1}) > 0$ for all l. We compute

$$\begin{aligned}
&\mathbb{P}_{x_0}\big(N_t^\varepsilon \geq N, X_{J_l}^\varepsilon = x_l, \forall l \in \{0, \dots, N\}\big)\\
&\quad = \int_0^t \exp\left(-\frac{\mathfrak{T}(x_0, x_1)}{\varepsilon}\right) \exp\big(-c^\varepsilon(x_0)s_0\big)\, ds_0 \cdots\\
&\qquad \cdots \int_0^{t-s_0-\dots-s_{N-2}} \exp\left(-\frac{\mathfrak{T}(x_{N-1}, x_N)}{\varepsilon}\right) \exp\big(-c^\varepsilon(x_{N-1})s_{N-1}\big)\, ds_{N-1}\\
&\quad \leq \exp\left(-\frac{\mathfrak{T}(x_0, x_1) + \dots + \mathfrak{T}(x_{N-1}, x_N)}{\varepsilon}\right) \int_0^t ds_0 \cdots \int_0^{t-s_0-\dots-s_{N-2}} ds_{N-1}\\
&\quad = \frac{t^N}{N!} \exp\left(-\frac{\mathfrak{T}(x_0, x_1) + \dots + \mathfrak{T}(x_{N-1}, x_N)}{\varepsilon}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{x_0}\big(N_t^\varepsilon \geq N, X_{J_l}^\varepsilon = x_l, \forall l \in \{0, \dots, N\}\big)\\
&\qquad \leq -\mathfrak{T}(x_0, x_1) - \dots - \mathfrak{T}(x_{N-1}, x_N) \leq -N \inf_{i,j \in E} \mathfrak{T}(i, j).
\end{aligned}$$

Since the number of choices of $x_0, x_1, \dots, x_N \in E$ is finite, we have proved Lemma 4.5. □

We can now prove that X^ε satisfies (a strong version of) the full LDP.

THEOREM 4.6. *For any measurable set F of $\mathbb{D}([0, T], E)$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in F\big) \leq -\inf_{\varphi \in F} I_T(\varphi) \leq -\inf_{\varphi \in \bar{F}} I_T(\varphi).$$

In particular, X^ε satisfies the large deviation principle with rate ε^{-1} and rate function I_T.

PROOF. Let $F \subset \mathbb{D}([0, T], E)$ be measurable. Set, for any positive integer n,

$$F_n^+ = \{\varphi \in F \mid N_\varphi \geq n\} \quad \text{and} \quad F_n^- = F \setminus F_n^+.$$

We have

$$\mathbb{P}_i\big(X^\varepsilon \in F_n^-\big) = \sum_{\ell=0}^{n-1} \sum_{(x_1,\dots,x_\ell) \in E^\ell} \mathbb{P}_i\big(\{X_{J_k}^\varepsilon = x_k, \forall k \leq N_t^\varepsilon, N_t^\varepsilon = \ell\} \cap F\big).$$

According to the computations made in the proof of Lemma 4.5, we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_i\big(X_{J_k}^\varepsilon = x_k, \forall k \leq N_T^\varepsilon, N_T^\varepsilon = \ell\big) \leq -\left(\mathfrak{T}(i, x_1) + \sum_{k=1}^{\ell-1} \mathfrak{T}(x_k, x_{k+1})\right),$$

which leads to

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in F_n^-\big) \le \max\left\{-\left(\mathfrak{T}(i,x_1) + \sum_{k=1}^{\ell-1} \mathfrak{T}(x_k,x_{k+1})\right)\right\},$$

where the maximum is taken with respect to $\ell \in \{0,\dots,n-1\}$ and to the elements $(x_1,\dots,x_\ell)$ of $\bigcup_{\ell=0}^n E^\ell$ such that

$$\big\{X^\varepsilon_{J_k} = x_k, \forall k \le N_T^\varepsilon, N_T^\varepsilon = \ell\big\} \cap F \ne \varnothing.$$

This implies that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in F_n^-\big) \le -\inf_{\varphi\in F} I_T(\varphi).$$

Finally, using Lemma 4.5, we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_i\big(X^\varepsilon \in F\big) \le \max\Big(-\inf_{\varphi\in F} I_T(\varphi), -C_n\Big).$$

The result is now obtained by sending n to infinity. $\square$

4.3. *Varadhan's lemma and convergence to the variational problem.* The next theorem corresponds to Theorem 2.7 in the discrete case situation. As seen above, I_T is not a good rate function, which prevents us from applying directly Varadhan's lemma. This is the place where we need the assumption (4.2) on $\mathfrak{T}$. Our result makes use of the sequence $(\varepsilon_k)_{k\ge 1}$ constructed in Lemma 2.6, which holds true without modification in our discrete case. To avoid heavy notation, we shall write $\mathcal{M}_s(i)$ for $\mathcal{M}_s(\{i\})$, where $\mathcal{M}_s$ is the measure constructed in Lemma 2.6.

THEOREM 4.7. *For all (t,i) in $(0,+\infty)\times E$,*

$$\begin{aligned} V(t,i) &:= \lim_{k\to\infty} \varepsilon_k \log u^{\varepsilon_k}(t,i) \\ &= \sup_{\varphi\in\mathbb{D}([0,t],E)\ s.t.\ \varphi_0=i} \bigg\{-h(\varphi_t) + \int_0^t \sum_{j\in E} R(\varphi_s,j)\mathcal{M}_{t-s}(j)\,ds \\ &\qquad - \sum_{0<s\le t} \mathfrak{T}(\varphi_{s-},\varphi_s)\bigg\}. \end{aligned} \tag{4.4}$$

PROOF. Let us fix a positive time t. To avoid heavy notation, we shall write ε instead of ε_k in this proof. Since the only part of the proof of Varadhan's lemma relying on the compactness of the level sets of the rate function is the upper bound, we restrict ourself to this bound. As in the proof of Theorem 2.7, let $a \in \mathbb{R}$ be smaller than the right-hand side of (4.4) and C satisfying $|\Phi(\varphi) - h(\varphi(t))| \le C(t+1)$. We define K as the (nonnecessarily compact) level set

$$K = \big\{\varphi \in \mathbb{D}\big([0,t],E\big) \mid \varphi_0 = i, I_t(\varphi) \le C(t+1) - a\big\}.$$

First, as in the proof of Theorem 2.7, we deduce from the LDP that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_i\biggl[\exp\biggl(\frac{1}{\varepsilon}\biggl(-h_\varepsilon(X_t^\varepsilon)+\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\biggr)\biggr)\mathbb{1}_{X^\varepsilon\in K^c}\biggr]$$

$$\le C(t+1)+\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\bigl(X^\varepsilon\in K^c\bigr)\le C(t+1)-\inf_{x\in K^c} I(x)\le a. \tag{4.5}$$

Second, by the definition of $I_t(\varphi)$, we have

$$N_\varphi \le \frac{I_t(\varphi)}{\min_{i,j\in E}\mathfrak{T}(i,j)}. \tag{4.6}$$

Hence, according to (4.6), there exists N such that for all φ in K, $N_\varphi \le N$. Fix $\gamma > 0$. We deduce that $K = K_{N,\delta} \cup L_{N,\delta}$, where

$$K_{N,\gamma} = \Bigl\{\varphi \in K \bigm| N_\varphi \le N \text{ and } \inf_l\bigl(t_{l+1}^\varphi - t_l^\varphi\bigr) \ge \gamma\Bigr\}$$

and

$$L_{N,\gamma} = \Bigl\{\varphi \in K \bigm| N_\varphi \le N \text{ and } \inf_l\bigl(t_{l+1}^\varphi - t_l^\varphi\bigr) < \gamma\Bigr\}.$$

Let $\Delta_\beta(T)$ be the set of subdivisions of $[0,t]$ with mesh greater that β. Since, for all $\varphi \in K_{N,\gamma}$ and all $\beta < \gamma$, we have

$$\inf_{s\in\Delta_\beta(T)}\max_{1\le l\le |s|}\sup_{u,v\in[s_l,s_{l+1})}|\varphi_u-\varphi_v| = 0,$$

we deduce from Arzela–Ascoli's theorem for the Skorokhod space that $K_{N,\gamma}$ is compact.

Fix $\delta > 0$. We deduce that there exist $n \ge 1$ and $\varphi_1^\gamma, \dots, \varphi_n^\gamma \in K_{N,\gamma}$ such that

$$K_{N,\gamma} = \bigcup_{l=1}^n G_{\varphi_l^\gamma},$$

where the neighborhood $G_{\varphi_l^\gamma}$ of φ_l^γ is chosen such that, for ε small enough,

$$-h\bigl(\varphi_l^\gamma(t)\bigr)+\Phi_\varepsilon\bigl(\varphi_l^\gamma\bigr)+\delta \ge \sup_{\varphi\in G_{\varphi_l^\gamma}} -h_\varepsilon\bigl(\varphi^\gamma(t)\bigr)+\Phi_\varepsilon(\varphi). \tag{4.7}$$

Because of Lemma 4.4, we can also assume without loss of generality that I_t is constant on $G_{\varphi_l^\gamma}$ for all l. Moreover, for all l in $\{1,\dots,n\}$ and ε small enough, we have

$$\Phi\bigl(\varphi_l^\gamma\bigr)-\delta \le \Phi_\varepsilon\bigl(\varphi_l^\gamma\bigr) \le \Phi\bigl(\varphi_l^\gamma\bigr)+\delta. \tag{4.8}$$

Following the lines of Theorem 2.7, we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_i\biggl[\exp\biggl(\frac{1}{\varepsilon}\biggl(-h_\varepsilon(X_t^\varepsilon)+\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\biggr)\biggr)\mathbb{1}_{X^\varepsilon\in K_{N,\gamma}}\biggr]$$

$$
\begin{aligned}
&\le \max_{1\le l\le n}\left\{-h\bigl(\varphi_l^\gamma(t)\bigr)+\int_0^t\sum_{j\in E}R\bigl(\varphi_l^\gamma(s),j\bigr)\mathcal{M}_{t-s}(j)\,ds+2\delta-I_t\bigl(\varphi_l^\gamma\bigr)\right\}\\
&\le \max_{\varphi\in K_{N,\gamma}}\left\{-h\bigl(\varphi(t)\bigr)+\int_0^t\sum_{j\in E}R\bigl(\varphi(s),j\bigr)\mathcal{M}_{t-s}(j)\,ds-I_t(\varphi)\right\}+2\delta. \qquad (4.9)
\end{aligned}
$$

We now prove a similar inequality when X^ε lies in $L_{N,\gamma}$. We first introduce some notation: for any $\varphi\in\mathbb{D}([0,T],E)$, in the case where $t^\varphi_{N_\varphi}\le t-\gamma$, we define

$$
\Gamma^\gamma\varphi(s)=\begin{cases}\varphi(s) & \text{if } s\in\bigl[t_l^\varphi,t_{l+1}^\varphi\bigr)\\ & \text{for } 0\le l\le N_\varphi \text{ and } t^\varphi_{l+1}-t^\varphi_l\ge\gamma,\\ \varphi\bigl(t^\varphi_{\inf\{l\le j\le N_\varphi \mid t^\varphi_{j+1}-t^\varphi_j\ge\gamma\}}\bigr) & \text{if } s\in\bigl[t_l^\varphi,t_{l+1}^\varphi\bigr)\\ & \text{for } 0\le l\le N_\varphi \text{ and } t^\varphi_{l+1}-t^\varphi_l<\gamma,\end{cases}
$$

with the convention that $t_0^\varphi=0$ and $t^\varphi_{N_\varphi+1}=t$. In the case where $t-\gamma<t^\varphi_{N_\varphi}\le t$, we set $\Gamma^\gamma\varphi:=\Gamma^\gamma\widetilde{\varphi}$ where

$$
\widetilde{\varphi}:=\begin{cases}\varphi(t) & \text{if } s\in[t-\gamma,t],\\ \varphi(s) & \text{if } s<t-\gamma.\end{cases}
$$

Now, for all φ in $L_{N,\gamma}$, $\Gamma^\gamma\varphi$ lies in $K_{N,\gamma}$. Indeed, by construction, $\Gamma^\gamma\varphi$ has inter-jumps times larger than γ and, because of Hypothesis (4.3), $I_t(\Gamma^\gamma\varphi)\le I_t(\varphi)$ since $\Gamma^\gamma\varphi$ is obtained by suppressing jumps in φ. Hence, we get

$$
\begin{aligned}
&\mathbb{E}_i\left[\exp\left(\frac{1}{\varepsilon}\left(-h_\varepsilon\bigl(X_t^\varepsilon\bigr)+\int_0^t R^\varepsilon\bigl(X_s^\varepsilon,v_{t-s}^\varepsilon\bigr)\,ds\right)\right)\mathbb{1}_{X^\varepsilon\in L_{N,\gamma}}\right]\\
&\quad\le\sum_{l=1}^n\mathbb{E}\left[\exp\left(\frac{1}{\varepsilon}\left(-h_\varepsilon\bigl(X_t^\varepsilon\bigr)+\int_0^t R^\varepsilon\bigl(X_s^\varepsilon,v_{t-s}^\varepsilon\bigr)\,ds\right)\right)\mathbb{1}_{X^\varepsilon\in L_{N,\gamma}}\mathbb{1}_{\Gamma^\gamma X^\varepsilon\in G_{\varphi_l^\gamma}}\right].
\end{aligned}
$$

For all φ in $L_{N,\gamma}$, we necessarily have $\Gamma^\gamma\varphi(t)=\varphi(t)$, hence, it follows from the definition of Γ^γ that

$$
\begin{aligned}
&-h_\varepsilon\bigl(\varphi(t)\bigr)+\int_0^t R^\varepsilon\bigl(\varphi(s),v_{t-s}^\varepsilon\bigr)\,ds\\
&\qquad=-h_\varepsilon\bigl(\Gamma^\gamma\varphi(t)\bigr)+\int_0^t R^\varepsilon\bigl(\Gamma^\gamma\varphi(s),v_{t-s}^\varepsilon\bigr)\,ds\\
&\qquad\quad+\sum_{t^\varphi_{j+1}-t^\varphi_j<\gamma}\int_{t_j^\varphi}^{t^\varphi_{j+1}}\bigl(R^\varepsilon\bigl(\varphi(s),v_{t-s}^\varepsilon\bigr)-R^\varepsilon\bigl(\Gamma^\gamma\varphi(s),v_{t-s}^\varepsilon\bigr)\bigr)\,ds\\
&\qquad\le-h_\varepsilon\bigl(\Gamma^\gamma\varphi(t)\bigr)+\int_0^t R^\varepsilon\bigl(\Gamma^\gamma\varphi(s),v_{t-s}^\varepsilon\bigr)\,ds+2NM\gamma,
\end{aligned}
$$

where the constant M comes from Assumption (2d). Using this last inequality, (4.7) and (4.8), we get

$$\begin{aligned}
&\mathbb{E}_i\Big[\exp\Big(\frac{1}{\varepsilon}\Big(-h_\varepsilon(X_t^\varepsilon)+\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\Big)\Big)\mathbb{1}_{x\in L_{N,\gamma}}\Big]\\
&\quad\le \sum_{l=1}^n \mathbb{E}\Big[\exp\Big(\frac{1}{\varepsilon}\Big(-h_\varepsilon(\Gamma^\gamma X_t^\varepsilon)+\int_0^t R^\varepsilon(\Gamma^\gamma X_0^\varepsilon, v_{t-s}^\varepsilon)\,ds\Big)\Big)\\
&\qquad\times\exp\Big(\frac{2}{\varepsilon}NM\gamma\Big)\mathbb{1}_{x\in L_{N,\gamma}}\mathbb{1}_{\Gamma^\gamma X^\varepsilon\in G_{\varphi_l^\gamma}}\Big]\\
&\quad\le \sum_{l=1}^n \exp\Big(\frac{1}{\varepsilon}\Big(-h(\varphi_l^\gamma(t))+\int_0^t\sum_{j\in E}R(\varphi_l^\gamma(s),j)\mathcal{M}_{t-s}(j)\,ds+2\delta\Big)\Big)\\
&\qquad\times\exp\Big(\frac{2}{\varepsilon}NM\gamma\Big)\mathbb{P}(X^\varepsilon\in L_{N,\gamma},\Gamma^\delta X^\varepsilon\in G_{\varphi_l^\gamma}).
\end{aligned}$$

It follows from Theorem 4.6 that

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\varepsilon\log\mathbb{E}_i\Big[\exp\Big(\frac{1}{\varepsilon}\Big(-h_\varepsilon(X_t^\varepsilon)+\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\Big)\Big)\mathbb{1}_{X^\varepsilon\in L_{N,\gamma}}\Big]\\
&\quad\le\max_{1\le l\le n}\Big(-h(\varphi_l^\gamma(t))+\int_0^t\sum_{j\in E}R(\varphi_l^\gamma(s),j)\mathcal{M}_{t-s}(j)\,ds\\
&\qquad+2\delta+2NM\gamma-\inf_{\varphi\in A_{\varphi_l^\gamma}}I_t(\varphi)\Big)
\end{aligned}$$

(with the convention $\inf_{\varphi\in\varnothing}I_t(\varphi)=+\infty$), with

$$A_{\varphi_l^\gamma}=\{\varphi\in L_{N,\gamma}\mid\Gamma^\gamma\varphi\in G_{\varphi_l^\gamma}\}.$$

Now, using Lemma 4.4 and $I_t(\Gamma^\gamma\varphi)\le I_t(\varphi)$, we have for all l such that $A_{\varphi_l^\gamma}\ne\varnothing$,

$$I_t(\varphi_l^\gamma)=\inf_{\varphi\in A_{\varphi_l^\gamma}}I_t(\Gamma^\gamma\varphi)\le\inf_{\varphi\in A_{\varphi_l^\gamma}}I_t(x),$$

which gives

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\varepsilon\log\mathbb{E}_i\Big[\exp\Big(\frac{1}{\varepsilon}\Big(-h_\varepsilon(X_t^\varepsilon)+\int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\Big)\Big)\mathbb{1}_{X^\varepsilon\in L_{N,\gamma}}\Big]\\
&\quad\le\max_{1\le l\le n}\Big(-h(\varphi_l^\gamma(t))+\int_0^t\sum_{j\in E}R(\varphi_l^\gamma(s),j)\mathcal{M}_{t-s}(j)\,ds\\
&\qquad+2\delta+2NM\gamma-I_t(\varphi_l^\gamma)\Big)
\end{aligned}$$

$$\leq \max_{\varphi\in K_{N,\gamma}} \left\{-h(\varphi(t)) + \int_0^t \sum_{j\in E} R(\varphi(s), j)\mathcal{M}_{t-s}(j)\,ds - I_t(\varphi)\right\} + 2\delta + 2NM\gamma.$$

Combining the last inequality with (4.5) and (4.9), we obtain

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_i\left[\exp\left(\frac{1}{\varepsilon}\left(-h_\varepsilon(X_t^\varepsilon) + \int_0^t R^\varepsilon(X_s^\varepsilon, v_{t-s}^\varepsilon)\,ds\right)\right)\right] \\
&\leq \max\left\{a; \max_{\varphi\in K_{N,\gamma}}\left[-h(\varphi(t)) + \int_0^t \sum_{j\in E} R(\varphi(s), j)\mathcal{M}_{t-s}(j)\,ds - I_t(\varphi)\right] \right.\\
&\qquad \left. + 2\delta + 2NM\gamma\right\} \\
&\leq \sup_{\varphi\in\mathbb{D}([0,t],E) \text{ s.t. } \varphi_0=i} \left\{-h(\varphi(t)) + \int_0^t \sum_{j\in E} R(\varphi(s), j)\mathcal{M}_{t-s}(j)\,ds - I_t(\varphi)\right\} \\
&\qquad + 2\delta + 2NM\gamma.
\end{aligned}$$

Since δ and γ are arbitrary, we have proved Theorem 4.7. $\square$

Our next goal is to obtain a version of Theorem 2.8. Since its proof makes use of the translation invariance of Brownian motion, we cannot use the same method. In particular, we will see that the function $t \mapsto \varepsilon \log u^\varepsilon(t, i)$ may not be uniformly Lipschitz for particular initial conditions. Hence we shall prove directly the Lipschitz regularity of the limit V. For this, we first need the following lemma.

LEMMA 4.8. *For all subsequence $(\varepsilon_k)_{k\geq 1}$ as in Theorem 4.7, the limit $V(t, i)$ of $\varepsilon_k \log u_k^\varepsilon(t, i)$ satisfies, for all $t > 0$ and all $i \neq j \in E$,*

$$V(t, i) \geq V(t, j) - \mathfrak{T}(i, j).$$

In particular, this inequality is satisfied for all $t \geq 0$ if and only if $h(i) \leq h(j) + \mathfrak{T}(i, j)$ for all $i \neq j$.

PROOF. Fix $i \neq j$, $t > 0$ and $\eta > 0$. Because of (4.4), we can choose a function $\hat{\varphi} \in \mathbb{D}([0, t], E)$ such that $\hat{\varphi}_0 = j$ and

$$V(t, j) \leq \eta - h(\hat{\varphi}_t) + \int_0^t \sum_{k\in E} R(\hat{\varphi}_s, k)\mathcal{M}_{t-s}(k)\,ds - \sum_{0<s\leq t} \mathfrak{T}(\hat{\varphi}_{s-}, \hat{\varphi}_s).$$

For all $n \in \mathbb{N}$ such that $\frac{1}{n} < t_1^{\hat{\varphi}} \wedge t$, we define $\varphi^{(n)} \in \mathbb{D}([0, t], E)$ as

$$\varphi_s^{(n)} = \begin{cases} i & \text{if } 0 \leq s < 1/n, \\ \hat{\varphi}_s & \text{if } 1/n \leq s \leq t. \end{cases}$$

Using (4.4) again, we have

$$
\begin{aligned}
&\mathfrak{T}(i,j)+V(t,i)\\
&\quad\geq V\big(t,\varphi_t^{(n)}\big)+\int_0^t\sum_{k\in E}R\big(\varphi_s^{(n)},k\big)\mathcal{M}_{t-s}(j)\,ds-\sum_{1/n<s\leq t}\mathfrak{T}\big(\varphi_{s-}^{(n)},\varphi_s^{(n)}\big)\\
&\quad\geq V(t,\hat{\varphi}_t)-\frac{M}{n}+\int_{1/n}^t\sum_{k\in E}R(\hat{\varphi}_s,k)\mathcal{M}_{t-s}(j)\,ds-\sum_{0<s\leq t}\mathfrak{T}(\hat{\varphi}_{s-},\hat{\varphi}_s)\\
&\quad\geq V(t,j)-\eta-2\frac{M}{n},
\end{aligned}
$$

where the constant M comes from Assumption (2d). This concludes the proof letting $\eta\to 0$ and $n\to+\infty$. □

We can now state our result on the regularity of V.

THEOREM 4.9. *For all subsequence $(\varepsilon_k)_{k\geq 1}$ as in Theorem 4.7, the limit $V(t,i)$ of $\varepsilon_k\log u^{\varepsilon_k}(t,i)$ is Lipschitz with respect to the time variable t on $(0,+\infty)$. In addition, if $h(i)\leq h(j)+\mathfrak{T}(i,j)$ for all $i\neq j$, the function V is Lipschitz on $\mathbb{R}_+$.*

PROOF. Fix $t\geq 0$ and $i\in E$. We shall write ε for ε_k to avoid heavy notation. For all $\delta>0$ and $\varepsilon>0$, proceeding as in the proof of Theorem 2.8, Markov's property entails

$$
e^{-\frac{M\delta}{\varepsilon}}\mathbb{E}_i\big[u^\varepsilon\big(t,X_\delta^\varepsilon\big)\big]\leq u^\varepsilon(t+\delta,i)\leq e^{\frac{M\delta}{\varepsilon}}\mathbb{E}_i\big[u^\varepsilon\big(t,X_\delta^\varepsilon\big)\big]. \tag{4.10}
$$

We can now estimate the distribution of X_δ^ε as follows. Let N^ε be the number of jumps of X^ε on the time interval $[0,\delta]$. For all $j\neq i$,

$$
\begin{aligned}
\delta e^{-|E|\delta}e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}&\leq\mathbb{P}_i\big(X_\delta^\varepsilon=j,N^\varepsilon=1\big)\\
&=\int_0^\delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}e^{-c^\varepsilon(i)s}e^{-c^\varepsilon(j)(\delta-s)}\,ds\leq\delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}},
\end{aligned}
$$

where $|E|$ is the cardinality of E. Similarly,

$$
1-|E|\delta\leq\mathbb{P}_i\big(X_\delta^\varepsilon=i,N^\varepsilon=0\big)\leq 1
$$

and

$$
0\leq\mathbb{P}_i\big(N^\varepsilon\geq 2\big)\leq\sum_{j\neq i}c_\varepsilon(j)e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}\int_0^\delta(\delta-s)\,ds\leq\frac{|E|^2}{2}\delta^2.
$$

Putting all these inequalities together, we obtain

$$\big(1-|E|\delta\big)u^\varepsilon(t,i)+\sum_{j\neq i}\delta e^{-|E|\delta}e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}u^\varepsilon(t,j)$$

$$\leq \mathbb{E}_i\big[u^\varepsilon\big(t,X^\varepsilon_\delta\big)\big]\leq u^\varepsilon(t,i)+\sum_{j\neq i}\delta e^{-\frac{\mathfrak{T}(i,j)}{\varepsilon}}u^\varepsilon(t,j)+\delta^2\bar{C}\frac{|E|^2}{2},$$

where $\bar{C}$ is the right-hand side of the main result in Lemma 4.1. Taking ε log of both sides of (4.10), using the last inequality and sending ε to 0, we obtain

$$\max\Big\{V(t,i);\max_{j\neq i}V(t,j)-\mathfrak{T}(i,j)\Big\}-M\delta$$

$$\leq V(t+\delta,i)\leq \max\Big\{V(t,i);\max_{j\neq i}V(t,j)-\mathfrak{T}(i,j)\Big\}+M\delta. \tag{4.11}$$

Since $\delta>0$ was arbitrary, the result follows from Lemma 4.8. $\square$

4.4. *Detailed study of the variational problem in the finite case.* Assume as above that E is a finite set. Our goal here is to study the limit problem

$$V(t,i)=\sup_{\varphi(0)=i}\Big\{-h\big(\varphi(t)\big)+\int_0^t\int_{\mathbb{R}^r}R\big(\varphi(u),y\big)\mathcal{M}_{t-u}(dy)\,du-I_t(\varphi)\Big\},$$

$$(t,i)\in\mathbb{R}_+\times E. \tag{4.12}$$

A direct adaptation of Theorem 4.7 allows to obtain the dynamic programming version of (4.12): for any $0\leq s\leq t$, the limit of $\varepsilon_k\log u_{\varepsilon_k}$ satisfies

$$V(t,i)=\sup_{\varphi(s)=i}\Big\{V\big(s,\varphi(t)\big)+\int_s^t\int_{\mathbb{R}^r}R\big(\varphi(u),y\big)\mathcal{M}_{s+t-u}(dy)\,du-I_{s,t}(\varphi)\Big\},$$

where, for all φ in $\mathbb{D}([s,t],E)$

$$I_{s,t}(\varphi)=\sum_{u\in(s,t]}\mathfrak{T}\big(\varphi(u-),\varphi(u)\big).$$

In all this section, we shall assume that there exists a constant $\kappa>0$ such that, when $\varepsilon\to 0$,

$$\|R^\varepsilon-R\|_{L^\infty(E\times\mathbb{R}_+)}=o\bigg(\exp\Big(-\frac{\kappa}{\varepsilon}\Big)\bigg). \tag{4.13}$$

This is, for example, satisfied for (1.2).

In the sequel, we make use of the following assumptions on the dynamical systems related to problem (4.1). For all $A\subset E$, we define the dynamical system in $\mathbb{R}^A_+:=\{(u_i,i\in A):u_i\geq 0 \text{ for all } i\in A\}$ denoted S_A by

$$\dot{u_i}=u_iR_i\bigg(\sum_{j\in A}\Psi_p(j)u_j,1\leq p\leq r\bigg),\qquad i\in A, \tag{4.14}$$

where $R_i(v)$ stands for $R(i,v)$.

HYPOTHESIS (H). For all $A \subset E$, let Eq_A be the set of steady states of S_A. Assume that all element of Eq_A are hyperbolic steady states and S_A admits a unique steady state $u^*_A = (u^*_{A,i})_{i \in A}$ such that, for all i in A,

$$u^*_{A,i} = 0 \quad \Longrightarrow \quad R_i\left(\sum_{j \in E} \Psi_p(j) u^*_{A,j}, 1 \le p \le r\right) < 0.$$

Assume also that there exists a strict Lyapunov function $L_A : \mathbb{R}^A_+ \to \mathbb{R}$ for the dynamical system S_A, which means that L_A is C^1, admits a unique global minimizer on $\mathbb{R}^A_+$ and satisfies, for any solution $u(t)$ to S_A,

$$\frac{dL_A(u(t))}{dt} = \sum_{i \in A} \frac{\partial L_A}{\partial u_i}(u(t)) u_i(t) R_i\left(\sum_{j \in A} \Psi_p(j) u_j(A), 1 \le p \le r\right) < 0$$

for all $t \ge 0$ such that $u(t) \notin \mathrm{Eq}_A$.

The hyperbolicity assumption means that for all steady state u^*, $u^*_i = 0$ implies that

$$R_i\left(\sum_{j \in E} \Psi_p(j) u^*_j, 1 \le p \le r\right) \ne 0.$$

For finite dimensional dynamical systems, it is well known that the hyperbolicity condition is generic under perturbation (see [15]). Since hyperbolic equilibria are isolated, Hypothesis (H) implies that Eq_A is finite. The global minimizer of L_A is necessarily a stable steady state of S_A, hence it must be u^*_A (since all the other steady states are not stable). In addition, the equilibrium u^*_A is globally asymptotically stable, in the sense that, for all initial condition $u(0) = (u_i(0))_{i \in A}$ in $(0, +\infty)^A$, the solution to (4.14) converges to u^*_A. Indeed, since $\dot{u}_i / u_i$ is uniformly bounded, $u(0) \in (0, +\infty)^A$ implies that $u(t) \in (0, +\infty)^A$ for all $t > 0$. Now, the existence of a strict Lyapunov function implies that $u(t)$ converges to an equilibrium u^*. If $u^* \ne u^*_A$, because of Hypothesis (H), there exists i such that $u^*_i = 0$ and $\dot{u}_i(t)/u_i(t) \ge \frac{1}{2} R_i(\sum_{j \in E} \Psi_p(j) u^*_j, 1 \le p \le r) > 0$ for all t large enough. This is a contradiction with the convergence of $u(t)$ to u^*.

General classes of dynamical systems satisfying Hypothesis (H) have been given in [12]. We give here two examples.

EXAMPLE 1. Our first example corresponds to indirect competition for environmental resources and is an extension of the chemostat model (1.3):

$$R_i(v) = -d_i + c_i \sum_{p=1}^{r} \frac{\alpha_p \Psi_p(i)}{1 + v_p},$$

where d_i, c_i, α_p are positive real numbers satisfying $c_i \sum_{p=1}^{r} \alpha_p \Psi_p(i) > d_i$, for all i in E. Here, the Lyapounov functions of Hypothesis (H) are given by

$$L : u \in \mathbb{R}^{|A|} \to \sum_{i \in A} c_i \log\left(\frac{c_i}{1 + \sum_{i \in A} \Psi_p(i) u_i}\right) - \sum_{i \in A} u_i.$$

EXAMPLE 2. This example corresponds to direct competition of Lotka–Volterra (or logistic) type: we assume that $E = \{1, \dots, r\}$ and

$$R_i(v) = r_i - v_i,$$

with the hypothesis that there exist positive constants $c_1, \dots, c_r$ such that

$$c_i \Psi_i(j) = c_j \Psi_j(i) \qquad \forall i, j \in E \quad \text{and}$$
$$\sum_{i,j \in E} x_i x_j c_i \Psi_i(j) > 0 \qquad \forall x \in \mathbb{R}^r \setminus \{0\}.$$

In this example the matrix $(\Psi_i(j))_{i,j \in E}$ is interpreted as a competition matrix between the different types of individuals. Here, the Lyapunov functions of Hypothesis (H) are given by

$$L : u \in \mathbb{R}^{|A|} \to \frac{1}{2} \sum_{i,j \in A} c_i \Psi_i(j) u_i u_j - \sum_{i \in A} u_i r_i c_i.$$

Hypothesis (H) implies a key property given in the next lemma.

LEMMA 4.10. *Assume Hypothesis* (H). *For all $A \subset E$ and all $\rho > 0$ small enough, the first hitting time $t_A^*(u(0), \rho)$ of the ρ-neighborhood of u_A^* by a solution $u(t)$ to S_A satisfies*

$$t_A^*(u(0), \rho) \le C_\rho^* \Bigl(1 + \sup_{i \in A} - \log u_i(0)\Bigr)$$

for some constant C_ρ^* only depending on ρ.

This lemma means that, when one coordinate $u_i(0)$ of $u(0)$ is close to zero, the time needed to converge to u_A^* grows linearly with the logarithm of $(u_i(0))^{-1}$.

PROOF. Let A be a nonempty subset of E and u be a solution of the dynamical system S_A. Without loss of generality, we can assume that $L_A(u_A^*) = 0$, that is, $\min_{u \in \mathbb{R}_+^E} L_A(u) = 0$. Let $\mathcal{S}_A$ be the set of $u \in \mathbb{R}_+^A$ such that $\bar{u} \in \mathcal{S}$, where $\bar{u} \in \mathbb{R}_+^E$ is obtained from u by setting to zero the coordinates with indices in $E \setminus A$ and the set $\mathcal{S}$ was defined in Lemma 4.1. Let $\mathcal{U}^* = (\mathrm{Eq}_A \cap \mathcal{S}_A) \setminus \{u_A^*\}$.

Step 1. Decrease of $L_A(u(t))$ between the visits of two neighborhoods of steady states.

Let

$$\underline{d} := \min_{u^*, v^* \in \mathcal{U}^* \cup \{u_A^*\}} |u^* - v^*|$$

and $\alpha_0 > 0$ such that

$$\sum_{i \in A} \partial_i L_A(x) x_i R_i \left(\sum_{j \in A} \Psi_p(j) x_j, 1 \le p \le r \right) < -\alpha_0$$
$$\forall x \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B\big(u^*, \underline{d}/4\big).$$

Hence, setting $\|R\|_{\mathcal{H}} := \sup_{i \in E, x \in \mathcal{H}} |R(i,x)|$ with $\mathcal{H}$ defined in Assumption (2d), the decrease of $L_A(u(t))$ between two visits by $u(t)$ of two distinct balls $B(u^*, \underline{d}/4)$ for $u^* \in \mathcal{U}^* \cup \{u_A^*\}$ is at least $\frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}}$. Now, let $\delta < \underline{d}/4$ be small enough to have, for all $u^* \in \mathcal{U}^*$,

$$\sup_{u \in B(u^*, \delta)} L_A(u) - \inf_{u \in B(u^*, \delta)} L_A(u) < \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}}.$$

Hence, defining for all $k \in \mathbb{N}$,

$$\mathcal{S}_A^k := \left\{ u \in \mathcal{S}_A \;\middle|\; k \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} \le L_A(u) < (k+1) \frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} \right\},$$

the solution $u(t)$ can only visit at most one ball of type $B(u^*, \delta)$ for some $u^* \in \mathcal{U}^*$ during its travelling time through $\mathcal{S}_A^k$.

Step 2. Time spent in $\mathcal{S}_A^k$.

According to Hypothesis (H), for any steady state $u^* \in \mathcal{U}^*$, we have

$$R_i \left(\sum_{j \in A} \Psi_p(j) u_j^*, 1 \le p \le r \right) > 0$$
$$\text{for at least one } i\big(u^*\big) \in A \text{ such that } u_{i(u^*)}^* = 0.$$

Consequently, reducing $\delta > 0$ if necessary, there exists a positive real number R_- such that, for all $u^* \in \mathcal{U}^*$,

$$\forall x \in B\big(u^*, \delta\big), \qquad R_{i(u^*)} \left(\sum_{j \in A} \Psi_p(j) u_j^*, 1 \le p \le r \right) > R_-. \tag{4.15}$$

In addition, since L_A is a strict Lyapunov function, there exists a positive real number α such that

$$\sum_{i \in A} \partial_i L_A(x) x_i R_i \left(\sum_{j \in A} \Psi_p(j) x_j, 1 \le p \le r \right) < -\alpha$$
$$\forall x \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B\big(u^*, \delta\big). \tag{4.16}$$

Now, let c be a positive real number satisfying $c^{-1} < \alpha^{-1}\|R\|_{\mathcal{H}}$. Let k_0 such that $u_0 \in \mathcal{S}_A^{k_0}$. For any $0 \le k \le k_0$, let t_k be the hitting time of $\mathcal{S}^k$ and t_{-1} the first hitting time of $B(u_A^*, \delta)$, and define for all $k \ge 0$,

$$F_k(t) = \begin{cases} L_A\big(u(t)\big) & \text{if } u(t) \notin \bigcup_{u^* \in \mathcal{U}^* \cup \{u_A^*\}} B\big(u^*, \delta\big) \\ & \text{for all } t \in [t_k, t_{k-1}], \\ L_A\big(u(t)\big) - c \log u_{i(u^*)}(t) & \text{if } \exists t \in [t_k, t_{k+1}], \exists u^* \in \mathcal{U}^* \\ & \text{such that } u(t) \in B\big(u^*, \delta\big). \end{cases}$$

Note that Step 1 implies that, for all $k \ge 0$, there exists at most one $u^* \in \mathcal{U}^*$ such that $u(t) \in B(u^*, \delta)$ for some $t \in [t_k, t_{k+1}]$. Using (4.16), (4.15), the fact that $\frac{d}{dt} L_A(u(t)) \le 0$ and the definition of $\|R\|_{\mathcal{H}}$, we have for all $t \in [t_k, t_{k+1}]$,

$$\frac{dF_k(t)}{dt} \le (-cR_-) \vee \big(-\alpha + c\|R\|_{\mathcal{H}}\big) < 0.$$

Therefore, for all $t \in [t_k, t_{k+1}]$,

$$F_k(t) \le (k+1)\frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} - c \inf_{i \in A} \log u_i(t_k).$$

In addition it follows from Lemma 4.1 that

$$F_k(t) \ge k\frac{\alpha_0 \underline{d}}{2\|R\|_{\mathcal{H}}} - \log\bigg(\frac{v_{\max} + A(|E|-1)e^{-\gamma/\varepsilon}}{\Psi_{\min}}\bigg).$$

Setting $k = k_0$, we deduce that there exists a constant $C_{k_0-1} > 0$ such that

$$t_{k_0-1} \le C_{k_0-1}\Big(1 - \inf_{i \in A} \log u_i(0)\Big)$$

and, since $|\frac{\dot{u}_i}{u_i}| \le \|R\|_{\mathcal{H}}$,

$$-\inf_{i \in A} \log u_i(t_{k_0-1}) \le -\inf_{i \in A} \log u_i(0) + \|R\|_{\mathcal{H}} C_{k_0-1}\Big(1 - \inf_{i \in A} \log u_i(0)\Big).$$

Proceeding by induction, it follows that there exist constants C_k and D_k depending only on k_0 and $u(0)$ such that, for all $k \ge 1$,

$$t_{k-1} - t_k \le C_k\Big(1 - \inf_{i \in A} \log u_i(0)\Big) \quad \text{and}$$

$$-\inf_{i \in A} \log u_i(t_{k-1}) \le D_k\Big(1 - \inf_{i \in A} \log u_i(0)\Big).$$

Similarly, there exist constants C_k and D_k such that

$$t_{-1} - t_0 \le C_0\Big(1 - \inf_{i \in A} \log u_i(0)\Big) \quad \text{and}$$

$$-\inf_{i \in A} \log u_i(t_{-1}) \le D_0\Big(1 - \inf_{i \in A} \log u_i(0)\Big).$$

Since L is a strict Lyapunov function, for all $\rho < \delta$, the time needed to enter $B(u^*_A, \rho)$ starting from any point in $B(u^*_A, \delta)$ is bounded by a constant depending only on ρ. The result follows. $\square$

This lemma entails the following property.

PROPOSITION 4.11. *Assume* (4.13) *and Hypothesis* (H) *and let* $(\varepsilon_k)_{k\geq 1}$ *be as in Theorem* 4.9. *For any* $t \geq 0$, *let*

$$u_i(t) = \begin{cases} u^*_{\{V(t,\cdot)=0\},i} & \text{if } V(t,i) = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then there exists $\rho_t > 0$ *such that, for all* $\gamma \in (0, \rho_t)$,

$$\sup_{s\in[\gamma,\rho_t], i\in E} \left| u^{\varepsilon_k}(t+s, i) - u_i(t) \right| \xrightarrow[k\to+\infty]{} 0. \tag{4.17}$$

In particular, for all $s \in (t, t+\rho_t]$, $v^{\varepsilon_k}_s$ *converges to* $F(\{V(t,\cdot) = 0\})$, *where the convergence is uniform in all compact subsets of* $(t, t+\rho_t]$ *and where*

$$F(A) = \left(\sum_{j=1}^{r} \Psi_p(j) u^*_{A,j} \right)_{1\leq p\leq r} \qquad \forall A \subset E.$$

In addition, the weak limit $\mathcal{M}_s$ *of* $\delta_{v_{\varepsilon_k}(s)}$ *obtained in Lemma* 2.6 *satisfies*

$$\mathcal{M}_s = \delta_{F(\{V(t,\cdot)=0\})} \qquad \text{for almost all } s \in (t, t+\rho_t)$$

and the function $t \mapsto F(\{V(t,\cdot) = 0\})$ *is right continuous.*

Note that because the set E is finite, the range of the function F is finite, and thus, assuming that $t \mapsto F(\{V(t,\cdot) = 0\})$ is right continuous implies that the union of the time intervals where this function is constant is equal to $\mathbb{R}_+$. We emphasize that this is not true in general for measurable functions taking values in finite sets.

PROOF OF PROPOSITION 4.11. Let us fix $t \geq 0$ and define $A = \{i \in E \mid V(t,i) = 0\}$. Since, at time t, we have $V(t,i) = \lim_{k\to\infty} \varepsilon_k \log u^{\varepsilon_k}(t,i)$, it follows that, for all $\delta > 0$, for all k large enough,

$$u^{\varepsilon_k}(t,i) \geq e^{-\frac{\delta}{\varepsilon_k}} \qquad \forall i \in A. \tag{4.18}$$

In addition, there exists $\alpha > 0$ such that, for k large enough,

$$u^{\varepsilon_k}(t,i) \leq e^{-\frac{\alpha}{\varepsilon_k}} \qquad \forall i \in E \setminus A. \tag{4.19}$$

We define $(u^{\varepsilon_k}_i(s), s \in \mathbb{R}_+)_{i\in E}$ such that $u^{\varepsilon_k}_i(s) = 0$ for all $i \in E \setminus A$ and $(u^{\varepsilon_k}_i(s), s \in \mathbb{R}_+)_{i\in A}$ is the solution of the dynamical system S_A, with initial conditions $u^{\varepsilon_k}_i(0) = u^{\varepsilon_k}(t,i)$, for all $i \in A$. According to (4.14) and (4.1), we have, for any time s and any i in A,

$$
\begin{aligned}
&\big|u^{\varepsilon_k}(t+s,i)-u_i^{\varepsilon_k}(s/\varepsilon_k)\big| \\
&\quad\le \frac{1}{\varepsilon_k}\int_0^s \bigg|u^{\varepsilon_k}(t+\theta,i)R^{\varepsilon_k}\big(i,v_{t+\theta}^{\varepsilon_k}\big) \\
&\qquad - u_i^{\varepsilon_k}(\theta/\varepsilon_k)R_i\bigg(\sum_{j\in A}\Psi_p(j)u_j^{\varepsilon_k}(\theta/\varepsilon_k), 1\le p\le k\bigg)\bigg|\,d\theta \\
&\qquad + \int_0^s \sum_{j\in E} e^{-\frac{\Upsilon(i,j)}{\varepsilon_k}}\big|u^{\varepsilon_k}(\theta,j)-u^{\varepsilon_k}(\theta,i)\big|\,d\theta.
\end{aligned}
$$

Using Hypotheses (2d) and (4.13), in conjunction with Lemma 4.1, we get for all $s\le 1$

$$
\big|u^{\varepsilon_k}(t+s,i)-u_i^{\varepsilon_k}(s/\varepsilon_k)\big| \le \frac{D}{\varepsilon_k}\int_0^s \big|u^{\varepsilon_k}(t+\theta,i)-u_i^{\varepsilon_k}(\theta/\varepsilon_k)\big|\,d\theta + Ce^{-\frac{\gamma\wedge\kappa}{\varepsilon_k}},
$$

with γ defined in Lemma 4.1 and some positive constants C and D. Hence, the Gronwall lemma entails that

$$
\big|u^{\varepsilon_k}(t+s,i)-u_i^{\varepsilon_k}(s/\varepsilon_k)\big| \le \frac{C}{\varepsilon_k}\exp\bigg(\frac{Ds-\gamma}{\varepsilon_k}\bigg).
$$

Proceeding similarly for $i\in E\setminus A$, we deduce from (4.19) that

$$
\begin{aligned}
&\big|u^{\varepsilon_k}(t+s,i)-u_{\varepsilon_k}(t,i)\big| \\
&\quad\le \frac{1}{\varepsilon_k}\int_0^s u^{\varepsilon_k}(t+\theta,i)R\big(i,v_{t+\theta}^{\varepsilon_k}\big)\,d\theta + Ce^{-\frac{\gamma\wedge\kappa}{\varepsilon_k}} \\
&\quad\le \frac{M}{\varepsilon_k}\int_0^s \big|u^{\varepsilon_k}(t+\theta,i)-u_{\varepsilon_k}(t,i)\big|\,d\theta + \frac{M}{\varepsilon_k}e^{-\frac{\alpha}{\varepsilon_k}} + Ce^{-\frac{\gamma\wedge\kappa}{\varepsilon_k}},
\end{aligned}
$$

where the constant M comes from Assumption (2d). Using again Gronwall's lemma, we deduce (modifying the positive constants D and C if necessary) that

$$
\sup_{i\in E}\big|u^{\varepsilon_k}(t+s,i)-u_i^{\varepsilon_k}(s/\varepsilon_k)\big| \le \frac{C}{\varepsilon_k}\exp\bigg(\frac{Ds-\zeta}{\varepsilon_k}\bigg), \tag{4.20}
$$

for some positive constant ζ.

Now, for all $i\in A$, let us compare $u^{\varepsilon_k}(t+s,i)$ with the expected limit $u_{A,i}^*$ which gives

$$
\big|u^{\varepsilon_k}(t+s,i)-u_{A,i}^*\big| \le \big|u^{\varepsilon_k}(t+s,i)-u_i^{\varepsilon_k}(s/\varepsilon_k)\big| + \big|u_i^{\varepsilon_k}(s/\varepsilon_k)-u_{A,i}^*\big|.
$$

According to Lemma 4.10, we have, for any positive real number ρ,

$$
\frac{s}{\varepsilon_k} > C_\rho^*\Big(1+\sup_{i\in A}-\log u_i^{\varepsilon_k}(0)\Big) \quad\Longrightarrow\quad \big|u_i^{\varepsilon_k}(s/\varepsilon_k)-u_{A,i}^*\big| < \rho.
$$

However, according to (4.18), for all $\delta > 0$, for k large enough,

$$-\log u_i^{\varepsilon_k}(0) \le \frac{\delta}{\varepsilon_k} \qquad \forall i \in A.$$

This last inequality gives, for all k large enough,

$$s > 2\delta C_\rho^* \quad \Longrightarrow \quad \left|u_i^{\varepsilon_k}(s/\varepsilon_k) - u_{A,i}^*\right| < \rho.$$

This entails in conjunction with (4.20) that

$$\limsup_{k\to+\infty} \sup_{s\in[2\delta C_\rho^*, \zeta/2D]} \left|u^{\varepsilon_k}(t+s,i) - u_{A,i}^*\right| < \rho \qquad \forall i \in E. \tag{4.21}$$

Since ρ and δ were arbitrary, we have proved (4.17). The remaining statements of Proposition 4.11 follow easily. □

COROLLARY 4.12. *Assume* (4.13) *and Hypothesis* (H). *Any limit* V *of* $\varepsilon_k \log u^{\varepsilon_k}$ *along a subsequence as in Theorem* 4.9 *satisfies that the function* $t \mapsto F(\{V(t,\cdot)=0\})$ *is right continuous*, $V(0,i) = -h(i)$ *for all* $i \in E$ *and for all* $t \ge 0$,

$$V(t,i) = \sup_{\varphi(0)=i} \left\{ -h\big(\varphi(t)\big) + \int_0^t R\big(\varphi(u), F\big(\{V(t-u,\cdot)=0\}\big)\big)\,du - I_t(\varphi) \right\},$$

and its dynamic programming version

$$\begin{aligned} V(t,i) = \sup_{\varphi(s)=i} \bigg\{ & V\big(s,\varphi(t)\big) + \int_s^t R\big(\varphi(u), F\big(\{V(t+s-u,\cdot)=0\}\big)\big)\,du \\ & - I_{s,t}(\varphi) \bigg\}. \end{aligned} \tag{4.22}$$

In addition, the problem (4.22) *admits a unique solution such that* $t \mapsto F(\{V(t,\cdot)=0\})$ *is right continuous. In particular, the full sequence* $(\varepsilon \log u^\varepsilon)_{\varepsilon>0}$ *converges to this unique solution when* $\varepsilon \to 0$.

PROOF. The only nonobvious consequence of Proposition 4.11 is the uniqueness of a solution to (4.22) with $t \mapsto F(\{V(t,\cdot)=0\})$ right continuous. To prove this, observe that, by continuity of V for $t > 0$ and using (4.11) for $t = 0$, the set

$$A = \{t \ge 0 \mid \text{there exists a unique solution up to time } t\}$$

cannot have a finite upper bound. □

We can now give the discrete Hamilton–Jacobi formulation of the variational problem (4.22).

THEOREM 4.13. *Assume* (4.13), *Hypothesis* (H) *and that* $h(i) \le h(j) + \mathfrak{T}(i,j)$ *for all* $i \neq j$. *Then the problem*

$$\dot V(t,i) = \sup\{R_j\big(F(\{V(t,\cdot)=0\})\big) \mid j \in E, V(t,j) - \mathfrak{T}(j,i) = V(t,i)\},$$
$$V(0,i) = h(i) \qquad \forall i \in E \tag{4.23}$$

(*with the convention* $\mathfrak{T}(i,i) = 0$) *admits a unique solution such that* $t \mapsto F(\{V(t,\cdot)=0\})$ *is right continuous*, *which is the unique solution to the variational problem* (4.22).

The result also extends to cases where the condition $h(i) \le h(j) + \mathfrak{T}(i,j)$ is not satisfied for some $i \neq j$. In this case, one must replace the initial condition in (4.23) by

$$V(0,i) = \max\Big\{-h(i); \max_{j\neq i} -h(j) - \mathfrak{T}(i,j)\Big\} \qquad \forall i \in E$$

and the solution of (4.23) coincides with the solution to the variational problem (4.22) for positive times only.

PROOF. We first prove the uniqueness part and then that the solution of (4.22) also solves (4.23).

Step 1: Uniqueness for (4.23).

Let $t \in [0,+\infty]$ be the largest time such that there is uniqueness for (4.23) up to time t and assume $t < \infty$. Let $i \in E$ be such that $V(t,i) = 0$. Since $\mathfrak{T}(j,i) > 0$ for all $j \neq i$ and because of the Lipschitz regularity of any solution V of (4.23), $\dot V(s,i) = R_i(F(\{V(t,\cdot)=0\}))$ for all $s \in [t, t+\delta_t]$ for some $\delta_t > 0$. Hence $V(s,i)$ is uniquely determined for all i such that $V(t,i) = 0$ and $s \in [t,t+\delta_t]$. We proceed similarly for all the $i' \in E$ such that $V(t,i') \in [-\inf_{i\neq j}\mathfrak{T}(i,j)/2, 0)$: their dynamics is determined only by $F(\{V(t,\cdot)=0\})$ and $V(s,i)$ for all i such that $V(t,i) = 0$, for $s \ge t$ in the time interval $[t,t+\delta_t]$. We obtain a similar result inductively for all the $i' \in E$ such that $V(t,i') \in [-k\inf_{i\neq j}\mathfrak{T}(i,j)/2, -(k-1)\inf_{i\neq j}\mathfrak{T}(i,j)/2)$ for all $k \ge 1$. This contradicts the finiteness of t, and hence uniqueness for (4.23) is proved.

Step 2: The function V *of Corollary* 4.12 *solves* (4.23).

Let $t \ge 0$ and $i \in E$ be fixed and let us prove that the solution V of (4.22) is right differentiable with respect to time at (t,i) with derivative given by (4.23). According to Corollary 4.12, we have for all $\delta > 0$

$$V(t+\delta,i) = \sup_{\varphi(t)=i}\Big\{V\big(t,\varphi(t+\delta)\big)$$
$$+ \int_t^{t+\delta} R\big(\varphi(u), F\big(\{V(t+\delta-u,\cdot)=0\}\big)\big)\,du - I_{t,t+\delta}(\varphi)\Big\} \tag{4.24}$$

and there exists $\delta_t > 0$, such that $F(\{V(t+\delta,\cdot)=0\}) = F(\{V(t,\cdot)=0\})$ for all $\delta < \delta_t$.

We know from Lemma 4.8 that $V(t,i) \geq V(t,j) - \mathfrak{T}(i,j)$ for all $i \neq j$. If the inequality is strict for all $j \neq i$, it is clear that the supremum in (4.24) is attained for the constant function $\varphi \equiv i$ for sufficiently small $\delta > 0$, and hence $\dot{V}(t+\delta,i) = R_i(F(\{V(t,\cdot)=0\})$, which entails (4.23).

If there exists $j \neq i$ such that $V(t,i) = V(t,j) - \mathfrak{T}(i,j)$, let $j^* \in E$ be such that $R_j(F(\{V(t,\cdot)=0\})$ is maximal among all $j \in E$ such that $V(t,i) = V(t,j) - \mathfrak{T}(i,j)$. Since the supremum in (4.24) cannot be attained for functions φ jumping two times or more in $[t, t+\delta]$ for $\delta > 0$ small enough, considering all possible choices for φ with less than one jump, one easily checks that, for $\delta > 0$ small enough,

$$\begin{aligned} V(t+\delta,i) &= \lim_{n\to+\infty} V\bigl(t, \varphi^{(n)}(t+\delta)\bigr) \\ &\qquad + \int_t^{t+\delta} R\bigl(\varphi^{(n)}(u), F\bigl(\{V(t,\cdot)=0\}\bigr)\bigr)\,du - \mathfrak{T}\bigl(i,j^*\bigr) \\ &= V\bigl(t,j^*\bigr) - \mathfrak{T}\bigl(i,j^*\bigr) + \delta R_{j^*}\bigl(F\bigl(\{V(t,\cdot)=0\}\bigr) \\ &= V(t,i) + \delta R_{j^*}\bigl(F\bigl(\{V(t,\cdot)=0\}\bigr), \end{aligned}$$

where $\varphi^{(n)}_{t+s} = i$ if $s < 1/n$ and $\varphi^{(n)}_{t+s} = j^*$ if $1/n \leq s \leq \delta$. Hence $\dot{V}(t+\delta,i) = R_{j^*}(F(\{V(t,\cdot)=0\})$, which gives the result. $\square$

We conclude with the full characterization of the limit of u^ε.

COROLLARY 4.14. *Assume* (4.13), *Hypothesis* (H) *and that* $h(i) \leq h(j) + \mathfrak{T}(i,j)$ *for all* $i \neq j$. *Then, the family* $(u^\varepsilon(t,i), t \geq 0, i \in E)_{\varepsilon>0}$ *converges locally weakly to* $(u_i(t), t \geq 0, i \in E)$ *defined for all* $i \in E$ *and* $t \geq 0$ *by*

$$u_i(t) = \begin{cases} u^*_{\{V(t,\cdot)=0\},i} & \text{if } V(t,i)=0, \\ 0 & \text{otherwise}. \end{cases}$$

More precisely, for all continuous $f : \mathbb{R}_+ \to \mathbb{R}^E$ *and all* $T > 0$,

$$\lim_{\varepsilon\to 0} \int_0^T \bigl\langle f(t), u^\varepsilon(t,\cdot)\bigr\rangle\,dt = \int_0^T \bigl\langle f(t), u(t)\bigr\rangle\,dt.$$

PROOF. It follows from Theorem 4.13 that, for all $s \geq 0$, the full family $(\delta_{v^\varepsilon_s})_{\varepsilon>0}$ converges to $\mathcal{M}_s$ when $\varepsilon \to 0$. Therefore, the proof of Proposition 4.11 is valid replacing the sequence $(\varepsilon_k)_{k\geq 1}$ by the family $(\varepsilon)_{\varepsilon>0}$. Hence, for all $t \geq 0$ and $\gamma \in (0, \rho_t)$,

$$\sup_{s\in[\gamma,\rho_t], i\in E} \bigl|u^\varepsilon(t+s,i) - u_i(t)\bigr| \xrightarrow[\varepsilon\to 0]{} 0. \tag{4.25}$$

In addition, one can easily check from the proof of Proposition 4.11 that $t \mapsto \rho_t$ is measurable (indeed, one can take $\alpha = \alpha_t = \frac{1}{2}\sup_{i,V(t,i)<0} V(t,i)$ in this proof).

We fix $T>0$, $f:\mathbb{R}_+\to\mathbb{R}^E$ continuous and $\delta>0$. We define for all $n\geq 1$

$$I_n := \left\{t\in[0,T] : \rho_t > \frac{1}{n}\right\}.$$

Since $\bigcup_{n\geq 1} I_n = [0,T]$, we can choose n such that $\mathrm{Leb}(I_n^c)\leq\delta$, where Leb denotes Lebesgue's measure on $\mathbb{R}$. Since f is continuous, we can assume, reducing n if necessary that $1/n<\delta$ and, for all $t\in[0,T]$, $|f(t)-f(t+1/n)|<\delta$. Now that n is fixed, we can use (4.25) and Lebesgue's dominated convergence theorem to find $\varepsilon_0>0$ such that, for all $\varepsilon<\varepsilon_0$,

$$\int_{I_n}\left|\left\langle f(t), u^\varepsilon\left(t+\frac{1}{n},\cdot\right)-u(t)\right\rangle\right| dt \leq \delta.$$

Combining all the previous estimates, we obtain for all $\varepsilon<\varepsilon_0$,

$$\begin{aligned}
&\left|\int_0^T \langle f(t), u^\varepsilon(t,\cdot)\rangle\,dt - \int_0^T\langle f(t), u(t)\rangle\,dt\right| \\
&\quad\leq \left|\int_0^T \langle f(t), u^\varepsilon(t,\cdot)\rangle\,dt - \int_0^T\left\langle f(t), u^\varepsilon\left(t+\frac{1}{n},\cdot\right)\right\rangle dt\right| \\
&\qquad + \left|\int_{I_n^c}\left\langle f(t), u^\varepsilon\left(t+\frac{1}{n},\cdot\right)\right\rangle dt\right| + \left|\int_{I_n^c}\langle f(t), u(t)\rangle\,dt\right| \\
&\qquad + \int_{I_n}\left|\left\langle f(t), u^\varepsilon\left(t+\frac{1}{n},\cdot\right)-u(t)\right\rangle\right| dt \\
&\quad\leq \bar{C}T\sup_{t\in[0,T]}\left|f(t)-f\left(t+\frac{1}{n}\right)\right| + 2\bar{C}\sup_{t\in[0,T]}|f(t)|\left(\frac{1}{n}+\mathrm{Leb}(I_n^c)\right)+\delta \\
&\quad\leq C\delta
\end{aligned}$$

for a constant C independent of δ, where $\bar{C}$ is the right-hand side of the main result in Lemma 4.1. Hence Corollary 4.14 is proved. $\square$

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