

# Adaptive density estimation based on a mixture of Gammas

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**Abstract:** We consider the problem of Bayesian density estimation on the positive semiline for possibly unbounded densities. We propose a hierarchical Bayesian estimator based on the gamma mixture prior which can be viewed as a location mixture. We study convergence rates of Bayesian density estimators based on such mixtures. We construct approximations of the local Hölder densities, and of their extension to unbounded densities, to be continuous mixtures of gamma distributions, leading to approximations of such densities by finite mixtures. These results are then used to derive posterior concentration rates, with priors based on these mixture models. The rates are minimax (up to a  $\log n$  term) and since the priors are independent of the smoothness, the rates are adaptive to the smoothness.

One of the novel feature of the paper is that these results hold for densities with polynomial tails. Similar results are obtained using a hierarchical Bayesian model based on the mixture of inverse gamma densities which can be used to estimate adaptively densities with very heavy tails, including Cauchy density.

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## 1. Introduction

### 1.1. Context: Posterior concentration rates in Bayesian nonparametric mixture models

Nonparametric density estimation using Bayesian models with a mixture prior distribution has been used extensively in practice due to their flexibility and available computational techniques using MCMC. In some cases their theoretical properties have been studied, and in particular the asymptotic behaviour of the associated posterior distribution. Posterior weak consistency has been studied quite systematically in particular by [17], but posterior concentration rates have been derived only for a small number of kernels. In the case of density estimation on  $[0,1]$  (or any compact interval of  $\mathbb{R}$ ) [11] has studied mixtures of Beta densities, and [10] have considered mixtures of triangular densities, Gaussian location mixtures have been considered by [5, 4, 7, 14, 12] and, more generally, power exponential kernels by [7, 13]. Location scale mixtures have been considered also by [1]. Apart from the latter paper, the posterior concentration rates have been obtained by the above authors are equal to the minimax estimation rate (up to a  $\log n$  term) over some collections of functional classes, showing

that nonparametric mixture models are not only flexible prior models, but they also lead to optimal procedures, in the frequentist sense.

The above results do not specifically address estimation of densities on  $\mathbb{R}^+$ , they do not cover fat tail densities and the posterior concentration rates have been obtained only under the condition that the densities are uniformly bounded.

In this paper we propose to estimate a possibly unbounded density supported on the positive semiline via a Bayesian approach using a Dirichlet Process mixture of Gamma densities as a prior distribution. The proposed prior distribution does not depend on regularity properties of the unknown density (such as the Hölder exponent) so the resulting posterior estimates are adaptive. Bayesian Gamma mixtures are widely used in practice, for instance, for pattern recognition [2] and for modelling the signal-to-noise ratio in wireless channels [15]. An algorithm for implementing a Gamma mixture with unknown number of components as well as aspects of the practical application of this model is given in [16].

The main purpose of the paper is to derive the conditions on the Gamma mixture prior so that the posterior distribution asymptotically concentrates at the optimal rate (up to a log factor) around the true density over smooth classes of densities. We derive the concentration rate of the posterior distribution when the unknown density belongs to a local Hölder class on  $(0, \infty)$  (see the formal definition below) adapting the techniques applied in Shen et al. [14], Rousseau [11] and Kruijer et al. [7] to the proposed mixture of Gamma densities. In particular, we will show that this mixture provides a good approximation for such functions. Secondly, we investigate the concentration rate of this posterior distribution for an unknown density on  $(0, \infty)$  that can be unbounded at 0, namely for a density  $x^{\alpha-1}h(x)$  for  $\alpha \in (0, 1)$  in a neighbourhood of 0 and for function  $h$  belonging to a locally Hölder class on  $(0, \infty)$ . A typical example of such behaviour is a Gamma density with the shape parameter between 0 and 1. One of the novel feature of the paper is that these results hold for densities with polynomial tails. Similar results are obtained for a hierarchical Bayesian model based on the mixture of inverse gamma densities which can be used to estimate adaptively densities with very heavy tails, including the Cauchy density.

For a bounded density, we use the lower bound on the rate of convergence for estimators of densities from the local Hölder class  $\mathcal{H}(\beta, L)$  which is  $n^{-\beta/(2\beta+1)}$  [9].

The paper is organised as follows. In Section 2 we define the prior distribution and study the concentration rate of the corresponding posterior distribution over an extension of the local Hölder class to possibly unbounded densities. We also discuss the choice of the base measure of the Dirichlet process prior as well as the hyperprior measure on the shape parameter of the Gamma distribution that lead to consistent estimation with the posterior concentration rate equal to the minimax optimal rate of convergence up to a log factor. The prior model based on the mixtures of inverse gammas and the corresponding posterior concentration result are given in Section 2.3.2. Numerical performance of the estimator is studied on simulated data for bounded and unbounded densities and on real data, with the results presented in Section 3. The proof of the main result is

given in Section 4, and the proofs of the auxiliary results are deferred to the appendix.

**1.2. Setup and notation**

Throughout the paper we assume that  $X^n = (X_1, \dots, X_n)$  is an  $n$ -sample from a distribution with density  $f$  on  $\mathcal{R}^+$  with respect to Lebesgue measure. We denote by  $\mathcal{F} = \{f \in L_1(\mathbb{R}^+); f : \mathbb{R}^+ \rightarrow \mathbb{R}^+; \int_{\mathbb{R}^+} f(x)dx = 1\}$  with  $L_1(\mathbb{R}^+)$  denoting the set of measurable and integrable functions on  $\mathbb{R}^+$ .

The aim is to estimate the unknown density  $f \in \mathcal{F}$  which we do using a Bayesian approach. We construct a prior probability on  $\mathcal{F}$ , by modelling  $f$  as a mixture of Gamma densities, see (2.1) below. The associated posterior distribution is denoted by  $\Pi(\cdot|X^n)$ . Let  $f_0$  be the true density of the  $X_i$ 's and we are interested in determining the posterior concentration rate  $\varepsilon_n = o(1)$  defined by

$$\Pi(B_{\varepsilon_n}|X^n) = 1 + o_{P_{f_0}}(1), \quad B_{\varepsilon_n} = \{f : \|f_0 - f\|_1 \leq \varepsilon_n\},$$

where  $\|\cdot\|_1$  is the  $L_1$  norm.

We denote by  $\mathcal{KL}(f_1, f_2)$  the Kullback-Leibler divergence between  $f_1$  and  $f_2$  and by  $V(f_1, f_2)$  the variance of log-densities ratio:

$$\begin{aligned} \mathcal{KL}(f_1, f_2) &= \int_0^\infty \log\left(\frac{f_1(x)}{f_2(x)}\right) f_1(x)dx, \\ V(f_1, f_2) &= \int_0^\infty \left[\log\left(\frac{f_1(x)}{f_2(x)}\right)\right]^2 f_1(x)dx - \mathcal{KL}(f_1, f_2)^2 \end{aligned}$$

and the square of the Hellinger distance by

$$D_H^2(f_1, f_2) = \int_{\mathbb{R}^+} (\sqrt{f_1} - \sqrt{f_2})^2(x)dx.$$

Throughout the paper  $f(\cdot) \gtrsim g(\cdot)$  (resp.  $f(\cdot) \lesssim g(\cdot)$ ) means that there exists a positive constant  $C$  such that  $f(\cdot) \geq Cg(\cdot)$  (resp.  $f(\cdot) \leq Cg(\cdot)$ ) and  $f(\cdot) \asymp g(\cdot)$  means that  $g(\cdot) \lesssim f(\cdot) \lesssim g(\cdot)$ .

In the following Section we present the main results of the paper.

**2. Main results**

We start with description of the mixture of Gamma distributions which underpins the construction of our prior model on  $\mathcal{F}$ .

**2.1. Prior model: Mixtures of Gamma distributions**

We consider the following Gamma mixture types of models:

$$f_{P,z}(x) = \int_0^\infty g_{z,\epsilon}(x)dP(\epsilon), \quad g_{z,\epsilon}(x) = x^{z-1}e^{-zx/\epsilon} \left(\frac{z}{\epsilon}\right)^z \frac{1}{\Gamma(z)}. \quad (2.1)$$

We consider  $(P, z) \sim \Pi = \Pi_1 \otimes \Pi_z$ , where  $\Pi_1$  is a probability on the set of discrete distributions over  $\mathbb{R}^+$  and  $\Pi_z$  is a probability distribution on  $\mathbb{R}^+ = [0, +\infty)$ . We also denote  $\mathbb{R}^{+*} = (0, +\infty)$ .

Hence the densities are represented by *location* Gamma mixtures, since in the above parametrization  $\epsilon$  is the mean of the Gamma distribution with parameters  $(z, z/\epsilon)$ . This particular parametrization leads to the variance equal to  $\epsilon^2/z$ , and as  $z$  goes to infinity, the Gamma distribution  $(z, z/\epsilon)$  can be approximated by a Gaussian random variable with mean  $\epsilon$  and variance  $\epsilon^2/z$ . This allows for precise approximation near 0 and more loose approximation in the tail. This parametrization has also been used in Wiper et al. [16].

The key to good approximation properties of a continuous density  $f$  by the gamma mixtures defined above is

$$K_z f(x) \rightarrow f(x) \text{ as } z \rightarrow \infty$$

where operator  $K_z$  is defined by

$$K_z f(x) = \int_0^\infty g_{z,\epsilon}(x) f(\epsilon) d\epsilon. \quad (2.2)$$

We explain in more details in Section 2.4, why mixtures of Gamma distributions as proposed here are flexible models for estimating smooth densities on  $\mathbb{R}^+$ . The general idea is that, as in the case of mixtures of Beta distributions in Rousseau [11] or mixtures of Gaussian distributions in Kruijer et al. [7], under regularity conditions with  $f$  verifying some Hölder - type condition with regularity  $\beta > 0$ , one can construct a probability density  $f_1$  on  $\mathbb{R}^+$  such that

$$|K_z f_1(x) - f(x)| \lesssim z^{-\beta/2}, \quad z \rightarrow +\infty$$

The continuous mixture  $K_z f_1$  can then be approximated by a discrete mixture with  $O(\sqrt{z} \log z)$  components.

We consider discrete priors on  $P$  and priors on  $z$  that satisfy the following condition:

**Condition** ( $\mathcal{P}$ ): The prior on  $z$ ,  $\Pi_z$  satisfies: for some constants  $c \geq c' > 0, c_0 > 0$  and  $\rho_z \geq 0$ ,

$$\begin{aligned} \Pi_z([x, 2x]) &\gtrsim e^{-c\sqrt{x}(\log x)^{\rho_z}}, & \Pi_z([x, +\infty)) &\lesssim e^{-c'\sqrt{x}(\log x)^{\rho_z}} & \text{as } x \rightarrow +\infty, \\ \Pi_z([0, x]) &\lesssim x^{c_0} & \text{for } x \rightarrow 0, \end{aligned} \quad (2.3)$$

We consider either of these two types of prior on  $P$ :

- **Dirichlet Prior** of  $P$ :  $P \sim DP(m, G)$  where  $DP(m, G)$  denotes the Dirichlet Process with mass  $m > 0$  and base probability measure  $G$  having positive and continuous density  $g$  on  $\mathbb{R}^{+*}$  satisfying:

$$x^{a_0} \lesssim g(x) \lesssim x^{a'_0} \text{ as } x \rightarrow 0, \text{ \& } x^{-a_1} \lesssim g(x) \lesssim x^{-a'_1} \text{ as } x \rightarrow +\infty \quad (2.4)$$

for some  $-1 < a'_0 \leq a_0$  and  $1 < a'_1 \leq a_1$ .

• **Finite mixture:**

$$P(d\epsilon) = \sum_{j=1}^K p_j \delta_{\epsilon_j}(d\epsilon), \quad K \sim \pi_K, \quad \epsilon_j \stackrel{iid}{\sim} G,$$

$$e^{-c_1 k(\log k)^{\rho_2}} \gtrsim \pi_K(k) \gtrsim e^{-c_2 k(\log k)^{\rho_2}} \quad \forall k \geq 2,$$

$$(p_1, \dots, p_k) \sim \mathcal{D}(\alpha_1, \dots, \alpha_k),$$

with  $G$  satisfying (2.4),  $\rho_2 \geq 0$ ,  $0 < c_1 \leq c_2 < +\infty$  and there exists  $\bar{m}$  such that

$$\sum_{i=1}^k \alpha_i \leq \bar{m}k, \quad \sum_{i=1}^k (-\log \alpha_i)_+ \leq \bar{m}k \log k \quad \forall k \geq 2.$$

Note that  $(\alpha_j, j \leq k)$  possibly all depend on  $k$ , but we omit the additional index  $k$  to simplify the notation.

Condition (P) is quite mild. It is satisfied for instance for  $\sqrt{z}$  following a *Gamma* distribution, in which case  $\rho_z = 0$ . The prior condition on the base measure (2.4) imposes fat tails on  $G$ . It is satisfied for instance if  $g(x) \propto x^{a_0}(1+x^2)^{-a_0-1}$  with  $a_0 > 0$ .

Note that, it appears from the proofs that if  $g(x) \leq e^{-cx}$  then posterior concentration rates would remain unchanged over the functional class described below, assuming that the true density  $f$  also has exponential type tails. Hence, when estimating such densities, Gamma densities satisfy the conditions required for  $G$  but inverse Gammas do not.

**2.2. Functional classes**

In this paper we are interested in estimating densities that are possibly unbounded at 0. To construct a class of such functions over which posterior concentration rates are derived, first we need a class of bounded functions.

Let  $\mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ ,  $\beta > 0$ ,  $\gamma \geq 0$  be the set of such functions  $f : \mathbb{R}^+ \rightarrow [0, \infty)$  which are  $r$  times continuously differentiable with  $r = \lceil \beta \rceil - 1$  and which satisfy for all  $x \in \mathbb{R}_+$  and  $y: y > -x$  and  $|y| \leq \Delta$ ,

$$\left| f^{(r)}(x+y) - f^{(r)}(x) \right| \leq L(x)|y|^{\beta-r}(1+|y|^\gamma), \quad f(x) \leq C_0, \quad (2.5)$$

defining  $r_0 = \lceil \beta/2 \rceil - 1$  for  $\beta > 2$  and  $r_0 = 0$  if  $\beta \leq 2$ ,

$$\int_0^\infty \left( \frac{x^j |f^{(j)}(x)|}{f(x)} \right)^{(2\beta+e)/j} f(x) dx \leq C_1, \quad j \leq r;$$

$$\int_0^\infty \left( \frac{(1+x^{\gamma+2r_0})x^\beta L(x)}{f(x)} \right)^2 f(x) dx \leq C_1 \quad (2.6)$$

for some  $e > 0$ . In the above definition  $L(\cdot)$  is a fixed positive function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ .

We also consider classes of densities unbounded around zero that are defined as follows. Let  $\mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ ,  $\alpha \in (0, 1]$ ,  $\beta > 0$ ,  $\gamma \geq 0$ , be the set of such functions  $f : \mathbb{R}^+ \rightarrow [0, \infty)$  such that function  $h(x) := x^{-(\alpha-1)}f(x)$  satisfies the following conditions:

- 1) function  $h$  is  $r$  times continuously differentiable with  $r = \lceil \beta \rceil - 1$  and such that for all  $x > 0$ ,  $y > -x$  and  $|y| \leq \Delta$ ,

$$\left| h^{(r)}(x+y) - h^{(r)}(x) \right| \leq L(x)|y|^{\beta-r}(1+|y|^\gamma), \quad h(x) \leq C_0 \quad (2.7)$$

- 2) for some  $e > 0$ ,

$$\int_0^\infty \left( \frac{x^j |h^{(j)}(x)|}{h(x)} \right)^{(2\beta+e)/j} x^{\alpha-1} h(x) dx \leq C_1, \quad j \leq r;$$

$$\int_0^\infty \left( \frac{L(x)x^\beta(1+x^{\gamma+2r_0})}{h(x)} \right)^2 x^{\alpha-1} h(x) dx \leq C_1, \quad (2.8)$$

where  $r_0 = \lceil \beta/2 \rceil - 1$  for  $\beta > 2$  and  $r_0 = 0$  if  $\beta \leq 2$ .

Note that in the case  $\alpha = 1$ , we recover the first functional class, namely

$$\mathcal{P}_1(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta) = \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta).$$

The rationale behind the functional class  $\mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  comes from the following lemma.

**Lemma 2.1.** *For any  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  and  $x > 0$ ,*

$$K_z f(x) = x^{\alpha-1} K_{z+1-\alpha} h(x/C_z)(1 + O_z)$$

where  $h(x) = x^{1-\alpha}f(x)$ ,  $C_z = 1 + (1 - \alpha)/z$  and  $O_z = \frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} - 1 = O(1/z)$  for large  $z$ .

Lemma 2.1 is a consequence of Lemma A.1 which is given in Appendix A.2.

**Remark 2.1.** 1. Note that in the above functional class  $\mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ ,  $f$  is bounded from above but is allowed converge to 0 as  $x$  goes to 0. Interestingly if  $f(x) = x^\tau h(x)$  for any  $\tau \geq 0$  and  $h$  satisfies (2.6) then condition (2.6) is satisfied by  $f$ .

2. If  $\log f$  is locally Hölder, with suitable integrability conditions, then  $f(x) \in \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ . Similarly, the same holds for  $h(x)$  bounded from below if  $f(x) = x^{\alpha-1}h(x) \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ .

3. The moment condition (2.6) is satisfied for a number of densities on the positive semiline (see Appendix C.2 for details):

- (a) for the Weibull distribution with density  $f(x) = C_{a,b}x^{a-1}e^{-x^b}$ ,  $a, b > 0$ , with  $\beta = b$  for  $a \in (0, 1]$ ; for  $a \geq 1$ ,  $\log f$  is Hölder with exponent  $b$ ;

- (b) for folded Student  $t$  distribution  $f(x) = c_\nu(1 + x^2)^{-(\nu+1)/2}I(x > 0)$  with  $\nu \geq 1$ ;
- (c) for a Frechet-type distribution with density  $f(x) = cx^{-b-1}e^{-x^{-b}}$ ,  $b > 0$ . In this case,  $\lim_{x \rightarrow 0} f(x) = 0$  so we can take  $\alpha = 1$ ,  $\gamma = 1$  and  $\beta \leq 1$ .

### 2.3. Posterior concentration rates

#### 2.3.1. Posterior concentration rate for the mixture of Gammas

Similarly to location mixtures of Gaussian densities, mixtures of Gamma densities provide a flexible tool to approximate smooth densities on  $\mathbb{R}^+$ , and using the representation of Lemma 2.1, to approximate smooth but possibly unbounded densities. The posterior concentration rates are presented in the following Theorem. In addition to the regularity conditions induced by the functional classes defined above we will need the following tail assumption:

$$\exists \rho_1 > 0 \ \& \ C_2 > 0 : \int_x^\infty y^2 f_0(y) dy \leq C_2(1 + x)^{-\rho_1}. \tag{2.9}$$

We denote by  $\mathcal{T}(\rho_1, C_2)$  the set of densities satisfying (2.9).

**Theorem 2.1.** *Consider the prior defined in Section 2.1 and assume that  $\mathbf{X}^n = (X_1, \dots, X_n)$  is a sample of independent observations identically distributed according to a probability  $P_0$  on  $\mathbb{R}^+$  having density  $f_0$  with respect to Lebesgue measure.*

*Then, for any  $\alpha_0 > 0$ ,  $\beta_1 \geq \beta_0 > 0$ ,  $L(\cdot)$ ,  $\gamma$ ,  $C_0$ ,  $C_1$ ,  $e$ ,  $\Delta$ ,  $C_2$  and  $\rho_1$ , there exists  $M > 0$  such that*

$$\sup_{\alpha \in [\alpha_0, 1]} \sup_{\beta \in [\beta_0, \beta_1]} \sup_{f_0 \in \mathcal{Q}_{\alpha, \beta}(\dots)} E_{P_0} \left[ \Pi \left( \|f - f_0\|_1 > M \frac{(\log n)^q}{n^{\beta/(2\beta+1)}} | \mathbf{X}^n \right) \right] = o(1) \tag{2.10}$$

*with  $q = (5\beta + 1)/(4\beta + 2)$  if  $\rho_z \leq 5/2$  and  $q = (2\rho_z\beta + 1)/(4\beta + 2)$  if  $\rho_z > 5/2$  and*

$$\mathcal{Q}_{\alpha, \beta}(\dots) = \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta) \cap \mathcal{T}(\rho_1, C_2).$$

Theorem 2.1 is proved in Section 4.1.

Note that in (2.10),  $f_0$  is supposed to satisfy (2.9) so that the supremum in  $f_0$  is taken over the intersection of

Note also that the tail assumption (2.9) is much weaker than the tail condition considered for mixtures of Gaussians as in [7] or Shen et al. [14] where exponential decay in the form  $e^{-c|x|^\tau}$  is assumed for some  $\tau > 0$ . Because the posterior concentration rates obtained in [7] or Shen et al. [14] are upper bounds, it is not clear that the exponential tail condition was necessary. It seems however that because Gamma densities have fatter tails than Gaussian densities, they allow for approximations of densities with fatter tails.

The theorem is proved following the approach of Ghosal et al. [3], so that first we construct an approximation of the true density  $f_0$  by a continuous mixture of Gammas in the form  $K_z f_1$  for some density  $f_1$  close to  $f_0$  and then approximate the continuous mixture by a discrete and finite mixture. This allows us to control the prior mass of Kullback-Leibler neighbourhoods. The tail condition (2.9) is used in this second step, similarly as the exponential tail condition used with Gaussian mixtures in [7] or Shen et al. [14]. It is to be noted however that the integrability conditions (2.2) in the definition of the functional classes  $\mathcal{P}_\alpha(\cdot)$  implicitly induce some tail constraints on  $f_0$ , see Remark 2.1.

### 2.3.2. Mixture of inverse Gamma distributions

Although (2.9) is a rather mild condition, it excludes fat tail distributions such as the folded Cauchy density whose density at infinity behaves like  $x^{-2}$ . The Bayesian model proposed here can be adapted to estimating fat tail densities at infinity as follows.

Let  $X_i \sim f_0$  with  $f_0(x) \lesssim x^{a+1}$  for small  $x$  and some  $a > 0$  and  $f_0(x) \lesssim x^{-b-1}$  for some  $b > 0$  when  $x$  goes to infinity. Then the density of  $1/X_i$ ,  $q_0(y) = y^{-2} f_0(1/y) \lesssim y^{-a-3}$  when  $y$  goes to infinity and  $q_0(y) \lesssim y^{b-1}$  when  $y$  goes to 0. Hence  $q_0$  satisfies the tail conditions both at 0 and infinity assumed in Theorem 2.1. Since  $\|q - q_0\|_1 = \|f - f_0\|_1$ , Theorem 2.1 implies that density  $q(x)$  can be estimated using the appropriately adapted prior, such that the corresponding prior on  $f$  satisfies the conditions stated in the theorem, with the same rate of convergence.

The prior for estimating  $f(x) = x^{-2} q(1/x)$  is a mixture of inverse Gammas:

$$f_{\bar{P},z}(x) = \int_0^\infty \bar{g}_{z,\xi}(x) d\bar{P}(\xi) \quad (2.11)$$

where

$$\bar{g}_{z,\xi}(x) = x^{-2} g_{z,1/\xi}(1/x) = x^{-z-1} e^{-z\xi/x} (z\xi)^z \frac{1}{\Gamma(z)}$$

which is the density of an inverse Gamma distribution.

**Condition ( $\bar{P}$ ):**

The hyperprior is  $(\bar{P}, z) \sim \bar{\Pi} = \bar{\Pi}_1 \otimes \Pi_z$ , where  $\Pi_z$  is a probability distribution on  $\mathbb{R}^+$  satisfying condition (2.3) and  $\bar{\Pi}_1$  is a probability on the set of discrete distributions over  $\mathbb{R}^+$  satisfying either of these two types of prior on  $P$ :

- **Dirichlet Prior** of  $\bar{P}$ :  $\bar{P} \sim DP(m, \bar{G})$  where  $DP(m, \bar{G})$  denotes the Dirichlet Process with mass  $m > 0$  and base probability measure  $\bar{G}$  having positive and continuous density  $\bar{g}$  on  $\mathbb{R}^{+*}$  satisfying:

$$y^{\bar{a}_0} \lesssim \bar{g}(y) \lesssim y^{\bar{a}'_0} \text{ as } y \rightarrow 0, \text{ \& } y^{-\bar{a}_1} \lesssim \bar{g}(y) \lesssim y^{-\bar{a}'_1} \text{ as } y \rightarrow +\infty \quad (2.12)$$

for some  $-1 < \bar{a}_0 \leq \bar{a}'_0$  and  $1 < \bar{a}_1 \leq \bar{a}'_1$  (note that this corresponds to the density  $x^{-2} \bar{g}(1/x)$  satisfying conditions (2.4) with  $\bar{a}_0 = a_1 - 2$  and  $\bar{a}_1 = a_0 + 2$ ).

• **Finite mixture:**

$$\begin{aligned} \bar{P}(d\xi) &= \sum_{j=1}^K p_j \delta_{\xi_j}(d\xi), \quad K \sim \pi_K, \quad \xi_j \stackrel{iid}{\sim} \bar{G}, \\ e^{-c_1 k (\log k)^{\rho_2}} &\gtrsim \pi_K(k) \gtrsim e^{-c_2 k (\log k)^{\rho_2}} \quad \forall k \geq 2, \\ (p_1, \dots, p_k) &\sim \mathcal{D}(\alpha_1, \dots, \alpha_k), \end{aligned}$$

with  $\bar{G}$  satisfying (2.12),  $\rho_2 \geq 0$ ,  $0 < c_1 \leq c_2 < +\infty$  and there exists  $\bar{m}$  such that

$$\forall k \geq 1, \quad \sum_{i=1}^k \alpha_i \leq k\bar{m}, \quad \sum_{i=1}^k (-\log \alpha_i)_+ \leq \bar{m}k \log k \quad \forall k \geq 2.$$

In particular, we have similar approximation properties for  $f$  in the following class:

$$\bar{\mathcal{P}}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta) = \{f : x^{-2}f(1/x) \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)\},$$

since for all  $x > 0$ , as  $z \rightarrow \infty$ , with  $q(x) = x^{-2}f(1/x)$ ,

$$\bar{K}_z f(x) \stackrel{def}{=} \int_0^\infty \bar{g}_{z,\xi}(x) f(\xi) d\xi = \frac{1}{x^2} \int_0^\infty g_{z,\epsilon}(1/x) q(\epsilon) d\epsilon \rightarrow \frac{1}{x^2} q(1/x) = f(x).$$

We summarize the result in the following corollary.

**Corollary 2.1.** *Consider the prior defined by (2.11) that satisfies condition ( $\bar{\mathcal{P}}$ ), and assume that  $\mathbf{X}^n = (X_1, \dots, X_n)$  is a sample of independent observations distributed according to a probability  $P_0$  on  $\mathbb{R}^+$  having density  $f_0$  with respect to Lebesgue measure.*

*Then, for any  $\alpha_0 > 0$ ,  $\beta_1 \geq \beta_0 > 0$ ,  $L(\cdot)$ ,  $\gamma$ ,  $C_0$ ,  $C_1$ ,  $e$ ,  $\Delta$ ,  $C_2$  and  $\rho_1$ , there exists  $M > 0$  such that*

$$\sup_{\alpha \in [\alpha_0, 1]} \sup_{\beta \in [\beta_0, \beta_1]} \sup_{f_0 \in \bar{\mathcal{Q}}_{\alpha, \beta}(\dots)} E_0 \left[ \mathbb{I} \left( \|f - f_0\|_1 > M n^{-\beta/(2\beta+1)} (\log n)^q | \mathbf{X}^n \right) \right] = o(1)$$

*with  $q = (5\beta + 1)/(4\beta + 2)$  if  $\rho_z \leq 5/2$  and  $q = (2\rho_z\beta + 1)/(4\beta + 2)$  if  $\rho_z > 5/2$  and*

$$\bar{\mathcal{Q}}_{\alpha, \beta}(\dots) = \{f : x^{-2}f(1/x) \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta) \cap \mathcal{T}(\rho_1, C_2)\}.$$

Note that the folded Cauchy density satisfies the conditions for  $f_0(x)$  required in the corollary.

**2.4. Approximation of densities by Gamma mixtures**

As in Rousseau [11], Kruijer et al. [7], one of the key elements in the proof of Theorem 2.1 is the approximation of a smooth density  $f$  by a continuous Gamma mixture of the form  $K_z f_1$  where  $f_1$  is a smooth function close to  $f$  which is of independent interest. Similarly to Rousseau [11], Kruijer et al. [7],  $f_1$  is constructed iteratively to be able to adapt to the smoothness of  $f_0$ . The

general idea is that  $K_z f_0(x)$  is a good approximation of  $f_0$  if  $f_0$  has smoothness  $\beta \leq 2$ , as in the Gaussian case, because the Gamma density  $g_{z,\epsilon}$  behaves like a Gaussian density when  $z$  goes to infinity. To approximate  $f_0$  with the correct order  $z^{-\beta/2}$  for  $\beta > 2$ , we need to correct for the error  $K_z f_0 - f_0$  and replace  $K_z f_0$  by  $K_z f_1$  where construction of  $f_1$  takes into account  $K_z f_0 - f_0$ . Thus, we iterate until the approximation error  $K_z f_k - f_k$  is of the required order,  $z^{-\beta/2}$ .

The above approximation scheme is described in the following proposition:

**Proposition 2.1.** *For all  $z > 1 - \alpha$ , for all  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ , there exist  $d_j \in \mathbb{R}$ ,  $j = 1, \dots, 2r_0$  such that the function  $\tilde{f}_{\beta,\alpha}(x) = C_z f_{\beta,\alpha}(C_z x)$  with  $C_z = (z - \alpha + 1)/z$  and*

$$f_{\beta,\alpha}(x) = f(x) - x^{\alpha-1} \sum_{j=1}^r \frac{d_j(z)}{j! z^{j/2}} x^j [x^{1-\alpha} f(x)]^{(j)}, \quad d_j(z) = d_j + O(1/z),$$

satisfies

$$\left| K_z \tilde{f}_{\beta,\alpha}(x) - f(x) \right| \leq z^{-\beta/2} R(x), \quad \int_0^\infty \frac{R(x)^2}{f(x)} dx \leq C_R, \quad (2.13)$$

where  $C_R$  depends on  $\alpha, \beta, L(\cdot), \gamma, C_0, C_1, e, \Delta$  only. Moreover, the probability density

$$\bar{f}_\beta = c_\beta \left( \tilde{f}_{\beta,\alpha} \mathbb{I}_{\tilde{f}_{\beta,\alpha} \geq \bar{f}/2} + \frac{\tilde{f}}{2} \mathbb{I}_{\tilde{f}_{\beta,\alpha} < \bar{f}/2} \right) \quad (2.14)$$

with  $\tilde{f}(x) = C_z f(C_z x)$  for all  $x \in \mathbb{R}^+$ , satisfies

$$D_H(K_z \bar{f}_\beta, f) \leq B z^{-\beta/2}, \quad \forall f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta) \quad (2.15)$$

where  $B$  depends only on  $\alpha, \beta, L(\cdot), \gamma, C_0, C_1, e, \Delta$ .

Note that in the approximations (2.13) and (2.15), when  $\alpha \in [\alpha_0, 1]$  with  $0 < \alpha_0 \leq 1$ , then  $C_R$  and  $B$  can be chosen depending on  $\alpha_0$  and not  $\alpha$ . Proposition 2.1 is proved in Section A.1.

### 3. Numerical results

#### 3.1. Prior model

In the following sections we will fit a Bayesian model to the data with following Dirichlet Process prior:

$$\begin{aligned} f(x) &= \sum_j p_j g_z(x|\epsilon_j), \quad \epsilon_j \stackrel{iid}{\sim} G, \quad p_j = V_j \prod_{i<j} (1 - V_i), \quad V_i \sim \text{Beta}(1, m), \\ G(x) &\propto [x^a I_{x \leq 1} + x^{-a} I_{x > 1}], \quad a > 1, \\ \Pi_z : \quad \sqrt{z} &\sim \Gamma(b, c), \quad b, c > 0. \end{aligned} \quad (3.1)$$

It is easy to check that Condition (P) holds for this prior. We use default choices of free parameters  $m = 1$ ,  $a = 2$  and  $b = c = 1$ , however we check sensitivity with respect to these parameters.

To sample from the posterior we use the slice sampling algorithm of Kalli et al. [6]. Introducing the auxiliary variables  $\mathbf{u} = (u_1, \dots, u_n)$  the uniform random truncation variables and  $\mathbf{c} = (c_1, \dots, c_n)$  the allocation variables so that the full likelihood is written as

$$L_n(\mathbf{X}^n, \mathbf{u}, \mathbf{c}) = \prod_{i=1}^n \frac{\mathbb{I}_{u_i \leq p_{c_i}}}{p_{c_i}} g_{z, \epsilon_{c_i}}(X_i) p_{c_i}, \quad p_j = V_j \prod_{l < j} (1 - V_l)$$

allows to use a Gibbs sampler algorithm based on the following full conditional distributions

- $[\epsilon_j | \dots] \stackrel{ind}{\sim} G(\epsilon_j) e^{-z S_j / \epsilon_j} \epsilon_j^{-z n_j}, \quad S_j = \sum_{c_i=j} X_i, \quad n_j = \sum_{i=1}^n \mathbb{I}_{c_i=j}$
- $[V_j | \dots \text{exclude } u] \stackrel{ind}{\sim} \text{Beta}(n_j + 1, \sum_{l>j} n_l + m)$
- $[u_i | \dots] \stackrel{ind}{\sim} \mathcal{U}(0, p_{c_i})$
- $p[c_i = j | \dots] \propto \mathbb{I}_{u_i \leq p_j} g_z(X_i | \epsilon_j)$
- $[z | \dots] \propto \frac{\pi_z(z) z^{n z}}{\Gamma(z)^n} e^{-z \sum_j S_j / \epsilon_j + z \sum_i \log X_i}$

where the full conditional distribution of  $z$  is sampled using a Metropolis-Hastings step with proposal

$$\Gamma \left( (b+n)/2, \sum_{j:n_j>0} \frac{S_j}{\epsilon_j} - n - \sum_i \log X_i + \sum_{j:n_j>0} n_j \log \epsilon_j \right). \quad (3.2)$$

To improve convergence when sampling from the conditional distribution of  $z$  at iteration  $i$ , we also use a proposal which is a mixture of the gamma density (3.2) with weight  $1 - w_z$  and of  $\Gamma(z^{(i-1)} B_z, B_z)$  with a small weight  $w_z$  and a large  $B_z$  where  $z^{(i-1)}$  is the value of  $z$  sampled at iteration  $i - 1$ . We use the default values  $w_z = 0.01$  and  $B_z = 10$ .

Another Metropolis-Hastings step is sampling from the full conditional distribution of  $\epsilon_j$  which uses proposal  $IG(a + z^{(i-1)} n_j, z^{(i-1)} S_j)$  at iteration  $i$  for  $j: n_j > 0$ .

### 3.2. Simulations

We simulate  $n = 1000$  observations from the true density  $f_0$ , and fit a Bayesian model with prior (3.1). We considered the following true densities.

1. Exponential:  $f_0(x) = e^{-x}, x > 0$  (Figure 1).
2. Folded Cauchy:  $f_0(x) = \frac{2}{\pi} \frac{1}{(1+x^2)}, x > 0$  (Figure 1).

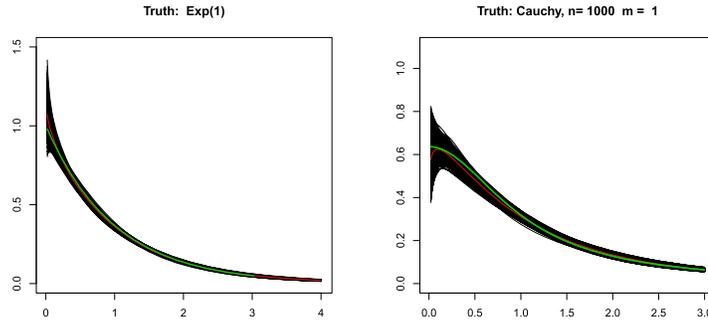


FIG 1. Left: exponential density. Right: folded Cauchy density

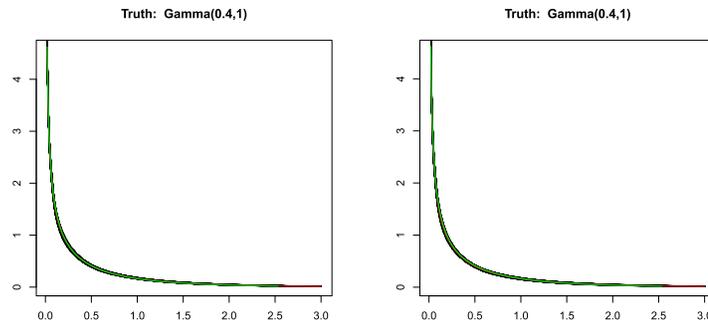


FIG 2.  $f_0(x) = \text{Gamma}(0.4, 1)$ . Left:  $m=0.1$ , right:  $m=1$

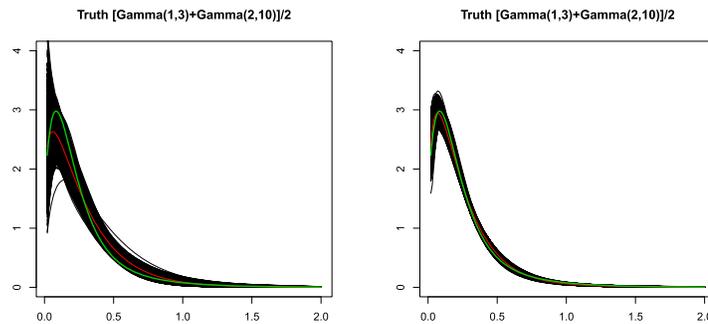


FIG 3. Density:  $0.5 \text{Gamma}(1,3) + 0.5 \text{Gamma}(2,10)$ . Left:  $n=100$ , right:  $n= 1000$

- 3. Unbounded:  $f_0(x)$  is  $\Gamma(0.4, 1)$  (Figure 2).
- 4. Mixture with different  $z$ :  $f_0(x)$  is  $0.5\Gamma(1, 3) + 0.5\Gamma(2, 10)$  (Figure 3).

Even though the folded Cauchy density does not satisfy the conditions of the theorem, we show that the proposed gamma mixture still provides a reasonably concentrating posterior distribution.

Distribution	$m$	$n$	50%	95%
Cauchy	$m = 1$	$n = 100$	0.187	0.303
Cauchy	$m = 1$	$n = 1000$	0.071	0.109
Cauchy	$m = 0.1$	$n = 1000$	0.073	0.108
Cauchy	$m = 10$	$n = 1000$	0.065	0.100
Exp(1)	$m = 1$	$n = 100$	0.0933	0.1873
Exp(1)	$m = 1$	$n = 1000$	0.0323	0.0605
Exp(1)	$m = 0.1$	$n = 1000$	0.0312	0.0609
Exp(1)	$m = 10$	$n = 1000$	0.0383	0.0727
Gamma mixture	$m = 1$	$n = 100$	0.1527	0.2549
Gamma mixture	$m = 1$	$n = 1000$	0.0944	0.1170
Gamma mixture	$m = 0.1$	$n = 1000$	0.0931	0.1184
Gamma mixture	$m = 10$	$n = 1000$	0.0692	0.0961
Gamma(0.4,1)	$m = 1$	$n = 100$	0.1274	0.2267
Gamma(0.4,1)	$m = 1$	$n = 1000$	0.0400	0.0631
Gamma(0.4,1)	$m = 0.1$	$n = 1000$	0.0321	0.0655
Gamma(0.4,1)	$m = 10$	$n = 1000$	0.0347	0.0658

TABLE 1

50% and 95% quantiles of the posterior distribution of  $\|f - f_0\|_1$ , for different values of the Dirichlet Process prior mass  $m$  and sample size  $n$ .

1000 thinned samples from the posterior distribution are plotted for each true density after at least 50000 burn in iterations, with the red line representing the mean of the posterior distribution and the green line representing the true density. Improvement of the concentration of the posterior distribution with increasing sample size is presented for the two component mixture in Figure 3 (see also Table 1 for other densities). For all considered true densities, including the unbounded one and the folded Cauchy density, the proposed gamma mixture model performs well. However, the value of the folded Cauchy density around 0 has high uncertainty. Sensitivity with respect to the choice of the free parameters was investigated for all densities, all leading to good performance (presented for the unbounded density  $Gamma(0.4, 1)$ ). We found that using  $m = 10$  leads to a high number of mixture components even in the cases  $\Gamma(0.4, 1)$  and  $Exp(1)$  (there is only one component if  $m = 1$  or  $m = 0.1$ ).

We study sensitivity of the quality of estimation with regard to the considered loss function, the  $L_1$  norm  $\|f - f_0\|_1$ , with respect to the DP mass parameter  $m$  and the sample size  $n$ . The median and the 95% quantile of the posterior distribution of  $\|f - f_0\|_1$  for the considered densities with different values of  $m$  (0.1, 1, 10) and different sample sizes for the default value  $m = 1$  ( $n = 100$  and  $n = 1000$ ) are presented in Table 1. The quantiles decrease with increasing sample size, and are little affected by changing the value of  $m$ .

### 3.3. Email arrival data

In this section we consider the data of the intervals between arrival times of emails modelled in [16] which consists of the interarrival times (minutes) of 203 E-mail messages (we are grateful to Fabrizio Ruggeri, one of the authors, who has kindly provided the data). The proposed location Gamma mixture (3.1) with

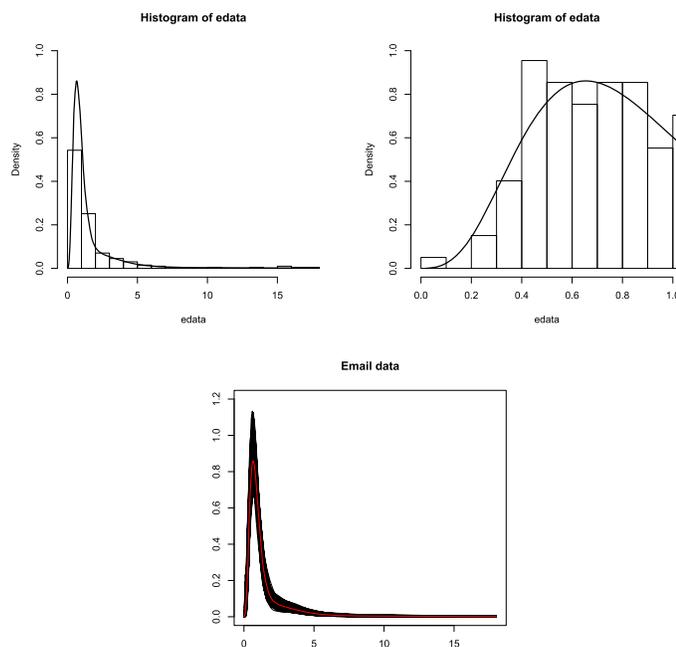


FIG 4. *Email arrival data. Top left: histogram of the data with the posterior mean density estimate (black line). Top right: zoom in of the histogram of the data with the posterior mean density estimate (black line). Bottom: draws from the posterior distribution with posterior mean (red line).*

the default choice of the free parameters was fitted to the data. 1250 samples from the posterior were used which are the last 25% of the 50000 iterations thinned by 10. The histogram of the data with superimposed posterior mean and the samples from the posterior distribution are shown in Figure 4. The histogram shows that the fit of the location mixture is similar to the fit of the location-scale mixture presented in [16]. The plot of the samples with the posterior indicates uncertainty about the values of the density around 0 as well as high uncertainty about the possible second mode around 3. We also present a zoom in into the neighbourhood of 0 which confirms the findings that the density is small around 0 and that the distribution of the email arrival times differs from exponential.

## 4. Proofs

### 4.1. Proof of Theorem 2.1

*Proof of Theorem 2.1.* The proof consists in verifying the assumptions of Theorem 2.1 of Ghosal et al. [3]. The first assumption, on the prior mass of Kullback - Leibler neighbourhood of the true density, is verified in Lemma 4.1. Note that

there is a small modification from condition (2.4) of Theorem 2.1 of Ghosal et al. [3] in that the bound on the variance of the log-likelihood ratio has an extra  $\log n$  term. This does not affect the conclusion since the variance term only need to be negligible compared to  $n\mathcal{KL}(f, K_z * P)^2$ . In Lemma 4.2 we control the  $L_1$  (and Hellinger) entropy of the sieves which are defined below.

Fix an arbitrary  $\zeta > 0$  to be defined later, and take a sieve  $Q_n = Q(\zeta\epsilon_n, J_n, a_n, b_n, \underline{z}, \bar{z})$  as defined by (4.2) in Lemma 4.2 with

$$\begin{aligned}\epsilon_n &= n^{-\beta/(2\beta+1)}[\log n]^{(5\beta+1)/(4\beta+2)}, & J_n &= J_0 n^{1/(2\beta+1)}[\log n]^{3\beta/(2\beta+1)}, \\ a_n &= \exp(-C(n(\log n)^{5\beta+1})^{1/(2\beta+1)}), & b_n &= \exp(C(n(\log n)^{5\beta+1})^{1/(2\beta+1)}), \\ \underline{z} &= \exp(-z_0(n(\log n)^{5\beta+1})^{1/(2\beta+1)}), & \bar{z} &= z_0 n^{2/(2\beta+1)}(\log n)^{2(q-\rho_z)}\end{aligned}$$

for some constants  $C, z_0$  and  $J_0$  large enough and  $q$  as defined in the theorem.

Lemma 4.2 implies that we need to verify whether these constants satisfy the following conditions:

$$\log N(\zeta\epsilon_n, Q_n, \|\cdot\|_1) \leq \tilde{c}n\epsilon_n^2, \quad \Pi(Q_n^c) \lesssim e^{-\tilde{c}n\epsilon_n^2} \quad (4.1)$$

for any  $\tilde{c} > 0$  with  $\epsilon_n = \epsilon_0\epsilon_n$  by choosing  $\epsilon_0$  large enough. This corresponds to choosing  $\varepsilon = \zeta\epsilon_n/5$  in Lemma 4.2.

The second inequality in (4.1) holds if

$$\begin{aligned}J_n \bar{G}((0, a_n)) &\lesssim e^{-cn\epsilon_n^2}, \\ J_n \bar{G}((a_n + b_n, \infty)) &\lesssim e^{-cn\epsilon_n^2}, \\ 1 - H([\underline{z}, \bar{z}]) &\lesssim e^{-cn\epsilon_n^2}, \\ \left(\frac{em}{J_n} \log(1/\varepsilon_n)\right)^{J_n} &\lesssim e^{-cn\epsilon_n^2}.\end{aligned}$$

In our case,  $\varepsilon_n = n^{-\gamma}(\log n)^t$  with  $\gamma = \beta/(2\beta + 1)$  and  $t = (5\beta + 1)/(4\beta + 2)$ . The last condition holds if

$$J_n(\log J_n - \log \log n + C) \geq cn^{1-2\gamma}[\log n]^{2t},$$

e.g. if  $J_n = Cn^{1-2\gamma}[\log n]^{2t-1} = Cn^{1/(2\beta+1)}[\log n]^{3\beta/(2\beta+1)}$ .

The first inequality in (4.1) holds if

$$J_n [\log \log(b_n/a_n) + \log(\bar{z}_n) + \log(1/\varepsilon_n)] + \log \log(\bar{z}_n/\underline{z}_n) \lesssim n\epsilon_n^2 = n^{1-2\gamma}[\log n]^{2t}$$

that is, if  $b_n/a_n \lesssim e^{Bn^A}$  for some  $A, B > 0$ ,  $\bar{z}_n \lesssim n^D$  for some  $D > 0$ ,

$$n^{1-2\gamma}[\log n]^{2t-1} \{\log n + \log \log n + C\} + \log n \leq Cn^{1-2\gamma}[\log n]^{2t}$$

which holds for large enough constant  $C$ , and if  $\log \log(\bar{z}_n/\underline{z}_n) \lesssim n^{1-2\gamma}[\log n]^{2t}$ . In our case,  $b_n/a_n = \exp(C(n(\log n)^{5\beta+1})^{1/(2\beta+1)}) \lesssim e^{Bn^A}$  with any  $A > 1/(2\beta + 1)$ , and  $\bar{z}_n \lesssim n^D$  with any  $D > 2/(2\beta + 1)$ .

Choosing  $\epsilon_0$  large enough in the definition of  $\epsilon_n$  completes the proof of (4.1), and hence the proof of Theorem 2.1.  $\square$

We extend the definition of  $K_z$  in the following way: for any distribution  $P$ , define

$$K_z * P(x) = \int_0^\infty g_{z,\epsilon}(x) dP(\epsilon).$$

If  $P$  has Lebesgue density  $f$ , then  $K_z * P(x) = K_z f(x)$ .

**Lemma 4.1.** *Assume that the probability density  $f_0 \in \mathcal{P}_\alpha(\beta, L, \gamma, C_0, C_1, e, \Delta)$  and that there exist  $C > 0$  and  $\rho_1 > 0$  such that*

$$\int_x^\infty y^2 f_0(y) dy \leq C(1+x)^{-\rho_1}.$$

*Then, for any  $\epsilon_0 > 0$ , there exist  $\kappa, C_p > 0$  such that*

$$\Pi \left( \mathcal{KL}(f, K_z * P) \leq \epsilon_n^2; V(f, K_z * P) \leq \epsilon_n^2 \log n \right) \geq C_p e^{-\kappa n^{1/(2\beta+1)} (\log n)^{2q}},$$

*for any prior satisfying condition  $(\mathcal{P})$  and  $n \geq 1$  where  $\epsilon_n = \epsilon_0 n^{-\beta/(2\beta+1)} (\log n)^q$ , with  $q$  defined in Theorem 2.1. The constants  $\kappa$  and  $C_p$  depend on  $\Pi$ ,  $\epsilon_0$  and on the constants defining the functional class.*

Lemma 4.1 is proved in Section 4.2.

As in Shen et al. [14], we control the entropy of the following approximating sets.

**Lemma 4.2.** *Fix  $\varepsilon > 0$ ,  $J \in \mathbb{N}$ ,  $a, b > 0$ ,  $0 < \underline{z} < \bar{z} < \infty$  and introduce the following class of densities:*

$$Q = Q(\varepsilon, J, a, b, \underline{z}, \bar{z}) = \left\{ f = \sum_{j=1}^\infty \pi_j g_{z, \epsilon_j} : \sum_{j>J} \pi_j < \varepsilon, z \in [\underline{z}, \bar{z}], \right. \\ \left. \epsilon_j \in [a, a+b] \text{ for } j = 1, \dots, J \right\}. \tag{4.2}$$

*Then, for  $\varepsilon \leq \sqrt{\bar{z}}$ ,*

$$\log N(5\varepsilon, Q, \|\cdot\|_1) \leq C + J \left[ \log \log \left( \frac{b}{a} \right) - 3 \log \varepsilon + 0.5 \log(\bar{z}) \right] + \log \log \left( \frac{\bar{z}}{\underline{z}} \right),$$

$$\Pi(Q^c) \leq \left( \frac{em}{J} \log(1/\varepsilon) \right)^J + J(1 - G([a, a+b])) + 1 - \Pi_z([\underline{z}, \bar{z}]),$$

*where  $\Pi$  is a prior satisfying condition  $(\mathcal{P})$ .*

The proof of Lemma 4.2 is given in Section 4.3. In the next section we prove Lemma 4.1.

#### 4.2. Proof of Lemma 4.1

Consider  $P_N$  the discrete distribution constructed in Lemma B.2, which we write as  $P_N = \sum_{j=1}^N p_j \delta_{u_j}$ , with  $N \leq N_0 \sqrt{z} (\log z)^{3/2}$ ,  $u_j \in [e_z, E_z]$ ,  $u_1 \leq u_2 \leq \dots \leq u_N$ ,  $u_{i+1} - u_i > z^{-A}$  and  $p_j > z^{-A}$  for some  $A$  and with  $e_z = z^{-a}$  and  $E_z = z^b$ , with  $a, b$  defined as in Lemma B.2.

Set  $U_j = [(u_j + u_{j-1})/2, (u_j + u_{j+1})/2]$ , with  $u_0 = u_1$  and  $u_{N+1} = u_N$ ,  $U_0 = \mathbb{R}^+ \setminus \cup_{j=1}^N U_j$  and define for  $A > 0$

$$\mathcal{P}_z = \{P : P(U_j)/p_j \in (1 - 2z^{-A}, 1 - z^{-A}) \quad \forall 1 \leq j \leq N\}.$$

Note that for all  $P \in \mathcal{P}_z$ ,  $P(U_0) \in (z^{-A}, 2z^{-A})$ .

Let  $z_n = n^{2/(2\beta+1)}(\log n)^t$  with  $t = 2q - 5$  if  $\rho_z \leq 5/2$  and  $t = 2(q - \rho_z)$  if  $\rho_z > 5/2$  and set  $I_n = (z_n, 2z_n)$ . Then for all  $z \in I_n$  and all  $P \in \mathcal{P}_{2z_n}$ , Lemma B.3 implies that

$$\begin{aligned} \mathcal{KL}(f, K_z * P) &\leq n^{-2\beta/(2\beta+1)}(\log n)^{-2\beta t+1} \asymp \epsilon_n^2, \\ V(f, K_z * P) &\leq C_2 n^{-2\beta/(2\beta+1)}(\log n)^{-2\beta t+2} \asymp \epsilon_n^2 \log n, \end{aligned}$$

if  $A$  is large enough (depending on  $\beta, L, \gamma, C_0, C_1, e, \Delta$ ).

To prove Lemma 4.1, we thus need to bound from below  $\Pi(I_n \times \mathcal{P}_n)$ . Denote  $\alpha_j = mG(U_j)$ ,  $j = 0, \dots, N$  with  $N \asymp \sqrt{z_n}(\log z_n)^{3/2}$  in the DP( $m, G$ ) type prior case. Note that for large  $u_{j-1} \gtrsim E_z$ ,

$$\begin{aligned} \alpha_j &= mG(U_j) = m \int_{(u_{j-1}+u_j)/2}^{(u_j+u_{j+1})/2} g(u) du \geq C \int_{(u_{j-1}+u_j)/2}^{(u_j+u_{j+1})/2} u^{-a_1} du \geq C[E_z]^{1-a_1} \\ &= Cz^{b(1-a_1)}, \end{aligned}$$

and similarly  $\alpha_j \leq Cz^{-b(a'_1-1)}$ . For small  $u_{j+1} \lesssim e_z$ ,

$$\alpha_j = mG(U_j) \geq C \int_{(u_{j-1}+u_j)/2}^{(u_j+u_{j+1})/2} u^{a'_0} du \geq C[e^z]^{a'_0+1} = Cz^{-a(a'_0+1)},$$

and similarly  $\alpha_j \leq Cz^{-a(a_0+1)}$ . Hence we have  $\alpha_j \geq Cz^{-B}$  with  $B = \min(a(a_0+1), b(a'_1-1))$ .

In particular, we have that  $\sum_j \alpha_j = m$  for the DP prior, and

$$\sum_{j=1}^N (-\log \alpha_j) \leq \sum_{j=1}^N B \log z = NB \log z \leq NB \log N.$$

Also, we have that

$$\sum_{i=1}^N \alpha_i (\log \alpha_i)_+ = \sum_{i=1}^N mp_i (\log mp_i)_+ \leq m \log m$$

Note that for  $x \in (0, 1]$ ,  $\Gamma(x) \leq x^{-1}$ , and for  $x > 1$ ,  $\Gamma(x) < \exp((x - 1/2) \log x - (x - 1))$  [8].

Adapting Lemma 6.1 of Ghosal et al. [3] to the case of hyperparameters  $\alpha_i$  of the Dirichlet distribution possibly greater than 1, we obtain:

$$\Pi(\mathcal{P}_n) \geq \frac{\Gamma(m)}{\prod_i \Gamma(\alpha_i)} z_n^{-A(\alpha_0-1)} 2^{-(\alpha_0-1)-} \int \prod_{i=1}^N \mathbb{I}_{x_i \in (p_i(1-2z_n^{-A}), p_i(1-z_n^{-A}))} x_i^{\alpha_i-1} dx_i$$

$$\begin{aligned} &\gtrsim z_n^{-A(\alpha_0-1+N)} \frac{\Gamma(m)}{\prod_i \Gamma(\alpha_i)} \prod_{i=1}^N p_i^{\alpha_i} \\ &\gtrsim z_n^{-A(N+m+BN \log N)} \gtrsim z_n^{-(B+1)AN \log N} \gtrsim e^{-(B+2)AN_0 \sqrt{z_n} (\log z_n)^{5/2}} \end{aligned}$$

Using condition  $(\mathcal{P})$  on  $\Pi_z$  we have that  $\Pi_z(I_n) \gtrsim e^{-c\sqrt{z_n}(\log z_n)^{\rho z}}$ , replacing  $z_n$  by its expression terminates the proof of Lemma 4.1 for the DP prior.

For the mixture prior satisfying  $(\mathcal{P})$ , we have

$$\begin{aligned} \Pi(\mathcal{P}_n) &\geq \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} z_n^{-A(\alpha_0-1)} 2^{-(\alpha_0-1)} \int \prod_{i=1}^N \mathbb{1}_{x_i \in (p_i(1-2z_n^{-A}), p_i(1-z_n^{-A}))} x_i^{\alpha_i-1} dx_i \\ &\gtrsim z_n^{-A(\alpha_0-1+N)} \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \prod_{i=1}^N p_i^{\alpha_i} \gtrsim B^N z_n^{-A(\alpha_0-1+N)} \\ &\times \exp\left(-A \log z_n \sum_{i:\alpha_i>1} \alpha_i + \sum_i (-\log \alpha_i)_+ - \sum_{i:\alpha_i>1} (\alpha_i - 1/2) \log \alpha_i\right) \\ &\gtrsim \exp(-C(N + N \log n)), \end{aligned}$$

which terminates the proof.

### 4.3. Proof of Lemma 4.2

Take any  $f \in Q$ , that is,  $f = \sum_{j=1}^\infty \pi_j g_{z, \epsilon_j}$  such that  $\sum_{j>J} \pi_j < \varepsilon$ ,  $z \in [\underline{z}, \bar{z}]$  and  $\epsilon_j \in [a, a + b]$  for  $j = 1, \dots, J$ .

Fix  $\delta_2 = \varepsilon/C$  for the constant  $C$  to be defined below, and  $\delta_0 = \varepsilon/\sqrt{2z}$ . Let  $\hat{A}$  be the following set  $\{a(1 + \delta)^k\}_{k=0}^K$  with  $K = K_z = \lceil \log(1 + b/a) / \log(1 + \delta) \rceil$ , and  $\hat{Z} = \{\underline{z}(1 + \delta_2)^\ell\}_{\ell=0}^L$  with  $L = \lceil \log(\bar{z}/\underline{z}) / \log(1 + \delta_2) \rceil$ . In particular, for any  $z \in [\underline{z}(1 + \delta_2)^\ell, \underline{z}(1 + \delta_2)^{\ell+1}]$  for some  $\ell \in \{0, 1, \dots, L\}$ ,  $\inf_{\hat{z} \in \hat{Z}} |\hat{z}/z - 1| \leq \delta_2$ . Let  $\hat{S}$  be an  $\varepsilon$ -net for  $S = \{(p_1, \dots, p_J) : p_j = \pi_j / (1 - \sum_{j=1}^J \pi_j) \forall j\}$ .

Define

$$\hat{Q} = \left\{ \begin{aligned} &\hat{f} = \sum_{j=1}^J \hat{\pi}_j g_{\hat{z}, \hat{\epsilon}_j} \text{ where } \hat{z} \in \hat{Z}, \quad |\hat{z}/z - 1| < \delta_2, \\ &\hat{\epsilon}_j \in \hat{A}, \quad j = 1, \dots, J, \quad \max_{j=1, \dots, J} |\hat{\epsilon}_j/\epsilon_j - 1| < \delta = \varepsilon/\sqrt{2z}, \\ &\hat{\pi} = (\hat{\pi}_j) \in \hat{S} \text{ and } \sum_{j=1}^J |\hat{\pi}_j - \tilde{\pi}_j| < \varepsilon, \quad \text{with } \tilde{\pi}_j = \pi_j / [\sum_{j=1}^J \pi_j] \end{aligned} \right\}.$$

Now we show that  $\hat{Q}$  is a  $5\varepsilon$ -net of  $Q$ :

$$\begin{aligned} \left\| \sum_{j=1}^J \hat{\pi}_j g_{\hat{z}, \hat{\epsilon}_j} - \sum_{j=1}^\infty \pi_j g_{z, \epsilon_j} \right\|_1 &\leq \left\| \sum_{j=1}^J \hat{\pi}_j g_{z, \hat{\epsilon}_j} - \sum_{j=1}^J \hat{\pi}_j g_{\hat{z}, \hat{\epsilon}_j} \right\|_1 + \left\| \sum_{j>J} \pi_j g_{z, \epsilon_j} \right\|_1 \\ &\quad + \left\| \sum_{j=1}^J \pi_j (g_{z, \hat{\epsilon}_j} - g_{z, \epsilon_j}) \right\|_1 + \left\| \sum_{j=1}^J (\hat{\pi}_j - \pi_j) g_{z, \hat{\epsilon}_j} \right\|_1 \end{aligned}$$

$$\leq \sum_{j=1}^J \hat{\pi}_j \|g_{z,\hat{\epsilon}_j} - g_{\hat{z},\hat{\epsilon}_j}\|_1 + \sum_{j>J} \pi_j + \sum_{j=1}^J \pi_j \|g_{z,\hat{\epsilon}_j} - g_{z,\epsilon_j}\|_1 + \sum_{j=1}^J |\hat{\pi}_j - \pi_j|.$$

The second term is less than  $\varepsilon$  by the definition of  $Q$ . The last term is bounded in the same way as in [14] by  $\sum_{j=1}^J |\hat{\pi}_j - \pi_j| \leq 2\varepsilon$ .

To bound the third term, we first bound the  $L_1$  distance between the two gamma densities using Lemma C.1:

$$\|g_{z,\hat{\epsilon}_j} - g_{z,\epsilon_j}\|_1 \leq \sqrt{2\mathcal{KL}(g_{z,\hat{\epsilon}_j}, g_{z,\epsilon_j})} \leq \sqrt{2z}\delta = \varepsilon$$

by the definition of  $\delta$ .

To bound the first term, we bound the Kullback-Leibler distance between the corresponding probability distributions:

$$\begin{aligned} \mathcal{KL}(g_{z,\epsilon}, g_{\hat{z},\epsilon}) &= \log\left(\frac{\hat{z}^{-\hat{z}}\Gamma(\hat{z})}{z^{-z}\Gamma(z)}\right) - (\hat{z} - z)[\Gamma'(z)/\Gamma(z) - \log z - 1] \\ &= 0.5(\hat{z} - z)^2[-z_c^{-1} + \psi_1(z_c)] \end{aligned}$$

for some  $z_c$  between  $z$  and  $\hat{z}$  where  $\psi_1(z) = (\log \Gamma(z))''$  is the trigamma function. It is known that as  $z \rightarrow 0$ ,  $\psi_1(z) = \gamma + z^{-2} + o(1)$ , and as  $z \rightarrow \infty$ ,  $\psi_1(z) = z^{-1} + 0.5z^{-2} + o(z^{-2})$  which implies that for both  $z$  large and small,

$$\psi_1(z) - 1/z = O(z^{-2}),$$

which implies that we can bound the Kullback-Leibler distance as

$$\mathcal{KL}(g_{z,\epsilon}, g_{\hat{z},\epsilon}) = 0.5(\hat{z} - z)^2[-z_c^{-1} + \psi_1(z_c)] \leq C(\hat{z}/z - 1)^2 \leq C\delta_2^2$$

for an appropriate constant  $C$ . Therefore, the first term is bounded by

$$\sum_{j=1}^J \hat{\pi}_j \|g_{z,\hat{\epsilon}_j} - g_{\hat{z},\hat{\epsilon}_j}\|_1 \leq \sum_{j=1}^J \hat{\pi}_j \sqrt{2\mathcal{KL}(g_{z,\hat{\epsilon}_j}, g_{\hat{z},\hat{\epsilon}_j})} \leq \sqrt{2C}\delta_2 = \varepsilon$$

by the definition of  $\delta_2$ .

Now we study cardinality of set  $\hat{A}$ . For each  $z \in [\underline{z}, \bar{z}]$ ,

$$K_z \leq 1 + \log(1 + b/a) / \log(1 + \varepsilon/\sqrt{2z}) \lesssim \frac{\log(b/a)}{\log(1 + \varepsilon/\sqrt{2z})} \lesssim \frac{\sqrt{\bar{z}} \log(b/a)}{\varepsilon}$$

for large  $b/a$  to due to  $\log(1 + x) \geq x(1 - 0.5/\sqrt{2})$  for  $x \leq 1/\sqrt{2}$  and assuming that  $\varepsilon \leq \sqrt{\bar{z}}$ .

Cardinality of  $\hat{S}$  is  $\lesssim \varepsilon^{-J}$  [14, proof of Proposition 2].

Then, for  $\varepsilon \leq \sqrt{\bar{z}}$ , the cardinality of  $\hat{Q}$  is bounded by

$$|\hat{Q}| \leq |\hat{S}| \sum_{\ell=1}^L |K_z|^J \lesssim \varepsilon^{-J} L \left[ \frac{\sqrt{\bar{z}} \log(b/a)}{\varepsilon} \right]^J \lesssim \left[ \frac{\log(b/a)\sqrt{\bar{z}}}{\varepsilon^2} \right]^J \frac{\log(\bar{z}/\underline{z})}{\varepsilon}$$

due to  $\delta_2 = C\varepsilon$  and by the definition of  $L$ .

Therefore, combining all the inequalities together, we obtain that

$$\left\| \sum_{j=1}^J \hat{\pi}_j g_{\bar{z}, \hat{\epsilon}_j} - \sum_{j=1}^{\infty} \pi_j g_{z, \epsilon_j} \right\|_1 \leq 5\varepsilon,$$

and hence  $\hat{Q}$  is a  $5\varepsilon$ -net of  $Q$ , with

$$\begin{aligned} & \log N(5\varepsilon, Q, \|\cdot\|_1) \\ & \leq C + J [\log \log(b/a) - 2 \log \varepsilon + 0.5 \log(\bar{z})] + \log \log(\bar{z}/\underline{z}) - \log \varepsilon \\ & \leq C + J [\log \log(b/a) - 3 \log \varepsilon + 0.5 \log(\bar{z})] + \log \log(\bar{z}/\underline{z}). \end{aligned}$$

The second inequality is proved following the same route as in the proof of Proposition 2 in Shen et al. [14].

For the Dirichlet process prior,

$$\begin{aligned} \Pi(Q^c) & \leq \Pi_z([\underline{z}, \bar{z}]^c) + JG([a, a+b]^c) + \Pi\left(\sum_{j>J} \pi_j > \varepsilon\right) \\ & \leq J(1 - G([a, a+b])) + 1 - \Pi_z([\underline{z}, \bar{z}]) + \left(\frac{em}{J} \log(1/\varepsilon)\right)^J. \end{aligned}$$

For the mixture prior,

$$\begin{aligned} \Pi(Q^c) & \leq \Pi_z([\underline{z}, \bar{z}]^c) + \sum_{k=1}^J \pi_K(k) [G([a, a+b]^c)]^k + \sum_{k=1}^{\infty} \pi_K(k) \Pi\left(\sum_{j=J+1}^k \pi_j > \varepsilon\right) \\ & \leq 1 - \Pi_z([\underline{z}, \bar{z}]) + JG([a, a+b]^c) + \sum_{k=J+1}^{\infty} C e^{-ck(\log k)^{\rho_2}} \\ & \leq 1 - \Pi_z([\underline{z}, \bar{z}]) + J[1 - G([a, a+b])] \\ & \quad + (\log J)^{-\rho_2} e^{-cJ(\log J)^{\rho_2}} \frac{c^{-1} C e^{-c(\log J)^{\rho_2}}}{1 - e^{-c(\log J)^{\rho_2}}} \\ & \lesssim 1 - \Pi_z([\underline{z}, \bar{z}]) + J[1 - G([a, a+b])] + (\log J)^{-\rho_2} e^{-cJ(\log J)^{\rho_2}} \end{aligned}$$

since  $\rho_2 \geq 0$ .

## Appendix A: Proof of Proposition 2.1 and related lemmas

### A.1. Proof of Proposition 2.1

The proof of the proposition is based on the ideas of Rousseau [11], Kruijer et al. [7] and Shen et al. [14]. First we prove (2.13) for  $f \in \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ , and then adapt it for the case  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ . The proof of (2.15) is then presented directly for  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ .

1. Proof of (2.13): Case  $\alpha = 1$

This case corresponds to  $f \in \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ . We can thus write

$$f(\epsilon) = \sum_{j=0}^r \frac{f^{(j)}(x)}{j!} (\epsilon - x)^j + R_1(\epsilon, x)$$

where  $|R_1(\epsilon, x)| \leq L(x)|\epsilon - x|^\beta$ . Then, using (C.7),

$$K_z f(x) = f(x)I_0(z) + \sum_{j=1}^r \frac{x^j f^{(j)}(x)}{j!z^{j/2}} \mu_j(z) + \frac{R_z(x)}{z^{\beta/2}},$$

$$|R_z(x)| \lesssim L(x)x^\beta \left(1 + \frac{x^\gamma}{z^{\gamma/2}}\right),$$

where  $I_0(z)$  is defined by (C.1) and  $\mu_k(z)$  are defined by (C.6). We then have

$$\Delta_z f(x) := K_z f(x) - f(x) = \frac{f(x)}{z-1} + \sum_{j=1}^r \frac{x^j f^{(j)}(x)}{j!z^{j/2}} \mu_j(z) + \frac{R_z(x)}{z^{\beta/2}},$$

so that if  $\beta \leq 2$ , and since  $\mu_1(z) = O(z^{-H})$  for any  $H > 0$ , we obtain

$$\Delta_z f(x) = \frac{f(x)}{z-1} + \frac{R_z(x)}{z^{\beta/2}} + O(z^{-H} x f^{(1)}(x)),$$

where the last term only appears if  $\beta > 1$  and Proposition 2.1 is verified for  $f \in \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ . If  $\beta \in (2, 4]$ , define

$$f_1(x) = f(x) - \frac{f(x)}{z-1} - \frac{x^2 f^{(2)}(x) \mu_2(z)}{2z},$$

then

$$K_z f_1(x) - f(x) = -\frac{\Delta_z f(x)}{z-1} - \frac{\mu_2(z) \Delta_z (x^2 f^{(2)}(x))}{2z} + \frac{R_z(x)}{z^{\beta/2}} + O(z^{-H} x f^{(1)}(x)).$$

Note that if  $f \in \mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ , with  $\beta > 2$ , the function  $x \rightarrow x^2 f^{(2)}(x)$  is  $r - 2$  times continuously differentiable and its derivatives are given by

$$(x^2 f^{(2)}(x))^{(l)} = x^2 f^{(2+l)}(x) + \sum_{j=1}^{2 \wedge l} C_l^j 2 \cdot \dots \cdot (2-j+1) x^{2-j} f^{(2+l-j)}(x) \quad (\text{A.1})$$

so that

$$\int_0^\infty \frac{(x^l (x^2 f^{(2)}(x))^{(l)})^2}{f(x)} dx \lesssim \sum_{j=2}^r \int_0^\infty \frac{(x^j f^{(j)}(x))^2}{f(x)} dx < +\infty.$$

Hence,

$$\begin{aligned} & \left| (x^2 f^{(2)})^{(r-2)}(x+y) - (x^2 f^{(2)})^{(r-2)}(x) \right| \\ & \lesssim \left| (x+y)^2 f^{(r)}(x+y) - x^2 f^{(r)}(x) \right| \\ & + \sum_{l=1}^{2 \wedge (r-2)} \left| (x+y)^{2-l} f^{(r-l)}(x+y) - x^{2-l} f^{(r-l)}(x) \right| \\ & \lesssim |y|(|y|+x) |f^{(r)}(x)| + (y^2 + x^2 + 1) |y|^{\beta-r+2} L(x) (1 + |y|^\gamma) \\ & + (|y|+x) |y| |f^{(3)}(x)| \end{aligned}$$

where the last term only appears if  $r = 3$ . Combined with (C.1) and (C.6), this leads to a remaining term in the control of  $\Delta_z(x^2 f^{(2)}(x))$  bounded by

$$\begin{aligned} R(x) &= Cz^{-(r-1)/2} \left[ x^r |f^{(r)}(x)| + (x^2 + 1)x^\beta L(x)(1 + x^\gamma) + x^3 |f^{(3)}(x)| \right] \\ &:= z^{-(r-1)/2} \tilde{R}(x) \end{aligned}$$

with  $\tilde{R}$  satisfying

$$\int_0^\infty \frac{\tilde{R}(x)^2}{f(x)} dx < +\infty.$$

$\tilde{R}$  thus behaves like  $R_z(x)$  and we can write

$$K_z f_1(x) - f(x) = -\frac{f(x)}{(z-1)^2} - \frac{\mu_2(z)x^2 f^{(2)}(x)}{z(z-1)} + z^{-\beta/2} R_{2,z}(x)$$

with

$$\int_0^\infty \frac{R_{2,z}(x)^2}{f(x)} dx < +\infty$$

uniformly in  $z$ . We can reiterate if  $\beta > 4$ . At the  $k - 1$  - th iteration

$$f_{k-1}(x) = \sum_{j=0}^{2k-2} d_{k-1,j}(z) \frac{x^j f^{(j)}(x)}{z^{j/2}}$$

with  $d_{k-1,0} = (z/(z-1))^{k-1}$  and for each  $j$

$$\epsilon^j f^{(j)}(\epsilon) = \sum_{l=0}^{r-j} \frac{(\epsilon-x)^l}{l!} \left\{ \sum_{t=0}^{l \wedge j} x^{j-t} f^{(j+l-t)}(x) a_{j,t} \right\} + \tilde{R}(\epsilon, x)$$

so that we can write

$$\begin{aligned} K_z f_{k-1}(x) - f(x) &= \frac{d_{k-1,0} f(x)}{z-1} \\ &+ \sum_{l=1}^{2k-2} \frac{x^l f^{(l)}(x)}{z^{l/2}} \left( \frac{d_{k-1,l}}{z-1} + \sum_{j=0}^{l-1} d_{k-1,j} \sum_{l'=1}^{l-j} \sum_{t=0}^{l' \wedge j} \frac{a_{j,t} \mu_{l'}(z)}{z^{t/2}} \right) \\ &+ \frac{x^{2k} f^{(2k)}(x)}{z^{k/2}} \sum_{j=0}^{2k-1} d_{k-1,j} \sum_{l'=1}^{2k-j} \sum_{t=0}^{l' \wedge j} \frac{a_{j,t} \mu_{l'}(z)}{z^{t/2}} \\ &+ z^{-\beta/2} \tilde{R}_{k,z}(x) \end{aligned}$$

with

$$\int_0^\infty \frac{\tilde{R}_{k,z}(x)^2}{f(x)} dx < +\infty$$

and we define

$$f_k(x) = \sum_{l=0}^{2k-2} \frac{x^l f^{(l)}(x)}{z^{l/2}} \left( d_{k-1,l} \frac{z}{z-1} - \sum_{j=0}^{l-1} d_{k-1,j} \sum_{l'=1}^{l-j} \sum_{t=0}^{l' \wedge j} \frac{a_{j,t} \mu_{l'}(z)}{z^{t/2}} \right)$$

$$-\frac{x^{2k} f^{(2k)}(x)}{z^{k/2}} \sum_{j=0}^{2k-1} d_{k-1,j} \sum_{l'=1}^{2k-j} \sum_{t=0}^{l' \wedge j} \frac{a_{j,t} \mu_{l'}(z)}{z^{t/2}}$$

which corresponds to  $f_{k-1} - \Delta_z f_{k-1}$  without the terms  $\tilde{R}_{k,z}(x)$ . The recursive relation is

$$d_{k,l} = d_{k-1,l} \frac{z}{z-1} - \sum_{j=0}^{l-1} d_{k-1,j} \sum_{l'=1}^{l-j} \sum_{t=0}^{l' \wedge j} \frac{a_{j,t} \mu_{l'}(z)}{z^{t/2}}$$

for  $l = 0, \dots, 2k$  with the convention that  $d_{k-1,2k} = 0$ . By construction when  $\beta \in (2k-2, 2k]$

$$\int_0^\infty \frac{(K_z f_k(x) - f(x))^2}{f(x)} dx \lesssim z^{-\beta},$$

that is, we iterate until  $k = r_0$ . Since  $\int_0^1 f(x) dx = 1$  and since

$$\|K_z f_k - f\|_1 \leq \sqrt{\|f\|_1 \int_0^\infty \frac{(K_z f_k(x) - f(x))^2}{f(x)} dx} \lesssim z^{-\beta/2},$$

we have that

$$\int K_z f_k(x) dx = 1 + O(z^{-\beta/2}) \Rightarrow \int_0^\infty f_k(x) dx = 1 + O(z^{-\beta/2}),$$

which proves (2.13) for  $\tilde{f}_{\beta,\alpha} = f_{\beta,\alpha} = f_k$ .

2. Proof of (2.13): Case  $\alpha \leq 1$

Now let  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  and denote  $h(x) = x^{1-\alpha} f(x)$ . Recall that  $C_z = (z - \alpha + 1)/z$  and  $\tilde{f}(x) = C_z f(C_z x)$ , so  $\tilde{f}$  is still a density. Note that  $C_z = (z - \alpha + 1)/z \rightarrow 1$  as  $z \rightarrow \infty$ . By Lemma A.1,

$$K_z \tilde{f}(x) = x^{\alpha-1} C_z^{\alpha-1} \frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} K_{z+1-\alpha} h(x)$$

and  $\frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} = 1 + O(1/z)$  so we can write  $C_z^{\alpha-1} \frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} = 1 + r(z)$ , where  $|r(z)| \leq c/z$  for sufficiently large  $z$ .

Applying case  $\alpha = 1$ , for  $\mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  to  $h(x)$  with  $k$  such that  $\beta \in (2k-2, 2k]$ , i.e.  $k = r_0$ , we have that there exists  $d_j \in \mathbb{R}$ ,  $j = 1, \dots, k$  such that the function

$$g_\beta(x) = h(x) - \sum_{j=1}^{2k} \frac{d_j(z)}{z^{j/2}} x^j h^{(j)}(x), \quad d_j(z) = d_j + O(1/z)$$

satisfies

$$|K_z g_\beta(x) - h(x)| \leq z^{-\beta/2} R_0(x).$$

Thus, we can define

$$\tilde{f}_{\beta,\alpha}(x) = (1 + r(z))^{-1} C_z^\alpha x^{\alpha-1} g_\beta(x C_z)$$

$$= (1+r(z))^{-1} C_z^\alpha x^{\alpha-1} \left[ h(xC_z) - \sum_{j=1}^r \frac{d_j(z)}{z^{j/2}} x^j C_z^j h^{(j)}(xC_z) \right],$$

which satisfies

$$\left| K_z \tilde{f}_{\beta,\alpha}(x) - f(x) \right| \leq z^{-\beta/2} x^{\alpha-1} R_0(x) (1 + O(1/z)), \quad (\text{A.2})$$

since

$$\begin{aligned} \left| K_z \tilde{f}_{\beta,\alpha}(x) - f(x) \right| &= \left| (1+r(z))^{-1} (1+r(z)) x^{\alpha-1} (K_{z+1-\alpha} g_\beta(x) - x^{\alpha-1} h(x)) \right| \\ &\leq x^{\alpha-1} |K_{z+1-\alpha} g_\beta(x) - h(x)| \leq z^{-\beta/2} x^{\alpha-1} R_0(x) (1 + O(1/z)), \end{aligned}$$

and the first part of (2.13) holds with  $R(x) \asymp x^{\alpha-1} R_0(x)$ . From the proof of case  $\alpha = 1$ , it follows that  $R_0(x)$  has terms proportional to  $(x^{2\ell} + 1)x^\beta(1+x^\gamma)L_h(x)$  with  $1 \leq \ell \leq r_0$  and  $x^j|h^{(j)}(x)|$  for  $1 \leq j \leq r$ . Therefore, the second part of (2.13)

$$\int_0^\infty \frac{R^2(x)}{f(x)} dx \asymp \int_0^\infty \frac{R_0^2(x)}{h^2(x)} f(x) dx < \infty$$

is satisfied since

$$\int_0^\infty \frac{[x^{2\ell} x^\beta (1+x^\gamma) L(x)]^2}{h^2(x)} f(x) dx < \infty, \quad \int_0^\infty \frac{[x^j |h^{(j)}(x)|]^2}{h^2(x)} f(x) dx < \infty$$

hold due to  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  and inequality  $\int g^2(x) d\mu \leq [\int g^p(x) d\mu]^{2/p}$  for  $p \geq 2$  for a probability measure  $\mu$ .

We now prove (2.15).

**3. Proof of (2.15): general case** We follow the same route as in Kruijer et al. [7]. We bound

$$\begin{aligned} D_H(K_z \bar{f}_\beta, f)^2 &\leq 2[D_H^2(K_z \bar{f}_\beta, c_\beta f) + D_H^2(f, c_\beta f)] \leq 2(1 - \sqrt{c_\beta})^2 \quad (\text{A.3}) \\ &+ 2c_\beta \int \left( \sqrt{K_z \left( \tilde{f}_{\beta,\alpha} \mathbb{I}_{\tilde{f}_{\beta,\alpha} \geq \bar{f}/2} + \frac{\tilde{f}}{2} \mathbb{I}_{\tilde{f}_{\beta,\alpha} < \bar{f}/2} \right)} - \sqrt{f(x)} \right)^2 dx. \end{aligned}$$

We first prove that

$$c_\beta^{-1} = \int_0^\infty \left( \tilde{f}_{\beta,\alpha}(x) \mathbb{I}_{\tilde{f}_{\beta,\alpha} \geq \bar{f}/2}(x) + \tilde{f}(x)/2 \mathbb{I}_{\tilde{f}_{\beta,\alpha} < \bar{f}/2}(x) \right) dx = 1 + O(z^{-\beta/2}). \quad (\text{A.4})$$

Define  $\tilde{h}(x) = h(C_z x)$  and

$$\mathcal{A}_1(a) = \left\{ x : \frac{x^j |\tilde{h}^{(j)}(x)|}{\tilde{h}(x)} \leq \delta \frac{z^{j/2}}{(\log z)^a}, j = 1, \dots, 2k, \frac{x^\beta L_{\tilde{h}}(x)}{\tilde{h}(x)} \leq \delta \frac{z^{\beta/2}}{(\log z)^a} \right\}, \quad (\text{A.5})$$

we have that for  $z$  large enough  $\{\tilde{f}_{\beta,\alpha} < \tilde{f}/2\} \subset \mathcal{A}_1(a)^c$  (that is, if  $\delta \sum_{j=1}^{2k} |d_{k,j}| > (\log z)^{-\alpha}/2$  with  $k = r_0$ ), so that

$$\begin{aligned} c_{\beta}^{-1} &\geq \int \tilde{f}_{\beta,\alpha}(x) dx = 1 + O(z^{-\beta/2}), \\ c_{\beta}^{-1} &= 1 + O(z^{-\beta/2}) + \int \mathbb{I}_{\tilde{f}_{\beta,\alpha} \leq \tilde{f}/2} \left( \frac{\tilde{f}}{2} - \tilde{f}_{\beta,\alpha} \right) dx \\ &\leq (1 + O(z^{-\beta/2})) + \int \mathbb{I}_{\tilde{f}_{\beta,\alpha} \leq \tilde{f}/2} \left( \tilde{f}(x) + x^{\alpha-1} \sum_{j=1}^{2k} \frac{|d_{k,j}|}{z^{j/2}} x^j |\tilde{h}^{(j)}(x)| \right) dx. \end{aligned}$$

Since

$$\begin{aligned} \tilde{F}(\{\tilde{f}_{\beta,\alpha} < \tilde{f}/2\}) &\leq \tilde{F}(\mathcal{A}_1(a)^c) \\ &\leq \sum_{j=1}^{2k} z^{-(2\beta+e)/2} (\log z)^{a(2\beta+e)/j} \int \frac{(x^{\alpha-1} x^j |\tilde{h}^{(j)}(x)|)^{(2\beta+e)/j}}{[\tilde{f}(x)]^{(2\beta+e)/j}} \tilde{f}(x) dx \\ &= O(z^{-\beta-e/4}) \end{aligned} \tag{A.6}$$

and since for all  $j = 1, \dots, 2k$

$$\begin{aligned} &\int_{\mathcal{A}_1(a)^c} x^{\alpha-1} x^j |\tilde{h}^{(j)}(x)| dx \\ &\leq \tilde{F}(\mathcal{A}_1(a)^c)^{(\beta-j)/\beta} \left[ \int \frac{[x^{\alpha-1} x^j |\tilde{h}^{(j)}(x)|]^{\beta/j}}{[\tilde{f}(x)]^{\beta/j}} \tilde{f}(x) dx \right]^{j/\beta} \\ &= O(\tilde{F}(\mathcal{A}_1(a)^c)^{(\beta-j)/\beta}) = O(z^{-(\beta-j)(1+e/(4\beta))}), \end{aligned} \tag{A.7}$$

which implies (A.4). We now bound the second term of the right hand side of (A.3).

$$\begin{aligned} &\int \left( \sqrt{K_z \tilde{f}_{\beta,\alpha} \mathbb{I}_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} + 0.5K_z \tilde{f} \mathbb{I}_{\tilde{f}_{\beta,\alpha} < \tilde{f}/2}} - \sqrt{f(x)} \right)^2 dx \\ &= \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} - \sqrt{f(x)} \right)^2 dx \\ &\quad + \int_{\tilde{f}_{\beta,\alpha} < \tilde{f}/2} \left( \sqrt{0.5K_z \tilde{f}(x)} - \sqrt{f(x)} \right)^2 dx \\ &\leq \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} - \sqrt{f(x)} \right)^2 dx \\ &\quad + 2 \int_{\mathcal{A}_1(a)^c} [0.5K_z \tilde{f}(x) + f(x)] dx \\ &\leq \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} - \sqrt{f(x)} \right)^2 dx \end{aligned}$$

$$+ 2 \int_{\mathcal{A}_1(a)^c} 0.5K_z \tilde{f}(x) dx + F(\mathcal{A}_1(a)^c)$$

since  $\{\tilde{f}_{\beta,\alpha} < \tilde{f}/2\} \subseteq \mathcal{A}_1(a)^c$ . Using (2.13), we have

$$\begin{aligned} & \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} - \sqrt{f(x)} \right)^2 dx \\ & \leq \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} - \sqrt{f(x)} \right)^2 dx \\ & \leq \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \frac{(K_z \tilde{f}_{\beta,\alpha}(x) - f(x))^2}{\left( \sqrt{K_z \tilde{f}_{\beta,\alpha}(x)} + \sqrt{f(x)} \right)^2} dx \\ & \leq z^{-\beta} \int_{\tilde{f}_{\beta,\alpha} \geq \tilde{f}/2} \frac{(x^{\alpha-1} R_0(x))^2}{f(x)} dx \\ & = O(z^{-\beta}). \end{aligned}$$

Now we consider the integral  $\int_{\mathcal{A}_1(a)^c} K_z \tilde{f}(x) dx$ . Note that

$$\begin{aligned} \int_{\mathcal{A}_1^c(a)} K_z \tilde{f}(x) dx &= \int \int \mathbb{I}_{\mathcal{A}_1^c(a)}(x) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx \\ &\leq \int \int \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx + \tilde{F}(\mathcal{A}_1(2a)^c) \\ &\leq \int \int \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx + O(z^{-\beta-e/2}) \end{aligned}$$

Using (C.5), we have that

$$\begin{aligned} & \int \int \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx \\ &= \int \int \mathbb{I}_{|\epsilon/x-1| \leq M\sqrt{\log z}/\sqrt{z}} \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx + O(z^{-H}), \end{aligned}$$

for any  $H > 0$  by choosing  $M$  large enough since

$$\begin{aligned} & \int \int \mathbb{I}_{|\epsilon/x-1| > M\sqrt{\log z}/\sqrt{z}} \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx \\ & \leq \int \mathbb{I}_{|v| > M\sqrt{\log z}/\sqrt{z}} (v+1)^{-1} \phi_{1/\sqrt{z}}(v) (1 + O(1/z)) \int (v+1) \tilde{f}((v+1)x) dx dv \\ & \leq (1 + M\sqrt{\log z}/\sqrt{z})^{-1} \int \mathbb{I}_{|v| > M\sqrt{\log z}/\sqrt{z}} \phi_{1/\sqrt{z}}(v) (1 + O(1/z)) dv \\ & = 2(1 + M\sqrt{\log z}/\sqrt{z})^{-1} (1 + O(1/z)) [1 - \Phi(M\sqrt{\log z}/\sqrt{z})] = O(z^{-H}), \end{aligned}$$

for the appropriate choice of  $M$ . We need only to study what happens if  $x \in \mathcal{A}_1^c(a)$ ,  $\epsilon \in \mathcal{A}_1(2a)$  and  $|\epsilon/x-1| \leq M\sqrt{\log z}/\sqrt{z}$ . We assume that  $z$  is large enough

so that  $M\sqrt{\log z/z} \leq 1/2$  and hence

$$x \leq \frac{\epsilon}{1 - M\sqrt{\log z/z}} \leq 2\epsilon.$$

For  $\epsilon \in \mathcal{A}_1(2a)$  and  $|\epsilon/x - 1| \leq M\sqrt{\log z}/\sqrt{z}$ , we have

$$\begin{aligned} |x^j \tilde{h}^j(x)| &= x^j |\tilde{h}^{(j)}(\epsilon) + \sum_{l=1}^{r-j} \epsilon^l \tilde{h}^{(j+l)}(\epsilon) \frac{(x/\epsilon - 1)^\ell}{\ell!} + O((x/\epsilon - 1)^{\beta-j} L_{\tilde{h}^j}(\epsilon))| \\ &\leq 2^j \left[ |\epsilon^j \tilde{h}^{(j)}(\epsilon)| + \sum_{l=1}^{r-j} |\epsilon^{\ell+j} \tilde{h}^{(j+l)}(\epsilon)| \frac{M^\ell (\log z)^{\ell/2}}{\ell! z^{\ell/2}} \right] \\ &\quad + O([z^{-1} \log z]^{(\beta-j)/2}) 2^j |L_{\tilde{h}^j}(\epsilon)| \\ &\leq 2^j \tilde{h}(\epsilon) \delta z^{j/2} (\log z)^{-2a} \left[ 1 + \sum_{l=1}^{r-j} \frac{M^\ell (\log z)^{\ell/2}}{\ell!} + O([\log z]^{(\beta-j)/2}) \right] \\ &\leq C_j \tilde{h}(\epsilon) \delta z^{j/2} [(\log z)^{-2a} + (\log z)^{-2a+(\beta-j)/2}]. \end{aligned}$$

We bound  $\tilde{h}(x)$  from below using  $\epsilon \in \mathcal{A}_1(2a)$  and  $|\epsilon/x - 1| \leq M\sqrt{\log z}/\sqrt{z}$ :

$$\begin{aligned} \tilde{h}(x) &\geq \tilde{h}(\epsilon) - \sum_{j=1}^r \frac{(M\sqrt{\log z/z})^j}{j!} x^j \tilde{h}^{(j)}(\epsilon) - 2(M\sqrt{\log z/z})^\beta x^\beta L_{\tilde{h}}(\epsilon) \\ &\geq \tilde{h}(\epsilon) \left( 1 - \sum_{j=1}^r \frac{(M\sqrt{\log z/z})^j}{j!} \frac{(2\epsilon)^j \tilde{h}^{(j)}(\epsilon)}{\tilde{h}(\epsilon)} - 2(M\sqrt{\log z/z})^\beta \frac{(2\epsilon)^\beta L_{\tilde{h}}(\epsilon)}{\tilde{h}(\epsilon)} \right) \\ &\geq \tilde{h}(\epsilon) \left( 1 - \sum_{j=1}^r \frac{(2M)^j}{j!} \delta (\log z)^{-2a+j/2} - 2(2M)^\beta \delta (\log z)^{-2a+\beta/2} \right) \\ &\geq \tilde{h}(\epsilon) (1 - O((\log z)^{-2a+\beta/2})) \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{f}(x) &= C_z^\alpha x^{\alpha-1} \tilde{h}(x) \geq C_z^\alpha (2\epsilon)^{\alpha-1} \tilde{h}(\epsilon) (1 - O((\log z)^{-2a+\beta/2})) \\ &= 2^{\alpha-1} \tilde{f}(\epsilon) (1 - O((\log z)^{-2a+\beta/2})) \end{aligned}$$

and

$$\frac{|x^j \tilde{h}^{(j)}(x)|}{\tilde{h}(x)} \leq \delta z^{j/2} (\log z)^{-a} [(\log z)^{-a+\beta/2} + 1].$$

In particular, it implies that if  $a \geq \beta/2$  then  $x \in \mathcal{A}_1(a)$ ; so for  $x \in \mathcal{A}_1^c(a)$ , we must have  $a < \beta/2$ . Therefore, using (A.1) and taking  $a \in (\beta/4, \beta/2)$ , we have

$$\begin{aligned} &\int \int \mathbb{I}_{|\epsilon/x-1| \leq M\sqrt{\log z}/\sqrt{z}} \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \mathbb{I}_{\mathcal{A}_1(2a)}(\epsilon) g_{z,\epsilon}(x) \tilde{f}(\epsilon) d\epsilon dx \\ &\leq 2^{\alpha-1} (1 - O((\log z)^{-2a+\beta/2}))^{-1} (1 + O(1/z))^{-1} \end{aligned}$$

$$\begin{aligned} & \times \int \int \mathbb{I}_{\mathcal{A}_1^c(a)}(x) \phi_{1/\sqrt{z}}(1-u) \tilde{f}(x) du dx \\ & \leq 2^{\alpha-1} (1 + O((\log z)^{-2\alpha+\beta/2})) (1 + O(1/z)) \tilde{F}(\mathcal{A}_1^c(a)) \\ & \lesssim z^{-\beta-e/4} (1 + O((\log z)^{-2\alpha+\beta/2})) (1 + O(1/z)), \end{aligned}$$

and (2.15) is proved.

**A.2. Adjustments for an unbounded density**

**Lemma A.1.** For any  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$ ,  $x > 0$ ,  $A \in (0, \infty]$ ,

$$\begin{aligned} \int_0^A g_{z,\epsilon}(x) f(\epsilon) d\epsilon &= x^{\alpha-1} \frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} \int_0^A g_{z-\alpha+1,\epsilon}(x/C_z) h(\epsilon) d\epsilon, \\ \int_0^A g_{z,\epsilon}(x) C_z f(C_z \epsilon) d\epsilon &= x^{\alpha-1} \frac{C_z^{\alpha-1} z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} \int_0^{C_z A} g_{z-\alpha+1,\epsilon}(x) h(\epsilon) d\epsilon, \end{aligned}$$

and

$$\frac{z^\alpha \Gamma(z-\alpha)}{\Gamma(z)} = 1 + O(1/z) \quad \& \quad C_z^{\alpha-1} = 1 + O(1/z) \quad \text{as } z \rightarrow \infty,$$

where  $h(x) = x^{1-\alpha} f(x)$  and  $C_z = \frac{z-\alpha+1}{z}$ .

*Proof of Lemma A.1.* Let  $f \in \mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  and denote  $h(x) = x^{1-\alpha} f(x)$ .

1. For large enough  $z$  and for any  $A \in (0, \infty]$ , denoting  $x_\alpha = C_z x$ , we have

$$\begin{aligned} \int_0^A g_{z,\epsilon}(x) f(\epsilon) d\epsilon &= \frac{1}{\Gamma(z)} \int_0^A x^{z-1} e^{-zx/\epsilon} \left(\frac{z}{\epsilon}\right)^z \epsilon^{\alpha-1} h(\epsilon) d\epsilon \\ &= x^\alpha \frac{\Gamma(z-\alpha+1)}{\Gamma(z)} (x_\alpha)^{z-\alpha+1} \int_0^A \frac{e^{-(z-\alpha+1)x_\alpha/\epsilon}}{\Gamma(z-\alpha+1)} \left(\frac{z-\alpha+1}{\epsilon}\right)^{z-\alpha+1} h(\epsilon) d\epsilon \\ &= x^{\alpha-1} \frac{z^\alpha \Gamma(z-\alpha+1)}{(z-\alpha+1)\Gamma(z)} \int_0^A g_{z-\alpha+1,\epsilon}(x_\alpha) h(\epsilon) d\epsilon \\ &= x^{\alpha-1} (1 + O(1/z)) \int_0^A g_{z-\alpha+1,\epsilon}(x_\alpha) h(\epsilon) d\epsilon \end{aligned}$$

since the Stirling formula implies

$$\frac{\Gamma(z-\alpha)}{\Gamma(z)} = z^{-\alpha} (1 + O(1/z)).$$

2. For large enough  $z$  and for any  $A \in (0, \infty]$ ,

$$\int_0^A g_{z,\epsilon}(x) C_z f(C_z \epsilon) d\epsilon = \frac{x^{z-1}}{\Gamma(z)} \int_0^A e^{-zx/\epsilon} \left(\frac{z}{\epsilon}\right)^z (C_z \epsilon)^{\alpha-1} h(C_z \epsilon) C_z d\epsilon$$

$$\begin{aligned}
 &= [v = C_z \epsilon] = \frac{x^{z-1} C_z^z}{\Gamma(z)} \int_0^{C_z A} e^{-zx C_z/v} \left(\frac{z}{v}\right)^z v^{\alpha-1} h(v) dv \\
 &= x^{\alpha-1} \frac{z^{\alpha-1} C_z^{\alpha-1} \Gamma(z+1-\alpha)}{\Gamma(z)} \frac{x^{z-\alpha}}{\Gamma(z+1-\alpha)} \\
 &\quad \times \int_0^{C_z A} e^{-(z+1-\alpha)x/v} \left(\frac{z+1-\alpha}{v}\right)^{z+1-\alpha} h(v) dv \\
 &= x^{\alpha-1} \frac{z^{\alpha-1} C_z^{\alpha-1} \Gamma(z+1-\alpha)}{\Gamma(z)} \int_0^{C_z A} g_{z+1-\alpha, \epsilon}(x) h(\epsilon) d\epsilon.
 \end{aligned}$$

Since  $C_z = 1 - \frac{1-\alpha}{z-\alpha+1} = 1 + O(z^{-1})$ , we also have  $C_z^{\alpha-1} = (1 + (1-\alpha)/z)^{\alpha-1} = 1 + O(1/z)$  for large  $z$ .

Therefore, the lemma is proved. □

## Appendix B: Approximation of densities by finite mixtures

### B.1. Construction of the discrete approximation

The construction of a discrete finite mixture and the lower bound on the prior mass of Kullback-Leibler neighbourhoods of a smooth density  $f$  are similar to Ghosal and van der Vaart [4] and Rousseau [11]. We first present the construction of the discrete distribution in Lemma B.1, then we control the Hellinger distance between  $f$  and the discrete approximation in Lemma B.2.

**Lemma B.1.** *Let  $e_z = z^{-a}$ ,  $E_z = z^b$  and  $H$  be a probability distribution on  $[e_z, E_z]$ . Then for all  $\kappa > 0$ , there exists  $N_0 > 0$  and a probability distribution  $P$  with at most  $\bar{N} = N_0 \sqrt{z} (\log z)^{3/2}$  supporting points such that: for all  $x \in [\tau_0 e_z, \tau_1 E_z]$  with  $0 < \tau_0 < 1 < \tau_1 < +\infty$*

$$|K_z * (H - P)(x)| \leq z^{-\kappa}, \quad \text{when } z \text{ is large enough} \tag{B.1}$$

*Proof of Lemma B.1.* The proof of the lemma is based on the ideas of Ghosal and van der Vaart [5] combined with some ideas of Rousseau [11]. We use Gaussian approximation (C.5). Set  $u = \epsilon/x$  and consider  $|u - 1| \leq M \sqrt{\log z/z} := \delta_z$  with  $M$  some arbitrarily large constant. Then writing  $h(u) = \log u + 1/u - 1$

$$\begin{aligned}
 \frac{z^z e^{-z/u}}{\Gamma(z) u^z} &= \frac{\sqrt{z} \exp(-z[\log u + 1/u - 1])}{\sqrt{2\pi}} (1 + R(z))^{-1} \\
 &= \frac{\sqrt{z}}{\sqrt{2\pi}(1 + R(z))} \left( 1 + \sum_{j=1}^N \frac{z^j h(u)^j}{j!} + R_N(u) \right)
 \end{aligned}$$

where  $|R_N(u)| \leq \frac{z^{N+1} h(u)^{N+1}}{(N+1)!}$ . Note that  $0 \leq h(u) \leq (u-1)^2$  when  $|u-1| \leq \delta_z$  and  $z$  is large enough. Hence for all,  $u$  such that  $|u-1| \leq \delta_z$ ,

$$|R_N(u)| \leq \frac{(M \log z)^{(N+1)}}{(N+1)!} \leq \frac{e^{-(N+1) \log(\frac{N+1}{M e \log z})}}{\sqrt{N+1}} \leq e^{-\tau(N+1)},$$

as soon as  $N + 1 > \tilde{N}_0 \log z$  with  $\tilde{N}_0$  large enough, for some  $\tau > 0$ . Choose  $r \geq 2$ , then a Taylor expansion of  $h(u)$  around 1 leads to

$$h(u) = \frac{(u - 1)^2}{2} + \sum_{j=3}^r h_j(u - 1)^j + R_h(u), \quad |R_h(u)| \leq \frac{C_h |u - 1|^{r+1}}{(r + 1)!}$$

with  $h_j = h^{(j)}(1)$ . This implies that

$$g_{z,\epsilon}(x) = \frac{Q_{N,z}(\epsilon, x)}{x} + \Delta_N(x),$$

where  $Q_{N,z}(\epsilon, x)$  is a polynomial function of  $\epsilon$  with degrees less than or equal to  $rN$  and

$$\Delta_N(\epsilon, x) = \frac{R_N(\epsilon/x)}{x} + O\left(\frac{N\sqrt{z}[|zR_h(\epsilon/x)| + |z^N R_h(\epsilon/x)|^N/N!]}{x}\right).$$

For all  $|\epsilon/x - 1| \leq \delta_z$ ,

$$\Delta_N(\epsilon, x) \lesssim \frac{e^{-\tau(N+1)}}{x} + \frac{z^{-r/2+1}(\log z)^{(r+1)/2}}{x}.$$

If  $|\epsilon/x - 1| \in (\delta_z, \delta)$  with  $\delta > 0$  arbitrarily small but fixed,  $h(\epsilon/x) \geq (\epsilon/x - 1)^2/4$  and

$$g_{z,\epsilon}(x) \leq \frac{2\sqrt{z} \exp\left(-\frac{z(\epsilon-x)^2}{4x^2}\right)}{x\sqrt{2\pi}}.$$

Split  $[e_z, E_z]$  into intervals in the form  $I_j = [e_z(1 + \delta_z/2)^j, e_z(1 + \delta_z/2)^{j+1}]$ , with  $j \leq J_z$  and

$$J_z = \lceil \log(E_z/e_z)/\log(1 + \delta_z/2) \rceil \lesssim \frac{(b+a)}{\delta_z} \log z \lesssim \sqrt{z}(\log z)^{1/2}$$

Following Lemma A1 of Ghosal and van der Vaart [5], since the functions  $\epsilon \rightarrow \epsilon^\ell$ ,  $\ell \leq rN$  are continuous over  $I_j$ , there exists a probability  $P_{j,N}$  with support included in  $I_j$  with at most  $rN + 1$  points in the support such that for all  $\ell \leq rN$

$$\int_{I_j} \epsilon^\ell dH_j(\epsilon) = \int_{I_j} \epsilon^\ell dP_{j,N}(\epsilon), \tag{B.2}$$

where  $H_j = H\mathbb{1}_{I_j}/H[I_j]$ . Construct  $P_N = \sum_j H(I_j)P_{j,N}$ , then  $P_N$  has support  $[e_z, E_z]$  and for all  $x$ ,

$$|K_z * H(x) - K_z * P_N(x)| \leq \sum_{j=1}^{J_z} H(I_j) \sup_{\epsilon \in I_j} |\Delta_N(\epsilon, x)|.$$

Let  $\epsilon \in I_j$  and  $x \in [e_z(1 + \delta_z/2)^{j-1}, e_z(1 + \delta_z/2)^{j+2}]$ , then  $x/\epsilon \leq (1 + \delta_z/2)^2 \lesssim 1 + 2\delta_z$  when  $z$  is large enough and

$$|\Delta_N(\epsilon, x)| \leq \frac{e^{-\tau(N+1)}}{x} + \frac{z^{-r/2+1}(\log z)^{(r+1)/2}}{x} \leq z^{-\kappa-1}$$

as soon as  $r/2 > a + \kappa + 1$ . If  $|\epsilon/x - 1| \in (2\delta_z, \delta)$

$$\begin{aligned} |K_z * H_j(x) - K_z * P_{j,N}(x)| &\leq K_z * H_j(x) + K_z * P_{j,N}(x) \\ &\leq \frac{4\sqrt{z} \exp\left(-\frac{M^2 \log z}{4}\right)}{x\sqrt{2\pi}} \lesssim z^{-(M^2-2)/4+a} \leq z^{-\kappa} \end{aligned}$$

as soon as  $M^2 > 4(\kappa + a) + 2$ . Finally if  $|\epsilon/x - 1| > \delta$ , using Lemma C.2

$$|K_z * H_j(x) - K_z * P_{j,N}(x)| \leq K_z * H_j(x) + K_z * P_{j,N}(x) \lesssim e^{-cz}$$

for some  $c > 0$ . This implies that for all  $x \in \mathbb{R}$ ,

$$|K_z * H(x) - K_z * P_N(x)| \leq z^{-\kappa}.$$

where  $P_N$  has at most  $N_0\sqrt{z}(\log z)^{3/2}$  supporting points in  $[e_z, E_z]$ , with  $N_0$  depending on  $\kappa, a, b$ .  $\square$

The following Lemma allows us to control the Kullback-Leibler divergence between  $f$  and  $K_z * P$ .

**Lemma B.2.** *Assume that  $f \in \mathcal{P}_\alpha(\beta, L, \gamma, C_0, C_1, e, \Delta)$ , and that there exist  $C_2 > 0$  and  $\rho_1 > 0$  such that*

$$\int_x^\infty y^2 f(y) dy \leq C_2(1+x)^{-\rho_1}.$$

*Let  $e_z = z^{-a}$  and  $E_z = z^b$  with  $a > (\alpha \wedge 1)^{-1}(\beta \vee (2\beta - 1))$  and  $b > (\beta \vee (2\beta - 1))/(\rho_1 + 2)$ . Then there exists*

$$P_N = \sum_{i=1}^N p_i \delta_{(u_i)}, \quad u_i \in [e_z, E_z], \quad N \leq N_0\sqrt{z}(\log z)^{3/2}$$

*such that*

$$D_H^2(K_z * P_N, K_z * \bar{f}_\beta) \lesssim z^{-\beta}, \quad D_H^2(K_z * P_N, f) \lesssim z^{-\beta}.$$

*Moreover, there exists  $A > 0$  such that we can choose  $u_1 \leq \dots \leq u_N$ ,  $u_i - u_{i-1} > z^{-A}$  and  $p_i > 3z^{-A}$  for all  $i \leq N$  as long as  $z$  is large enough.*

Note that it appears from the proof of Lemma B.2, that  $P_N$  can be chosen so that for all  $1 \leq \ell \leq J_z$  where  $J_z$  is such that  $e_z(1 + Mz^{-1/2}\sqrt{\log z})^{J_z+1} \geq E_z$  and  $e_z(1 + Mz^{-1/2}\sqrt{\log z})^{J_z} \leq E_z$ ,  $P_N(U_\ell) \geq 3z^{-A}$ .

*Proof of Lemma B.2.* Consider  $\bar{f}_\beta$  as defined in Proposition 2.1. First we approximate this function by a function supported on  $[e_z, E_z]$  so that both upper and lower approximation errors are bounded by  $O(z^{-\beta})$ . Recall that  $\tilde{h}(x) =$

$C_z(C_z x)^{1-\alpha} f_\beta(C_z x)$ . Since  $\int_0^{e_z} f(\epsilon) d\epsilon \lesssim e_z^\alpha$  for small  $e_z$ , then, by definition of  $f_\beta$ , we have

$$\begin{aligned} \int_0^{e_z} \bar{f}_\beta(\epsilon) d\epsilon &\lesssim e_z^\alpha + \sum_{j=1}^r \frac{\int_0^{e_z} \epsilon^{j+\alpha-1} |\tilde{h}^{(j)}(\epsilon)| d\epsilon}{z^{j/2}} \\ &\lesssim e_z^\alpha + \sum_{j=1}^r \sqrt{e_z^\alpha} z^{-j/2} \left( \int_0^{+\infty} \frac{(\epsilon \wedge 1)^{2j+\alpha-1} |\tilde{h}^{(j)}(\epsilon)|^2 d\epsilon}{\tilde{h}(\epsilon)} \right)^{1/2} \\ &\lesssim e_z^\alpha + \frac{e_z^{\alpha/2}}{\sqrt{z}} I(r > 0). \end{aligned}$$

We also have that

$$\begin{aligned} \int_{E_z}^{+\infty} \bar{f}_\beta(\epsilon) d\epsilon &\lesssim 1 - F(E_z) + \sum_{j=1}^r \frac{\int_{E_z}^{+\infty} \epsilon^{j+\alpha-1} |h^{(j)}(\epsilon)| d\epsilon}{z^{j/2}} \\ &\lesssim 1 - F(E_z) + \frac{\sqrt{1 - F(E_z)}}{\sqrt{z}} \end{aligned}$$

Therefore, for  $e_z = z^{-a}$ ,  $\bar{F}_\beta([0, e_z]) \leq C z^{-\beta}$  since  $a > \alpha^{-1}(\beta \vee (2\beta - 1))$ . For  $E_z = z^b$ , inequality  $\bar{F}_\beta([E_z, \infty)) \leq C z^{-\beta}$  is satisfied with  $b > (\beta \vee (2\beta - 1))/(\rho_1 + 2)$ . We thus have that  $\bar{F}_\beta[e_z, E_z] \geq 1 - O(z^{-\beta})$ . Define

$$f_\beta = \frac{\bar{f}_\beta \mathbb{I}_{[e_z, E_z]}}{\bar{F}_\beta[e_z, E_z]}$$

then

$$\begin{aligned} K_z \bar{f}_\beta(x) &= \bar{F}_\beta([e_z, E_z]) K_z f_\beta(x) + \int_0^{e_z} \bar{f}_\beta(\epsilon) g_{z,\epsilon}(x) d\epsilon \\ &+ \int_{E_z}^{+\infty} \bar{f}_\beta(\epsilon) g_{z,\epsilon}(x) d\epsilon \geq \bar{F}_\beta([e_z, E_z]) K_z f_\beta(x). \end{aligned}$$

It implies that

$$\begin{aligned} D_H^2(K_z \bar{f}_\beta, K_z f_\beta) &\leq 2 - 2 \int_0^\infty K_z f_\beta(x) dx \sqrt{\bar{F}_\beta([e_z, E_z])} \\ &\leq 2 - 2 \sqrt{1 - O(z^{-\beta})} \lesssim z^{-\beta}, \end{aligned}$$

so that

$$D_H^2(f, K_z f_\beta) \leq [D_H(f, K_z \bar{f}_\beta) + D_H(K_z \bar{f}_\beta, K_z f_\beta)]^2 \lesssim z^{-\beta}. \tag{B.3}$$

For an arbitrary  $\kappa > 0$ , which we will choose later, consider the discrete distribution  $P_N$  constructed in Lemma B.1, which we write as  $\tilde{P}_N = \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} \delta_{\epsilon_{j,i}}$ ,  $\epsilon_{j,i} \in [e_z(1 + \delta_z/2)^j, e_z(1 + \delta_z/2)^{j+1}]$ , with  $N = \sum_{j=1}^{J_z} N_j \leq N_0 \sqrt{z} (\log z)^{3/2}$  and  $\delta_z = M z^{-1/2} \sqrt{\log z}$ , where  $N_0 = N_0(\kappa)$  such that

$$|K_z * \tilde{P}_N(x) - K_z f_\beta(x)| \lesssim z^{-\kappa} \quad \forall x \in [e_z/2, 2E_z]. \tag{B.4}$$

Note that for  $e_z = z^{-a}$  and  $E_z = z^b$ ,  $J_z \leq (b+a)M^{-1}\sqrt{z \log z}$  and  $N_j \lesssim \log z$ . This implies that

$$\begin{aligned} D_H^2(K_z * \tilde{P}_N, K_z f_\beta) &\leq \frac{1}{2} \|K_z * \tilde{P}_N - K_z f_\beta\|_1 \\ &\lesssim z^{-\kappa} (2E_z - e_z/2) + \int_{[e_z/2, 2E_z]^c} (K_z * \tilde{P}_N + K_z f_\beta)(x) dx. \end{aligned}$$

For any distribution  $P$  with support  $[e_z, E_z]$ , by Lemma C.2 with  $u = \epsilon/x > 1 + \delta$ ,  $\delta = 1$  and  $c_1 = c(\delta)$ :

$$\begin{aligned} \int_0^{e_z/2} (K_z * P)(x) dx &= \int_0^{e_z/2} dx \int_{[e_z, E_z]} g_{z, \epsilon}(x) dP(\epsilon) \\ &\leq \int_0^{e_z/2} x^{c(\delta)z-1} dx \int_{[e_z, E_z]} \epsilon^{-c(\delta)z} dP(\epsilon) \\ &\lesssim z^{-1} (e_z/2)^{c(\delta)z} e_z^{-c(\delta)z} \int_{[e_z, E_z]} dP(\epsilon) \lesssim 2^{-c_1 z} z^{-1}. \end{aligned}$$

Similarly, applying Lemma C.2 with  $u = \epsilon/x < 1 - \delta$ ,  $\delta = 1/2$  and  $c_{0.5} = c(\delta)$ ,

$$\begin{aligned} \int_{2E_z}^\infty (K_z * P)(x) dx &= \int_{2E_z}^\infty dx \int_{[e_z, E_z]} g_{z, \epsilon}(x) dP(\epsilon) \\ &\leq \int_{2E_z}^\infty x^{-1} e^{-c(\delta)zx/E_z} dx \int_{[e_z, E_z]} dP(\epsilon) \\ &\lesssim E_z^{-1} \frac{e^{-2c(\delta)z}}{z/E_z} \int_{[e_z, E_z]} dP(\epsilon) \lesssim z^{-1} 2^{-2c_{0.5} z}. \end{aligned}$$

Hence choosing  $\kappa \geq 2\beta + b$  implies that

$$D_H^2(K_z * \tilde{P}_N, K_z \bar{f}_\beta) \leq z^{-\beta}, \quad D_H^2(K_z * \tilde{P}_N, f) \lesssim z^{-\beta}.$$

Let  $A > 0$  and construct the grid  $(u_\ell)_\ell$ :

$$u_\ell = e_z(1 + z^{-A})^\ell, \ell = 0, \dots, L, \quad L = \left\lceil \frac{\log E_z - \log e_z}{\log(1 + z^{-A})} \right\rceil \lesssim z^A \log z$$

Let  $P_N = \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} \delta_{u_{j,i}}$  be the probability on  $\mathbb{R}^+$  with supporting points  $u_{j,i}$  where  $u_{j,i}$  is the closest point to  $\epsilon_{j,i}$  on the grid  $(u_\ell, \ell \leq L)$ . If there are multiple  $u_{i,j}$  then we collapse the probabilities and without loss of generality we can assume that the  $u_{i,j}$  are all distinct. Define

$$U(u_\ell) = [(u_\ell + u_{\ell-1})/2, (u_\ell + u_{\ell+1})/2], \tag{B.5}$$

covering the interval  $[e_z, E_z]$ , with a suitable adjustment on the boundaries, and hence the corresponding sets  $U_{j,i} = U(u_{j,i})$ . By construction  $|u_{j,i}/\epsilon_{j,i} - 1| \leq z^{-A}$  and we have first that for  $x \in [e_z/2, 2E_z]$ ,

$$|K_z * P_N(x) - K_z * \tilde{P}_N(x)| \leq \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} |g_{z, u_{j,i}}(x) - g_{z, \epsilon_{j,i}}(x)|$$

$$\begin{aligned} &\leq \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} g_{z,\epsilon_{j,i}}(x) \exp(z^{-A+1}(1+x/\epsilon_{j,i})) \\ &\leq \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} g_{z,\epsilon_{j,i}}(x) \exp(z^{-A+1}(1+2E_z/e_z)) \\ &\leq K_z * \tilde{P}_N(x) [1 + Cz^{-A+1+b+a}] \end{aligned}$$

for large enough  $z$ , which implies

$$K_z * P_N(x) \leq K_z * \tilde{P}_N(x)(2 + Cz^{a+b+1-A}), \quad \forall x \in [e_z/2, 2E_z]. \tag{B.6}$$

Finally,

$$\begin{aligned} D_H^2(K_z * P_N, K_z * \tilde{P}_N) &\leq \frac{1}{2} \|K_z * P_N - K_z * \tilde{P}_N\|_1 \\ &\leq \frac{1}{2} \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} \int |g_{z,u_{j,i}}(x) - g_{z,\epsilon_{j,i}}(x)| dx \\ &\leq \sqrt{1/2} \sum_{j=1}^{J_z} \sum_{i=1}^{N_j} p_{j,i} z^{-A+1/2} = z^{-A+1/2}/\sqrt{2}, \end{aligned}$$

where the last inequality comes from Lemma C.1. By choosing  $A > 1/2 + \beta$ , Lemma B.2 is proved by re-indexing  $p_{i,j}$  as  $p_l$  and  $u_{i,j}$  as  $u_l$ ,  $l \leq N$ .  $\square$

**B.2. Kullback-Leibler neighbourhoods**

In the following Lemma we describe Kullback-Leibler neighbourhoods of  $f$  of size  $\epsilon_n^2$ .

**Lemma B.3.** *Assume that  $f \in \mathcal{P}_\alpha(\beta, L, \gamma, C_0, C_1, e, \Delta)$ , and that there exists  $C - 2$  and  $\rho_1 > 0$  such that*

$$\int_x^\infty y^2 f(y) dy \leq C_2(1+x)^{-\rho_1}.$$

Define  $P_N = \sum_{i=1}^N p_i \delta_{u_i}$  and  $A > 0$  as in Lemma B.2 and set

$$\mathcal{P}_z = \{P : P(U_i)/p_i \in (1 - 2z^{-2A}, 1 - z^{-2A}) \forall i = 1, \dots, N\}.$$

Then, if  $A$  is large enough, for all  $z$  large enough and all  $P \in \mathcal{P}_z$ ,

$$\mathcal{KL}(f, K_z * P) \leq z^{-\beta}(\log z); V(f, K_z * P) \leq z^{-\beta}(\log z)^2.$$

*Proof of Lemma B.3.* Let  $P_N$  be defined as in Lemma B.2. Using Lemma B2 of Shen et al. [14] with  $\lambda = z^{-A_1}$  and  $A_1 > 0$  to be defined later, we have that if  $P \in \mathcal{P}_z$ ,

$$\mathcal{KL}(f, K_z * P) \lesssim D_H^2(f, K_z * P)(1 + A_1 \log z)$$

$$\begin{aligned}
 & + \int_{f > z^{A_1} K_z * P} f(x) \log \left( \frac{f(x)}{K_z * P(x)} \right) dx \\
 & \lesssim D_H^2(f, K_z * P)(1 + A_1 \log z) \\
 & - \int_{[e_z, E_z] \cap \{f > z^{A_1} K_z * P\}} \log(K_z * P(x)) f(x) dx \\
 & + \int_{\{f > z^{A_1} K_z * P\} \cap [e_z, E_z]} f(x) (\log f(x))_+ dx \\
 & + \int_{[e_z, E_z]^c \cap \{f > z^{A_1} K_z * P\}} f(x) ((\log f(x))_+ - \log(K_z * P(x))) dx,
 \end{aligned}$$

and similarly for  $V(f, K_z * P)$ . The above computations imply that for all  $P \in \mathcal{P}_z$ , if  $A \geq \beta$

$$D_H^2(f, K_z * P) \lesssim z^{-\beta}.$$

First, we show that for any  $\kappa > 0$ ,  $\exists A, A_1$  such that if  $f(x) > z^{A_1} K_z * P(x)$  for  $x \in \mathcal{A}_1(0) \cap [e_z/2, 2E_z]$ , then  $f(x) \leq z^{-\kappa}$ , where  $\mathcal{A}_1(0)$  is defined in (A.5). Using Lemma C.1,

$$\begin{aligned}
 & |K_z * P(x) - K_z * P_N(x)| = \left| \int g_{z,u}(x) dP(u) - \sum_{i=1}^N p_i g_{z,u_i}(x) \right| \\
 & \leq \sum_{i=1}^N \left| \int_{U_i} g_{z,u_i}(x) dP(u) - p_i g_{z,u_i}(x) \right| + \sum_{i=1}^N \int_{U_i} |g_{z,u}(x) - g_{z,u_i}(x)| dP(u) \\
 & \leq \sum_{i=1}^N g_{z,u_i}(x) |P(U_i) - p_i| + \sum_{i=1}^N g_{z,u_i}(x) |e^{z^{1-2A}(x/u_i+1)} - 1| P(U_i) \\
 & \leq \sum_{i=1}^N g_{z,u_i}(x) p_i z^{-2A} + \sum_{i=1}^N p_i g_{z,u_i}(x) |e^{z^{1-2A}(x/u_i+1)} - 1| (1 - z^{-2A}) \\
 & \leq K_z * P_N(x) z^{-2A} + \sum_{i=1}^N p_i g_{z,u_i}(x) |e^{z^{1-2A}(x/u_i+1)} - 1|.
 \end{aligned}$$

Then,

$$K_z * P(x) \geq K_z * P_N (1 - z^{-2A}) - \sum_{i=1}^N p_i g_{z,u_i}(x) |e^{z^{-2A+1}(1+x/u_i)} - 1|. \tag{B.7}$$

By construction  $f_\beta \geq c_\beta \tilde{f}/2 \geq \tilde{f}/2(1 + o(1))$ , also on  $\mathcal{A}_1(0)$ ,

$$K_z h(x) \geq h(x) - \sum_{j=1}^r \frac{|\mu_j h^{(j)}(x) x^j|}{z^{j/2}} - z^{-\beta/2} |R_z(x)| \geq h(x)/2$$

which implies that

$$K_z \tilde{f}(x) = x^{\alpha-1} (1 + O(1/z)) K_{z+1-\alpha} h(x) \geq (1 + O(1/z)) f(x)/2.$$

Using Lemma B.1, on  $\mathcal{A}_1(0) \cap [e_z/2, 2E_z]$ , for arbitrarily chosen  $\kappa > 0$ ,

$$f(x) \leq 2K_z \tilde{f}(x) \leq 4K_z f_\beta(x) \lesssim K_z * P_N + z^{-\kappa}.$$

Moreover, for  $x \in [e_z/2, 2E_z]$ , using (B.7),

$$\sum_{i=1}^N p_i g_{z,u_i}(x) |e^{z^{-2A+1}(1+x/u_i)} - 1| \leq \sum_{i=1}^N p_i g_{z,u_i}(x) z^{-2A+1} (1 + 4z^{a+b})$$

which gives

$$K_z * P(x) \geq K_z * P_N (1 - z^{-2A} - 5z^{-2A+a+b+1}) \tag{B.8}$$

and hence  $K_z * P \geq CK_z * P_N$  as soon as  $A > (a + b + 1)/2$ . Hence on  $\mathcal{A}_1(0) \cap [e_z/2, 2E_z]$ ,  $f(x) > z^{A_1} K_z * P_N(x)$  with  $A_1 > 0$  only if  $f(x) \lesssim z^{-A_1} f(x) + z^{-\kappa}$ , i.e. if  $f(x) \lesssim z^{-\kappa}$ .

Now consider  $x$  such that  $f(x) \leq z^{A_1} K_z * P_N(x)$  and  $f(x) \geq 2z^{A_1} K_z * P(x)$ , i.e. such that  $K_z * P_N(x) \geq 2K_z * P(x)$ . Then, using (B.7),

$$2(1 - 2z^{-A})^{-1} \sum_j p_{i,j} g_{z,u_j}(x) |e^{z^{-A+1}(x/u_j+1)} - 1| \geq K_z * P_N(x).$$

If  $x/u_j \leq 2$  then  $|e^{z^{-A+1}(x/u_j+1)} - 1| \lesssim z^{-A+1}$  while if  $x > 2u_j$ ,

$$g_{z,u_j}(x) e^{z^{-A+1}(x/u_j+1)} \leq e^{-cz(x/u_j+1)}/x$$

for some  $c > 0$ . Therefore,

$$\begin{aligned} \sum_j p_j g_{z,u_j}(x) |e^{z^{-A+1}(x/u_j+1)} - 1| &\leq K_z * P_N z^{-A+1} + \sum_j p_j \frac{e^{-cz(x/u_j+1)}}{x} \mathbb{I}_{x>2u_j} \\ &\leq K_z * P_N z^{-A+1} + e_z^{-1} e^{-3cz}. \end{aligned}$$

Using (B.7), we have

$$\begin{aligned} K_z * P_N &\leq \sum_{i=1}^N p_i g_{z,u_i}(x) |e^{z^{-2A+1}(1+x/u_i)} - 1| (0.5 - z^{-2A})^{-1} \\ &\leq 4K_z * P_N z^{-A+1} + 4e_z^{-1} e^{-3cz} \end{aligned}$$

which in turn implies that

$$z^a e^{-3cz} \gtrsim K_z * P_N \geq z^{-A_1} f(x)$$

so that  $f(x) \lesssim z^{a+A_1} e^{-3cz}$ . In all cases, for all  $H \geq \kappa$ , by choosing  $A$  and  $A_1$  such that  $A + 1 - a - A_1 > H$  and so that (B.4) holds, we obtain that on  $\mathcal{A}_1(0) \cap [e_z/2, 2E_z]$  if  $2z^{A_1} K_z * P(x) \leq f(x)$  then  $f(x) \leq z^{-H}$ . For  $x$  such that  $\mathcal{A}_1(0) \cap [e_z/2, 2E_z]$  and  $z^{A_1} K_z * P(x) \leq f(x) \leq 2z^{A_1} K_z * P(x)$ ,  $f(x) \leq z^{-\kappa}$ .

Now we bound from below  $K_z * P(x)$ .

- Take  $x \in [e_z, E_z]$ , and let  $\ell$  be such that  $x \in [e_z(1 + \delta_z)^\ell, e_z(1 + \delta_z)^{\ell+1}]$  with  $e_z(1 + \delta_z)^{\ell+1} \leq E_z$  and  $\delta_z = M\sqrt{z^{-1} \log z}$ , then

$$\begin{aligned} K_z * P(x) &\geq P([e_z(1 + \delta_z)^\ell, e_z(1 + \delta_z)^{\ell+1}]) \frac{\sqrt{z}e^{-z\delta_z^2/2}}{\sqrt{2\pi}}(1 + o(1)) \\ &\gtrsim z^{-A+1/2-M^2/2}, \end{aligned}$$

since  $P(U_j) \geq p_j(1 - z^{-A})$  for all  $j$  and  $P([e_z(1 + \delta_z)^\ell, e_z(1 + \delta_z)^{\ell+1}]) \geq 3z^{-A}$ .

- If  $x < e_z$ ,

$$\begin{aligned} K_z * P(x) &\geq \frac{z^{-A-2}e^{-zx/e_z(1+\delta_z)}(zx/e_z(1 + \delta_z))^{z-1}}{e_z\Gamma(z)} \\ &\geq \exp(2z \log(x/e_z) - c \log z), \end{aligned}$$

when  $z$  is large enough, for some  $c > 0$ .

- If  $x > E_z$ ,

$$\begin{aligned} K_z * P(x) &\gtrsim \frac{e^{-zx/e_z}(zx/e_z)^{z-1}}{x\Gamma(z)} \\ &\geq \exp\left(-z\frac{x}{e_z} + (z-1)[\log(x) - \log(e_z)]\right) \\ &\gtrsim e^{-2zx/e_z}. \end{aligned}$$

Then, using Lemma B2 of Shen et al. [14] with  $\lambda = z^{-A_1}$ , we have, using  $\log f(x) \lesssim \log x$ ,

$$\begin{aligned} \mathcal{KL}(f, K_z * P) &\lesssim D_H^2(f, K_z * P)(1 + A_1 \log z) \\ &\quad + \int_{f > z^{A_1} K_z * P} f(x) \log(f(x)/K_z * P(x)) dx \\ &\lesssim D_H^2(f, K_z * P)(1 + A_1 \log z) + \log z \int_{[e_z, E_z] \cap \{f > z^{A_1} K_z * P\}} f(x) dx \\ &\quad + \int_{\{f > z^{A_1} K_z * P\} \cap [e_z, E_z]} f(x) (\log f(x))_+ dx \\ &\quad + z \int_0^{e_z} f(x) (|\log x| + \log z) dx + z^{a+1} \int_{E_z}^\infty f(x) x dx \\ &\lesssim z^{-\beta} \log z + z \log z F(0, e_z) + z^{-b(2+\rho_1)/2} + z^{a+1-b(1+\rho_1)} \\ &\quad + \log z \int_{\mathcal{A}_1(0) \cap [e_z, E_z] \cap \{f > z^{A_1} K_z * P\}} f(x) dx \\ &\quad + \log z \int_{\mathcal{A}_1(0)^c \cap [e_z, E_z] \cap \{f > z^{A_1} K_z * P\}} f(x) dx \\ &\lesssim z^{-\beta} \log z + z^{1-a\alpha} \log z + z^{-b(2+\rho_1)/2} + z^{a+1-b(1+\rho_1)} \end{aligned}$$

$$+ z^{b-\kappa} \log z + z^{-\beta-\epsilon/4} \log z$$

using  $F(\mathcal{A}_1(0)^c) \lesssim z^{-\beta-\epsilon/4}$  and

$$\begin{aligned} \int_{E_z}^\infty f(x)(\log f(x))_+ dx &\leq \sqrt{F[E_z, +\infty)} \sqrt{\int_1^\infty f(x) \log^2(x) dx} \lesssim E_z^{-1-\rho_1/2}, \\ \int_{E_z}^\infty x^k f(x) dx &\leq E_z^{k-2} \int_{E_z}^\infty x^2 f(x) dx \lesssim E_z^{k-2-\rho_1} \end{aligned}$$

for  $E_z > 1$  and  $k \in [0, 2)$ .

Choosing  $a, b$  and  $\kappa$  such that

$$a \geq (\beta + 1)/\alpha, \quad b \geq 2\beta/(2 + \rho_1), \quad b(1 + \rho_1) - a - 1 \geq \beta \quad \kappa \geq b + \beta,$$

we have that

$$\int f \log \left( \frac{f}{K_z * P} \right) \mathbb{I}_{f > z^{A_1} K_z * P} \lesssim z^{-\beta} (\log z).$$

Similarly,

$$\int f \left[ \log \left( \frac{f}{K_z * P} \right) \right]^2 \mathbb{I}_{f > z^{A_1} K_z * P} \lesssim z^{-\beta} (\log z)^2,$$

under the same constraints. □

**Appendix C: Some technical lemmas**

**Lemma C.1.** *For all  $\delta > 0$ , there exists  $C > 0$  such that for all  $\epsilon_1, \epsilon_2$  satisfying  $|\epsilon_1/\epsilon_2 - 1| < \delta$*

$$\|g_{z,\epsilon_1} - g_{z,\epsilon_2}\|_1 \leq \sqrt{2\mathcal{KL}(g_{z,\epsilon_1}, g_{z,\epsilon_2})} \leq \sqrt{2z}\delta, \quad g_{z,\epsilon_2}(x) \leq g_{z,\epsilon_1}(x) e^{z\delta(1+x/\epsilon_1)}.$$

*Proof of Lemma C.1.* Inequality

$$\|g_{z,\epsilon_1} - g_{z,\epsilon_2}\|_1 \leq \sqrt{2\mathcal{KL}(g_{z,\epsilon_1}, g_{z,\epsilon_2})}$$

holds due the inequality for the total variation distance to be upper bounded by  $\sqrt{2}$  times the square root of the Kullback-Leibler distance between the corresponding probability distributions.

The Kullback-Leibler distance between two densities  $g_{z,\epsilon_1}$  and  $g_{z,\epsilon_2}$  is

$$\begin{aligned} \mathcal{KL}(g_{z,\epsilon_1}, g_{z,\epsilon_2}) &= \int_0^\infty g_{z,\epsilon_1}(x) \log \left( \frac{g_{z,\epsilon_1}(x)}{g_{z,\epsilon_2}(x)} \right) dx \\ &= \int_0^\infty g_{z,\epsilon_1}(x) (zx[\epsilon_2^{-1} - \epsilon_1^{-1}] + z \log(\epsilon_2/\epsilon_1)) dx \end{aligned}$$

$$\begin{aligned}
 &= z(\epsilon_1/\epsilon_2 - 1 - \log(\epsilon_1/\epsilon_2)) \\
 &\leq z\delta^2
 \end{aligned}$$

due to condition  $|\epsilon_1/\epsilon_2 - 1| < \delta$  and inequality  $x - 1 - \log x \leq |x - 1|r/(1+r) \leq r^2$  if  $|x - 1| \leq r$ .

Moreover, for any  $x > 0$ ,

$$\begin{aligned}
 |\log g_{z,\epsilon_2}(x) - \log g_{z,\epsilon_1}(x)| &= |zx[\epsilon_1^{-1} - \epsilon_2^{-1}] + z \log(\epsilon_1/\epsilon_2)| \\
 &\leq \left| \frac{zx}{\epsilon_1} \delta + z \log(1 + \delta) \right|
 \end{aligned}$$

which implies that

$$g_{z,\epsilon_2}(x)/g_{z,\epsilon_1}(x) \leq \exp \left\{ \frac{zx}{\epsilon_1} \delta + z\delta \right\}$$

which completes the proof. □

**C.1. Properties of gamma densities**

In this section we present some technical computations which are used throughout the paper. We first present some identities on mixtures of Gamma densities, together with tail inequalities

**Lemma C.2.** *Let  $z > 0$  and  $x > 0$ , then*

$$I_0(z, x) := \int_0^\infty g_{z,\epsilon}(x) d\epsilon = 1 + \frac{1}{z - 1} \tag{C.1}$$

and for all  $k \geq 0$

$$I_k(z, x) := \int_0^\infty (\epsilon - x)^k g_{z,\epsilon}(x) d\epsilon = \frac{x^k z^z}{\Gamma(z)} \int_0^\infty \frac{(u - 1)^k e^{-z/u}}{u^z} du \tag{C.2}$$

Moreover for all  $\delta \in (0, 1)$  there exists  $c(\delta) > 0$  such that for all  $z$  large enough and  $u < 1 - \delta$ ,

$$\frac{z^z e^{-z/u}}{\Gamma(z) u^z} \leq e^{-c(\delta)z/u} \tag{C.3}$$

and for all  $u > 1 + \delta$

$$\frac{z^z e^{-z/u}}{\Gamma(z) u^z} \leq u^{-c(\delta)z}. \tag{C.4}$$

*Proof of Lemma C.2.* We have

$$\begin{aligned}
 I_0(z, x) &:= \int_0^\infty g_{z,\epsilon}(x) d\epsilon = \frac{x^{z-1} z^z}{\Gamma(z)} \int_0^\infty e^{-zx/\epsilon} \frac{1}{\epsilon^{(z-1)+1}} d\epsilon \\
 &= \frac{x^{z-1} z^z}{\Gamma(z)} (zx)^{-(z-1)} \Gamma(z - 1) = \frac{z}{z - 1} = 1 + \frac{1}{z - 1},
 \end{aligned}$$

which proves (C.1). For all  $k \geq 0$ ,

$$\begin{aligned} I_k(z, x) &:= \int_0^\infty (\epsilon - x)^k g_{z,\epsilon}(x) d\epsilon = \frac{x^{k-1} z^z}{\Gamma(z)} \int_0^\infty \left(\frac{\epsilon}{x} - 1\right)^k e^{-zx/\epsilon} \frac{x^{(z-1)+1}}{\epsilon^{(z-1)+1}} d\epsilon \\ &= \frac{x^k z^z}{\Gamma(z)} \int_0^\infty \frac{(u-1)^k e^{-z/u}}{u^z} du \end{aligned}$$

and (C.2) is verified. Now, note that when  $z$  is large

$$\begin{aligned} \frac{z^z e^{-z/u}}{\Gamma(z) u^z} &= \frac{\sqrt{z} \exp(-z[\log u + 1/u - 1])}{\sqrt{2\pi}} (1 + R(z))^{-1} \\ &= \frac{\sqrt{z} \exp(-\frac{z}{2}(1-u)^2(1+o(1)))}{\sqrt{2\pi}} (1 + R(z))^{-1} \end{aligned} \tag{C.5}$$

where  $R(z) = O(1/z)$  is the remainder term of the Stirling formula. When  $u < 1 - \delta$  the first inequality leads to (C.3) while when  $u > 1 + \delta$  it leads to (C.4).  $\square$

From Lemma C.2, we can deduce the following approximations:

**Lemma C.3.** *For all  $k \geq 0$  and  $x > 0$ ,*

$$I_k(z, x) = \frac{x^k}{z^{k/2}} (1 + R(z))^{-1} (\mu_k + O(z^{-H})) := \frac{x^k}{z^{k/2}} \mu_k(z), \quad \forall H > 0, \tag{C.6}$$

where  $\mu_k = \int_{\mathbb{R}} x^k \varphi(x) dx$  with  $\varphi$  the density of a standard Gaussian random variable. We also have

$$K_z f(x) = \sum_{j=0}^r \frac{f^{(j)}(x) x^j}{j! z^{j/2}} \mu_j(z) + z^{-\beta/2} R_z(x) \tag{C.7}$$

where

$$|R_z(x)| \leq C_{\beta,z} L(x) x^\beta \left[ 1 + \frac{x^\gamma}{z^{\gamma/2}} \right].$$

For all  $g(x) \leq C_1 + C_2 x^a$  for some  $a > 0$ , then

$$K_z g(x) \leq 2C_1 + 2C_2 x^a, \tag{C.8}$$

for  $z$  large enough and a fixed.

*Proof of Lemma C.3.* Lemma C.2 implies that

$$K_z f(x) = \sum_{j=0}^r \frac{f^{(j)}(x)}{j!} I_j(z, x) + R_z(x) = \sum_{j=0}^r \frac{f^{(j)}(x) x^j}{j! z^{j/2}} \mu_j(z) + z^{-\beta/2} R_z(x)$$

where

$$|R_z(x)| \leq C_\beta L(x) z^{\beta/2} [I_\beta(z, x) + I_{\beta+\gamma}(z, x)] \leq C_{\beta,z} L(x) x^\beta \left[ 1 + \frac{x^\gamma}{z^{\gamma/2}} \right].$$

Also if  $g(x)$  be a function bounded by  $C_1 + C_2x^a$  for some  $a > 0$ , then

$$\begin{aligned} K_z g(x) &\leq C_1 I_0(z) + C_2 \frac{x^{z-1} z^z}{\Gamma(z)} \int_0^\infty e^{-zx/\epsilon} \frac{1}{\epsilon^{(z-a-1)+1}} d\epsilon \\ &\leq 2C_1 + C_2 \frac{x^a \Gamma(z-a-1) z^z}{\Gamma(z) z^{z-a-1}} \leq 2C_1 + 2C_2 x^a, \end{aligned}$$

for  $z$  large enough and  $a$  fixed. □

**C.2. Examples of functions in  $\mathcal{P}_\alpha(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$**

In this section we verify conditions in Remark 2.1.

In Remark 2.1 we state that moment condition (2.6) is satisfied for Weibull distribution  $f(x) = Cx^{\alpha-1}e^{-cx^b}$  with  $\alpha, b, c > 0$ ; for folded Student t distribution  $f(x) = c_\nu(1+x^2)^{-(\nu+1)/2}$ ,  $x > 0$ ; for the Frechet-type distributions  $f(x) = bx^{-b-1}e^{-x^{-b}}$ ,  $b > 0$ .

*C.2.1. Weibull distribution*

Consider Weibull distribution with density  $f(x) = C_{\alpha,b}x^{\alpha-1}e^{-x^b}$  with  $a, b > 0$ . Assume first that  $a = \alpha \in (0, 1]$  and  $b \geq 1$ , then  $\ell(x) = \log h(x) = -x^b + \log C_{\alpha,b}$  which is infinitely differentiable. Take some integer  $r \geq 0$  such that  $b-r \in (0, 1]$  then  $\ell^{(r)}(x) = -(b)_r x^{b-r}$  where  $(x)_r = x(x-1)\dots(x-r+1)$  and  $\beta > r$ . We need to check that for  $j = 1, \dots, r$ ,

$$\begin{aligned} \int_0^\infty [x^j |\ell^{(j)}(x)|]^{(2\beta+e)/j} f(x) dx &\lesssim \int_0^\infty [x^b + x^j]^{(2\beta+e)/j} x^{\alpha-1} e^{-x^b} dx \\ &\lesssim \int_0^\infty x^{b(2\beta+e)/j + \alpha - 1} e^{-x^b} dx + \int_0^\infty x^{(2\beta+e) + \alpha - 1} e^{-x^b} dx = [z = x^b] \\ &\lesssim \int_0^\infty z^{(2\beta+e)/j + \alpha/b - 1} e^{-z} dz + \int_0^\infty z^{(2\beta+e) + \alpha/b - 1} e^{-z} dz \end{aligned}$$

which is finite since  $b(2\beta + e)/j + \alpha > 0$  for  $j = 1, 2, \dots, r$ .

Since  $b - r \in (0, 1]$ ,

$$|\ell^{(r)}(x+y) - \ell^{(r)}(x)| = (b)_r |(x+y)^{b-r} - x^{b-r}| \leq (b)_r |y|^{b-r}$$

due to inequality  $|z^A - w^A| \leq |z - w|^A$  for  $A \in (0, 1]$ . Here  $L_{\log}(x) = (b)_r$ ,  $\beta = b$  and  $\gamma = 0$ .

It is sufficient to check that

$$\begin{aligned} \int_0^\infty [(x^\beta + x^{2\beta})^2 f(x)] dx &\lesssim \int_0^\infty (x^{2b+\alpha-1} + x^{4b+\alpha-1}) e^{-x^b} dx \\ &\lesssim \int_0^\infty [z^{\alpha/b+1} + z^{\alpha/b+3}] e^{-z} dz < \infty \end{aligned}$$

which holds.

For  $a > 1$ , then we can take  $\alpha = 1$ , and the corresponding Weibull density belongs to  $\mathcal{P}(\beta, L(\cdot), \gamma, C_0, C_1, e, \Delta)$  due to the first part of Remark 2.1.

C.2.2. *Folded Student t distribution*

Now we take folded Student t distribution  $f(x) = c_\nu(1 + x^2)^{-(\nu+1)/2}$ ,  $x > 0$ . Then  $\alpha = 1$  and  $\ell(x) = -0.5(\nu + 1) \log(1 + x^2)$ , and the derivatives for large  $x$  are

$$\begin{aligned} \ell'(x) &= -(\nu + 1)x(1 + x^2)^{-1}, \quad \ell''(x) = -(1 + x^2)^{-1} + 2x^2(1 + x^2)^{-2}, \\ \ell^{(2j)}(x) &= \sum_{s=0}^j c_{2j,s} \frac{x^{2s}}{(1 + x^2)^{j+s}}, \quad \ell^{(2j+1)}(x) = \sum_{s=0}^j c_{2j+1,s} \frac{x^{1+2s}}{(1 + x^2)^{1+j+s}}, \end{aligned}$$

which is easy to prove by induction. Note that for any positive integer  $k$ ,  $|\ell^{(k)}(x)| \lesssim (1 + x^2)^{-k/2}$ .

Hence, for even derivatives,

$$\int_0^\infty [x^{2j} |\ell^{(2j)}(x)|]^{(2\beta+e)/(2j)} f(x) dx \lesssim \sum_{s=0}^j \int_0^\infty \left( \frac{x^2}{(1 + x^2)} \right)^{(1+s/j)(\beta+e/2)} f(x) dx$$

which is finite. Similarly, for odd derivatives,

$$\begin{aligned} &\int_0^\infty [x^{2j+1} |\ell^{(2j+1)}(x)|]^{(2\beta+e)/(2j+1)} f(x) dx \\ &\lesssim \sum_{s=0}^j \int_0^\infty \left( \frac{x^2}{(1 + x^2)} \right)^{(2\beta+e)(1+s+j)/(1+2j)} f(x) dx < \infty. \end{aligned}$$

Case  $r = 0$ :

$$\begin{aligned} |\ell(x + y) - \ell(x)| &= 0.5(\nu + 1) \left| \log \left( \frac{1 + (x + y)^2}{1 + x^2} \right) \right| \\ &\leq 0.5(\nu + 1) A^{-1} \left[ \frac{y(2x + y)}{1 + x^2} \right]^A I(y > 0) \\ &\quad + 0.5(\nu + 1) A^{-1} \left[ \frac{|y|(2x - |y|)}{1 + (x - |y|)^2} \right]^A I(y < 0) \\ &\leq 0.5(\nu + 1) A^{-1} \left[ \frac{2|y|(1 + |y|)}{(1 + x^2)^{1/2}} \right]^A I(y > 0) \\ &\quad + 0.5(\nu + 1) A^{-1} [2|y|(1 + |y|)]^A I(y < 0) \\ &\lesssim |y|^A (1 + |y|^A) \end{aligned}$$

using inequality  $\log(1+x) \leq x^A/A$  for any  $x \geq 0$  and any  $A > 0$ . Then,  $\beta = A$  for  $A \in (0, 1]$ ,  $\gamma = \beta = A$  and  $L_\ell(x) = C$ . Condition  $\int_0^\infty x^{2\beta}(1+x^2)^\gamma L_\ell^2(x) f(x) dx < \infty$  holds if

$$\int_0^\infty x^{2A}(1+x^{2A})(1+x^2)^{-(\nu+1)/2} dx \leq C + \int_1^\infty x^{4A}(1+x^2)^{-(\nu+1)/2} dx < \infty$$

i.e. if  $\beta = A < \nu/4$  (here  $r_0 = 0$ ).

Now fix a positive integer  $r$ . Since

$$\begin{aligned} |\ell^{(r)}(x+y) - \ell^{(r)}(x)| &\leq |y| \sup_{z \in \langle x, x+y \rangle} |\ell^{(r+1)}(z)| \lesssim |y| \sup_{z \in \langle x, x+y \rangle} (1+z^2)^{-(r+1)/2} \\ &\lesssim |y|(1+x^2)^{-(r+1)/2} I(y > 0) + |y|I(y < 0) \lesssim |y|. \end{aligned}$$

Therefore, for any integer  $r \geq 1$ , the first condition is satisfied with  $\beta = r + 1$ ,  $L_\ell(x) = C$  and  $\gamma = 0$ . Condition  $\int_0^\infty x^{2\beta}(1+x^{4r_0+2\gamma})L_\ell^2(x)f(x)dx < \infty$  holds if  $\beta = r + 1 < \nu/2$  and since  $\beta = r + 1 \geq 2$ , we also need  $2\beta + 4r_0 < \nu$ . Since  $r_0 = \lceil \beta/2 \rceil - 1 < \beta/2$  and  $\beta$  is an integer, we can write this condition as  $\beta = r + 1$  where  $a_r < \nu$  where for even  $r = 2k$   $a_r = 4r + 2$  and for odd  $r = 2k + 1$   $a_r = 4r$ . For instance,  $a_1 = 4$ ,  $a_2 = 10$ ,  $a_3 = 12$ ,  $a_4 = 18$  etc.

Therefore, the conditions on  $\beta$  and  $L_\ell(x)$  given  $\nu$  can be summarised as follows:  $L_\ell(x) = C$  and

- $\nu \in [1, 4]$ :  $\beta < \nu/4$ ,  $\gamma = \beta$ .
- $\nu \in (a_r, a_{r+1}]$ :  $\beta = r + 1$ ,  $\gamma = 0$ .

### C.2.3. Frechet distribution

Consider a Frechet-type distribution with density  $f(x) = c_b x^{-b-1} e^{-x^{-b}}$ ,  $x > 0$ , for some  $b > 0$ . This density does not belong to a logarithmic Hölder class. For simplicity we consider a bound of the type  $|f(x) - f(x+y)| \leq L(x)|y|^\beta(1+|y|^\gamma)$  with  $r = 0$ , i.e. with  $\beta \leq 1$ . Hence, for  $|y| \leq \Delta$ ,  $x > 0$  and  $x > -y$ ,

$$\begin{aligned} &|x^{-b-1}e^{-x^{-b}} - (x+y)^{-b-1}e^{-(x+y)^{-b}}| \\ &\leq |y| \sup_{z \in \langle x, x+y \rangle} [(b+1)z^{-b-2} + bz^{-2b-2}]e^{-z^{-b}} \\ &\leq (b+1)|y| \left[ \sup_{z \in \langle x, x+y \rangle} z^{-b-2}e^{-z^{-b}} + \sup_{z \in \langle x, x+y \rangle} z^{-2b-2}e^{-z^{-b}} \right]. \end{aligned} \tag{C.9}$$

For  $a = b + 2$  and for  $a = 2$ , consider  $\sup_{z \in \langle x, x+y \rangle} z^{-b-a}e^{-z^{-b}}$ . Function  $z^{-b-a}e^{-z^{-b}}$  achieves the maximum on the whole semiline at  $x_a^* = (1+a/b)^{-1/b}$ . Hence, if  $x_a^* \in \langle x, x+y \rangle$  then the supremum is achieved at this point. If  $\min(x, x+y) > x_a^*$  then the supremum is achieved at  $\min(x, x+y)$ , and if  $\max(x, x+y) < x_a^*$  then the supremum is achieved at  $\max(x, x+y)$ .

1.  $\max(x, x+y) < x_a^*$ . If  $y \leq 0$  then the condition is  $x < x_a^*$  and the supremum is  $x^{-b-a}e^{-x^{-b}}$ . If  $y > 0$ , then the supremum is

$$\begin{aligned} &\sup_{z \in \langle x, x+y \rangle} z^{-b-a}e^{-z^{-b}} = (x+y)^{-b-a}e^{-(x+y)^{-b}} \\ &\leq x^{-b-a}e^{-x^{-b}}(1+|y|)\max(1, 2bx^{-b-1}) \end{aligned}$$

using inequality

$$e^{x^{-b}-(x+y)^{-b}} \leq 1 + 2b|y|x^{-b-1}. \tag{C.10}$$

We can unite the upper bound as  $x^{-b-a}e^{-x^{-b}}(1+|y|)\max(1, 2bx^{-b-1})$ .

2.  $\min(x, x+y) > x_a^*$ . If  $y \geq 0$  then the condition is  $x > x_a^*$  and the supremum is  $x^{-b-a}e^{-x^{-b}}$ . If  $y < 0$  then

$$\sup_{z \in \langle x, x+y \rangle} z^{-b-a}e^{-z^{-b}} = (x+y)^{-b-a}e^{-(x+y)^{-b}} \leq (x-\Delta)^{-b-a}e^{-x^{-b}}$$

We can unite the upper bound as  $(x-\Delta)^{-b-a}e^{-x^{-b}}$ .

3.  $\min(x, x+y) \leq x_a^* \leq \max(x, x+y)$ , that is,  $|x-x_a^*| \leq |y| \leq \Delta$ . Let's write the supremum as a function of  $x$ ,  $y$  and  $\Delta$ :

$$\begin{aligned} \sup_{z \in \langle x, x+y \rangle} z^{-b-a}e^{-z^{-b}} &= x_a^{*-b-a}e^{-x_a^{*-b}} \leq (x-\Delta)^{-b-a}e^{-(x+|y|)^{-b}} \\ &\leq (x-\Delta)^{-b-a}e^{-x^{-b}}[1+|y|] \max(1, 2bx^{-b-1}) \end{aligned}$$

using (C.10).

To apply the bounds to the cases  $a = 2$  and  $a = b+2$ , note that  $\max(x, x+y) < x_a^*$  holds if  $x < x_a^* - \Delta$ ; and it holds for both values of  $a$  if  $x < x_{b+2}^* - \Delta$  since  $x_a^*$  decreases in  $a$ .

Then, the upper bound in (C.9) can be written as

$$\begin{aligned} &|x^{-b-1}e^{-x^{-b}} - (x+y)^{-b-1}e^{-(x+y)^{-b}}| \leq (b+1)|y|e^{-x^{-b}}[1+|y|] \times \\ &\times \max(1, 2bx^{-b-1}) [(x-\Delta)^{-b-a}I(x > x_{b+2}^* - \Delta) + x^{-b-a}I(x < x_{b+2}^* - \Delta)], \end{aligned}$$

i.e.  $\gamma = 1$  and

$$\begin{aligned} L(x) &= (b+1)e^{-x^{-b}} \max(1, 2bx^{-b-1}) \\ &\times \sum_{a \in \{2, b+2\}} [(x-\Delta)^{-b-a}I(x > x_{b+2}^* - \Delta) + x^{-b-a}I(x < x_{b+2}^* - \Delta)]. \end{aligned}$$

Now we check the integrability condition:

$$\begin{aligned} &\int_0^\infty (x^\beta(1+x)L(x))^2 f(x) dx \leq 2 \int_0^\infty (x^\beta(1+x) \max(1, x^{-b-1}))^2 \\ &\times \sum_{a \in \{2, b+2\}} [(x-\Delta)^{-b-a}x^{b+1}I(x > x_{b+2}^* - \Delta) \\ &+ x^{-b-a}I(x < x_{b+2}^* - \Delta)]^2 f(x) dx \\ &\leq 8 \sum_{a \in \{2, b+2\}} \int_{x_{b+2}^* - \Delta}^\infty [x^\beta \max(x, x^{-b-1})(x-\Delta)^{-b-a}x^{b+1}]^2 c_b x^{-b-1} e^{-x^{-b}} dx \\ &+ 8 \sum_{a \in \{2, b+2\}} \int_0^{x_{b+2}^* - \Delta} [x^\beta \max(x, x^{-b-1})x^{1-a}]^2 c_b x^{-b-1} e^{-x^{-b}} dx \\ &\leq 8 \sum_{a \in \{2, b+2\}} \int_0^{x_{b+2}^* - \Delta} x^{2(\beta-b-a)-1} c_b x^{-b-1} e^{-x^{-b}} dx \end{aligned}$$

$$+ 8 \sum_{a \in \{2, b+2\}} \int_{x_{b+2}^* - \Delta}^{\infty} ((x - \Delta)^{-b-a} x^{\beta+b+1})^2 c_b x^{-b-1} e^{-x^{-b}} dx$$

The first integral is finite. The second integral is finite if  $-2a + 2\beta + 4 < b$  for  $a = 2$  and for  $a = b + 2$ , i.e. for  $\beta < b/2 + 1$  which holds for any  $\beta \in (0, 1]$ .

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