

## SUPERMARKET MODEL ON GRAPHS

BY AMARJIT BUDHIRAJA<sup>1</sup>, DEBANKUR MUKHERJEE<sup>2</sup> AND RUOYU WU

*University of North Carolina, Chapel Hill, Brown University  
and University of Michigan*

We consider a variation of the supermarket model in which the servers can communicate with their neighbors and where the neighborhood relationships are described in terms of a suitable graph. Tasks with unit-exponential service time distributions arrive at each vertex as independent Poisson processes with rate  $\lambda$ , and each task is irrevocably assigned to the shortest queue among the one it first appears and its  $d - 1$  randomly selected neighbors. This model has been extensively studied when the underlying graph is a clique in which case it reduces to the well-known *power-of- $d$*  scheme. In particular, results of Mitzenmacher (1996) and Vvedenskaya *et al.* (1996) show that as the size of the clique gets large, the occupancy process associated with the queue-lengths at the various servers converges to a deterministic limit described by an infinite system of ordinary differential equations (ODE). In this work, we consider settings where the underlying graph need not be a clique and is allowed to be suitably sparse. We show that if the minimum degree approaches infinity (however slowly) as the number of servers  $N$  approaches infinity, and the ratio between the maximum degree and the minimum degree in each connected component approaches 1 uniformly, the occupancy process converges to the same system of ODE as the classical supermarket model. In particular, the asymptotic behavior of the occupancy process is insensitive to the precise network topology. We also study the case where the graph sequence is random, with the  $N$ th graph given as an Erdős–Rényi random graph on  $N$  vertices with average degree  $c(N)$ . Annealed convergence of the occupancy process to the same deterministic limit is established under the condition  $c(N) \rightarrow \infty$ , and under a stronger condition  $c(N)/\ln N \rightarrow \infty$ , convergence (in probability) is shown for almost every realization of the random graph.

**1. Introduction.** *Background and motivation.* In this paper, we analyze a variation of the supermarket model in which the servers can communicate with their neighbors and where the neighborhood relationships are described in terms of a suitable graph. Specifically, consider a graph  $G_N$  on  $N$  vertices, where the vertices

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represent single-server queues. Tasks with unit-exponential service time distributions arrive at each server as independent Poisson processes of rate  $\lambda$ , and each task is irrevocably assigned to the shortest queue among the one it first appears and its  $d - 1$  randomly selected neighbors.

The above model has been extensively investigated in the case where  $G_N$  is a clique. In that case, each task is assigned to the shortest queue among  $d \geq 2$  queues selected randomly from the entire system, which is commonly referred to as the “power-of- $d$ ” or JSQ( $d$ ) scheme. Since the servers are exchangeable when the underlying graph is a clique, the system is quite tractable via classical mean-field techniques. Results in Mitzenmacher [18, 20] and Vvedenskaya et al. [29] show that for any fixed value of  $d$ , as the size of the clique gets large, the occupancy process associated with the queue-lengths at the various servers converges to a deterministic limit described by an infinite system of ordinary differential equations (ODE). Moreover, even sampling as few as  $d = 2$  servers yields significant performance enhancements over purely random assignment ( $d = 1$ ) as  $N \rightarrow \infty$ . Specifically, when  $\lambda < 1$ , the probability that there are  $i$  or more tasks at a given queue in steady state is proportional to  $\lambda \frac{d^i - 1}{d - 1}$  as  $N \rightarrow \infty$ , and thus exhibits super-exponential decay in  $\lambda$  as opposed to exponential decay for the random assignment policy.

However, in many service systems the “ $d$  choices” might be geographically constrained [10, 11], and when a task arrives at any specific server, it becomes difficult, if not impossible, to fetch instantaneous state information from an arbitrarily selected  $d - 1$  servers. This might give rise to a constrained network architecture that can be captured in terms of a graph. Moreover, executing a task commonly involves the use of some data, and storing such data for all possible tasks on all servers will typically require an excessive amount of storage capacity [30, 32]. The above issues motivate consideration of sparser graph topologies where tasks that arrive at a specific server  $i$  can only be forwarded to a subset of the servers  $\mathcal{N}_i$  that possess the data required to process the tasks. For the tasks that arrive at server  $i$ , the  $d - 1$  random choices must come from  $\mathcal{N}_i$ . The subset  $\mathcal{N}_i$  containing the peers of server  $i$  can be thought of as neighbors in some graph  $G_N$ . Although the above scenario corresponds to a setting with directed graphs and in our paper we consider the case of undirected graphs, our results extend in a straightforward manner to the setting of directed graphs; see Remarks 4 and 7 for detailed discussions. While considering load balancing schemes with sparse topologies is desirable from applications perspectives, the corresponding mathematical formulation, that results in systems that in general will not be exchangeable or have simple Markovian state descriptors, puts us outside the range of classical mean-field techniques, leading to a fairly uncharted territory from methodological standpoint, as further discussed below.

*Related work.* The study of the JSQ( $d$ ) scheme in the context of large-scale queueing networks was initiated by Mitzenmacher [18, 20] and Vvedenskaya et

al. [29]. Since then, this scheme along with its many variations have been studied extensively in [1, 4–6, 9, 12, 16, 17, 19, 22, 33] and many more. In the context of load balancing problems on graphs, [11, 27] examines the performance on certain fixed-degree graphs and in particular ring topologies. Their results demonstrate that the flexibility to forward tasks to a few neighbors, or even just one, with possibly shorter queues significantly improves the performance in terms of the waiting time and tail distribution of the queue length. This is similar to the power-of-two effect in the setting of cliques, but the results in [11, 27] also establish that the performance is sensitive to the underlying graph topology, and that selecting from a fixed set of  $d - 1$  neighbors typically does not match the performance of resampling  $d - 1$  alternate servers for each incoming task from the entire population. Recently, Mukherjee et al. [21] study the join-the-shortest queue (JSQ) policy on graphs, where each task joins the shortest queue among the one it first appears and *all* its neighbors, and establishes that asymptotically, the performance of the JSQ policy on a clique can be achieved by much sparser topologies, provided the graph is suitably random in Erdős–Rényi sense. We will contrast the results of the current paper with those obtained in [21] in greater detail in Section 2; see Remark 3. Relevant from a high level, queueing system topologies with limited flexibility were examined using quite different techniques by Tsitsiklis and Xu [25, 26] in a dynamic scheduling framework (as opposed to the load balancing context). We refer to [28] for a recent survey on scalable load balancing algorithms.

If tasks do not get served and never depart but simply accumulate, then our model as described above amounts to a so-called balls-and-bins problem on a graph. Viewed from that angle, a close counterpart of our problem is studied in Kenthapadi and Panigrahy [13], where in our terminology, each arriving task is routed to the shortest of  $d \geq 2$  randomly selected neighboring queues. In this setup, they show that if each vertex in the underlying graph has degree  $\Theta(N^\varepsilon)$ , where  $\varepsilon$  is not too small, the maximum number of balls in a bin scales as  $\log(\log(N))/\log(d) + O(1)$ . This scaling is the same as in the case when the underlying graph is a clique [2]. In a more recent paper by Peres, Talwar and Weider [23] the balls-and-bins problem has been analyzed in the context of a  $(1 + \beta)$ -choice process, where each ball goes to a random bin with probability  $1 - \beta$  and to the lesser loaded of the two bins corresponding to the nodes of a random edge of the graph with probability  $\beta$ . In particular, for this process they show that the difference between the maximum number of balls in a bin and the typical number of balls in the bins is  $O(\log(N)/\sigma)$ , where  $\sigma$  is the edge expansion property of the underlying graph. We refer to [31] for a recent survey on the balls-and-bins literature.

*Main contributions.* In most of the load balancing literature on systems of single-server queues mentioned above, the primary tool has been a convenient occupancy measure representation for the collection of queue-length processes associated with the various servers. Specifically, under the assumption of exponential service time distributions, the number of queues with queue length at least  $i$  at

time  $t$  denoted by  $Q_i(t)$ , for  $i = 1, 2, \dots$  forms a Markov process. This occupancy process  $\mathbf{Q}(\cdot) = (Q_1(\cdot), Q_2(\cdot), \dots)$  is then analyzed using classical mean-field techniques as the number of servers becomes large. The fundamental challenge in the analysis of load balancing on arbitrary graph topologies is that one cannot reduce the study to that for the state occupancy process since it is no longer a Markov process. In general, one needs to keep track of the evolution of the number of tasks at each vertex along with the information on neighborhood relationships. This is a significant obstacle in using tools from classical mean-field analysis for such systems. Consequently, results for load balancing queuing systems on general graphs have to date remained scarce. To the best of our knowledge, this is the first work to study rigorously the limits of the JSQ( $d$ ) occupancy process for nontrivial graph topologies (i.e., other than a clique).

In [21], where the tasks are assigned to the shortest queue among *all* the neighbors, the authors used a stochastic coupling to compare the occupancy process for an arbitrary graph topology with that for the clique, and establish that under suitable assumptions on the *well-connectedness* of the graph topology, the occupancy processes and their diffusion scaled versions have to the same weak limits as for the clique. Loosely speaking, for the first convergence, the well-connectedness requires that for any  $\varepsilon > 0$ , the neighborhood of any collection of  $\varepsilon N$  vertices contains  $N - o(N)$  vertices. This ensures that on any finite time interval, the fraction of tasks not assigned to servers with the “fluid-scaled minimum queue length” is arbitrarily small. Thus for large  $N$  the occupancy process becomes nearly indistinguishable from that in a clique. The coupling in [21] is particularly tailored for schemes where on any finite time interval, most of the arrivals are assigned to one of the fluid-scaled shortest queues. For the setting considered in the current work where a fixed number of servers are probed at each arrival, developing analogous coupling methods appears to be challenging. To see this, observe that when all neighbors are probed at arrivals, it is clear that the queue lengths will be better balanced (in the sense of stochastic majorization) for a clique than any other graph topology. In contrast, for the JSQ( $d$ ) scheme with fixed  $d$ , even this basic property, namely that the performance of the system will be “optimal” if the topology is a clique, is not clear. In this paper, we take a very different approach, and analyze the evolution of the queue-length process at an arbitrary tagged server as the system size becomes large. The main ingredient is a careful analysis of local occupancy measures associated with neighborhood of each server and to argue that under suitable conditions their asymptotic behavior is the same for all servers.

Our first result establishes that under fairly mild conditions on the graph topology  $G_N$  (diverging minimum degree and a degree regularity condition, see Condition 1 and also Remark 1), for suitable initial occupancy measure, for any fixed  $d \geq 2$ , the global occupancy state process for the JSQ( $d$ ) scheme on  $G_N$  has the same weak limit as that on a clique, as the number of vertices  $N$  becomes large (see Theorem 2.1). Also, we show that the propagation of chaos property holds for this system, in the sense that the queue lengths at any finite collection of tagged

servers are statistically asymptotically independent, and the queue-length process for each server converges in distribution (in the path space) to the corresponding McKean–Vlasov process (see Theorem 2.2). We note that the class of graphs for which the above results hold includes arbitrary  $d(N)$ -regular graphs, where  $d(N) \rightarrow \infty$  as  $N \rightarrow \infty$ . As an immediate consequence of these results, we obtain that the same asymptotic performance of a JSQ( $d$ ) scheme on cliques can be achieved by a much sparser graph in which the number of connections is reduced by almost a factor  $N$ . Such a result provides a significant improvement on network connectivity requirements and gives important insights for sparse network design.

When the graph sequence  $\{G_N\}_{N \geq 1}$  is random with  $G_N$  given as an Erdős–Rényi random graph (ERRG) with average degree  $c(N)$ , we establish that for any  $c(N)$  that diverges to infinity with  $N$ , the annealed law of the occupancy process converges weakly to the same limit as in the case of a clique. For convergence of the quenched law, we require a somewhat more stringent growth condition on the average degree. Specifically, we show that if  $c(N)/\log(N) \rightarrow \infty$  as  $N \rightarrow \infty$ , then for almost every realization of the random graph the quenched law of the state occupancy process converges to the same limit as for the case of a clique. Thus the above results show that the asymptotic performance for cliques can be achieved by much sparser topologies, even when the connections are random.

In the classical setting of weakly interacting particle systems, one considers a collection of  $N$  stochastic processes on a clique, given as the solution of  $N$  coupled stochastic differential equations, where the evolution of any particle at a given time instant depends on its own state and the empirical measure of all particles at that moment (see [14, 15, 24] and references therein). The asymptotic behavior of the associated state occupancy measures have been well studied, including the law of large numbers, propagation of chaos properties, central limit theorems and large and moderate deviation principles. However, there is much less work for systems on general graphs except for some recent results for weakly interacting diffusions on Erdős–Rényi random graphs. Annealed law of large numbers and central limit theorems for such systems have been established in [3] and quenched law of large numbers has been shown in [8]. However, these works do not study queuing systems of the form considered here.

*Organization of the paper.* The rest of the paper is organized as follows. In Section 2, we present the main results of this paper along with some remarks and discussion. Sections 2.1 and 2.2 contain the results for sequence of deterministic and random graphs, respectively. The proofs of the results in Section 2 are presented in Section 3. Finally, we conclude with a discussion of future research directions in Section 4.

*Notation.* Let  $[N] \doteq \{1, \dots, N\}$  for  $N \in \mathbb{N}$ . For any graph  $G_N = (V_N, E_N)$ , where  $V_N$  is a finite set of vertices and  $E_N \subset V_N \times V_N$  is the set of edges, and  $i, j \in V_N$ , let  $\xi_{ij}^N = 1$  if  $(i, j) \in E_N$  and 0 otherwise. In this work, throughout

$V_N = [N]$ ,  $G_N$  is undirected, namely  $\xi_{ij}^N = \xi_{ji}^N$ , and  $E_N$  will be allowed to be random, in which case  $\xi_{ij}^N$  will be random variables. Let  $\mathbb{N}_0 \doteq \mathbb{N} \cup \{0\}$ . For a set  $A$ , denote by  $|A|$  the cardinality. For a Polish space  $\mathbb{S}$ , denote by  $\mathbb{D}([0, \infty), \mathbb{S})$  the space of right continuous functions with left limits from  $[0, \infty)$  to  $\mathbb{S}$ , endowed with the Skorokhod topology. For functions  $f : [0, \infty) \rightarrow \mathbb{R}$ , let  $\|f\|_{*,t} \doteq \sup_{0 \leq s \leq t} |f(s)|$ . We will use  $\kappa, \kappa_1, \kappa_2, \dots$  for various nonnegative finite constants. The distribution of  $\mathbb{S}$ -valued random variable  $X$  will be denoted as  $\mathcal{L}(X)$ . For  $x \in \mathbb{S}$ , denote by  $\delta_x$  the Dirac measure at the point  $x$ . When the underlying graph is nonrandom, expectations will be denoted by “ $\mathbb{E}$ ,” and when the graphs are random, the notation “ $\mathbb{E}$ ” will be used to denote the expectation (which integrates also over the randomness of the graph topology).

**2. Model description and main results.** Let  $\{G_N = (V_N, E_N)\}_{N \geq 1}$  be a sequence of simple graphs where recall that  $V_N = [N]$ . The graph  $G_N$  corresponds to a system with  $N$  servers, where each vertex in the graph represents a server and edges in the graph define the neighborhood relationships. Tasks arrive at the various servers as independent Poisson processes of rate  $\lambda$ . Each server has its own queue with an infinite buffer. Fix  $d \in \mathbb{N}$ ,  $d \geq 2$ . When a task appears at a server  $i$ , it is immediately assigned to the server with the shortest queue among server  $i$  and  $d - 1$  servers selected uniformly at random from its neighborhood in  $G_N$ . If there are multiple such servers, one of them is chosen uniformly at random. Arrivals to any server having less than  $d - 1$  neighbors in  $G_N$  can be assigned in an arbitrary fashion among that server and its neighbors, for example, to itself (i.e., without probing the queue length at any other server). The tasks have independent unit-mean exponentially distributed service times. The service order at each of the queues is taken to be oblivious to the actual service time requirements.

Let  $X_i^N(t)$  be the number of tasks at the  $i$ th server at time instant  $t$ , starting from some a.s. finite  $X_i^N(0)$ , and  $q_j^N(t)$  be the fraction of servers with queue length at least  $j$  in the  $N$ th system at time  $t$ ,  $i \in [N]$ ,  $j = 1, 2, \dots$ , namely

$$(2.1) \quad q_j^N(t) \doteq \frac{1}{N} \sum_{i=1}^N \sum_{k=j}^{\infty} \mathbf{1}_{\{X_i^N(t)=k\}}, \quad t \geq 0, j \in \mathbb{N}_0.$$

Let  $\mathbf{q}^N(t) \doteq (q_i^N(t))_{i \in \mathbb{N}_0}$ . Then  $\mathbf{q}^N \doteq \{\mathbf{q}^N(t)\}_{0 \leq t < \infty}$  is a process with sample paths in  $\mathbb{D}([0, \infty), S)$  where  $S = \{\mathbf{q} \in [0, 1]^{\mathbb{N}} : q_0 = 1, q_i \geq q_{i+1} \ \forall i \in \mathbb{N}_0, \text{ and } \sum_i q_i < \infty\}$  is equipped with the  $\ell_1$  topology.

We will now introduce a convenient representation for the evolution of the queue length processes in the  $N$ th system. We begin by introducing some notation. For  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{N}_0^d$ , let  $b(\mathbf{x})$  represent the probability that given  $d$  servers chosen with queue lengths  $\mathbf{x}$ , the job is sent to the first server in the selection. Recalling that the job is sent to the shortest queue with ties resolved by selecting at random,

the precise definition is as follows:

$$(2.2) \quad b(\mathbf{x}) \doteq \sum_{k=1}^d \frac{1}{k} \mathbf{1}_{\{x_1 = \min_{i \in [d]} \{x_i\}, |\operatorname{argmin}\{x_i\}| = k\}}.$$

Note that (i)  $b(\mathbf{x})$  is symmetric in  $(x_2, \dots, x_d)$ , (ii)  $b(\mathbf{x}) \in [0, 1]$  and (iii)  $b(\mathbf{x})$  is 1-Lipschitz in  $\mathbf{x} \in \mathbb{N}_0^d$ . Denote by  $D_i^N$  the number of neighbors of a vertex  $i$  in  $G_N$ . Let  $\mathcal{N}_i$  be i.i.d. Poisson processes of rate 1, corresponding to service completions, and  $\tilde{\mathcal{N}}_i$  be i.i.d. Poisson random measures on  $[0, \infty) \times \mathbb{R}_+$  with intensity  $\lambda ds dy$ . Assume that  $\{\mathcal{N}_i, \tilde{\mathcal{N}}_i\}$  are mutually independent. Thanks to the Poisson splitting property, the evolution of  $X_i^N(t)$  can be written as follows:

$$(2.3) \quad \begin{aligned} X_i^N(t) &= X_i^N(0) - \int_0^t \mathbf{1}_{\{X_i^N(s-) > 0\}} \mathcal{N}_i(ds) \\ &\quad + \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{0 \leq y \leq C_i^N(s-)\}} \tilde{\mathcal{N}}_i(ds dy), \end{aligned}$$

where

$$(2.4) \quad \begin{aligned} C_i^N(t) &= \mathbf{1}_{\{D_i^N < d-1\}} \bar{b}_i^N((X_k^N(t))_{k \in [N]}, (\xi_{kl}^N)_{k,l \in [N]}) \\ &\quad + \mathbf{1}_{\{D_i^N \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \\ &\quad \times b(X_i^N(t), X_{j_2}^N(t), \dots, X_{j_d}^N(t)) \\ &\quad + (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2}^N \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \\ &\quad \times b(X_i^N(t), X_{j_2}^N(t), \dots, X_{j_d}^N(t)) \\ &\quad + \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2}^N < d-1\}} \xi_{ij_2}^N \bar{b}_{ij_2}^N((X_k^N(t))_{k \in [N]}, (\xi_{kl}^N)_{k,l \in [N]}), \\ \alpha^N(i; j_2, j_3, \dots, j_d) &\doteq \frac{\xi_{ij_2}^N \xi_{ij_3}^N \dots \xi_{ij_d}^N}{D_i^N (D_i^N - 1) \dots (D_i^N - d + 2)} \\ \mathcal{S}_i^N &\doteq \{(j_2, \dots, j_d) \in [N]^{d-1} : (i, j_2, \dots, j_d) \text{ are distinct}\}. \end{aligned}$$

Here,  $\bar{b}_i^N$  and  $\bar{b}_{ij}^N$  are measurable functions with

$$(2.5) \quad \begin{aligned} &\bar{b}_i^N((X_k^N(t))_{k \in [N]}, (\xi_{kl}^N)_{k,l \in [N]}), \\ &\bar{b}_{ij}^N((X_k^N(t))_{k \in [N]}, (\xi_{kl}^N)_{k,l \in [N]}) \in [0, D_i^N + 1], \end{aligned}$$

which define the rules of assigning tasks when  $D_i^N < d - 1$  or  $D_j^N < d - 1$ , respectively. Precise form of these functions will not be important in our analysis.

The second term in the expression for  $C_i^N(t)$  gives the probability that a job arriving at server  $i$  (with  $D_i^N \geq d - 1$ ) is in fact assigned to server  $i$  itself, which will happen if server  $i$  is one of the queues with minimal queue length among the  $d - 1$  randomly selected neighbors and itself, and it is the winner of the tie between queues with minimal queue-lengths in the selection. The third term corresponds to the probability that a job arriving at some other server (say  $j_2$ , with  $D_{j_2}^N \geq d - 1$ ) is assigned to server  $i$ , which will happen if  $i$  is a neighbor of  $j_2$ , server  $i$  is among the random selection of  $d - 1$  neighbors of  $j_2$ , it is also among the queues with minimal queue-length in the selection, and it wins the tie-breaker among queues with minimal queue-length in the selection. We note that although  $C_i^N(t)$  takes a complicated form, it essentially depends on the local empirical queue-length and degree distributions of the neighbors, which allows the implementation of a mean-field type argument. This is one of the key advantages of the Poisson random measure representation in equation (2.3).

2.1. *Scaling limits for deterministic graph sequences.* In this section, we will consider arbitrary deterministic graph sequences, and establish a scaling limit when the graphs satisfy a certain “regularity” condition as formulated in Condition 1 below. For any graph  $G$ , let  $d_{\min}(G)$  and  $d_{\max}(G)$  denote the minimum and maximum degree, respectively.

CONDITION 1 (Regularity of degrees). The sequence  $\{G_N\}_{N \geq 1}$  satisfies the following:

- (i)  $d_{\min}(G_N) \rightarrow \infty$  as  $N \rightarrow \infty$ .
- (ii)  $\max_{i \in [N]} \left| \sum_{j \in [N], j \neq i} \frac{\xi_{ji}^N}{D_j^N} - 1 \right| \rightarrow 0$  as  $N \rightarrow \infty$ .

REMARK 1. Condition 1(ii) holds if, for example,  $d_{\max}(G_N)/d_{\min}(G_N) \rightarrow 1$  as  $N \rightarrow \infty$ , since

$$\frac{d_{\min}(G_N)}{d_{\max}(G_N)} \leq \frac{D_i^N}{d_{\max}(G_N)} \leq \sum_{j \in [N], j \neq i} \frac{\xi_{ji}^N}{D_j^N} \leq \frac{D_i^N}{d_{\min}(G_N)} \leq \frac{d_{\max}(G_N)}{d_{\min}(G_N)}$$

for each  $i \in [N]$ . But Condition 1(ii) also allows  $G_N$  to have degrees of very different orders in different components of the graph. For example, if  $\{C_k^N\}_{k \geq 1}$  denote the connected components of  $G_N$ , then Condition 1(ii) is satisfied if

$$\sup_{k \geq 1} \left| \frac{d_{\min}(C_k^N)}{d_{\max}(C_k^N)} - 1 \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Our first result establishes under Condition 1, the convergence of the occupancy state process  $q^N$  to the same deterministic limit as for the classical JSQ( $d$ ) policy (i.e., the case when  $G_N$  is a clique), as  $N \rightarrow \infty$ .



**THEOREM 2.1** (Convergence of global occupancy states). *Assume that the sequence of graphs  $\{G_N\}_{N \geq 1}$  satisfies Condition 1, and  $\{X_i^N(0) : i \in [N]\}$  is i.i.d. with  $\mathbb{P}(X_i^N(0) \geq j) = q_j^\infty$ ,  $j = 1, 2, \dots$ , for some  $\mathbf{q}^\infty \in S$ . Then on any finite time interval, the occupancy state process  $\mathbf{q}^N(\cdot)$  converges weakly with respect to Skorohod  $J_1$  topology to the deterministic limit  $\mathbf{q}(\cdot)$  given by the unique solution to the set of ODE:*

$$(2.6) \quad \frac{dq_i(t)}{dt} = \lambda[(q_{i-1}(t))^d - (q_i(t))^d] - (q_i(t) - q_{i+1}(t)), \quad i = 1, 2, \dots,$$

and  $\mathbf{q}(0) = \mathbf{q}^\infty$ .

**REMARK 2.** We make the following observations:

(i) Unique solvability of the system of equations (2.6) is a consequence of Lipschitz continuity of the right side. Specifically, define the function  $\mathbf{F}(\cdot) = (F_1(\cdot), F_2(\cdot), \dots)$  on  $S$  as

$$F_i(\mathbf{q}) = \lambda(q_{i-1}^d - q_i^d) - (q_i - q_{i+1}), \quad i = 1, 2, \dots,$$

with  $\mathbf{q} \in S$  and  $F_i(\mathbf{q})$  being the  $i$ th component of  $F(\mathbf{q})$ . It is easily seen that  $F$  is Lipschitz on  $S$  (equipped with the  $\ell_1$  distance). Standard results then imply that the system of ODE defined by  $d\mathbf{q}(t)/dt = \mathbf{F}(\mathbf{q})$  admits a unique solution.

(ii) The above result shows in particular that the evolution of the asymptotic global occupancy process as described by (2.6) coincides with that when the underlying graph is a clique, that is, when each arriving task can probe any set of  $d$  servers. Thus under Condition 1, the system exhibits the same asymptotic transient performance even when the underlying graph is much sparser. As an immediate corollary we see that (2.6) describes the asymptotic system occupancy process associated with arbitrary  $d(N)$ -regular graphs as long as  $d(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .

**REMARK 3.** Now we contrast Condition 1 with the condition introduced in [21] for the JSQ policy on a graph to behave as that on a clique. We note that Condition 1 relies only on local properties of the graph, and in particular may hold even when, for example, the graph contains several connected components of sizes that grow to infinity with  $N$ . In contrast, the condition in [21] requires that any two  $\Theta(N)$ -sized component must share  $\Theta(N)$  cross-edges, which does not hold in many networks with connectivity governed by spatial attributes, such as geometric graphs. In this sense, Condition 1 includes much broader class of graphs including arbitrary  $d(N)$ -regular graphs with  $d(N) \rightarrow \infty$ , as mentioned above. On the other hand, our condition requires the minimum degree in the graph to diverge to infinity, whereas [21] allows any  $o(N)$  vertices to have bounded degree (or degree zero). As noted in the Introduction, it is easy to see that the queue length process of the JSQ policy on a clique is better balanced (in stochastic majorization sense) than on any other graph. This is also reflected by the fact that the sufficient criterion for

fluid optimality as developed in [21] is monotone with respect to edge addition. Specifically, let  $\{G_N = (V_N, E_N)\}_{N \geq 1}$  be a graph sequence which satisfies the sufficient criterion in [21] for the limit of the occupancy process coincides with that for cliques. Then [21] shows that for any graph sequence  $\{\bar{G}_N = (V_N, \bar{E}_N)\}_{N \geq 1}$  with  $E_N \subseteq \bar{E}_N$ , the limit of the occupancy process also coincides with that for cliques. The above property is not immediate for systems considered in the current work since adding edges arbitrarily may result in violating Condition 1(ii).

REMARK 4. Although in this paper we consider the case of undirected graphs, that is, we assume  $\xi_{ij}^N = \xi_{ji}^N$  for all  $i, j \in [N]$ , we note that the results naturally extends to the directed graph scenario. Indeed, most of the proofs go through unchanged without the symmetry assumption for  $\xi$  and remaining require minor modification. In this remark, we discuss how Condition 1 and the simplified condition in Remark 1 need to be modified in the case of directed graphs in order for the conclusion of Theorem 2.1 to hold.

For  $i \in [N]$ , denote  $D_{i,\text{in}}^N \doteq \sum_{j \in [N]} \xi_{ji}^N$  and  $D_{i,\text{out}}^N \doteq \sum_{j \in [N]} \xi_{ij}^N$ . Also, for a graph  $G_N$ , denote  $d_{\min}^{\text{in}}(G_N) \doteq \min_i D_{i,\text{in}}^N$  and  $d_{\max}^{\text{in}}(G_N) \doteq \max_i D_{i,\text{in}}^N$ , and define  $d_{\min}^{\text{out}}(G_N)$  and  $d_{\max}^{\text{out}}(G_N)$  similarly. Assume that whenever a task arrives at a server  $i \in [N]$ , it is immediately assigned to the server with the shortest queue among server  $i$  and  $d - 1$  servers selected uniformly at random from the set of vertices  $\{j : \xi_{ij}^N = 1\}$  (ties are broken arbitrarily). As before, arrivals to any server with  $D_{i,\text{out}}^N < d - 1$  can be assigned arbitrarily. Now, the conclusion of Theorem 2.1 holds when the graph sequence  $\{G_N\}_{N \geq 1}$  of directed graphs satisfies the following condition:

**Condition 1'** (Criteria for directed graphs). The graph sequence  $\{G_N\}_{N \geq 1}$  satisfies the following:

- (i)  $\min\{d_{\min}^{\text{in}}(G_N), d_{\min}^{\text{out}}(G_N)\} \rightarrow \infty$  as  $N \rightarrow \infty$ .
- (ii)  $\max_{i \in [N]} |\sum_{j \in [N], j \neq i} \frac{\xi_{ji}^N}{D_{j,\text{out}}^N} - 1| \rightarrow 0$  as  $N \rightarrow \infty$ .

Similar to the condition discussed in Remark 1, a weaker but simpler sufficient condition implying Condition 1'(ii) is the following:

$$\frac{\min\{d_{\min}^{\text{in}}(G_N), d_{\min}^{\text{out}}(G_N)\}}{\max\{d_{\max}^{\text{in}}(G_N), d_{\max}^{\text{out}}(G_N)\}} \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

Our second result gives the joint asymptotic behavior of queue length processes for any finite collection of servers. In particular, it shows that the propagation of chaos holds, that is, the queue length processes for any finite collection of servers are asymptotically statistically independent. Recall the sequence of Poisson processes  $\{\mathcal{N}_i\}$ , Poisson random measures  $\{\tilde{\mathcal{N}}_i\}$ , and the function  $b$ .

THEOREM 2.2 (Evolution of tagged servers). Assume that the sequence of graphs  $\{G_N\}_{N \geq 1}$  satisfies Condition 1, and  $\{X_i^N(0) : i \in [N]\}$  is i.i.d. with

$\mathbb{P}(X_i^N(0) \geq j) = q_j^\infty$ ,  $j = 1, 2, \dots$ , for some  $\mathbf{q}^\infty \in S$ . Then the following convergence results hold:

(i) On any finite time interval, the queue length process  $X_i^N(\cdot)$  at server  $i$  converges weakly with respect to Skorohod  $J_1$  topology to the following McKean–Vlasov process:

$$\begin{aligned}
 X_i(t) &= X_i(0) - \int_0^t \mathbf{1}_{\{X_i(s-) > 0\}} \mathcal{N}_i(ds) \\
 &+ \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{\{0 \leq y \leq C_i(s-)\}} \tilde{\mathcal{N}}_i(ds dy), \\
 C_i(t) &= d \int_{\mathbb{N}^{d-1}} b(X_i(t), x_2, \dots, x_d) \mu_t(dx_2) \cdots \mu_t(dx_d),
 \end{aligned}
 \tag{2.7}$$

where  $\mu_t = \mathcal{L}(X_i(t))$  and  $\mu_0[j, \infty) = q_j^\infty$  for  $t \geq 0$  and  $j \in \mathbb{N}_0$ .

(ii) For any  $m$ -tuple  $(i_1, \dots, i_m) \in \mathbb{N}^m$  with  $i_j \neq i_k$  whenever  $j \neq k$ ,

$$\mathcal{L}(X_{i_1}^N(\cdot), \dots, X_{i_m}^N(\cdot)) \rightarrow \mu^{\otimes m},$$

as probability measures on  $\mathbb{D}([0, \infty) : \mathbb{N}_0^m)$  where  $\mu$  is the probability law of  $X_1(\cdot)$  in part (i).

(iii) For any  $i \in \mathbb{N}$ , the process  $\mu^{i,N}$  denoting the occupancy measure process for the neighborhood of the  $i$ th server, defined as

$$\mu_t^{i,N} \doteq \frac{1}{D_i^N + 1} \sum_{j \in [N], j \neq i} \xi_{ij}^N \delta_{X_j^N(t)} + \frac{1}{D_i^N + 1} \delta_{X_i^N(t)}, \quad t \geq 0,
 \tag{2.8}$$

converges weakly with respect to Skorohod  $J_1$  topology to the deterministic limit  $\mu_\cdot$ , where for  $t \geq 0$ ,  $\mu_t$  is as in part (i).

REMARK 5. We note the following.

(i) The existence and uniqueness of solutions to (2.7) can be proved by standard arguments using the boundedness and Lipschitz property of the functions  $b$  and  $x \mapsto \mathbf{1}_{\{x > 0\}}$  on  $\mathbb{N}_0$ .

(ii) Using the propagation of chaos property and the fact that  $\{X_i(t) : i \in [N]\}$  are i.i.d., it follows that the limit of the global occupancy measure at any time instant  $t$  is in fact the law of  $X_i(t)$  for any fixed  $i$ . Therefore,

$$\mu_t[j, \infty) = \mathbb{P}(X_i(t) \geq j) = q_j(t), \quad j \in \mathbb{N}_0, i \in \mathbb{N} \text{ and } t \geq 0.$$

2.2. *Scaling limits for random graph sequences.* Next, we will consider the scenario when the underlying graph topology is random. We consider asymptotics of both annealed and quenched laws of the occupancy process and the queue length process at any tagged server. The following is our main condition in the study of the annealed law.

CONDITION 2 (Diverging mean degree).  $\{G_N\}_{N \geq 1}$  is a sequence of Erdős–Rényi random graphs (ERRG) where any two vertices share an edge with probability  $p_N$ , and  $Np_N \rightarrow \infty$  as  $N \rightarrow \infty$ .  $\{G_N\}_{N \geq 1}$  is independent of  $\{X_j^N(0), \mathcal{N}_i, \tilde{\mathcal{N}}_i, j \in [N], N \in \mathbb{N}, i \in \mathbb{N}\}$ .

THEOREM 2.3 (Asymptotics of annealed law). *Assume that the sequence of graphs  $\{G_N\}_{N \geq 1}$  satisfies Condition 2, and  $\{X_i^N(0) : i \in [N]\}$  is i.i.d. with  $\mathbb{P}(X_i^N(0) \geq j) = q_j^\infty, j = 0, 1, 2, \dots$ , for some  $\mathbf{q}^\infty \in S$ . Then the following hold:*

(i) For any  $T \in (0, \infty)$ ,

$$(2.9) \quad \sup_{N \geq 1} \max_{i \in [N]} \sqrt{Np_N} \mathbf{E} \|X_i^N - X_i\|_{*,T}^2 < \infty,$$

where  $X_i$  is as defined in (2.7).

(ii) For any  $m$ -tuple  $(i_1, \dots, i_m) \in \mathbb{N}^m$  with  $i_j \neq i_k$  whenever  $j \neq k$ ,

$$\mathcal{L}(X_{i_1}^N(\cdot), \dots, X_{i_m}^N(\cdot)) \rightarrow \mu^{\otimes m},$$

as probability measures on  $\mathbb{D}([0, \infty) : \mathbb{N}_0^m)$  where  $\mu$  is as in Theorem 2.2.

(iii) For any  $i \in \mathbb{N}$ , the law of the neighborhood occupancy measure process defined as in (2.8) converges weakly in Skorohod  $J_1$  topology to the deterministic limit  $\mu$ .

REMARK 6. We make the following observations.

1. In contrast to standard convergence results for weakly interacting diffusions (see, e.g., [24] or [3]), the estimate in (2.9) gives a rate of convergence of  $\sqrt{Np_N}$  instead of  $Np_N$ . The reason for this can be seen from the proof which shows that the bound for the quantity  $\mathbf{E} \|X_i^N - X_i\|_{*,T}^2$  is controlled by  $\mathbf{E} |C_i^N(s) - C_i(s)|$  rather than  $\mathbf{E} |C_i^N(s) - C_i(s)|^2$ , due to the form of indicator function in the evolution of  $X_i^N$  (cf. (2.3)).

2. Condition needed for Theorem 2.3 should be contrasted with that for Theorems 2.1 and 2.2. In particular, for the study of the annealed law asymptotics we only need information on the average degree rather than on the maximal and minimal degree of the graph.

3. All the limit theorems established in the current work have the feature that as long as there is interaction between “enough” particles the asymptotic behavior is same as that of a fully connected system. In settings where the interaction graph is very sparse, one expects different types of asymptotic behavior. Consider, for example, one extreme case when the graph is a collection of disjoint cliques of size  $d$ . In this case, the system decomposes into i.i.d. copies of a JSQ system with  $d$  servers and the limit behavior is very different. For example, a propagation of chaos result of the form in Theorem 2.2(ii) is clearly false.

We will now consider the asymptotic behavior of the quenched law of the occupancy process. For this we formulate a condition that is stronger than the one used in the study of the annealed asymptotics.

**CONDITION 3** (Condition for quenched limit).  $\{G_N\}_{N \geq 1}$  is a sequence of Erdős–Rényi random graphs, such that in  $G_N$  any two vertices share an edge with probability  $p_N$ , and  $Np_N/\ln(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .  $\{G_N\}_{N \geq 1}$  is independent of  $\{X_j^N(0), \mathcal{N}_i, \bar{\mathcal{N}}_i, j \in [N], N \in \mathbb{N}, i \in \mathbb{N}\}$ .

The following theorem provides, under the above condition, the asymptotic behavior of the quenched law.

**THEOREM 2.4** (Asymptotics of quenched law). *Assume that the sequence of graphs  $\{G_N\}_{N \geq 1}$  satisfies Condition 3, and  $\{X_i^N(0) : i \in [N]\}$  is i.i.d. with  $\mathbb{P}(X_i^N(0) \geq j) = q_j^\infty, j = 0, 1, 2, \dots$ , for some  $\mathbf{q}^\infty \in S$  for all  $N$ . Then the convergence results as stated in Theorems 2.1 and 2.2 hold for almost every realization of the random graph sequence.*

**REMARK 7.** Consider the directed ERRG where any two distinct vertices  $i, j$  have a directed edge from  $i$  to  $j$ , independently of all other distinct pairs of vertices, with probability  $p_N$ . Suppose conditions analogous to Conditions 2 and 3 hold for this directed ERRG. Then, by a minor modification of the proofs, the conclusions of Theorems 2.3 and 2.4 continue to hold. In fact, some arguments get simpler since instead of the identity  $\xi_{ij}^N = \xi_{ji}^N$  we have the independence of  $\xi_{ij}^N$  and  $\xi_{ji}^N$ .

### 3. Proofs.

3.1. *Proofs for deterministic graph sequences.* An overview of the proof idea is as follows. First, note that the queue length process at any two vertices can be exactly coupled to evolve identically if the occupancy measure of the corresponding neighborhoods are indistinguishable. The main step is to show that if the graph sequence satisfies Condition 1, then the local occupancy measure associated with the neighborhood of every server over any finite time interval converges to the same limit as for the global occupancy measure, which in turn is the same as that when the whole system uses the ordinary JSQ( $d$ ) policy and the graph is a clique. This ensures that the rate of arrival (exogenous + forwarded from the neighboring vertices) to a typical server is (asymptotically) the same as that in the clique case. Thus, the law of the number of tasks at each server, and consequently the global occupancy measure, converge to the same limit. For technical convenience, we will provide the proof of Theorem 2.2 first, and then use that to establish Theorem 2.1.

We will define the limiting processes  $(X_i(\cdot))_{i \geq 1}$  and the pre-limit processes  $(X_i^N(\cdot))_{i \geq 1}$  on the same probability space by taking the same sequence of Poisson processes  $\{\mathcal{N}_i\}$  and Poisson random measures  $\{\bar{\mathcal{N}}_i\}$  in both cases. Also, take  $X_i^N(0) = X_i(0)$  for all  $i \in [N]$ ,  $N \geq 1$ . Using Condition 1, we can find a  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$ ,

$$(3.1) \quad d_{\min}(G_N) \geq d, \quad \sup_{i \in [N]} \left| \sum_{j \in [N], j \neq i} \frac{\xi_{ji}^N}{D_j} - 1 \right| \leq \frac{1}{2},$$

$$\sup_{i \in [N]} \sup_{t \in [0, T]} |C_i^N(t)| \leq 2d.$$

For the rest of this section, we will assume that  $N \geq N_0$  and, therefore, in particular, the first and fourth terms in the definition of  $C_i^N(s)$  are zero and the indicators in the second and third terms can be replaced by 1. We will frequently suppress  $N$  in the notation  $D_i^N$  and  $\xi_{ij}^N$  and write them as  $D_i$  and  $\xi_{ij}$ , respectively. We begin with the following lemma. The proof is given at the end of the subsection.

LEMMA 3.1. *Let, for  $i \in [N]$  and  $s \in [0, T]$ ,*

$$U_s \doteq \mathbb{E} \left( \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \right)$$

and

$$V_s \doteq \mathbb{E} \left( \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(j_2; i, j_3, \dots, j_d) \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \right).$$

Under the conditions of Theorem 2.1, there exists  $K \in (0, \infty)$  such that for every  $s \in [0, T]$  and  $i \in [N]$ ,

$$(3.2) \quad U_s \leq \frac{K}{d_{\min}(G_N)},$$

$$V_s \leq \frac{K}{d_{\min}(G_N)} \left( \sum_{j=1, j \neq i}^N \frac{\xi_{ji}}{D_j} \right)^2.$$

PROOF OF THEOREM 2.2. Fix any  $i \in \mathbb{N}$  and  $T > 0$ . From (2.3) and (2.7), using Cauchy–Schwarz and Doob’s inequalities, we have for any fixed  $t \in [0, T]$

and  $N \geq i$ ,

$$\begin{aligned}
 & \mathbb{E} \|X_i^N - X_i\|_{*,t}^2 \\
 & \leq \kappa_1 \mathbb{E} \int_0^t |\mathbf{1}_{\{X_i^N(s) > 0\}} - \mathbf{1}_{\{X_i(s) > 0\}}|^2 ds \\
 & \quad + \kappa_1 \mathbb{E} \left( \int_0^t |\mathbf{1}_{\{X_i^N(s) > 0\}} - \mathbf{1}_{\{X_i(s) > 0\}}| ds \right)^2 \\
 & \quad + \kappa_1 \mathbb{E} \int_{[0,t] \times \mathbb{R}_+} |\mathbf{1}_{\{0 \leq y \leq C_i^N(s)\}} - \mathbf{1}_{\{0 \leq y \leq C_i(s)\}}|^2 ds dy \\
 (3.3) \quad & \quad + \kappa_1 \mathbb{E} \left( \int_{[0,t] \times \mathbb{R}_+} |\mathbf{1}_{\{0 \leq y \leq C_i^N(s)\}} - \mathbf{1}_{\{0 \leq y \leq C_i(s)\}}| ds dy \right)^2 \\
 & \leq \kappa_1 \int_0^t \mathbb{E} |X_i^N(s) - X_i(s)|^2 ds + \kappa_1 \mathbb{E} \left( \int_0^t |X_i^N(s) - X_i(s)| ds \right)^2 \\
 & \quad + \kappa_1 \int_0^t \mathbb{E} |C_i^N(s) - C_i(s)| ds + \kappa_1 \mathbb{E} \left( \int_0^t |C_i^N(s) - C_i(s)| ds \right)^2 \\
 & \leq \kappa_2 \int_0^t \mathbb{E} |X_i^N(s) - X_i(s)|^2 ds + \kappa_2 \int_0^t \mathbb{E} |C_i^N(s) - C_i(s)| ds
 \end{aligned}$$

for some  $\kappa_1, \kappa_2 \in (0, \infty)$ , where the last line uses (3.1) and the fact that  $0 \leq \frac{C_i(s)}{d} \leq 1$ .

Now we analyze the difference  $|C_i^N(s) - C_i(s)|$  in (3.3). Note that by adding and subtracting terms we have

$$\begin{aligned}
 (3.4) \quad & |C_i^N(s) - C_i(s)| \\
 & \leq |C_i^N(s) - C_i^{N,1}(s)| + |C_i^{N,1}(s) - C_i^{N,2}(s)| + |C_i^{N,2}(s) - C_i(s)|,
 \end{aligned}$$

where

$$\begin{aligned}
 & C_i^{N,1}(s) \\
 & = \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) \\
 & \quad + (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(j_2; i, j_3, \dots, j_d) b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))
 \end{aligned}$$

and

$$\begin{aligned}
 C_i^{N,2}(s) & = \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \frac{C_i(s)}{d} \\
 & \quad + (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(j_2; i, j_3, \dots, j_d) \frac{C_i(s)}{d}.
 \end{aligned}$$

We now analyze each term in (3.4). In particular, we will use the Lipschitz property of  $b$  to handle the term  $|C_i^N - C_i^{N,1}|$ , and then use the i.i.d. property of  $X_i$ 's to handle the term  $|C_i^{N,1} - C_i^{N,2}|$ .

First, consider  $|C_i^N(s) - C_i^{N,1}(s)|$ . From the Lipschitz property of  $b$  and the definition of  $\alpha^N$ , we have

$$\begin{aligned} & \mathbb{E}|C_i^N(s) - C_i^{N,1}(s)| \\ & \leq \mathbb{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\ & \quad \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \\ & \quad + (d - 1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(j_2; i, j_3, \dots, j_d) \\ & \quad \left. \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \right] \\ & \leq \max_{j \in [N]} \mathbb{E}|X_j^N(s) - X_j(s)| \left( d + (d - 1) d \sum_{j_2 \in [N], j_2 \neq i} \frac{\xi_{j_2 i}}{D_{j_2}} \right). \end{aligned}$$

From (3.1), we have

$$(3.5) \quad \mathbb{E}|C_i^N(s) - C_i^{N,1}(s)| \leq \kappa_3 \max_{j \in [N]} \mathbb{E}|X_j^N(s) - X_j(s)|$$

for some  $\kappa_3 \in (0, \infty)$ . Next, we consider  $|C_i^{N,1}(s) - C_i^{N,2}(s)|$ . It follows from Cauchy–Schwarz inequality that

$$\begin{aligned} & \mathbb{E}|C_i^{N,1}(s) - C_i^{N,2}(s)|^2 \\ & \leq 2 \mathbb{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\ & \quad \left. \times \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \\ & \quad + 2(d - 1)^2 \mathbb{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(j_2; i, j_3, \dots, j_d) \right. \\ & \quad \left. \times \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \\ & \leq \kappa_4 (U_s + V_s), \end{aligned}$$

where  $U_s, V_s$  are as in Lemma 3.1. From Lemma 3.1 and (3.1), we obtain

$$(\mathbb{E}|C_i^{N,1}(s) - C_i^{N,2}(s)|)^2$$



$$(3.6) \quad \leq \mathbb{E}|C_i^{N,1}(s) - C_i^{N,2}(s)|^2$$

$$(3.7) \quad \leq \frac{\kappa_5}{d_{\min}(G_N)} + \frac{\kappa_5}{d_{\min}(G_N)} \left( \sum_{j=1, j \neq i}^N \frac{\xi_{ji}}{D_j} \right)^2 \leq \frac{\kappa_6}{d_{\min}(G_N)}.$$

Finally, we consider  $|C_i^{N,2}(s) - C_i(s)|$ . Using the fact that  $0 \leq \frac{C_i(s)}{d} \leq 1$ , we have

$$(3.8) \quad \begin{aligned} \mathbb{E}|C_i^{N,2}(s) - C_i(s)| &\leq \mathbb{E} \left[ \frac{(d-1)C_i(s)}{d} \left| \sum_{j \in [N], j \neq i} \frac{\xi_{ji}}{D_j} - 1 \right| \right] \\ &\leq (d-1) \left| \sum_{j \in [N], j \neq i} \frac{\xi_{ji}}{D_j} - 1 \right|. \end{aligned}$$

Since  $|X_i^N(s) - X_i(s)|$  is nonnegative integer-valued, we have  $|X_i^N(s) - X_i(s)| \leq |X_i^N(s) - X_i(s)|^2$ . Combining this and (3.3)–(3.8) yields

$$\begin{aligned} &\max_{i \in [N]} \mathbb{E} \|X_i^N - X_i\|_{*,t}^2 \\ &\leq \kappa_7 \int_0^t \max_{i \in [N]} \mathbb{E} \|X_i^N - X_i\|_{*,s}^2 ds \\ &\quad + \kappa_7 \left( \frac{1}{(d_{\min}(G_N))^{1/2}} + \max_{i \in [N]} \left| \sum_{j \in [N], j \neq i} \frac{\xi_{ji}}{D_j} - 1 \right| \right). \end{aligned}$$

From Gronwall’s lemma and Condition 1, we have

$$(3.9) \quad \lim_{N \rightarrow \infty} \max_{i \in [N]} \mathbb{E} \|X_i^N - X_i\|_{*,T}^2 = 0,$$

which gives Theorem 2.2(i).

Given part (i), the proof of propagation of chaos property as stated in Theorem 2.2(ii) follows from standard arguments (cf. [24]), and hence is omitted. Also, having established the asymptotic result in Theorem 2.2(i), the proof of convergence of local occupancy measures as stated in Theorem 2.2(iii) can be established using similar arguments as in [3], Corollary 3.3.  $\square$

We now complete the proof of Theorem 2.1.

**PROOF OF THEOREM 2.1.** From the asymptotic result in (3.9) it follows (cf. [24] and [3], Corollary 3.3(b)) that  $\mathbf{q}^N(\cdot)$  converges weakly with respect to Skorohod  $J_1$  topology to the deterministic limit  $\tilde{\mathbf{q}}(\cdot)$  given by  $\tilde{q}_j(t) = \mu_i[j, \infty) = \mathbb{P}(X_i(t) \geq j)$  for all  $j \in \mathbb{N}_0$  and  $t \geq 0$ . However, we provide a proof here for completeness. Fix  $T < \infty$  and consider random measures  $\mu^N \doteq \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(\cdot)}$  and

$\bar{\mu}^N \doteq \frac{1}{N} \sum_{i=1}^N \delta_{X_i(\cdot)}$ , on  $\mathbb{S} \doteq \mathbb{D}([0, T] : \mathbb{N}_0)$ , where  $X_i$  is introduced in (2.7). Denote by  $d_{BL}(\cdot, \cdot)$  the bounded Lipschitz metric:

$$d_{BL}(v_1, v_2) \doteq \sup_{\|f\|_{BL} \leq 1} \left| \int_{\mathbb{S}} f dv_1 - \int_{\mathbb{S}} f dv_2 \right|,$$

$$\|f\|_{BL} \doteq \max \left\{ \|f\|_{\infty}, \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \right\}.$$

From (3.9), we have

$$\begin{aligned} \mathbb{E} d_{BL}(\mu^N, \bar{\mu}^N) &\leq \mathbb{E} \sup_{\|f\|_{BL} \leq 1} \frac{1}{N} \sum_{i=1}^N |f(X_i^N) - f(X_i)| \\ &\leq \frac{1}{N} \sum_{i=1}^N \mathbb{E} \|X_i^N - X_i\|_{*,T} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ , which implies  $d_{BL}(\mu^N, \bar{\mu}^N) \rightarrow 0$  in probability. Also note that  $\bar{\mu}^N \rightarrow \mu$  in probability by independence of  $\{X_i\}$ . Therefore  $\mu^N \rightarrow \mu$  in probability as  $N \rightarrow \infty$ . One can easily check that  $\sup_{N \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \|\mathbf{q}^N(t)\|_{\ell_1}^2 < \infty$ , from which we have  $\mathbf{q}^N \rightarrow \tilde{\mathbf{q}}$  in probability as  $N \rightarrow \infty$ .

Next, in order to prove the theorem it suffices to show that  $\tilde{\mathbf{q}}$  satisfies the system of ODE in (2.6). Define  $f_j(x) = \mathbf{1}_{\{x \geq j\}}$ ,  $j = 1, 2, \dots$ . Then equation (2.7) yields

$$\begin{aligned} \mathbb{E} f_j(X_i(t)) &= \mathbb{E} f_j(X_i(0)) + \int_0^t \mathbb{E} \mathbf{1}_{\{X_i(s) > 0\}} (f_j(X_i(s) - 1) - f_j(X_i(s))) ds \\ &\quad + \lambda d \int_0^t \int_{\mathbb{N}^{d-1}} \mathbb{E} [b(X_i(s), x_2, \dots, x_d) (f_j(X_i(s) + 1) \\ &\quad - f_j(X_i(s)))] \mu_s(dx_2) \cdots \mu_s(dx_d) ds \\ &= \mathbb{E} f_j(X_i(0)) - \int_0^t \mathbb{E} f_j(X_i(s)) - f_{j+1}(X_i(s)) ds \\ &\quad + \lambda d \int_0^t \int_{\mathbb{N}^{d-1}} \mathbb{E} [b(j - 1, x_2, \dots, x_d) (f_{j-1}(X_i(s)) \\ &\quad - f_j(X_i(s)))] \mu_s(dx_2) \cdots \mu_s(dx_d) ds. \end{aligned}$$

Since  $\mathbb{E}[f_j(X_i(t))] = \tilde{q}_j(t)$  for  $j = 1, 2, \dots$ , we obtain

$$\begin{aligned} (3.10) \quad \tilde{q}_j(t) &= \tilde{q}_j(0) - \int_0^t (\tilde{q}_j(s) - \tilde{q}_{j+1}(s)) ds + \lambda d \int_0^t (\tilde{q}_{j-1}(s) - \tilde{q}_j(s)) \\ &\quad \times \int_{\mathbb{N}^{d-1}} b(j - 1, x_2, \dots, x_d) \mu_s(dx_2) \cdots \mu_s(dx_d) ds. \end{aligned}$$

Using (2.2) and the fact that  $\tilde{q}_j(t) = \mu_t[j, \infty) = \mathbb{P}(X_i(t) \geq j)$ ,  $j = 1, 2, \dots$ , we have

$$\begin{aligned} & d(\tilde{q}_{j-1}(s) - \tilde{q}_j(s)) \int_{\mathbb{N}^{d-1}} b(j-1, x_2, \dots, x_d) \mu_s(dx_2) \cdots \mu_s(dx_d) \\ &= d(\tilde{q}_{j-1}(s) - \tilde{q}_j(s)) \sum_{k=1}^d \frac{1}{k} \binom{d-1}{k-1} (\tilde{q}_{j-1}(s) - \tilde{q}_j(s))^{k-1} (\tilde{q}_j(s))^{d-k} \\ &= \sum_{k=0}^d \binom{d}{k} (\tilde{q}_{j-1}(s) - \tilde{q}_j(s))^k (\tilde{q}_j(s))^{d-k} - (\tilde{q}_j(s))^d \\ &= (\tilde{q}_{j-1}(s))^d - (\tilde{q}_j(s))^d. \end{aligned}$$

Therefore (3.10) can be written as

$$\tilde{q}_j(t) = \tilde{q}_j(0) - \int_0^t (\tilde{q}_j(s) - \tilde{q}_{j+1}(s)) ds + \lambda \int_0^t [(\tilde{q}_{j-1}(s))^d - (\tilde{q}_j(s))^d] ds.$$

This shows that  $\tilde{q}$  satisfies the system of ODE in (2.6) and completes the proof of Theorem 2.1.  $\square$

**PROOF OF LEMMA 3.1.** We first show the first inequality in (3.2). Observe that

$$\begin{aligned} U_s = & \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \sum_{(k_2, \dots, k_d) \in \mathcal{S}_i^N} [\alpha^N(i; j_2, j_3, \dots, j_d) \alpha^N(i; k_2, k_3, \dots, k_d)] \\ & \times \mathbb{E} \left[ \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right. \\ & \left. \times \left( b(X_i(s), X_{k_2}(s), \dots, X_{k_d}(s)) - \frac{C_i(s)}{d} \right) \right]. \end{aligned}$$

Now observe that since  $\{X_i(0) : i \in [N]\}$  are i.i.d., we have  $\{X_i(s) : i \in [N]\}$  are also i.i.d. for any fixed  $s > 0$ . Thus,

$$\begin{aligned} (3.11) \quad & \mathbb{E} \left[ \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right. \\ & \left. \times \left( b(X_i(s), X_{k_2}(s), \dots, X_{k_d}(s)) - \frac{C_i(s)}{d} \right) \right] = 0 \end{aligned}$$

when  $(i, j_2, k_2, \dots, j_d, k_d)$  are distinct. Therefore, we have

$$(3.12) \quad U_s \leq \sum \alpha^N(i; j_2, j_3, \dots, j_d) \alpha^N(i; k_2, k_3, \dots, k_d),$$

where the summation is taken over

$$(3.13) \quad \begin{aligned} \hat{\mathcal{S}}_i^N \doteq & \{(j_2, \dots, j_d) \in \mathcal{S}_i^N, (k_2, \dots, k_d) \in \mathcal{S}_i^N, \\ & (j_2, k_2, \dots, j_d, k_d) \text{ are not distinct}\} \end{aligned}$$

and the inequality follows since  $0 \leq b \leq 1$  and  $0 \leq \frac{C_i(s)}{d} \leq 1$ . Since the total number of combinations in (3.13) such that  $(\xi_{ij_2}\xi_{ij_3} \cdots \xi_{ij_d})(\xi_{ik_2}\xi_{ik_3} \cdots \xi_{ik_d}) = 1$  is no more than

$$(3.14) \quad \left[ (d-1)! \binom{D_i}{d-1} \right]^2 - (2d-2)! \binom{D_i}{2d-2} \leq \kappa_1 D_i^{2d-3},$$

we can bound (3.12) by

$$\frac{\kappa_1 D_i^{2d-3}}{D_i^2 (D_i - 1)^2 \cdots (D_i - d + 2)^2} \leq \kappa_2 \frac{1}{D_i} \leq \frac{\kappa_2}{d_{\min}(G_N)}.$$

This gives the first bound in (3.2).

Next, we show the second bound in (3.2). From (3.11), it follows from the same argument used for (3.12) that

$$(3.15) \quad V_s \leq \sum \alpha^N(j_2; i, j_3, \dots, j_d) \alpha^N(k_2; i, k_3, \dots, k_d),$$

where the summation is taken over (3.13). Since for fixed  $(j_2, k_2) \in \bar{S}_i$ , where

$$(3.16) \quad \bar{S}_i \doteq \{(j, k) \in [N]^2 : j \neq i, k \neq i\},$$

the total number of combinations in (3.13) such that  $(\xi_{j_2i}\xi_{j_2j_3} \cdots \xi_{j_2j_d})(\xi_{k_2i}\xi_{k_2k_3} \cdots \xi_{k_2k_d}) = 1$  is no more than

$$(3.17) \quad \begin{aligned} & \left[ (d-2)! \binom{D_{j_2}-1}{d-2} \right] \left[ (d-2)! \binom{D_{k_2}-1}{d-2} \right] \\ & - \left[ (d-2)! \binom{D_{j_2}-2}{d-2} \right] \left[ (d-2)! \binom{D_{k_2}-d}{d-2} \right] \\ & \leq \kappa_3 (D_{j_2}^{d-3} D_{k_2}^{d-2} + D_{j_2}^{d-2} D_{k_2}^{d-3}), \end{aligned}$$

where the second term in the first line corresponds to choosing distinct  $j_3, \dots, j_d$  from  $D_{j_2} - 2$  neighbors (excluding  $i, k_2$ ) of  $j_2$  and then choosing distinct  $k_3, \dots, k_d$  from  $D_{k_2} - d$  neighbors (excluding  $i, j_2, \dots, j_d$ ) of  $k_2$ . Now, we can bound (3.15) by

$$\begin{aligned} & \sum_{(j_2, k_2) \in \bar{S}_i} \frac{\kappa_3 (D_{j_2}^{d-3} D_{k_2}^{d-2} + D_{j_2}^{d-2} D_{k_2}^{d-3}) \xi_{j_2i} \xi_{k_2i}}{D_{j_2} (D_{j_2} - 1) \cdots (D_{j_2} - d + 2) D_{k_2} (D_{k_2} - 1) \cdots (D_{k_2} - d + 2)} \\ & \leq \kappa_4 \sum_{(j_2, k_2) \in \bar{S}_i} \left( \frac{\xi_{j_2i} \xi_{k_2i}}{D_{j_2}^2 D_{k_2}} + \frac{\xi_{j_2i} \xi_{k_2i}}{D_{j_2} D_{k_2}^2} \right) \\ & \leq \kappa_4 \frac{2}{d_{\min}(G_N)} \left( \sum_{j=1, j \neq i}^N \frac{\xi_{ji}}{D_j} \right)^2. \end{aligned}$$

This completes the proof.  $\square$

3.2. *Proofs for random graph sequences.* In this section, we give the proofs of Theorems 2.3 and 2.4. As in the proof of Theorem 2.2, we will define the limiting processes  $(X_i(\cdot))_{i \geq 1}$  and the pre-limit processes  $(X_i^N(\cdot))_{i \geq 1}$  on the same probability space by taking identical sequence of Poisson processes  $\{N_i\}$  and Poisson random measures  $\{\tilde{N}_i\}$  in both cases. The random graph sequence  $\{G_N\}$  will also be given on this common probability space and is taken to be independent of the Poisson processes and Poisson random measures. Finally, we take  $X_i^N(0) = X_i(0)$  for all  $i \in [N]$ ,  $N \geq 1$ . Once again, we will frequently suppress  $N$  in the notation  $D_i^N$  and write it as  $D_i$ . We begin with three lemmas that will be used in the proof. Let, for  $s \geq 0$ ,

$$(3.18) \quad U_s^A \doteq \mathbf{E} \left( \left[ \mathbf{1}_{\{D_i^N \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \times \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \right)$$

and

$$(3.19) \quad V_s^A \doteq \mathbf{E} \left( \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2}^N \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \times \left( b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d} \right) \right]^2 \right).$$

Note that the dependence of  $U_s^A$  and  $V_s^A$  on  $i$  is suppressed in the notation. The next lemma provides uniform bounds on  $U_s^A$  and  $V_s^A$ .

LEMMA 3.2. *Fix  $T \geq 0$ . Under the conditions of Theorem 2.3, there exists  $\kappa \in (0, \infty)$  such that for every  $s \in [0, T]$  and  $i \in [N]$ ,*

$$U_s^A \leq \frac{\kappa}{Np_N} \quad \text{and} \quad V_s^A \leq \frac{\kappa}{Np_N} + \frac{\kappa}{(Np_N)^2}.$$

The proof of Lemma 3.2 follows along similar lines as the proof of Lemma 3.1; however, note that the expectations in (3.18) and (3.19) are taken also over the randomness of the graph topology, and thus we need additional arguments. Proof of Lemma 3.2 is provided at the end of this subsection.

The next lemma is taken from [3].

LEMMA 3.3 ([3], Lemma 5.2). *Let  $G_N$  be an ERRG with connection probability  $p_N$ . Then*

$$\mathbf{E} \left( \sum_{j \in [N], j \neq i} \frac{\xi_{ij}^N}{D_j^N} \mathbf{1}_{\{D_j^N > 0\}} - 1 \right)^2 \leq \frac{4}{Np_N} + 2e^{-Np_N}, \quad i \in [N].$$

The following lemma provides useful moment bounds on  $|X_i^N - X_i|$  and its proof is given at the end of this subsection.

LEMMA 3.4. *Fix  $T \geq 0$ . Under the conditions of Theorem 2.3,*

$$\sup_{N \geq 1} \max_{i \in [N]} \mathbf{E} \|X_i^N - X_i\|_{*,T}^4 < \infty.$$

We now present the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. Fix any  $i \in \mathbb{N}$  and  $T > 0$ . From (2.3) and (2.7), by adding and subtracting the compensator, using Cauchy–Schwarz and Doob’s inequalities we have for any fixed  $t \in [0, T]$

$$\begin{aligned} & \mathbf{E} \|X_i^N - X_i\|_{*,t}^2 \\ & \leq \kappa_1 \mathbf{E} \int_0^t |\mathbf{1}_{\{X_i^N(s) > 0\}} - \mathbf{1}_{\{X_i(s) > 0\}}|^2 ds \\ & \quad + \kappa_1 \mathbf{E} \left( \int_0^t |\mathbf{1}_{\{X_i^N(s) > 0\}} - \mathbf{1}_{\{X_i(s) > 0\}}| ds \right)^2 \\ & \quad + \kappa_1 \mathbf{E} \int_{[0,t] \times \mathbb{R}_+} |\mathbf{1}_{\{0 \leq y \leq C_i^N(s)\}} - \mathbf{1}_{\{0 \leq y \leq C_i(s)\}}|^2 ds dy \\ & \quad + \kappa_1 \mathbf{E} \left( \int_{[0,t] \times \mathbb{R}_+} |\mathbf{1}_{\{0 \leq y \leq C_i^N(s)\}} - \mathbf{1}_{\{0 \leq y \leq C_i(s)\}}| ds dy \right)^2 \\ (3.20) \quad & \leq \kappa_1 \int_0^t \mathbf{E} |X_i^N(s) - X_i(s)|^2 ds + \kappa_1 \mathbf{E} \left( \int_0^t |X_i^N(s) - X_i(s)| ds \right)^2 \\ & \quad + \kappa_1 \int_0^t \mathbf{E} |C_i^N(s) - C_i(s)| ds + \kappa_1 \mathbf{E} \left( \int_0^t |C_i^N(s) - C_i(s)| ds \right)^2 \\ & \leq \kappa_1(1 + T) \int_0^t \mathbf{E} |X_i^N(s) - X_i(s)|^2 ds \\ & \quad + \kappa_1 \int_0^t \mathbf{E} |C_i^N(s) - C_i(s)| ds \\ & \quad + \kappa_1 T \int_0^t \mathbf{E} |C_i^N(s) - C_i(s)|^2 ds \end{aligned}$$

for some  $\kappa_1 \in (0, \infty)$ . Define  $C_i^{N,1}(s)$  and  $C_i^{N,2}(s)$  by

$$\begin{aligned} C_i^{N,1}(s) &= \mathbf{1}_{\{D_i < d-1\}} \bar{b}_i \left( (X_k^N(s))_{k \in [N]}, (\xi_{kl})_{k,l \in [N]} \right) \\ & \quad + \mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \end{aligned}$$

$$\begin{aligned} &\times b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) \\ &+ (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \\ &\times b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) \\ &+ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1\}} \xi_{ij_2} \bar{b}_{ij_2} ((X_k^N(s))_{k \in [N]}, (\xi_{kl})_{k, l \in [N]}) \end{aligned}$$

and

$$\begin{aligned} C_i^{N,2}(s) &= \mathbf{1}_{\{D_i < d-1\}} \bar{b}_i((X_k^N(s))_{k \in [N]}, (\xi_{kl})_{k, l \in [N]}) \\ &+ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \frac{C_i(s)}{d} \\ &+ (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \frac{C_i(s)}{d} \\ &+ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1\}} \xi_{ij_2} \bar{b}_{ij_2} ((X_k^N(s))_{k \in [N]}, (\xi_{kl})_{k, l \in [N]}). \end{aligned}$$

By adding and subtracting terms, we have (3.4) and

$$\begin{aligned} &|C_i^N(s) - C_i(s)|^2 \\ (3.21) \quad &\leq 3|C_i^N(s) - C_i^{N,1}(s)|^2 \\ &+ 3|C_i^{N,1}(s) - C_i^{N,2}(s)|^2 + 3|C_i^{N,2}(s) - C_i(s)|^2. \end{aligned}$$

Here, although one has  $\mathbf{E}|C_i^N(s) - C_i(s)| \leq (\mathbf{E}|C_i^N(s) - C_i(s)|^2)^{1/2}$ , in order to get the desired rate  $\sqrt{Np_N}$  in (2.9), we have to estimate  $\mathbf{E}|C_i^N(s) - C_i(s)|$  more carefully through (3.4).

Let us consider  $|C_i^N(s) - C_i^{N,1}(s)|$  and  $|C_i^N(s) - C_i^{N,1}(s)|^2$  first. We claim that for  $m = 1, 2$ , there exists some  $\kappa_2 \in (0, \infty)$  such that

$$\begin{aligned} &\mathbf{E}|C_i^N(s) - C_i^{N,1}(s)|^m \\ &\leq \kappa_2 \mathbf{E}|X_i^N(s) - X_i(s)|^m \\ (3.22) \quad &+ \kappa_2 \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)|^m \right] \\ &+ \kappa_2 \left( \frac{1}{Np_N} + e^{-Np_N} \right)^{1/2}. \end{aligned}$$

To see this, note that from the Lipschitz property of  $b$  and the definition of  $\mathcal{S}_i^N$  we have

$$\begin{aligned}
 & \mathbf{E}|C_i^N(s) - C_i^{N,1}(s)| \\
 & \leq \mathbf{E}\left[\mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\
 & \quad \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \\
 & \quad + (d-1) \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \\
 & \quad \left. \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \right] \\
 & = d\mathbf{E}\left[\mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\
 & \quad \left. \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \right] \\
 & \leq d\mathbf{E}|X_i^N(s) - X_i(s)| + d(d-1) \\
 & \quad \times \mathbf{E}\left[\mathbf{1}_{\{D_i \geq d-1\}} \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)|\right],
 \end{aligned}$$

where in obtaining the equality we have used the exchangeability property:

$$\begin{aligned}
 & \mathcal{L}(\xi_{ij_2}, \xi_{ij_3}, \dots, \xi_{ij_d}, D_i, X_i^N(s), X_i(s), X_{j_2}^N(s), X_{j_2}(s), \\
 & X_{j_3}^N(s), X_{j_3}(s), \dots, X_{j_d}^N(s), X_{j_d}(s)) \\
 (3.23) \quad & = \mathcal{L}(\xi_{j_2i}, \xi_{j_2j_3}, \dots, \xi_{j_2j_d}, D_{j_2}, X_{j_2}^N(s), X_{j_2}(s), \\
 & X_i^N(s), X_i(s), X_{j_3}^N(s), X_{j_3}(s), \dots, X_{j_d}^N(s), X_{j_d}(s))
 \end{aligned}$$

for  $(j_2, \dots, j_d) \in \mathcal{S}_i^N$ . Therefore, the claim (3.22) holds for  $m = 1$ . Next, we verify (3.22) when  $m = 2$ . Note that

$$\mathbf{E}|C_i^N(s) - C_i^{N,1}(s)|^2 \leq 2R_i^{N,1}(s) + 2(d-1)^2R_i^{N,2}(s),$$

where

$$\begin{aligned}
 R_i^{N,1}(s) & \doteq \mathbf{E}\left[\mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\
 & \quad \left. \times [b(X_i^N(s), X_{j_2}^N(s), \dots, X_{j_d}^N(s)) - b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))] \right]^2,
 \end{aligned}$$



$$R_i^{N,2}(s) \doteq \mathbf{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \times [b(X_i^N(s), X_{j_2}^N(s), \dots, X_{j_d}^N(s)) - b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))] \right]^2.$$

From the Lipschitz property of  $b$ , the definition of  $\mathcal{S}_i^N$  and Cauchy–Schwarz inequality we have

$$\begin{aligned} R_i^{N,1}(s) &\leq \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\ &\quad \left. \times (|X_i^N(s) - X_i(s)| + |X_{j_2}^N(s) - X_{j_2}(s)| + \dots + |X_{j_d}^N(s) - X_{j_d}(s)|) \right]^2 \\ &= \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \left( |X_i^N(s) - X_i(s)| \right. \right. \\ &\quad \left. \left. + (d-1) \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)| \right) \right]^2 \\ &\leq 2\mathbf{E} |X_i^N(s) - X_i(s)|^2 + 2(d-1)^2 \\ &\quad \times \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)|^2 \right]. \end{aligned}$$

From the Cauchy–Schwarz inequality  $(\sum a_i b_i)^2 \leq (\sum a_i)(\sum a_i b_i^2)$  for nonnegative  $a_i$ 's, we have

$$\begin{aligned} R_i^{N,2}(s) &\leq \mathbf{E} \left\{ \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right] \right. \\ &\quad \times \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right. \\ &\quad \left. \left. \times [b(X_i^N(s), X_{j_2}^N(s), \dots, X_{j_d}^N(s)) - b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))] \right]^2 \right\} \\ &= \mathbf{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right. \\ &\quad \left. \times [b(X_i^N(s), X_{j_2}^N(s), \dots, X_{j_d}^N(s)) - b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))] \right]^2 \end{aligned}$$

$$\begin{aligned}
 & + \mathbf{E} \left\{ \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j \geq d-1\}} \frac{\xi_{ji}}{D_j} - 1 \right] \right. \\
 & \times \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right. \\
 & \left. \left. \times [b(X_i^N(s), X_{j_2}^N(s), \dots, X_{j_d}^N(s)) - b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s))]^2 \right] \right\} \\
 & \doteq R_i^{N,3}(s) + R_i^{N,4}(s),
 \end{aligned}$$

where the equality follows by adding and subtracting one in the first term. From the Lipschitz property of  $b$ , the definition of  $\mathcal{S}_i^N$  and the exchangeability property (3.23) we have

$$\begin{aligned}
 R_i^{N,3}(s) & \leq d \mathbf{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right. \\
 & \quad \times (|X_i^N(s) - X_i(s)|^2 + |X_{j_2}^N(s) - X_{j_2}(s)|^2 + \dots \\
 & \quad \left. + |X_{j_d}^N(s) - X_{j_d}(s)|^2) \right] \\
 & = d \mathbf{E} \left[ \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_i \geq d-1\}} \alpha^N(i; j_2, j_3, \dots, j_d) \right. \\
 & \quad \times (|X_i^N(s) - X_i(s)|^2 + |X_{j_2}^N(s) - X_{j_2}(s)|^2 + \dots \\
 & \quad \left. + |X_{j_d}^N(s) - X_{j_d}(s)|^2) \right] \\
 & \leq d \mathbf{E} |X_i^N(s) - X_i(s)|^2 + d(d-1) \\
 & \quad \times \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)|^2 \right].
 \end{aligned}$$

From the fact that  $\|b\|_\infty \leq 1$ , we have

$$\begin{aligned}
 R_i^{N,4}(s) & \leq \mathbf{E} \left\{ \left| \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1 \right| \right. \\
 & \quad \left. \times \left[ 4 \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \right] \right\} \\
 & \leq 4 \mathbf{E} \left\{ \left| \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1 \right| \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} \right\}
 \end{aligned}$$

$$\leq \kappa_3 \left( \frac{1}{Np_N} + e^{-Np_N} \right)^{1/2},$$

where the last inequality follows from Lemma 3.3 and Condition 2. Combining the above estimates on  $R_i^{N,k}(s)$  for  $k = 1, 2, 3, 4$  gives the claim (3.22) when  $m = 2$ .

Now using the exchangeability property:

$$\mathcal{L}(\xi_{ij}, D_i, X_j^N(s), X_j(s)) = \mathcal{L}(\xi_{ji}, D_j, X_i^N(s), X_i(s)), \quad i \neq j,$$

we have for  $m = 1, 2$ ,

$$\begin{aligned} & \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \sum_{j \in [N], j \neq i} \frac{\xi_{ij}}{D_i} |X_j^N(s) - X_j(s)|^m \right] \\ &= \mathbf{E} \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j \geq d-1\}} \frac{\xi_{ji}}{D_j} |X_i^N(s) - X_i(s)|^m \right] \\ &\leq \mathbf{E} \left[ \left( \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1 \right) |X_i^N(s) - X_i(s)|^m \right] \\ &\quad + \mathbf{E} |X_i^N(s) - X_i(s)|^m \\ &\leq \left[ \mathbf{E} \left( \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1 \right)^2 \mathbf{E} |X_i^N(s) - X_i(s)|^{2m} \right]^{1/2} \\ &\quad + \mathbf{E} |X_i^N(s) - X_i(s)|^m \\ &\leq \kappa_4 \left( \frac{1}{Np_N} + e^{-Np_N} \right)^{1/2} + \mathbf{E} |X_i^N(s) - X_i(s)|^m, \end{aligned}$$

where the second inequality follows from Cauchy–Schwarz inequality and the last line follows from Lemmas 3.3 and 3.4. Combining this, (3.22) with the fact that  $|X_i^N(s) - X_i(s)| \leq |X_i^N(s) - X_i(s)|^2$  gives

$$\begin{aligned} & \mathbf{E} |C_i^N(s) - C_i^{N,1}(s)| + \mathbf{E} |C_i^N(s) - C_i^{N,1}(s)|^2 \\ (3.24) \quad & \leq \kappa_5 \mathbf{E} |X_i^N(s) - X_i(s)|^2 + \kappa_5 \left( \frac{1}{Np_N} + e^{-Np_N} \right)^{1/2}. \end{aligned}$$

Next, we consider  $|C_i^{N,1}(s) - C_i^{N,2}(s)|^2$ . From the inequality  $(a + b)^2 \leq 2a^2 + 2b^2$ , it follows that

$$\begin{aligned} & (\mathbf{E} |C_i^{N,1}(s) - C_i^{N,2}(s)|)^2 \\ (3.25) \quad & \leq \mathbf{E} |C_i^{N,1}(s) - C_i^{N,2}(s)|^2 \\ & \leq 2U_s^A + 2(d - 1)^2 V_s^A \leq \frac{\kappa_6}{Np_N} + \frac{\kappa_6}{(Np_N)^2}, \end{aligned}$$

where  $U_s^A$  and  $V_s^A$  were introduced in (3.18) and (3.19) and the last inequality is from Lemma 3.2.

Finally, we consider  $|C_i^{N,2}(s) - C_i(s)|^2$ . Note that  $C_i^{N,2}(s)$  can be rewritten as

$$\begin{aligned}
 C_i^{N,2}(s) &= \mathbf{1}_{\{D_i < d-1\}} \bar{b}_i((X_k^N(t))_{k \in [N]}, (\xi_{kl})_{k,l \in [N]}) \\
 &\quad + \mathbf{1}_{\{D_i \geq d-1\}} \frac{C_i(s)}{d} + (d-1) \\
 &\quad \times \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j \geq d-1\}} \frac{\xi_{ji}}{D_j} \frac{C_i(s)}{d} \\
 &\quad + \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij} \bar{b}_{ij}((X_k^N(t))_{k \in [N]}, (\xi_{kl})_{k,l \in [N]}).
 \end{aligned}$$

Using the Cauchy–Schwarz inequality and the fact that  $0 \leq \frac{C_i(s)}{d} \leq 1$ , we have

$$\begin{aligned}
 &\mathbf{E}|C_i^{N,2}(s) - C_i(s)|^2 \\
 &\leq 5\mathbf{E}[\mathbf{1}_{\{D_i < d-1\}}(D_i + 1)]^2 + 5\mathbf{E}\left[\mathbf{1}_{\{D_i < d-1\}} \frac{C_i(s)}{d}\right]^2 \\
 &\quad + 5\mathbf{E}\left[(d-1) \sum_{j \in [N], j \neq i} \mathbf{1}_{\{0 < D_j < d-1\}} \frac{\xi_{ji}}{D_j} \frac{C_i(s)}{d}\right]^2 \\
 &\quad + 5\mathbf{E}\left[\frac{(d-1)C_i(s)}{d} \left| \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1 \right|\right]^2 \\
 (3.26) \quad &\quad + 5\mathbf{E}\left[\sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij}(D_i + 1)\right]^2 \\
 &\leq 5(d^2 + 1)\mathbf{P}(D_i < d-1) \\
 &\quad + 5(d-1)^2\mathbf{E}\left[\sum_{j \in [N], j \neq i} \mathbf{1}_{\{0 < D_j < d-1\}} \frac{\xi_{ji}}{D_j}\right]^2 \\
 &\quad + 5(d-1)^2\mathbf{E}\left[\sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j > 0\}} \frac{\xi_{ji}}{D_j} - 1\right]^2 \\
 &\quad + 5\mathbf{E}\left[\sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij}(D_i + 1)\right]^2.
 \end{aligned}$$

Note that on the right-hand side of (3.26), the second term can be bounded by the last term as follows:

$$\begin{aligned} \mathbf{E} \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{0 < D_j < d-1\}} \frac{\xi_{ji}}{D_j} \right]^2 &\leq \mathbf{E} \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ji} \right]^2 \\ &\leq \mathbf{E} \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij} (D_i + 1) \right]^2. \end{aligned}$$

For the last term in (3.26), we have

$$\begin{aligned} &\mathbf{E} \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij} (D_i + 1) \right]^2 \\ &\leq \mathbf{E} \left\{ \left[ \sum_{j \in [N], j \neq i} \mathbf{1}_{\{D_j < d-1\}} \xi_{ij} (D_i + 1)^2 \right] \left[ \sum_{j \in [N], j \neq i} \xi_{ij} \right] \right\} \\ &= \sum_{j \in [N], j \neq i} \mathbf{E} [\mathbf{1}_{\{D_j < d-1\}} \xi_{ij} (D_i + 1)^2 D_i] \\ &= \sum_{j \in [N], j \neq i} \mathbf{E} [\mathbf{1}_{\{D_j - \xi_{ij} + 1 < d-1\}} (D_i - \xi_{ij} + 2)^2 (D_i - \xi_{ij} + 1)] p_N \\ &\leq \kappa_7 (N - 1) \mathbf{P}(D_i < d) (N p_N + 1)^3 p_N, \end{aligned}$$

where the first inequality follows from Cauchy–Schwarz inequality, the second equality follows by conditioning on  $\xi_{ij} = 1$ , and the last inequality follows from independence, Condition 2 and moment estimates of binomial random variables. Furthermore, note that

$$\begin{aligned} \mathbf{P}(D_i < d) &= \sum_{k=0}^{d-1} \binom{N-1}{k} p_N^k (1-p_N)^{N-1-k} \\ (3.27) \quad &\leq \kappa_8 (1-p_N)^{N-d} [1 + N p_N + \dots + (N p_N)^{d-1}] \\ &\leq \kappa_9 [1 + (N p_N)^{d-1}] e^{-(N-d)p_N}. \end{aligned}$$

Combining the above four estimates with Lemma 3.3 gives

$$\begin{aligned} &(\mathbf{E} |C_i^{N,2}(s) - C_i(s)|)^2 \\ (3.28) \quad &\leq \mathbf{E} |C_i^{N,2}(s) - C_i(s)|^2 \leq \kappa_0 [1 + (N p_N)^{d+3}] e^{-N p_N} \\ &\quad + \kappa_0 \left( \frac{1}{N p_N} + e^{-N p_N} \right). \end{aligned}$$

Combining (3.4), (3.20), (3.21), (3.24), (3.25), (3.28) and Condition 2 gives us

$$\max_{i \in [N]} \sqrt{N p_N} \mathbf{E} \|X_i^N - X_i\|_{*,t}^2 \leq \kappa \int_0^t \max_{i \in [N]} \sqrt{N p_N} \mathbf{E} \|X_i^N - X_i\|_{*,s}^2 ds + \kappa.$$

Part (i) of the theorem now follows from Gronwall’s lemma.

The proof of propagation of chaos property as stated in Theorem 2.3(ii) follows now from standard arguments (cf. [24]), and hence is omitted. Also, having proved Theorem 2.3(i), the proof of convergence of local occupancy measures as stated in Theorem 2.3(iii) can be established using similar arguments as in [3], Corollary 3.3.  $\square$

We now complete the proof of Theorem 2.4.

**PROOF OF THEOREM 2.4.** In order to prove the theorem it suffices, in view of Theorems 2.1 and 2.2, to show that if  $\{G_N\}$  satisfies Condition 3, then it satisfies Condition 1 a.s.

Using the Chernoff inequality (cf. [7], Theorem 2.4), it follows that for every  $x \geq 0$  and  $N \in \mathbb{N}$ ,

$$\mathbf{P}(|D_i^N - \mathbf{E}D_i^N| \geq x) \leq 2 \exp\left\{-\frac{x^2}{2\mathbf{E}D_i^N + 2x/3}\right\}.$$

Let  $k(N) \doteq Np_N/\ln(N)$ . Note that by Condition 3,  $k(N) \rightarrow \infty$  as  $N \rightarrow \infty$ . Since  $\mathbf{E}D_i^N = (N - 1)p_N$  taking  $x = x(N) = \ln(N)(k(N))^{3/4}$  in the above expression yields, for some  $\kappa_1 \in (0, \infty)$ ,

$$\begin{aligned} \mathbf{P}(|D_i^N - Np_N| \geq x(N)) &\leq \mathbf{P}(|D_i^N - \mathbf{E}D_i^N| \geq x(N) - p_N) \\ (3.29) \qquad \qquad \qquad &\leq 2 \exp\left\{-\frac{(x(N) - p_N)^2}{2(N - 1)p_N + 2(x(N) - p_N)/3}\right\} \\ &\leq \kappa_1 \exp\left\{-\kappa_1 \frac{(x(N))^2}{Np_N}\right\}, \end{aligned}$$

for sufficiently large  $N$ . Thus

$$(3.30) \qquad \mathbf{P}\left(\bigcup_{i \in [N]} \{|D_i^N - Np_N| \geq x(N)\}\right) \leq \kappa_1 N \exp\left\{-\kappa_1 \frac{(x(N))^2}{Np_N}\right\}.$$

From the choice of  $x(N)$ , we have  $(x(N))^2/[Np_N \ln(N)] \rightarrow \infty$ , as  $N \rightarrow \infty$ . Therefore, the right-hand side of (3.30) is summable over  $N$ . From the Borel–Cantelli lemma, we conclude a.s., for all sufficiently large  $N$ ,

$$|D_i^N - Np_N| \leq x(N), \qquad i \in [N]$$

and, therefore, for all such  $N$ ,

$$(3.31) \qquad Np_N - x(N) \leq d_{\min}(G_N) \leq d_{\max}(G_N) \leq Np_N + x(N).$$

Finally, observe that

$$\frac{x(N)}{Np_N} = \frac{\ln(N)(k(N))^{3/4}}{k(N) \ln(N)} = \frac{1}{(k(N))^{1/4}} \rightarrow 0 \qquad \text{as } N \rightarrow \infty.$$

Combining the two displays,  $d_{\min}(G_N) \rightarrow \infty$  and

$$\frac{d_{\max}(G_N) - d_{\min}(G_N)}{d_{\min}(G_N)} = \frac{2x(N)}{Np_N - x(N)} \rightarrow 0,$$

as  $N \rightarrow \infty$ . This together with Remark 1 shows that Condition 1 holds for  $\{G_N\}$  a.s., completing the proof of Theorem 2.4.  $\square$

We now complete the proof of Lemma 3.2. We begin with the following lemma from [3].

LEMMA 3.5 ([3], Lemma 5.1). *Let  $X$  be a Binomial random variable with number of trials  $N$  and probability of success  $p$ . Let  $q \doteq 1 - p$ . Then for each  $m \in \mathbb{N}$ ,*

$$\mathbb{E}\left[\mathbf{1}_{\{X>0\}} \frac{1}{(2X)^m}\right] \leq \mathbb{E}\frac{1}{(X+1)^m} \leq \frac{m^m}{(N+1)^m p^m}.$$

PROOF OF LEMMA 3.2. As before, we will omit the superscript in  $\xi_{ij}$ 's and  $D_i$ 's for notational convenience. We first show (3.18). From the independence between  $\{X_i\}$  and  $\{\xi_{ij}\}$ , it follows that

$$\begin{aligned} U_s^A &= \sum_{(j_2, \dots, j_d) \in \mathcal{S}_i^N} \sum_{(k_2, \dots, k_d) \in \mathcal{S}_i^N} \mathbf{E}[\mathbf{1}_{\{D_i \geq d-1\}} \alpha^N(i; j_2, j_3, \dots, j_d) \\ &\quad \times \alpha^N(i; k_2, k_3, \dots, k_d)] \\ &\quad \times \mathbf{E}\left[\left(b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d}\right) \right. \\ &\quad \left. \times \left(b(X_i(s), X_{k_2}(s), \dots, X_{k_d}(s)) - \frac{C_i(s)}{d}\right)\right]. \end{aligned}$$

Noting that

$$\begin{aligned} (3.32) \quad &\mathbf{E}\left[\left(b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d}\right) \right. \\ &\quad \left. \times \left(b(X_i(s), X_{k_2}(s), \dots, X_{k_d}(s)) - \frac{C_i(s)}{d}\right)\right] = 0 \end{aligned}$$

when  $(i, j_2, k_2, \dots, j_d, k_d)$  are distinct, we have

$$\begin{aligned} (3.33) \quad U_s^A &= \sum \mathbf{E}[\mathbf{1}_{\{D_i \geq d-1\}} \alpha^N(i; j_2, j_3, \dots, j_d) \alpha^N(i; k_2, k_3, \dots, k_d)] \\ &\quad \times \mathbf{E}\left[\left(b(X_i(s), X_{j_2}(s), \dots, X_{j_d}(s)) - \frac{C_i(s)}{d}\right) \right. \\ &\quad \left. \times \left(b(X_i(s), X_{k_2}(s), \dots, X_{k_d}(s)) - \frac{C_i(s)}{d}\right)\right] \\ &\leq \mathbf{E}\left[\sum \mathbf{1}_{\{D_i \geq d-1\}} \alpha^N(i; j_2, j_3, \dots, j_d) \alpha^N(i; k_2, k_3, \dots, k_d)\right], \end{aligned}$$

where the summation is taken over the collection  $\hat{\mathcal{S}}_i^N$  defined in (3.13) and the inequality follows since  $0 \leq b \leq 1$  and  $0 \leq \frac{C_i(s)}{d} \leq 1$ . As noted in (3.14), the total number of combinations in (3.13) such that  $(\xi_{ij_2}\xi_{ij_3} \cdots \xi_{ij_d})(\xi_{ik_2}\xi_{ik_3} \cdots \xi_{ik_d}) = 1$  is no more than  $\kappa_1 D_i^{2d-3}$ , and thus we can bound (3.33) by

$$\mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \frac{\kappa_1 D_i^{2d-3}}{D_i^2 (D_i - 1)^2 \cdots (D_i - d + 2)^2} \right] \leq \kappa_2 \mathbf{E} \left[ \mathbf{1}_{\{D_i > 0\}} \frac{1}{D_i} \right] \leq \frac{2\kappa_2}{Np_N},$$

where the last inequality uses Lemma 3.5. This gives the first inequality in Lemma 3.2.

Next, we show the second inequality in Lemma 3.2. From the independence between  $\{X_i\}$  and  $\{\xi_{ij}\}$  and (3.32), it follows from the same argument used for (3.33) that

$$(3.34) \quad V_s^A \leq \mathbf{E} \left[ \sum \mathbf{1}_{\{D_{j_2} \geq d-1\}} \mathbf{1}_{\{D_{k_2} \geq d-1\}} \alpha^N(j_2; i, j_3, \dots, j_d) \times \alpha^N(k_2; i, k_3, \dots, k_d) \right],$$

where the summation is taken over  $\hat{\mathcal{S}}_i^N$  defined in (3.13). As noted in (3.17), for fixed  $(j_2, k_2) \in \bar{\mathcal{S}}_i$  with  $\bar{\mathcal{S}}_i$  as in (3.16), the total number of combinations in  $\hat{\mathcal{S}}_i^N$  such that

$$(\xi_{j_2i}\xi_{j_2j_3} \cdots \xi_{j_2j_d})(\xi_{k_2i}\xi_{k_2k_3} \cdots \xi_{k_2k_d}) = 1$$

is no more than  $\kappa_3(D_{j_2}^{d-3}D_{k_2}^{d-2} + D_{j_2}^{d-2}D_{k_2}^{d-3})$  we can bound (3.34) by

$$(3.35) \quad \begin{aligned} & \mathbf{E} \left[ \sum_{(j_2, k_2) \in \bar{\mathcal{S}}_i} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \mathbf{1}_{\{D_{k_2} \geq d-1\}} \right. \\ & \times \left. \frac{\kappa_3(D_{j_2}^{d-3}D_{k_2}^{d-2} + D_{j_2}^{d-2}D_{k_2}^{d-3})\xi_{j_2i}\xi_{k_2i}}{D_{j_2}(D_{j_2} - 1) \cdots (D_{j_2} - d + 2)D_{k_2}(D_{k_2} - 1) \cdots (D_{k_2} - d + 2)} \right] \\ & \leq \kappa_4 \sum_{(j_2, k_2) \in \bar{\mathcal{S}}_i} \mathbf{E} \left[ \mathbf{1}_{\{D_{j_2} \geq d-1\}} \mathbf{1}_{\{D_{k_2} \geq d-1\}} \left( \frac{\xi_{j_2i}\xi_{k_2i}}{D_{j_2}^2 D_{k_2}} + \frac{\xi_{j_2i}\xi_{k_2i}}{D_{j_2} D_{k_2}^2} \right) \right] \\ & = 2\kappa_4 \sum_{(j, k) \in \bar{\mathcal{S}}_i} \mathbf{E} \left[ \mathbf{1}_{\{D_j \geq d-1\}} \mathbf{1}_{\{D_k \geq d-1\}} \frac{\xi_{ji}\xi_{ki}}{D_j^2 D_k} \right]. \end{aligned}$$

Now for  $(j, k) \in \bar{\mathcal{S}}_i$  with  $j \neq k$ , we have

$$\begin{aligned} & \mathbf{E} \left[ \mathbf{1}_{\{D_j \geq d-1\}} \mathbf{1}_{\{D_k \geq d-1\}} \frac{\xi_{ji}\xi_{ki}}{D_j^2 D_k} \right] \\ & = \mathbf{E} \left[ \mathbf{1}_{\{\xi_{jk}=1\}} \mathbf{1}_{\{D_j \geq d-1\}} \mathbf{1}_{\{D_k \geq d-1\}} \frac{\xi_{ji}\xi_{ki}}{D_j^2 D_k} \right] \end{aligned}$$



$$\begin{aligned}
 & + \mathbf{E} \left[ \mathbf{1}_{\{\xi_{jk}=0\}} \mathbf{1}_{\{D_j \geq d-1\}} \mathbf{1}_{\{D_k \geq d-1\}} \frac{\xi_{ji} \xi_{ki}}{D_j^2 D_k} \right] \\
 & \leq \mathbf{E} \left[ \frac{\xi_{ji} \xi_{ki}}{(D_j - \xi_{jk} + 1)^2 (D_k - \xi_{jk} + 1)} \right] \\
 & \quad + \mathbf{E} \left[ \mathbf{1}_{\{D_j - \xi_{jk} > 0\}} \mathbf{1}_{\{D_k - \xi_{jk} > 0\}} \frac{\xi_{ji} \xi_{ki}}{(D_j - \xi_{jk})^2 (D_k - \xi_{jk})} \right] \\
 & = \mathbf{E} \left[ \frac{\xi_{ji}}{(D_j - \xi_{jk} + 1)^2} \right] \mathbf{E} \left[ \frac{\xi_{ki}}{D_k - \xi_{jk} + 1} \right] \\
 & \quad + \mathbf{E} \left[ \mathbf{1}_{\{D_j - \xi_{jk} > 0\}} \frac{\xi_{ji}}{(D_j - \xi_{jk})^2} \right] \mathbf{E} \left[ \mathbf{1}_{\{D_k - \xi_{jk} > 0\}} \frac{\xi_{ki}}{D_k - \xi_{jk}} \right],
 \end{aligned}$$

where the last equality follows from independence between  $(\xi_{ji}, D_j - \xi_{jk})$  and  $(\xi_{ki}, D_k - \xi_{jk})$ . Using exchangeability and Lemma 3.5, we have

$$\begin{aligned}
 \mathbf{E} \left[ \frac{\xi_{ji}}{(D_j - \xi_{jk} + 1)^2} \right] &= \frac{1}{N-2} \sum_{l \in [N], l \neq j, k} \mathbf{E} \left[ \frac{\xi_{jl}}{(D_j - \xi_{jk} + 1)^2} \right] \\
 &= \frac{1}{N-2} \mathbf{E} \left[ \frac{D_j - \xi_{jk}}{(D_j - \xi_{jk} + 1)^2} \right] \\
 &\leq \frac{1}{N-2} \mathbf{E} \left[ \frac{1}{D_j - \xi_{jk} + 1} \right] \leq \frac{1}{(N-2)(N-1)p_N}.
 \end{aligned}$$

Similarly, one can verify that

$$\begin{aligned}
 \mathbf{E} \left[ \frac{\xi_{ki}}{D_k - \xi_{jk} + 1} \right] &\leq \frac{1}{N-2}, \\
 \mathbf{E} \left[ \mathbf{1}_{\{D_j - \xi_{jk} > 0\}} \frac{\xi_{ji}}{(D_j - \xi_{jk})^2} \right] &\leq \frac{4}{(N-2)(N-1)p_N}, \\
 \mathbf{E} \left[ \mathbf{1}_{\{D_k - \xi_{jk} > 0\}} \frac{\xi_{ki}}{D_k - \xi_{jk}} \right] &\leq \frac{1}{N-2}.
 \end{aligned}$$

Combining these gives us

$$\mathbf{E} \left[ \mathbf{1}_{\{D_j \geq d-1\}} \mathbf{1}_{\{D_k \geq d-1\}} \frac{\xi_{ji} \xi_{ki}}{D_j^2 D_k} \right] \leq \frac{5}{(N-2)^2 (N-1) p_N} \quad \text{when } j \neq k.$$

Also note that the summation in (3.35) when  $j = k$  is

$$\begin{aligned}
 \sum_{j=1, j \neq i}^N \mathbf{E} \left[ \mathbf{1}_{\{D_j \geq d-1\}} \frac{\xi_{ji}}{D_j^3} \right] &= \sum_{j=1, j \neq i}^N \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \frac{\xi_{ij}}{D_i^3} \right] \\
 &= \mathbf{E} \left[ \mathbf{1}_{\{D_i \geq d-1\}} \frac{1}{D_i^2} \right] \leq \frac{4}{(N p_N)^2},
 \end{aligned}$$

where the first equality uses exchangeability and the inequality uses Lemma 3.5. Combining these two estimates with (3.35) gives

$$V_s^A \leq \kappa_5 \frac{N^2}{(N-2)^2(N-1)p_N} + \kappa_5 \frac{1}{(Np_N)^2} \leq \frac{\kappa_6}{Np_N} + \frac{\kappa_6}{(Np_N)^2}$$

for some  $\kappa_5, \kappa_6 \in (0, \infty)$ . This completes the proof of Lemma 3.2.  $\square$

Finally, we complete the proof of Lemma 3.4.

PROOF OF LEMMA 3.4. As before, we will omit the superscript in  $\xi_{ij}$ 's and  $D_i$ 's for notational convenience. Fix  $i \in \mathbb{N}$ . From (2.3) and (2.7), using Cauchy–Schwarz and Doob’s inequalities, we have for any fixed  $t \in [0, T]$ ,

$$\begin{aligned} \mathbf{E} \|X_i^N - X_i\|_{*,t}^4 &\leq \kappa_1 \int_0^t \mathbf{E} |X_i^N(s) - X_i(s)|^4 ds \\ (3.36) \qquad &+ \kappa_1 \int_0^t \mathbf{E} |C_i^N(s) - C_i(s)|^2 ds \\ &+ \kappa_1 \int_0^t \mathbf{E} |C_i^N(s) - C_i(s)|^4 ds. \end{aligned}$$

Recall the definition of  $C_i^N(s)$  and  $C_i(s)$  from (2.4) and (2.7). From the bound  $\|b\|_\infty \leq 1$  and (2.5), for  $s \in [0, T]$  we have  $|C_i(s)| \leq d$  and

$$\begin{aligned} \mathbf{E} |C_i^N(s)|^4 &\leq \mathbf{E} \left[ \mathbf{1}_{\{D_i < d-1\}}(D_i + 1) \right. \\ &+ (d-1) \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \frac{\xi_{j_2 i}}{D_{j_2}} \\ (3.37) \qquad &+ \left. \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1\}} \xi_{i j_2} (D_i + 1) \right]^4 \\ &\leq \kappa_2 + \kappa_2 \mathbf{E} \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \frac{\xi_{j_2 i}}{D_{j_2}} \right]^4 \\ &+ \kappa_2 \mathbf{E} \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1\}} \xi_{i j_2} (D_i + 1) \right]^4. \end{aligned}$$

Here, the second term on the right-hand side can be written as

$$\begin{aligned} &\kappa_2 \mathbf{E} \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1, D_i > 0\}} \frac{D_i}{D_{j_2}} \frac{\xi_{j_2 i}}{D_i} \right]^4 \\ &\leq \kappa_2 \mathbf{E} \left\{ \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1, D_i > 0\}} \left( \frac{D_i}{D_{j_2}} \right)^4 \frac{\xi_{j_2 i}}{D_i} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1, D_i > 0\}} \frac{\xi_{j_2 i}}{D_i} \right]^3 \Big\} \\
 & \leq \kappa_2 \mathbf{E} \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} \geq d-1\}} \frac{D_i^3 \xi_{j_2 i}}{D_{j_2}^4} \\
 & = \kappa_2 \sum_{j_2 \in [N], j_2 \neq i} \mathbf{E} \left[ \mathbf{1}_{\{D_{j_2} - \xi_{j_2 i} + 1 \geq d-1\}} \frac{(D_i - \xi_{j_2 i} + 1)^3}{(D_{j_2} - \xi_{j_2 i} + 1)^4} \right] p_N \\
 & = \kappa_2 \sum_{j_2 \in [N], j_2 \neq i} \mathbf{E} \left[ \mathbf{1}_{\{D_{j_2} - \xi_{j_2 i} + 1 \geq d-1\}} \frac{1}{(D_{j_2} - \xi_{j_2 i} + 1)^4} \right] \mathbf{E}[D_i - \xi_{j_2 i} + 1]^3 p_N \\
 & \leq \kappa_3 (N - 1) \frac{1}{(N - 1)^4 p_N^4} (N p_N + 1)^3 p_N \leq \kappa_4,
 \end{aligned}$$

where the second line uses Holder’s inequality, the fourth line follows by conditioning on  $\xi_{j_2 i} = 1$ , the fifth line follows from independence, and the last line uses Lemma 3.5 and moment estimates of binomial random variables. Following the similar argument, we can write the last term in (3.37) as

$$\begin{aligned}
 & \kappa_2 \mathbf{E} \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1, D_i > 0\}} D_i (D_i + 1) \frac{\xi_{i j_2}}{D_i} \right]^4 \\
 & \leq \kappa_2 \mathbf{E} \left\{ \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1, D_i > 0\}} D_i^4 (D_i + 1)^4 \frac{\xi_{i j_2}}{D_i} \right] \right. \\
 & \quad \times \left. \left[ \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1, D_i > 0\}} \frac{\xi_{i j_2}}{D_i} \right]^3 \right\} \\
 & \leq \kappa_2 \mathbf{E} \sum_{j_2 \in [N], j_2 \neq i} \mathbf{1}_{\{D_{j_2} < d-1\}} D_i^3 (D_i + 1)^4 \xi_{i j_2} \\
 & = \kappa_2 \sum_{j_2 \in [N], j_2 \neq i} \mathbf{E} [\mathbf{1}_{\{D_{j_2} - \xi_{i j_2} + 1 < d-1\}} (D_i - \xi_{i j_2} + 1)^3 (D_i - \xi_{i j_2} + 2)^4] p_N \\
 & = \kappa_2 \sum_{j_2 \in [N], j_2 \neq i} \mathbf{E} [\mathbf{1}_{\{D_{j_2} - \xi_{i j_2} + 1 < d-1\}}] \mathbf{E} [(D_i - \xi_{i j_2} + 1)^3 (D_i - \xi_{i j_2} + 2)^4] p_N \\
 & \leq \kappa_5 (N - 1) \mathbf{P}(D_i < d) (N p_N + 1)^7 p_N.
 \end{aligned}$$

Combining the above three estimates with (3.27) and using Condition 2, we have  $\mathbf{E}|C_i^N(s)|^4 \leq \kappa_6$ . It then follows from (3.36) that

$$\mathbf{E} \|X_i^N - X_i\|_{*,t}^4 \leq \kappa_7 \int_0^t \mathbf{E} \|X_i^N - X_i\|_{*,s}^4 ds + \kappa_7.$$

The result then follows from Gronwall’s inequality.  $\square$

**4. Conclusion.** We have considered the JSQ( $d$ ) policy in large-scale systems where the servers communicate with their neighbors and the neighborhood relationships are described in terms of a suitable graph. We have developed sufficient criteria for arbitrary graph sequences so that asymptotically the evolution of the occupancy process on any finite time interval is indistinguishable from that for the case when the graph is a clique. We have also considered sequence of Erdős–Rényi random graphs and established sufficient criteria in terms of the growth rates of the average degree that ensure the annealed and quenched limit of the occupancy process on any finite time interval to coincide with that in the clique.

The long time behavior of the occupancy measure process associated with the above graph sequences is an important and challenging open question. Long time properties of the JSQ( $d$ ) scheme have been well studied in the case of a clique. For example, in [18, 20], it is shown that  $\pi^N$ , the stationary measure of the occupancy process of the  $N$ th system, converges in distribution to  $\delta_{q^*}$ , where  $q^*$  is the unique fixed point of the limiting deterministic dynamical system  $q(\cdot)$ . Roughly speaking, such a result says that the limits  $t \rightarrow \infty$  and  $N \rightarrow \infty$  can be interchanged. Based on Theorems 2.1–2.4, it is natural to conjecture that a similar interchangeability also holds for more general graphs considered in this work. However, the setting here is significantly harder, in particular, the occupancy process is not any more a Markov process. One may conjecture that with  $\pi^N$  replaced by the time asymptotic limit of the law of occupancy process, the convergence  $\pi^N \rightarrow \delta_{q^*}$  still holds. However, currently even the existence of such a time asymptotic limit is not clear.

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A. BUDHIRAJA  
DEPARTMENT OF STATISTICS  
AND OPERATIONS RESEARCH  
UNIVERSITY OF NORTH CAROLINA  
CHAPEL HILL, NORTH CAROLINA 27599  
USA  
E-MAIL: [budhiraj@email.unc.edu](mailto:budhiraj@email.unc.edu)  
URL: <http://abudhiraja.web.unc.edu/>

D. MUKHERJEE  
DIVISION OF APPLIED MATHEMATICS  
BROWN UNIVERSITY  
182 GEORGE STREET  
PROVIDENCE, RHODE ISLAND 02906  
USA  
E-MAIL: [debankur\\_mukherjee@brown.edu](mailto:debankur_mukherjee@brown.edu)  
URL: <https://sites.google.com/site/debankurm/>

R. WU  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MICHIGAN  
530 CHURCH STREET  
ANN ARBOR, MICHIGAN 48109  
USA  
E-MAIL: [ruoyu@umich.edu](mailto:ruoyu@umich.edu)  
URL: <https://sites.google.com/site/ruoyuwu90/>