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Asymptotic direction for random walks in mixing random environments*

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Abstract

We prove that every random walk in a uniformly elliptic random environment satisfying the cone-mixing condition and a non-effective polynomial ballisticity condition with high enough degree has an asymptotic direction.

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1 Introduction

Random walk in random environment is a simple but powerful model for a variety of phenomena including homogenization in disordered materials [11], DNA chain replication [3], crystal growth [21] and turbulent behavior in fluids [14]. Nevertheless, challenging and fundamental questions about it remain open (see [23] for a general overview). In the multidimensional setting a widely open question is to establish relations between the environment at a local level and the long time behavior of the random walk. Interesting progress has been achieved specially in the case in which the movement takes place on the hypercubic lattice $\mathbb{Z}^d$ and the environment is i.i.d., establishing relations between directional transience, ballisticity and the existence of an asymptotic direction and the law of the environment in finite regions. To a great extent, these arguments are no longer valid when the i.i.d. assumption is dropped.

In this article we focus on the problem of finding local conditions on the environment which ensure the existence of a deterministic asymptotic direction for the random walk model in contexts where the environment is not necessarily i.i.d. In [13], Simenhaus proved that for i.i.d. elliptic environments, whenever the random walk is directionally transient in a open set of directions, it has an asymptotic direction. As it will be shown

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in Section 3, there exist environments which are stationary and ergodic, but not i.i.d., and where the random walk is directionally transient in direction l for every l in an open subset of $\mathbb{S}^d$, and for which there does not exist a deterministic asymptotic direction. Therefore, some kind of mixing condition or some condition stronger than directional transience in an open subset of directions of $\mathbb{S}^{d-1}$ should be imposed on the environment. Here we will impose both a mixing condition and a transience condition related to the polynomial ballisticity condition introduced in [1]. The polynomial ballisticity condition of [1] is essentially defined as the requirement that the probability that the random walk exits through the back and lateral sides of a box with lateral sides of a given length L and width CL^3, for some constant C, decays polynomially fast. This polynomial condition is *effective* in the sense that it can in principle be verified on finite sets of $\mathbb{Z}^d$, as opposed to *non-effective* ballisticity conditions which require information on infinite sets. Here we establish the existence of an asymptotic direction for random walks in random environments which are uniformly elliptic, are cone-mixing [5], and satisfy a uniform non-effective version of the polynomial ballisticity condition introduced in [1] with high enough degree of the decay. The term *uniform* means that a uniform polynomial decay on the probability conditioned on the environment outside the finite box is required. It will be also shown (see Section 3), that there exist environments *almost* satisfying the above assumptions which are directionally transient and for which there exists at least in a weak sense an asymptotic direction, but have a vanishing velocity. Here the term *almost* is used because in these examples the uniform non-effective polynomial ballisticity condition is satisfied with a low degree. This shows that somehow, while the cone-mixing and uniform non-effective polynomial conditions we will impose do imply the existence of an asymptotic direction, they might not necessarily imply the existence of a non-vanishing velocity (this should be compared with the i.i.d. situation in dimensions $d \geq 2$, where it is known that the polynomial condition implies ballisticity [1], and that transience in direction l for every l in an open subset of $\mathbb{S}^{d-1}$ implies the existence of an asymptotic direction [13]).

In [5], the existence of a strong law of large numbers is established for random walks in cone-mixing environments which also satisfy a version of Kalikow condition, and under an additional assumption of existence of certain moments of approximate regeneration times (which are not stopping times). It is known that Kalikow condition is a strictly stronger assumption than the polynomial condition we assume in this article [20, 1]. On the other hand, the moment condition assumption of the approximate regeneration times of [5] is unsatisfactory in the sense that it is in general difficult to verify if for a given random environment it is true or not. Furthermore, as it will be shown in Section 3, there exist examples of random walks in a random environment satisfying the cone-mixing assumption for which the law of large numbers is not satisfied, while an asymptotic direction exists. From this point of view, establishing the existence of an asymptotic direction under mild ballisticity conditions, is also a first step in the direction of obtaining scaling limit theorems for random walks in cone-mixing environments through ballisticity conditions weaker than Kalikow condition, and without any kind of assumption on the moments of approximate regeneration times or of the position of the random walk at these times. On the other hand, in [12], a strong law of large numbers is proved for random walks which satisfy Kalikow condition and Dobrushin-Shlosman's strong mixing assumption. The Dobrushin-Shlosman strong mixing assumption is stronger than cone-mixing, both because it implies cone-mixing in every direction and because it corresponds to a decay of correlations which is exponential.

In Section 2, we will introduce the main notations, assumptions and state the main result of this article. In Section 3, we will present the two mentioned examples of random walks in random environments which exhibit behavior which is not observed in the i.i.d.

case, giving an idea of the kind of limitations imposed by the framework of Theorem 2.1. In Section 4, the meaning of the uniform non-effective polynomial condition and its relation to other ballisticity conditions will be discussed. In Section 5, we will show that the uniform non-effective polynomial condition implies that the probability that the random walk never exits a cone is positive. This will be used in Section 6 to prove that the position of the random walk at the approximate regeneration times have finite moments of order two. In Section 7, Theorem 2.1 will be proved using coupling with i.i.d. random variables. In Appendix A, three lemmas will be proved that are used in Section 6 to prove the finitness of the moment of order two of the position of the random walk at the approximate regeneration times . In Appendix B, it will be shown that the cone mixing condition together with stationarity, implies ergodicity.

2 Notations, assumptions and main result

Here we will introduce the main notations and assumptions in order to state the main results of this article.

2.1 Notations

For $x \in \mathbb{R}^d$, we denote by $|x|_1$, $|x|_2$ and $|x|_\infty$ its l_1, l_2 and l_∞ norms, respectively. For each integer $d \geq 1$, we consider the $2d-$dimensional simplex $\mathcal{P}_d := \{z \in (\mathbb{R}^+)^{2d} : \sum_{i=1}^{2d} z_i = 1\}$ and $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\} = \{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$. We define the *environmental space* $\Omega := (\mathcal{P}_d)^{\mathbb{Z}^d}$ and endow it with its canonical σ-algebra. Now, for a fixed $\omega = (\omega(y) : y \in \mathbb{Z}^d) \in \Omega$, with $\omega(y) = (\omega(y,e) : e \in U) \in \mathcal{P}_d$, and a fixed $x \in \mathbb{Z}^d$, we consider the Markov chain $\{X_n : n \geq 0\}$ with state space $\mathbb{Z}^d$ starting from x defined by the transition probabilities

$$P_{x,\omega}[X_{n+1} = X_n + e \mid X_n] = \omega(X_n, e) \qquad \text{for} \quad e \in U. \tag{2.1}$$

We denote by $P_{x,\omega}$ the law of this Markov chain and call it a random walk in the environment ω. Consider a law $\mathbb{P}$ defined on Ω. We call $P_{x,\omega}$ the *quenched law* of the random walk starting from x. Furthermore, we define the semi-direct product probability measure on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ by

$$P_x[A \times B] := \int_A P_{x,\omega}[B] d\mathbb{P}$$

for each Borel-measurable set A in Ω and B in $(\mathbb{Z}^d)^{\mathbb{N}}$, and call it the *annealed or averaged law* of the random walk in random environment. We will also define for each $x \in \mathbb{Z}^d$ the canonical space-shift $\vartheta_x : \Omega \to \Omega$ as

$$\vartheta_x \omega(y) := \omega(x + y). \tag{2.2}$$

The law $\mathbb{P}$ of the environment is said to be *i.i.d.* if the random variables $(\omega(x) : x \in \mathbb{Z}^d)$ are i.i.d. under $\mathbb{P}$ and *stationary* if for every finite subset $B \subset \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$ the joint law of $(\omega(x) : x \in B)$ is equal to the joint law of $(\vartheta_y \omega(x) : x \in B)$. We also say that it is *elliptic* if for every $x \in \mathbb{Z}^d$ and $e \in U$ one has that $\mathbb{P}[\omega(x,e) > 0] = 1$ while uniformly elliptic if there exists a $\kappa > 0$ such that $\mathbb{P}[\omega(x,e) \geq \kappa] = 1$ for every $x \in \mathbb{Z}^d$ and $e \in U$.

For each $A \subset \mathbb{Z}^d$ we define

$$\partial A := \{z \in \mathbb{Z}^d : z \notin A, \text{ there exists some } y \in A \text{ such that } |y - z|_1 = 1\}.$$

Define the stopping time

$$T_A := \inf\{n \geq 0 : X_n \notin A\}.$$

We will call the elements of the set $\mathbb{S}^{d-1}$ *directions*. For a given direction $l \in \mathbb{S}^{d-1}$ and $a \geq 0$ we also define

$$
\begin{aligned}
T_a^l &:= \inf\{n \geq 0 : X_n \cdot l \geq a\} && (2.3)\\
\bar{T}_a^l &:= \inf\{n \geq 0 : X_n \cdot l > a\} && (2.4)\\
\hat{T}_a^l &:= \inf\{n \geq 0 : X_n \cdot l < -a\} && (2.5)
\end{aligned}
$$

along with

$$\widetilde{T}_{-a}^l := \inf\{n \geq 0 : X_n \cdot l \leq -a\}. \quad (2.6)$$

The following concepts will play an important role in this article:

1. **(Directional transience)** We say that a random walk in random environment is *transient in direction* l if P_0-a.s. one has that

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

   If the random walk is transient in some direction l, we will say that the random walk is *directionally transient*.
2. **(Ballisticity)** We say that a random walk in random environment is *ballistic in direction* l if

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

   If the random walk is ballistic in some direction l, we will say that it is *ballistic*.
3. **(Asymptotic direction)** On the other hand, we say that a deterministic vector $\hat{v} \in \mathbb{S}^{d-1}$ is an *asymptotic direction* if P_0-a.s. one has that

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = \hat{v}.$$

In the case in which the environment is elliptic and i.i.d., it is known that whenever a random walk is ballistic necessarily a law of large numbers is satisfied and in fact $\lim_{n\to\infty} \frac{X_n}{n} = v \neq 0$ is deterministic [7]. Furthermore, in the uniformly elliptic i.i.d. case, it is still an open question to establish whether or not in dimensions $d \geq 2$, every directionally transient random walk is ballistic (see [1]). As already mentioned, for elliptic i.i.d. environments, Simenhaus established [13] the existence of an asymptotic direction whenever the random walk is transient in direction l for every l in an open subset of $\mathbb{S}^{d-1}$.

Throughout the rest of this article, most constants will be denoted by $c_1, c_2, \ldots$ and will be ordered according to their appearance.

2.2 Main assumptions

Here we discuss the three main assumptions throughout this article: uniform ellipticity in a given direction, cone-mixing and the uniform non-effective polynomial ballisticity condition.

Condition (UE). Let $\kappa > 0$. We say that $\mathbb{P}$ is *uniformly elliptic with respect to* l, denoted by $(UE)|l$, if it is elliptic, and if the jump probabilities of the random walk are larger than 2κ in those directions which for which the projection of l is not zero. In other words if $\mathbb{P}[\omega(0,e) > 0] = 1$ for $e \in U$ and if

$$\mathbb{P}\left[\min_{e\in\mathcal{E}} \omega(0,e) \geq 2\kappa\right] = 1,$$

where

$$\mathcal{E} := \{sgn(l_i)e_i : i = 1, \ldots, d\}\backslash\{0\}, \quad (2.7)$$

where $(l_i : i = 1, \ldots, d)$ are the coordinates of l and by convention $sgn(0) = 0$.

We will now introduce the cone-mixing condition for the environment $\mathbb{P}$, as in [5]. We fix a direction $l \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$ and a number $\alpha > 0$. Let R be a rotation such that

$$R(e_1) = l. \tag{2.8}$$

To define the cone, it will be useful to consider for each $i \in \{2, \ldots, d\}$,

$$\bar{l}_{+i} := \frac{l + \alpha R(e_i)}{|l + \alpha R(e_i)|_2} \qquad \text{and} \qquad \bar{l}_{-i} := \frac{l - \alpha R(e_i)}{|l - \alpha R(e_i)|_2}. \tag{2.9}$$

The cone $C(x, l, \alpha)$ (with vertex x) is defined as

$$C(x, l, \alpha) := \bigcap_{i=2}^{d} \left\{ z \in \mathbb{R}^d : (z - x) \cdot \bar{l}_{+i} \geq 0, \ (z - x) \cdot \bar{l}_{-i} \geq 0 \right\}. \tag{2.10}$$

Let us now define Φ as the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that $\lim_{r\to\infty} \phi(r) = 0$.

Condition (CM). We say that a stationary probability measure $\mathbb{P}$ satisfies the *cone-mixing condition* with respect to $\alpha > 0$, $l \in \mathbb{S}^{d-1}$ and $\phi \in \Phi$, denoted $(CM)_{\alpha,\phi}|l$, if for every $r > 0$ and pair of events A, B, where $\mathbb{P}[A] > 0$, $A \in \sigma\{\omega(z) : z \cdot l \leq 0\}$, and $B \in \sigma\{\omega(z) : z \in C(rl, l, \alpha)\}$, it holds that

$$\left| \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} - \mathbb{P}[B] \right| \leq \phi\left(r \frac{|l|_1}{|l|_2} \right).$$

We will see that every cone-mixing measure $\mathbb{P}$ is necessarily ergodic. On the other hand, a cone-mixing environment can be such that the jump probabilities are highly dependent along certain directions.

We now introduce an assumption which is closely related to the effective polynomial ballisticity condition introduced in [1]. Given $L, L' > 0$, $x \in \mathbb{Z}^d$ and $l \in \mathbb{S}^{d-1}$ we define the boxes

$$B_{L,L',l}(x) := x + R\left((-L, L) \times (-L', L')^{d-1} \right) \cap \mathbb{Z}^d, \tag{2.11}$$

where R is defined in (2.8). The *positive boundary* of $B_{L,L',l}(x)$, denoted by $\partial^+ B_{L,L',l}(x)$, is

$$\partial^+ B_{L,L',l}(x) := \partial B_{L,L',l}(x) \cap \{z : (z - x) \cdot l \geq L\}, \tag{2.12}$$

Define also the half-space

$$H_{x,l} := \{y \in \mathbb{Z}^d : y \cdot l < x \cdot l\}$$

and the corresponding σ-algebra of the environment on that half-space

$$\mathcal{H}_{x,l} := \sigma\{\omega(y) : y \in H_{x,l}\}.$$

Condition (UWP). For $M \geq 1$ and $\mathfrak{c} > 0$, we say that the *uniform non-effective polynomial condition* $(UWP)_{M,\mathfrak{c}}|l$ is satisfied if for all $y \in H_{0,l}$ one has that

$$\lim_{L\to\infty} L^M \operatorname{ess\,sup} P_0 \left[X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0), T_{B_{L,\mathfrak{c}L,l}(0)} < T^l_{y\cdot l} | \mathcal{H}_{y,l} \right] = 0, \tag{2.13}$$

where the essential supremum is taken over all the coordinates $(\omega(x) : x \cdot l \leq y \cdot l)$ under the measure $\mathbb{P}$.

It is possible to show that for i.i.d. environments, this condition is implied by Sznitman's (T') condition [19], and it is equivalent to the polynomial condition introduced in [1] (which is an effective version of the polynomial condition introduced above).

2.3 Main result and overview

Define $\mathbb{S}_q^{d-1} := \left\{ \frac{l}{|l|_2} : l \in \mathbb{Z}^d \backslash \{0\} \right\}$. We can now state our main result.

Theorem 2.1. *Let $l \in \mathbb{S}_q^{d-1}$, $M > 6d$, $\mathfrak{c} > 0$, $0 < \alpha \leq \min\{\frac{1}{9}, \frac{1}{3\mathfrak{c}}\}$ and $\phi \in \Phi$. If a random walk in a random environment with stationary law satisfies the uniform ellipticity condition $(UE)|l$, the cone-mixing condition $(CM)_{\alpha,\phi}|l$ and the uniform non-effective polynomial condition $(UWP)_{M,\mathfrak{c}}|l$ (the three conditions defined in Subsection 2.2), then P_0-a.s. there exists a deterministic asymptotic direction. In other words, there exists a $\hat{v} \in \mathbb{S}^{d-1}$ such that P_0-a.s. one has that*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = \hat{v}.$$

In a way, Theorem 2.1 shows that if the i.i.d. assumption of Simenhaus [13] is weakened to cone-mixing, while the directional transience condition of [13] is strengthened to the uniform non-effective polynomial condition, we still can guarantee the existence of an asymptotic direction.

A key step to prove Theorem 1.1 will be to establish that the probability that the random walk never exits a cone is positive through the use of renormalization type ideas, and only assuming the uniform non-effective polynomial condition and uniform ellipticity. Using this fact, we will define approximate regeneration times as in [5], showing that they have finite moments of order larger than one when we also assume cone-mixing. This part of the proof will require careful and tedious computations. Once this is done, the existence of an asymptotic direction can be deduced using for example the coupling approach of [5].

3 Examples of directionally transient random walks without an asymptotic direction or with a vanishing velocity

We will present two examples of random walks in random environment which exhibit the framework of validity of the hypothesis of Theorem 2.1. In both examples the environment is not i.i.d. (see [2], for an example showing that the $0-1$ law conjecture for i.i.d. environments is not satisfied if this assumption is dropped).

The first example indicates that the hypothesis of Theorem 2.1 might not necessarily imply a strong law of large numbers with a non-vanishing velocity. The second example will show that one cannot prove the existence of an asymptotic direction without either some kind of mixing hypothesis on the environment or some ballisticity condition.

Throughout, p will be a non-deterministic random variable taking values in $(0,1)$ such that there exists a unique $\varkappa \in (1/2, 1)$ with the property that

$$E[\rho^{\varkappa}] = 1 \quad \text{and} \quad E[\rho^{\varkappa} \ln^+ \rho] < \infty, \tag{3.1}$$

where $\rho := (1-p)/p$.

3.1 Random walk with a vanishing velocity but with an asymptotic direction

Let $\{p_i : i \in \mathbb{Z}\}$ be i.i.d. copies of p. Define an i.i.d. sequence of random variables $\{\omega_i : i \in \mathbb{Z}\}$ with $\omega_i = \{\omega_i(e_1), \omega_i(-e_1), \omega_i(e_2), \omega_i(-e_2)\}$, by

$$\omega_i(e_2) = \omega_i(-e_2) = \frac{1}{4},$$

$$\omega_i(e_1) = \frac{p_i}{2} \quad \text{and} \quad \omega_i(-e_1) = \frac{1-p_i}{2}.$$

Now consider the random environment $\omega = \{\omega((i,j)) : (i,j) \in \mathbb{Z}^2\}$ defined by

$$\omega((i,j)) := \omega_i \quad \text{for all} \quad i, j \in \mathbb{Z}.$$

We will call $\mathbb{P}_1$ the law of the above environment and Q_1 the annealed law of the corresponding random walk starting from $(0,0)$.

Theorem 3.1. *Consider a random walk in a random environment with law $\mathbb{P}_1$ on $\mathbb{Z}^2$. Then, the following are satisfied:*

(i) Q_1*-a.s.*

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty.$$

(ii) Q_1*-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

(iii) *In Q_1-probability*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = e_1.$$

(iv) *The law Q_1 satisfies the uniform polynomial condition $(UWP)_{M,\mathfrak{c}}$ with $M = \varkappa - \frac{1}{2-\varepsilon}$ and $\mathfrak{c} = 1$, where ε is an arbitrary number in the interval $(0, 2-\frac{1}{\varkappa})$.*

Proof. Part (i). Note that

$$P_{0,\omega}[Y_{n+1} = Y_n + e \mid Y_n] = \tilde{\omega}(Y_n, e),$$

where $e = e_1, -e_1$ or $e = (0,0)$, and for $x \in \mathbb{Z}$ we define

$$\tilde{\omega}(x,e) := \begin{cases} \frac{p_x}{2} & \text{if } e = e_1 \\ \frac{1-p_x}{2} & \text{if } e = -e_1 \\ \frac{1}{2} & \text{if } e = 0. \end{cases}$$

By (3.1) and the fact that p is non-deterministic, it follows that $\tilde{E}_1[\ln[\tilde{\rho}_0]] < 0$, where $\tilde{\rho}_0 := \tilde{\omega}(0,-e_1)/\tilde{\omega}(0,e_1)$ and $\tilde{E}_1$ denotes the corresponding expectation. Now, from the transience condition in [23] Theorem 2.1.2 (which is a generalization of Solomon's [15] result for random walks with holding times) one has that Q_1-a.s.

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty.$$

Part (ii). Note that

$$\frac{X_n}{n} = \frac{Y_n e_1 + (X_n \cdot e_2)e_2}{n},$$

where $\{Y_n : n \geq 0\}$ is the projection of the random walk in the direction e_1 defined in part (i). Now, using the strong law of large numbers for this projection ([23], Theorem 2.1.9), and the fact that $(X_n \cdot e_2)$ is a random walk which moves with the same probability in both directions, we conclude that Q_1-a.s.

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

Part (iii). For each $n \geq 0$, we define the random variables $N_1(n)$ and $N_2(n)$ as horizontal and vertical steps performed by the walk $\{X_m : m \geq 0\}$ up to time n, respectively. Under the quenched law, both of them are Binomial-distributed with parameters n and $1/2$. In what follows, whenever there is no risk of confusion, we will remove in the writing the dependence on n and write N_1 and N_2 in place of $N_1(n)$ and $N_2(n)$. For each $\varepsilon > 0$, we have to estimate the probability

$$Q_1\left[\left|\frac{X_n}{|X_n|_2} - e_1\right|_2 > \varepsilon\right] = Q_1\left[\left|\frac{\frac{(X_n\cdot e_1)}{n^{\varkappa}}e_1 + \frac{(X_n\cdot e_2)}{n^{\varkappa}}e_2}{\sqrt{\frac{(X_n\cdot e_1)^2}{n^{2\varkappa}} + \frac{(X_n\cdot e_2)^2}{n^{2\varkappa}}}} - e_1\right|_2 > \varepsilon\right]. \tag{3.2}$$

Clearly, $X_n \cdot e_2$ under the annealed law has the same law of a one-dimensional simple symmetric random walk $\{Z_m : m \geq 0\}$ at time $m = N_2(n)$, whose law we call $\tilde{P}$. Note that $\tilde{P}$-a.s. $N_2/n \to 1/2$ as $n \to \infty$. Therefore, since $\varkappa > 1/2$ we see that

$$Q_1\left[\lim_{n\to\infty}\frac{X_n\cdot e_2}{n^{\varkappa}}=0\right]=\tilde P\left[\lim_{n\to\infty}\frac{Z_{N_2}}{N_2^{\varkappa}}\frac{1}{2^{\varkappa}}=0\right]=1.$$

On the other hand, by Theorem 1.1 of [8] (see also [10]), we see that

$$\lim_{n\to\infty}\frac{X_n\cdot e_1}{\sqrt{(X_n\cdot e_1)^2}}=1$$

in distribution, and hence also in Q_1- probability. It follows that for each $\varepsilon>0$, the left hand-side of (3.2) tends to 0 as $n\to\infty$.

Part (iv). Notice that

$$Q_1[X_{T_{B_{L,L,l}(0)}}\notin\partial^+B_{L,L,l}(0)]\le Q_1[\widetilde T^{e_1}_{-L}<T^{e_1}_L]+Q_1[T^{e_2}_L\wedge\widetilde T^{e_2}_{-L}<T^{e_1}_L]. \tag{3.3}$$

The first probability in the right-most side of (3.3) has an exponential bound in L. Observe that the second probability in the right-hand side of (3.3) is less than or equal to

$$Q_1[T^{e_2}_L\wedge\widetilde T^{e_2}_{-L}\le L^{2-\varepsilon}]+Q_1[L^{2-\varepsilon}<T^{e_1}_L]. \tag{3.4}$$

Now, for the first term in the above decomposition we have that

$$Q_1[T^{e_2}_L\wedge\widetilde T^{e_2}_{-L}\le L^{2-\varepsilon}]\le Q_1[T^{e_2}_L\le L^{2-\varepsilon}]+Q_1[\widetilde T^{e_2}_{-L}\le L^{2-\varepsilon}].$$

Now

$$Q_1[T^{e_2}_L\le L^{2-\varepsilon}]\le\sum_{n=L}^{[L^{2-\varepsilon}]}\widehat P[S_n\ge L], \tag{3.5}$$

where we denote by $\widehat P$ the law of the one-dimensional random walk $\{S_n:n\ge0\}$ starting from 0 which at each step jumps to the right with probability $1/4$, the the left with probability $1/4$ and does not move with probability $1/2$. Thus $\{S_n:n\ge0\}$ is a martingale with respect to its canonical filtration with increments bounded by 1. Therefore, using Azuma-Hoeffding inequality (see for example [22, (E14.2)]) for each term in (3.5) we get that

$$Q_1[T^{e_2}_L\le L^{2-\varepsilon}]\le\frac{1}{c_1}\exp\{-c_1L^{\varepsilon}\},$$

for some constant $c_1>0$ (which does not depend on L). An analogue bound holds for $Q_1[\widetilde T^{e_2}_{-L}\le L^{2-\varepsilon}]$. We end up concluding that there is a constant c_1' such that for all $L\ge1$

$$Q_1[T^{e_2}_L\wedge\widetilde T^{e_2}_{-L}\le L^{2-\varepsilon}]\le\frac{1}{c_1'}\exp\{-c_1'L^{\varepsilon}\}. \tag{3.6}$$

For the second term in the right-hand side of (3.4), we use [8, Theorem 1.3]. To this end, we denote by $\hat P$ the law of underlying one-dimensional random walk corresponding to the annealed law of $\{X_n\cdot e_1:n\ge0\}$. One has that there exists a positive constant c_1'' such that

$$Q_1[L^{2-\varepsilon}<T^{e_1}_L]\le Q_1[X_{[L^{2-\varepsilon}]}\cdot e_1<L]\le\hat P[Y_{[L^{2-\varepsilon}]}<L]\le c_1''L^{-(\varkappa-1/(2-\varepsilon))}. \tag{3.7}$$

Observe now that in view of inequality (3.3), the estimates (3.6) and the right-most bound of (3.7), the proof is complete. □

3.2 Directionally transient random walk without an asymptotic direction

Let $\{p_i:i\in\mathbb{Z}\}$ and $\{p_j':j\in\mathbb{Z}\}$ be two independent i.i.d. copies of p. Following a similar procedure as in the previous example, we consider in the lattice $\mathbb{Z}^2$ the canonical vectors e_1 and e_2, and define the random environment $\omega=\{\omega((i,j)):(i,j)\in\mathbb{Z}^2\}$ by

$$\omega_{(i,j)}(e_1) = \frac{p_i}{2} \quad \text{and} \quad \omega_{(i,j)}(-e_1) = \frac{1}{2} - \frac{p_i}{2}$$

together with

$$\omega_{(i,j)}(e_2) = \frac{p'_j}{2} \quad \text{and} \quad \omega_{(i,j)}(-e_2) = \frac{1}{2} - \frac{p'_j}{2}.$$

We call $\mathbb{P}_2$ the law of the above environment and Q_2 the annealed law of the corresponding random walk starting from $(0,0)$.

The following theorem shows that in dimension $d = 2$, there exist directionally transient random walks, with vanishing velocity, having a random asymptotic direction in the distributional sense and satisfying condition (T). It should be pointed out that the example could be easily generalized to dimensions $d \geq 2$, but for the sake of being concise and clear we have written it only for $d = 2$. On the other hand, it should also be pointed out that the environment in this example is ergodic, but it is not cone-mixing.

Theorem 3.2. *Consider a random walk in a random environment with law $\mathbb{P}_2$ on $\mathbb{Z}^2$. Then, the following are satisfied.*

(i) *Let $l \in \mathbb{S}^{d-1}$. Then $l \cdot e_1 \geq 0$ and $l \cdot e_2 \geq 0$ if and only if Q_2-a.s.*

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

(ii) Q_2*-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

(iii) *There exists a non-deterministic $\hat{v}$ such that*

$$\frac{X_n}{|X_n|_2} \to \hat{v}.$$

in distribution.

(iv) *We have that*

$$\overline{\lim}_{L\to\infty} L^{-1} \log Q_2[X_{T_{B_{L,2L,l}(0)}} \notin \partial^+ B_{L,2L,l}(0)] < 0, \tag{3.8}$$

where $l = (1/\sqrt{2}, 1/\sqrt{2})$. Thus, condition $(T)|l$ of [18] is satisfied.

Proof. Part (i). It is enough to prove that Q_2-a.s.

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty \text{ and } \lim_{n\to\infty} X_n \cdot e_2 = \infty.$$

Both assertions follow from an argument similar to the one used in part (i) of Theorem 3.1, [23, Theorem 2.1.2] and (3.1).

Part (ii). This proof is similar to part (ii) of Theorem 3.1 .

Part (iii). For $j = 1, 2$ we define $T_{0,j} := 0$,

$$T_{1,j} := \inf\{n \geq 0 : \ (X_n - X_0) \cdot e_j > 0 \text{ or } (X_n - X_0) \cdot e_j < 0\}$$

and recursively for $i \geq 2$

$$T_{i,j} := T_{1,j} \circ \theta_{T_{i-1,j}} + T_{i-1,j},$$

where for each $n \geq 0$, θ_n denotes the canonical time-shift on $(Z^d)^{\mathbb{N}}$. Define for each $n \geq 0$, $Y_{n,j} := X_{T_{n,j}} \cdot e_j$. Now note that $\{Y_{n,1} : n \geq 0\}$ and $\{Y_{n,2} : n \geq 0\}$ are independent with their transition probabilities at each site $(i,j) \in \mathbb{Z}^d$ determined by the random variables $\{p_i : i \in \mathbb{Z}\}$ and $\{p_j : j \in \mathbb{Z}\}$, respectively. Furthermore, for $j \in \{1, 2\}$, the strong law of large numbers implies that Q_2-a.s.

$$\lim_{n\to\infty} \frac{T_{n,j}}{n} = 2. \tag{3.9}$$

Now by [8, Theorem 1.1] (see also [10]) we know that there exist constants K_3 and K_4 such that

$$\left(\frac{Y_{n,1}}{n^{\varkappa}}, \frac{Y_{n,2}}{n^{\varkappa}}\right) \to \left(\left(\frac{K_3}{S_1}\right)^{\varkappa}, \left(\frac{K_4}{S_2}\right)^{\varkappa}\right)$$

in distribution, where S_1 and S_2 stand for two independent completely asymmetric stable laws of index $\varkappa$, which are positive. Using (3.9) we can see that

$$\frac{X_n}{|X_n|_2} = \frac{\frac{(X_n \cdot e_1)}{n^{\varkappa}} e_1 + \frac{(X_n \cdot e_2)}{n^{\varkappa}} e_2}{\sqrt{\frac{(X_n \cdot e_1)^2}{n^{2\varkappa}} + \frac{(X_n \cdot e_2)^2}{n^{2\varkappa}}}} \to \frac{\left(\frac{K_3}{S_1}\right)^{\varkappa} e_1 + \left(\frac{K_4}{S_2}\right)^{\varkappa} e_2}{\sqrt{\left(\frac{K_3}{S_1}\right)^{2\varkappa} + \left(\frac{K_4}{S_2}\right)^{2\varkappa}}}$$

in distribution, which shows that the limit $\hat{v}$ is random.

Part (iv). We will first prove that

$$\overline{\lim_{L\to\infty}} L^{-1} \log Q_2[\widetilde{T}^{e_j}_{-\tilde{c}L} < T^{e_j}_{cL}] < 0$$

for arbitrary positive cnumber $\mathfrak{c}_1$ and $\mathfrak{c}_2$ [cf. (2.3) and (2.6)]. We will prove this only in the case $j = 1$ since the case $j = 2$ is similar. Following the notation introduced in part (i) of Theorem 3.1 and denoting the greatest integer function by $[\cdot]$, we see that it is sufficient to prove that there exists a positive constant K_5 such that

$$Q_2[Y_n \text{ hits } -[\mathfrak{c}_1 L] + 1 \text{ before } [\mathfrak{c}_2 L] + 1] \leq \frac{1}{\hat{C}} \exp\{-K_5 L\}. \tag{3.10}$$

To this end, for a fixed random environment ω, define

$$\mathfrak{V}^L := P_{i,\omega}[Y_n \text{ hits } -[\mathfrak{c}_1 L] + 1 \text{ before } [\mathfrak{c}_2 L] + 1].$$

It is a standard fact that (see for example [4, Section 12 of Chapter 1])

$$\mathfrak{V}^L = \frac{\exp\{\sum_{-[\mathfrak{c}_1 L+1],0}\} + \ldots + \exp\{\sum_{-[\mathfrak{c}_1 L]+1,[\mathfrak{c}_2 L]}\}}{1 + \exp\{\sum_{-[\mathfrak{c}_1 L]+1,-[\mathfrak{c}_1 L]+2}\} + \ldots + \exp\{\sum_{-[\mathfrak{c}_1 L]+1,[\mathfrak{c}_2 L]}\}},$$

where we have adopted the notation $\sum_{z<m\leq z'} := \sum_{z<m\leq z'} \log \rho(m)$ and $\rho(m) := (1 - p_m)/p_m$. A slight variation of the argument in page 744 of [18] completes the proof of claim (3.10). Now note that (see Figure 1)

$$Q_2[X_{T_{B_{L,2L,l}(0)}} \notin \partial^+ B_{L,2L,l}(0)] \leq Q_2[\widetilde{T}^{e_1}_{-\frac{\sqrt{2}}{2}L} < T^{e_1}_{\frac{3}{\sqrt{2}}L}] + Q_2[\widetilde{T}^{e_2}_{-\frac{\sqrt{2}}{2}L} < T^{e_2}_{\frac{3}{\sqrt{2}}L}].$$

In virtue of the claim (3.10) the last expression has an exponential bound and this finishes the proof. □

4 Preliminary discussion

In this section we will derive some important relations that are satisfied between the uniform non-effective polynomial condition and other ballisticity conditions, including Kalikow condition. In Subsection 4.1 we will show that the uniform non-effective polynomial condition is weaker than the conditional form of Kalikow condition introduced in [6]. In Subsection 4.2, we will prove that the uniform non-effective polynomial condition in a given direction l implies the uniform non-effective polynomial condition in a open subset of $\mathbb{S}^{d-1}$ containing l, with a lower degree.

4.1 Uniform non-effective polynomial condition and its relation with other directional transience conditions

Here we will discuss the relationship between the uniform non-effective polynomial condition and other transience conditions. Furthermore we will show that the uniform

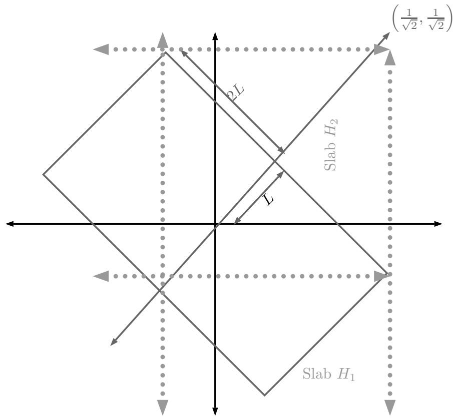

Figure 1: A geometric sketch of the bound for $Q_2[X_{T_{B_{L,2L,l}(0)}} \notin \partial^+ B_{L,2L,l}(0)]$.

non-effective polynomial condition is weaker than the conditional version of Kalikow condition introduced by Comets-Zeitouni in [5].

For reasons that will become clear in the next section, the following definition, which is actually weaker than the uniform non-effective polynomial condition introduce in Subsection 2.2 and to the polynomial condition of [1], will be useful.

Condition (WP). Let $l \in \mathbb{S}^{d-1}$, $M \geq 1$ and $\mathfrak{c} > 0$. We say that the *non-effective polynomial condition* or the *weak polynomial condition* $(WP)_{M,\mathfrak{c}}|l$ is satisfied if

$$\overline{\lim}_{L\to\infty} L^M P_0[X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0)] = 0.$$

It is straightforward to see that $(UWP)_{M,\mathfrak{c}}|l$ implies $(WP)_{M,\mathfrak{c}}|l$. Also, it should be pointed out, that for a fixed $\gamma \in (0,1)$, if both in the uniform and non-uniform non-effective polynomial conditions the polynomial decay is replaced by a stronger stretched exponential decay of the form e^{-L^γ}, one would obtain a condition defined on rectangles equivalent to condition $(T)_\gamma$ introduced by Sznitman in [19], and also a conditional version of it. On the other hand, as we will see now, the uniform non-effective polynomial condition is implied by Kalikow condition as defined in [5] for environments which are not necessarily i.i.d. Let us recall this definition. For V a finite, connected subset of $\mathbb{Z}^d$, with $0 \in V$, we let

$$\mathfrak{F}_{V^c} = \sigma\{\omega(z,\cdot) : z \notin V\}.$$

The *Kalikow random walk* is the Markov chain $\{X_n : n \geq 0\}$ with state space in $V \cup \partial V$ defined by the transition probabilities

$$\widehat{P}_V(x, x+e) := \begin{cases} \frac{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n=x\}}\omega(x,e)|\mathfrak{F}_{V^c}]}{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n=x\}}|\mathfrak{F}_{V^c}]} & \text{for} \quad x \in V \text{ and } e \in U, \\ 1 & \text{for} \quad x \in \partial V \text{ and } e = 0. \end{cases}$$

We denote by $\hat{P}_{y,V}$ the law of this random walk starting from $y \in V \cup \partial V$, and by $\hat{E}_{y,V}$ the corresponding expectation. The importance of Kalikow random walk stems from the fact that

$$X_{T_V} \text{ has the same law under } \widehat{P}_{0,V} \text{ and under } P_0[\cdot|\mathfrak{F}_{V^c}] \tag{4.1}$$

(see ([9])). Let $l \in \mathbb{S}^{d-1}$. We say that Kalikow condition with respect to the direction l is satisfied if there exists a positive constant δ such that $\mathbb{P}$-a.s.

$$\inf_{V,x\in V} \widehat{d}_V(x) \cdot l \geq \delta,$$

where

$$\widehat{d}_V(x) := \widehat{E}_{x,V}[X_1 - X_0] = \sum_{e\in U} e\widehat{P}_V(x, x+e) \tag{4.2}$$

denotes the drift of Kalikow random walk at x, and the infimum runs over all finite connected subsets V of $\mathbb{Z}^d$ such that $0 \in V$. The following result shows that Kalikow condition is indeed stronger than the uniform non-effective polynomial criterion.

Proposition 4.1. *Let $l \in \mathbb{S}^{d-1}$. Assume Kalikow condition with respect to l. Then there exists an $\mathfrak{c} > 0$ such that for all $y \in H_{0,l}$ one has that*

$$\overline{\lim}_{L\to\infty} L^{-1} \operatorname{ess\,sup} \log P_0[X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0), T_{B_{L,\mathfrak{c}L,l}(0)} < T^l_{y\cdot l} | \mathcal{H}_{y,l}] < 0, \tag{4.3}$$

where the supremum is taken in the same sense as in (2.13). In particular, Kalikow condition with respect to direction l implies $(UWP)_{M,\mathfrak{c}}|l$ for all $M > 0$.

Proof. Suppose that Kalikow condition is satisfied with a constant $\delta > 0$. We will first assume that $y \cdot l \in (-L, 0)$. Let $\mathfrak{r} > 1$. For $y \in H_{0,l}$ and $L \geq 1$ consider the box

$$V := R\left([y\cdot l, L] \times \left(-\frac{\mathfrak{r}}{\delta}L, \frac{\mathfrak{r}}{\delta}L\right)^{d-1}\right) \cap \mathbb{Z}^d.$$

Using (4.1) we find that

$$\begin{aligned}
&P_0[X_{T_{B_{L,\frac{\mathfrak{r}}{\delta}L,l}(0)}} \notin \partial^+ B_{L,\frac{\mathfrak{r}}{\delta}L,l}(0), T_{B_{L,\frac{\mathfrak{r}}{\delta}L,l}(0)} < T^l_{y\cdot l} | \mathfrak{F}_{V^c}] \\
&\leq P_0\left[\max_{j:2\leq j\leq d} X_{T_V}\cdot R(e_j) \geq \frac{\mathfrak{r}}{\delta}L, |X_{T_V}\cdot l| < L \middle| \mathfrak{F}_{V^c}\right] \\
&= \widehat{P}_{0,V}\left[\max_{j:2\leq j\leq d} X_{T_V}\cdot R(e_j) \geq \frac{\mathfrak{c}}{\delta}L, |X_{T_V}\cdot l| < L\right].
\end{aligned} \tag{4.4}$$

Notice that on the set

$$\left\{X_{T_V}\cdot R(e_j) \geq \frac{\mathfrak{r}}{\delta}L \text{ for some } j\right\} \cap \{|X_{T_V}\cdot l| < L\},$$

one has $\widehat{P}_{0,V}$-a.s. that

$$T_V \geq \left[\frac{\mathfrak{r}L}{\delta}\right].$$

Thus, by means of the martingale $\{M_n^V : n \geq 0\}$ defined by

$$M_n^V := X_n - X_0 - \sum_{j=0}^{n-1} \hat{d}_V(X_j),$$

(where $\hat{d}_V(x)$ is defined in (4.2)) which has bounded increments (indeed bounded by 2) we can see that on $\{T_V \geq \left[\frac{\mathfrak{c}L}{\delta}\right]\}$, by Kalikow condition, we have that for L large enough that $\widehat{P}_{0,V}$-a.s.

$$M^V_{[\frac{\mathfrak{r}L}{\delta}]}\cdot l < L - \left(\frac{\mathfrak{r}L}{\delta} - 1\right)\delta < \frac{(1-\mathfrak{r})L}{2}. \tag{4.5}$$

Now, using Azuma-Hoeffding inequality [22, (E14.2)] and (4.5) we obtain that

$$\begin{aligned}
&\widehat{P}_{0,V}[X_{T_V}\cdot R(e_j) > \frac{\mathfrak{r}}{\delta}L \text{ for some } j, X_{T_V}\cdot l \leq L] \leq \widehat{P}_{0,V}\left[T_V > \frac{\mathfrak{r}L}{\delta}\right] \\
\leq &\widehat{P}_{0,V}\left[M^V_{[\frac{\mathfrak{r}L}{\delta}]}\cdot(-l) > (\mathfrak{r}-1)L/2\right] \leq \exp\{-c_2 L\},
\end{aligned} \tag{4.6}$$

for a suitable $c_2 > 0$. Finally, coming back to (4.4), we can then conclude that

$$\overline{\lim}_{L\to\infty} L^{-1} \operatorname{ess\,sup} \log P_0[X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0), T_{B_{L,\mathfrak{c}L,l}(0)} < T^l_{y\cdot l} | \mathcal{H}_{y,l}] < 0,$$

where $\mathfrak{c} = \frac{\mathfrak{r}}{\delta}$. Let us now assume that $y \cdot l \le -L$. By Lemma 1.1 in [17] we know that there exists a positive constant ψ depending on δ such that for every finite connected subset V of $\mathbb{Z}^d$ with $0 \in V$

$$e^{-\psi X_n \cdot l}$$

is a supermartingale with respect to the canonical filtration of the walk under Kalikow law $\widehat{P}_{0,V}$. Thus, we have that

$$\widehat{P}_{0,V}[X_{T_V} \cdot l \le -L] \le \exp\{-\psi L\}$$

by means of the stopping time theorem applied at time T_V. By an argument similar to the one developed for the case $y \cdot l \in (-L, 0)$, we can finish the estimate in the case $y \cdot l \le -L$. □

4.2 Polynomial decay implies polynomial decay in an open set of directions

In this subsection we prove that whenever $(WP)_{M,\mathfrak{c}}|l$ holds, for prescribed positive numbers M and $\mathfrak{c}$, then there is a set an open subset O of $\mathbb{S}^{d-1}$ containing l such that for every $l' \in O$ the polynomial condition $(WP)_{M',\mathfrak{c}'}|l'$ is satisfied for some M' and $\mathfrak{c}'$. More precisely, we will prove the following.

Proposition 4.2. *Let $\mathfrak{c} > 0$ and $M > 6(d-1)$. Assume that condition $(WP)_{M,\mathfrak{c}}|l$ is satisfied. Then there exists an open subset O of $\mathbb{S}^{d-1}$ containing l such that for all $l' \in O$ we have that $(WP)_{N,2\mathfrak{c}}|l'$ is satisfied with $N = \frac{M}{3} - 1$.*

Proof of Proposition 4.2. Note that it will be enough to prove the result for $l' \in \{l_{\pm i} : i \in \{2, \dots, d\}\}$ [cf. (2.9)]. We will just give the proof for direction l_{-2}, the other cases being analogous. Throughout the proof we pick $\alpha \in (0,1)$ and we define the angle

$$\beta := \arctan(\alpha). \tag{4.7}$$

Consider the rotation A on $\mathbb{R}^d$, whose plane of rotation is the one generated by e_1 and e_2, and whose axis of rotation is the line perpendicular to this plane passing through the origin. Define

$$A' := AR.$$

Consider now the box

$$C_L := A'\left([-\lambda_1 L, \lambda_2 L] \times [-\lambda_3 L, \lambda_3 L]^{d-1}\right) \cap \mathbb{Z}^d$$

where

$$\lambda_1 := 2\cos\beta - |\cos\beta - c\sin\beta|,$$
$$\lambda_2 := |\cos\beta - c\sin\beta|$$

and

$$\lambda_3 := \sin\beta + c\cos\beta.$$

The dimensions of the box C_L are chosen exactly so that as shown in Figure 2, we have that

$$X_{T_{C_L}} \notin \partial^+ C_L \Rightarrow X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0), \tag{4.8}$$

where the positive boundary $\partial^+ C_L$ is defined in (2.12). Now, since by assumption the weak polynomial condition $(WP)_{M,\mathfrak{c}}|l$ is satisfied, we know that there is a constant c_3 such that for all $L > 0$ one has that

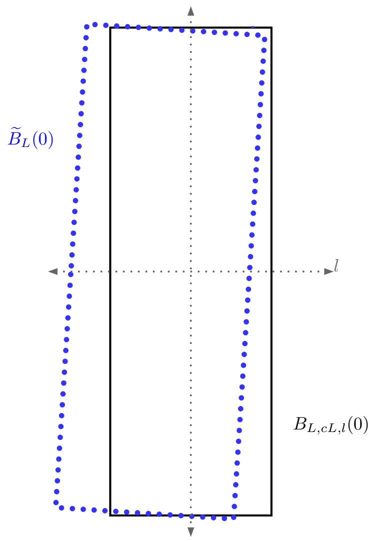

Figure 2: The choice of boxes.

$$P_0[X_{T_{B_{L,\mathfrak{c}L,l}(0)}} \notin \partial^+ B_{L,\mathfrak{c}L,l}(0)] \leq c_3 L^{-M}. \tag{4.9}$$

Now, (4.8) and (4.9) imply that for all $L > 0$ it is true that

$$P_0[X_{T_{C_L}} \notin \partial^+ C_L] \leq c_3 L^{-M}. \tag{4.10}$$

Also, note that although the origin is equidistant from oposite sides of the box C_L which are not perpendicular to l, it is not its center. Therefore, inequality (4.10) does not a priori imply the polynomial condition $(WP)_{M,\mathfrak{c}}|l$. To obtain it, we will need to consider the following box,

$$C'_L := A'\left([-\lambda_1 L, \lambda_1 L] \times [-m\lambda_3 L, m\lambda_3 L]^{d-1}\right) \cap \mathbb{Z}^d,$$

where

$$m := \left\lceil \frac{\lambda_1}{\lambda_2} \right\rceil.$$

Note that now the origin is the center of C'_L. Furthermore, m is the proportion between the distance to the right and to the left of the origin in the box C_L. We will derive now a polynomial decay for the probability to exit the box C'_L through sides different from $\partial_+ C'_L$, starting from 0. The general strategy to follow will be to stack translations of the box C_L up inside of C'_L, and then the Markov property along for the quenched law of the random walk, ensuring that the walk exits from the successive translations of the smaller boxes through the front sides. Introduce recursively the sequence of stopping times

$$T_1 = T_{C_L},$$

and for $i > 1$

$$T_i = T_{i-1} + T_1 \circ \theta_{T_{i-1}}.$$

Now, note that to ensure that the random walk exits at time $T_{C'_L}$ through $\partial^+ C'_L$, it is enough that it exits through the corresponding positive boundaries of translations of the box C_L, m successive times. To be precise, defining for $x \in \mathbb{Z}^d$ the box $C_L(x) := C_L + x$, we have that

$$P_0[X_{T_{C'_L}} \in \partial^+ C'_L] \geq P_0\left[X_{T_1} \in \partial^+ C_L(0), \left(X_{T_1} \in \partial^+ C_L(X_{T_1})\right) \circ \theta_{T_1}, \left(X_{T_1} \in \partial^+ C_L(X_{T_2})\right) \circ \theta_{T_2}, \ldots, \left(X_{T_1} \in \partial^+ C_L(X_{T_{m-1}})\right) \circ \theta_{T_{m-1}}\right]. \tag{4.11}$$

To estimate the right-hand side of (4.11), we define the set

$$F_1 := \partial^+ C_L(0),$$

and for $i > 1$ recursively the sets

$$F_i := \bigcup_{y \in F_1} \partial^+ C_L(y).$$

Define also $G_0 := \Omega$ and for $i \geq 1$ the events

$$G_i := \left\{\omega \in \Omega : P_{y,\omega}\left[X_{T_1} \in \partial^+ C_L(y)\right] \geq 1 - L^{-\frac{M}{2}}, \text{ for each } y \in F_i\right\}.$$

Note that (4.11) implies that

$$P_0[X_{T_{C'_L}} \in \partial^+ C'_L] \geq P_0\left[X_{T_1} \in \partial^+ C_L(0), \left(X_{T_1} \in \partial^+ C_L(X_{T_1})\right) \circ \theta_{T_1}, \left(X_{T_1} \in \partial^+ C_L(X_{T_2})\right) \circ \theta_{T_2}, \ldots, \left(X_{T_1} \in \partial^+ C_L(X_{T_{m-1}})\right) \circ \theta_{T_{m-1}} \mathbb{1}_{G_{m-1}}\right]. \tag{4.12}$$

By the Markov property applied at time T_{m-1} and the definition of G_{m-1}, we get that the right-hand side of (4.12) is equal to

$$\begin{aligned} &\sum_{y \in F_3} \mathbb{E}\big[P_{0,\omega}\big[X_{T_1} \in \partial^+ C_L(0), \left(X_{T_1} \in \partial^+ C_L(X_{T_1})\right) \circ \theta_{T_1}, \ldots, \\ &\qquad \ldots, \left(X_{T_1} \in \partial^+ C_L(X_{T_{m-2}})\right) \circ \theta_{T_{m-2}}\big] \\ &\times P_{y,\omega}[X_{T_{C_L(y)}} \in \partial^+ C_L(y)] \mathbb{1}_{G_{m-1}} \mathbb{1}_{A_{m-1}(y)}\big] \\ &\geq (1 - L^{-\frac{M}{2}}) \left(P_0\left[X_{T_1} \in \partial^+ C_L(0), \left(X_{T_1} \in \partial^+ C_L(X_{T_1})\right) \circ \theta_{T_1}, \ldots\right.\right. \\ &\ldots, \left(X_{T_1} \in \partial^+ C_L(X_{T_{m-2}})\right) \circ \theta_{T_{m-2}}\big] - \mathbb{P}[(G_{m-1})^c]\big), \end{aligned} \tag{4.13}$$

where $A_{m-1}(y) := \{X_{T_{m-1}} = y\}$. Repeating the above argument, we conclude from (4.12) and (4.13) that

$$P_0[X_{T_{C'_L}} \in \partial^+ C'_L] \geq (1 - L^{-\frac{M}{2}})^m - \sum_{i=1}^{m-1} (1 - L^{-\frac{M}{2}})^{m-i} \mathbb{P}[(G_i)^c]. \tag{4.14}$$

To this end, we observe that Chebyshev's inequality and our hypothesis imply that for each $1 \leq i \leq m$ it is true that

$$\mathbb{P}[(G_i)^c] \leq \sum_{y \in F_1} \mathbb{P}[P_{y,\omega}[X_{T_{C_L(y)}} \notin \partial^+ C_L(y)] > L^{-\frac{M}{2}}] \leq |F_i| L^{-\frac{M}{2}} \leq (2i\mathfrak{c}L)^{d-1} L^{-\frac{M}{2}}, \tag{4.15}$$

where in the last inequality we have the fact that $|F_i| \leq (2i\mathfrak{c}L)^{d-1}$. Combining the estimates in (4.14) with (4.15) and using the assumption $M \geq 6(d-1)$ we see that there is a constant $c_4 > 0$ such that for all $L > 0$ it is true that

$$P_0[X_{T_{C'_L}} \notin \partial^+ C'_L] \leq c_4 L^{-\frac{M}{3}}.$$

This proof can be finished by choosing $\alpha > 0$ as any number such that $m\lambda_3/\lambda_1 \leq 2$. □

5 Exit probability of the random walk out of a cone

Here we will provide a uniform control on the probability that a random walk starting form the vertex of a cone stays inside the cone forever. This will trun out to be one of the key steps in the proof of Theorem 2.1. It will be useful to this end to define for $l \in \mathbb{S}^{d-1}$ and $\alpha > 0$,

$$D' := \inf\{n \geq 0 : X_n \notin C(X_0, l, \alpha)\}. \tag{5.1}$$

Proposition 5.1. *Let $l \in \mathbb{S}^{d-1}$, $\mathfrak{c} > 0$ and $M > 6d - 3$. Suppose that $(WP)_{M,\mathfrak{c}}|l$ holds. Then there exists a positive constant $c_5(d) > 0$ such that $P_0[D' = \infty] > c_5(d)$.*

During the rest of this section we will prove this proposition. We will first need to introduce some notation in Subsection 5.1. In Subsection 5.2, we will prove an auxiliary lemma, while Subsection 5.3, we will finish the proof of Proposition 5.1.

5.1 Notations

Let $l' \in \mathbb{S}^{d-1}$ and choose a rotation R' on $\mathbb{R}^d$ with the property

$$R'(e_1) = l'.$$

For each $x \in \mathbb{Z}^d$, real numbers $m > 0$, $\mathfrak{c} > 0$ and integer $i \geq 0$ we define the box

$$B_i(x) := B_{2^{m+i}, 2\mathfrak{c}2^{m+i}, l'}(x)$$

[cf. (2.11)]. We also need slabs perpendicular to direction l'. Set

$$V_0(x) := x + R'\left([-2^m, 2^m] \times \mathbb{R}^{d-1}\right) \cap \mathbb{Z}^d$$

and for $i \geq 1$,

$$V_i(x) := x + R'\left(\left[-2^m, \sum_{j=0}^{i} 2^{m+j}\right] \times \mathbb{R}^{d-1}\right) \cap \mathbb{Z}^d.$$

The positive part of the boundary for this set is defined as

$$\partial^+ V_i(x) := \partial V_i(x) \cap \left\{x + R'\left(\left(\sum_{j=0}^{i} 2^{m+j}, \infty\right) \times \mathbb{R}^{d-1}\right)\right\}.$$

Furthermore, we will define recursively a sequence of stopping times as follows. First, let

$$T_0 := T_{B_0(X_0)}.$$

and for $i \geq 1$

$$T_i := T_{B_i(X_{T_{i-1}})} \circ \theta_{T_{i-1}} + T_{i-1}.$$

We define also the first time of entrance of the random walk to the hyperplane $R'\left((-\infty, 0) \times \mathbb{R}^{d-1}\right)$ by

$$D_{l'} := \inf\{n \geq 0 : X_n \cdot l' < 0\}.$$

5.2 Auxiliary lemma

Here we will prove the following lemma which will be the first step in the proof of Proposition 5.1.

Lemma 5.2. *Let $\mathfrak{c} > 0$ and $N > 2(d-1)$. Assume that $(WP)_{N,2\mathfrak{c}}|l'$ is satisfied. Then, for all $m \in \mathbb{N}$ and $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geq 2^m\}$, we have that*

$$P_x[D_{l'} = \infty] \geq y(m)$$

where $y(m)$ does not depend on l' and satisfies

$$\lim_{m\to\infty} y(m) = 1. \tag{5.2}$$

Proof. From the fact that $(WP)_{N,2\mathfrak{c}}|l'$ holds, we can assume that there exists a constant c_6 such that for all positive integers i and m one has that

$$P_0[X_{T_{B_i(0)}} \in \partial^+ B_i(0)] \geq 1 - c_6 2^{-N(m+i)}. \tag{5.3}$$

By stationarity, we have for $x \in \mathbb{Z}^d$:

$$P_x[X_{T_{B_i(x)}} \in \partial^+ B_i(x)] \geq 1 - c_6 2^{-N(m+i)}. \tag{5.4}$$

Throughout this proof, let us choose $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geq 2^m\}$. As it will become clear soon, it will be useful to estimate for $i \geq 1$ the following probability

$$I_i := P_x[X_{T_{V_i(x)}} \in \partial^+ V_i(x)]. \tag{5.5}$$

In view of (5.4), we have

$$I_0 \geq P_x[X_{T_{B_0(x)}} \in \partial^+ B_0(x)] \geq 1 - c_6 2^{-Nm} \geq 1 - c_6 2^{-N\frac{m}{2}}.$$

We will estimate I_i for $i \geq 1$ recursively. Let us first estimate I_1. Note that

$$I_1 \geq P_x[X_{T_0} \in \partial^+ B_0(X_0), (X_{T_{B_1(X_0)}} \in \partial^+ B_1(X_0)) \circ \theta_{T_0}]. \tag{5.6}$$

Using the strong Markov property at time T_0 we then see that

$$I_1 \geq \sum_{y \in \partial^+ B_0(x)} \mathbb{E}\left[P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), X_{T_0} = y] \times P_{y,\omega}[X_{T_{B_1(y)}} \in \partial^+ B_1(y)] \mathbb{1}_{G_0}\right], \tag{5.7}$$

where

$$G_0 := \{w \in \Omega : P_{y,\omega}[X_{T_{B_1(y)}} \in \partial^+ B_1(y)] > 1 - 2^{-N\frac{m}{2}}, \text{ for all } y \in \partial^+ B_0(x)\}.$$

Thus, it is clear that

$$I_1 \geq \left(1 - 2^{-N\frac{m}{2}}\right)\left(P_x[X_{T_0} \in \partial^+ B_0(X_0)] - \mathbb{P}[(G_0)^c]\right). \tag{5.8}$$

Notice that by (5.4) and Chebyshev's inequality

$$\begin{aligned}
\mathbb{P}[(G_0)^c] &\leq \sum_{y \in \partial^+ B_0(x)} \mathbb{P}[P_{y,\omega}[X_{T_{B_1(y)}} \notin \partial^+ B_1(y)] \geq 2^{-N\frac{m}{2}}] \\
&\leq \sum_{y \in \partial^+ B_0(x)} P_y[X_{T_{B_1(y)}} \notin \partial^+ B_1(y)] 2^{N\frac{m}{2}} \\
&= |\partial^+ B_0(x)| 2^{N\frac{m}{2}}] P_0[X_{T_{B_1(0)}} \notin \partial^+ B_1(0)] \\
&\leq c_6 (2\mathfrak{c} 2^{m+1})^{d-1} 2^{N\left(\frac{m}{2} - (m+1)\right)} \\
&\leq c_6 (2\mathfrak{c} 2^{m+1})^{d-1} 2^{-N\frac{m}{2}}.
\end{aligned} \tag{5.9}$$

Plugging the bound (5.9) into (5.8) we see that

$$I_1 \geq (1 - 2^{-N\frac{m}{2}})(1 - 2^{-N\frac{m}{2}} - c_6 (2\mathfrak{c} 2^{m+1})^{d-1} 2^{-N\frac{m}{2}}). \tag{5.10}$$

Hereafter we can do the general recursive procedure. For this end, we define for $i \geq 1$,

$$J_i := P_0[X_{T_0} \in \partial^+ B_0(X_0), (X_{T_{B_1(X_0)}} \in \partial^+ B_1(X_0)) \circ \theta_{T_0}, \ldots, (X_{T_{B_i(X_0)}} \in \partial^+ B_i(X_0)) \circ \theta_{T_{i-1}}]. \tag{5.11}$$

It is straightforward that $I_i \geq J_i$. Furthermore, through induction on $i \geq 1$, we will establish that

$$J_i \geq (1 - 2^{-N\frac{(m+i-1)}{2}}) \left\{ J_{i-1} - 2^{-N\frac{(m+i-1)}{2}} \left(\sum_{j=0}^{i-1} 2\mathfrak{c} 2^{(m+j)+1} \right)^{d-1} \right\}. \tag{5.12}$$

To prove this, we first define extended boundary of the pile of boxes at a given step as

$$F_0 := \partial^+ B_0(x),$$

and for $i \geq 2$

$$F_{i-1} := \partial \left\{ \cup_{y \in F_{i-2}} B_{i-1}(y) \right\} \cap \{x + R'((2^{m+i-1}, \infty) \times \mathbb{R}^{d-1}))\}.$$

Using these notations, we can apply the strong Markov property to (5.11) at time T_{i-1}, to get that

$$J_i = \sum_{y \in F_{i-1}} \mathbb{E} \left[P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), \ldots \right.$$
$$\left. \ldots, (X_{T_{B_{i-1}(X_0)}} \in \partial^+ B_{i-1}(X_0)) \circ \theta_{T_{i-2}}, X_{T_{i-1}} = y] P_{y,\omega}[X_{T_{B_i(y)}} \in \partial^+ B_i(y)] \right].$$

Following the same strategy used to deduce (5.10), it will be convenient to introduce for each $i \geq 2$ the event

$$G_{i-1} := \{\omega \in \Omega : P_{y,\omega}[X_{T_{B_i(y)}} \in \partial^+ B_i(y)] > 1 - 2^{-N\frac{(m+i-1)}{2}}, \text{ for all } y \in F_{i-1}\}.$$

Inserting the indicator function of the event G_{i-1} into (5.11) we get that

$$\begin{aligned} J_i \quad &\geq \sum_{y \in F_{i-1}} \mathbb{E} \Big[P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), \ldots (X_{T_{B_{i-1}(X_0)}} \in \partial^+ B_{i-1}(X_0)) \circ \theta_{T_{i-2}}, X_{T_{i-1}} = y] \\ &\times \quad P_{y,\omega}[X_{T_{B_i(y)}} \in \partial^+ B_i(y)] \mathbb{1}_{G_{i-1}} \Big]. \end{aligned}$$

By the same kind of estimation as in (5.8), we have

$$J_i \geq (1 - 2^{-N\frac{(m+i-1)}{2}}) \left(J_{i-1} - \mathbb{P}[(G_{i-1})^c] \right). \tag{5.13}$$

We need to get an estimate for $\mathbb{P}[(G_{i-1})^c]$. We do it repeating the argument given in (5.9). Let us first remark that

$$|F_{i-1}| \leq \left(\sum_{j=0}^{i-1} 2\mathfrak{c} 2^{(m+j)+1} \right)^{d-1}. \tag{5.14}$$

Indeed, the case in which $l' = e_1$ gives the maximum number for $|F_{i-1}|$. Keeping (5.14) and (5.4) in mind we get that

$$\begin{aligned} P_x[(G_{i-1})^c] &\leq \sum_{y \in F_{i-1}} \mathbb{P} \left[P_{y,\omega} \left[X_{T_{B_i(y)}} \notin \partial^+ B_i(y) \right] \geq 2^{-N\frac{(m+i-1)}{2}} \right] \\ &\leq \sum_{y \in F_{i-1}} P_y[X_{T_{B_i(y)}} \notin \partial^+ B_i(y)] 2^{N\frac{(m+i-1)}{2}} \\ &\leq \left(\sum_{j=0}^{i-1} 2\mathfrak{c} 2^{(m+j)+1} \right)^{d-1} c_6 2^{-N\frac{(m+i-1)}{2}}. \end{aligned} \tag{5.15}$$

Therefore, combining (5.15) and (5.13) we prove claim (5.12). Iterating (5.12) backward, from a given integer i, we have got

$$J_i \geq J_1 \left[\prod_{h=1}^{i-1} (1 - 2^{-N\frac{(m+h)}{2}}) \right] - \sum_{j=1}^{i-1} a_j 2^{-N\frac{m+j}{2}} \prod_{k=j}^{i-1} (1 - 2^{-N\frac{(m+k)}{2}}), \tag{5.16}$$

where we have used for short

$$a_j := c_6 \left(\sum_{i=0}^{j} \mathfrak{c} 2^{(m+i)+1} \right)^{d-1} \leq c_6 (2\mathfrak{c})^{d-1} 2^{(m+j+2)(d-1)}.$$

The same argument used to derive (5.10) can be repeated to conclude that

$$J_1 \geq (1 - 2^{-N\frac{m}{2}})(1 - 2^{-N\frac{m}{2}} - c_6(2\mathfrak{c}2^{m+1})^{d-1} 2^{-N\frac{m}{2}}). \tag{5.17}$$

Replacing the right hand side of (5.17) into (5.16), and together with the fact $I_i \geq J_i$, we see that

$$I_i \geq \left[\prod_{h=0}^{i-1} (1 - 2^{-N\frac{m+h}{2}}) \right] (1 - 2^{-N\frac{m}{2}}) - \sum_{j=0}^{i-1} c_6 a_j 2^{-N\frac{(m+j)}{2}} \prod_{k=j}^{i-1} (1 - 2^{-N\frac{(m+k)}{2}}). \tag{5.18}$$

Now we can finish the proof. First, observe that

$$P_x[D_{l'} = \infty] \geq I_\infty,$$

where as a matter of definition

$$I_\infty := \lim_{i \to \infty} I_i$$

(this limit exists, because it is the limit of a decreasing sequence of real numbers bounded from below). By the condition $N > 2(d-1)$, we get that for each $m \geq 1$ one has that for all $j \geq 1$,

$$a_j \; 2^{-\frac{N(m+j)}{2}} \leq c_6 (8\mathfrak{c})^{d-1} 2^{-\vartheta\frac{(m+j)}{2}},$$

where ϑ stands for the positive number so that $N = 2(d-1) + \vartheta$. Thus all the products and series in (5.18) converge and we have that for all $m \geq 1$ and $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geq 2^m\}$

$$P_x[D_{l'} = \infty] \geq y(m),$$

where

$$y(m) := \left[\prod_{h=0}^{\infty} (1 - 2^{-N\frac{(m+h)}{2}}) \right] (1 - 2^{-N\frac{m}{2}}) - \sum_{j=0}^{\infty} a_j 2^{-N\frac{(m+j)}{2}} \prod_{k=j}^{\infty} (1 - 2^{-N\frac{(m+k)}{2}}). \tag{5.19}$$

Clearly for each $m \geq 1$, $y(m)$ does not depend on the direction l' and $\lim_{m\to\infty} y(m) = 1$, which completes the proof. □

5.3 Proof of Proposition 5.1

We will now prove Proposition 5.1 using Lemma 5.2. Before this, we need a definition of geometric nature. We will say that a sequence $(x_0, \ldots, x_n)$ of lattice points is a *path* if for every $1 \leq i \leq n-1$, one has that x_i and x_{i-1} are nearest neighbors. Furthermore, we say that this path is *admissible* if for every $1 \leq i \leq n-1$ one has that

$$(x_i - x_{i-1}) \cdot l \neq 0.$$

Now, assume $(WP)_{M,\mathfrak{c}}|l$, where $M > 6(d-1)+3$ which is the condition of the statement of Proposition 5.1. We appeal to Proposition 4.2 and assumption $(WP)_{M,\mathfrak{c}}|l$ to choose an $\alpha > 0$ such that for all $i \in \{2, \ldots, d\}$

$$(WP)_{N,2\mathfrak{c}}|l_{\pm i}$$

is satisfied with

$$N := \frac{M}{3} - 1 > 2(d-1). \tag{5.20}$$

From now on, let m be any natural number satisfying

$$y(m) > 1 - \frac{1}{2(d-1)}, \tag{5.21}$$

where $y(m)$ is the function given in Lemma 5.2 (we now by (5.2) of Lemma 5.2 that this is possible). Note that there exists a constant

$$c_7 = c_7(d) \tag{5.22}$$

such that for all $x \in \mathbb{Z}^d$ contained in $C(R(2^m e_1), l, \alpha)$ and such that $|R(2^m e_1) - x|_1 \leq 1$ one has that there exists an admissible path with at most $c_7 2^m$ lattice points joining 0 and x. We denote this path by

$$(0, y_1, \dots, y_n = x)$$

noting that $n \leq c_7 2^m$.

The general idea to finish the proof is to push forward the walk up to site x with the help of uniform ellipticity in direction l and then make use of Lemma 5.2 to ensure that the walk remains inside the cone. More precisely, by (5.20) and Lemma 5.2 we can conclude that for all $2 \leq i \leq d$ one has that

$$P_x[D_{l_{i+}} = \infty] \geq y(m), \tag{5.23}$$

along with

$$P_x[D_{l_{i-}} = \infty] \geq y(m). \tag{5.24}$$

Define the event

$$A_n := \{(X_0, \dots, X_n) = (0, y_1, \dots, y_n)\}.$$

Notice that

$$P_0[D' = \infty] \geq P_0\left[A_n, (D_{l_{i-}} = \infty) \circ \theta_n, (D_{l_{i+}} = \infty) \circ \theta_n \text{ for } 2 \leq i \leq d\right]. \tag{5.25}$$

On the other hand, by definition of the annealed law, together with the strong Markov property we have that

$$\begin{aligned} &P_0[A_n, (D_{l_{i-}} = \infty) \circ \theta_n, (\ D_{l_{i+}} = \infty) \circ \theta_n \text{ for } 2 \leq i \leq d] \\ &= \mathbb{E}\left[P_{0,\omega}[A_n], P_{x,\omega}[D_{l_{i-}} = \infty, D_{l_{i+}} = \infty \text{ for } 2 \leq i \leq d]\right]. \end{aligned} \tag{5.26}$$

Using the uniform ellipticity condition $(UE)|l$, along with (5.23) and (5.24), we can see that (5.26) is bounded from below by

$$(2\kappa)^{c_7 2^m} \left(1 - 2(d-1)(1 - y(m))\right), \tag{5.27}$$

where c_7 is defined in (5.22). By virtue of our choice of m in (5.21), we see then that

$$c_8 := (2\kappa)^{c_7 2^m} \left(1 - 2(d-1)(1 - y(m))\right) > 0. \tag{5.28}$$

Finally, in view of the inequalities (5.25), (5.26) the bound (5.27) and (5.21), it follows that

$$P_0[D' = \infty] \geq c_8.$$

6 Polynomial control of positions at the approximate regeneration times

In this section, we define approximate regeneration times which will depend on a distance parameter $\mathcal{L} > 0$ as done in [5]. We will then show that these times, assuming the polynomial condition $(UWP)_{M,\mathfrak{c}}|l$ for M large enough, and the cone-mixing condition, when scaled by $\kappa^{\mathcal{L}}$, define positions which have a finite second moment.

6.1 Preliminaries

We recall the definition of the approximate regeneration times given in [5]. Let $W := \mathcal{E} \cup \{0\}$ [cf. (2.7)] and endow the space $W^{\mathbb{N}}$ with the canonical σ-algebra $\mathcal{W}$ generated by the cylinder sets. For fixed $\omega \in \Omega$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots) \in W^{\mathbb{N}}$, we denote by $P_{\omega,\varepsilon}$ the law of the Markov chain $\{X_n\}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$, so that $X_0 = 0$ and with transition probabilities defined for $z \in \mathbb{Z}^d$, $e, |e| = 1$ as

$$P_{\omega,\varepsilon}[X_{n+1} = z + e | X_n = z] = \mathbb{1}_{\{\varepsilon_n = e\}} + \frac{\mathbb{1}_{\{\varepsilon_n = 0\}}}{1 - \kappa|\mathcal{E}|}[\omega(z, e) - \kappa \mathbb{1}_{\{e \in \mathcal{E}\}}].$$

Call $E_{\omega,\epsilon}$ the corresponding expectation. Define also the product measure Q, which to each sequence of the form $\varepsilon \in W^{\mathbb{N}}$ assigns the probability $Q(\varepsilon_1 = e) := \kappa$, if $e \in \mathcal{E}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}|$, and denote by E_Q the corresponding expectation. Here, without loss of generality we choose the ellipticity constant κ so that $\kappa|\mathcal{E}| < 1$.

Now let $\mathfrak{G}$ be the σ-algebra on $(\mathbb{Z}^d)^{\mathbb{N}}$ generated by cylinder sets, while $\mathfrak{F}$ be the σ-algebra on Ω generated by cylinder sets. Then, we can define for fixed ω the measure

$$\overline{P}_{0,\omega} := Q \otimes P_{\omega,\varepsilon}$$

on the space $(W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{W} \times \mathfrak{G})$, and also

$$\overline{P}_0 := \mathbb{P} \otimes Q \otimes P_{\omega,\varepsilon}$$

on $(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathfrak{F} \times \mathcal{W} \times \mathfrak{G})$, denoting by $\bar{E}_{0,\omega}$ and $\bar{E}_0$ the corresponding expectations. A straightforward computation makes us conclude that the law of $\{X_n\}$ under $\bar{P}_{0,\omega}$ coincides with its law under $P_{0,\omega}$ and that its law under $\overline{P}_0$ coincides with its law under P_0.

Let q be a positive real number such that for all $1 \le i \le d$,

$$u_i := l_i q$$

is an integer. Define now the vector $u := (u_1, \dots, u_d)$. From now on, we fix a particular sequence $\overline{\varepsilon}$ in $\mathcal{E}$ of length $p := |u|_1$,

$$\overline{\varepsilon} := (\overline{\varepsilon}_1, \dots, \overline{\varepsilon}_p),$$

whose components are defined as

$$\begin{gathered}
\overline{\varepsilon}_1 = \overline{\varepsilon}_2 = \cdots = \overline{\varepsilon}_{|u_1|} := \text{ sgn}(u_1)e_1, \\
\overline{\varepsilon}_{|u_1|+1} = \overline{\varepsilon}_{|u_1|+2} = \cdots = \overline{\varepsilon}_{|u_1|+|u_2|} := \text{ sgn}(u_2)e_2 \\
\vdots \\
\overline{\varepsilon}_{p-|u_d|+1} = \cdots = \overline{\varepsilon}_p := \text{ sgn}(u_d)e_d.
\end{gathered}$$

Without loss of generality we can assume that $l_1 \neq 0$. And by taking α small enough that

$$\overline{\varepsilon}_1, \overline{\varepsilon}_1 + \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_1 + \cdots + \overline{\varepsilon}_p = u$$

are inside the cone $\mathcal{C}(0, l, \alpha)$. For $\mathcal{L} \in p\mathbb{N}$ consider the sequence $\bar{\varepsilon}^{(\mathcal{L})}$ of length $\mathcal{L}$, defined as the concatenation $\mathcal{L}/p$ times with itself of the sequence $\bar{\varepsilon}$, so that

$$\bar{\varepsilon}^{(\mathcal{L})} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_p, \dots, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_p).$$

Consider the filtration $\mathcal{G} := \{\mathcal{G}_n : n \ge 0\}$ where

$$\mathcal{G}_n := \sigma\{(\varepsilon_i, X_i) : i \le n\}.$$

Define $S_0 := 0$,

$$S_1 := \inf\{n \geq \mathcal{L} : \; X_{n-\mathcal{L}} \cdot u > \max\{X_m \cdot u : m < n - \mathcal{L}\}, \\ (\varepsilon_{n-\mathcal{L}}, \dots, \varepsilon_{n-1}) = \overline{\varepsilon}^{(\mathcal{L})}\}$$

together with

$$R_1 := D' \circ \theta_{S_1} + S_1.$$

We can now recursively define for $k \geq 1$,

$$S_{k+1} := \inf\{n \geq R_k : \; X_{n-\mathcal{L}} \cdot u > \max\{X_m \cdot u : m < n - \mathcal{L}\}, \\ (\varepsilon_{n-\mathcal{L}}, \dots, \varepsilon_{n-1}) = \overline{\varepsilon}^{(\mathcal{L})}\}$$

and

$$R_{k+1} := D' \circ \theta_{S_{k+1}} + S_{k+1}.$$

Clearly,

$$0 = S_0 \leq S_1 \leq R_1 \leq \cdots \infty,$$

the inequalities are strict if the left member of the corresponding inequality is finite, and the sequences $\{S_k : k \geq 0\}$ and $\{R_k : k \geq 0\}$ are $\mathcal{G}$-stopping times. On the other hand, we can check that $\bar{P}_0$-a.s. one has that $S_1 < \infty$ along with the fact that $\bar{P}_0$-a.s. it is true that

$$\{\lim X_n \cdot l = \infty\} \cap \{R_k < \infty\} \implies S_{k+1} < \infty. \tag{6.1}$$

Put

$$K := \inf\{k \geq 1 : S_k < \infty, R_k = \infty\}$$

and define the *approximate regeneration time*

$$\tau^{(\mathcal{L})} := S_K. \tag{6.2}$$

The random variable $\tau^{(\mathcal{L})}$ is the first time n such that: at time $n - \mathcal{L}$ it reached a record in direction l; then it moves $\mathcal{L}$ steps in the direction l by means of the action of $\overline{\varepsilon}^{(\mathcal{L})}$; and finally after time n, never exits the cone $C(X_n, l, \alpha)$.

The following lemma is required to show that the approximate renewal times are $\bar{P}_0$-a.s. finite.

Lemma 6.1. *Let $l \in \mathbb{S}_q^{d-1}$, $M \geq d+1$ and $\mathfrak{c} > 0$ and assume that $(UWP)_{M,\mathfrak{c}}|l$ is satisfied. Then the random walk is transient in direction l.*

Proof. The proof can be obtained following for example the argument presented in page 517 of [19], through the use of Borel-Cantelli and the fact that for any $M > 0$ we have that

$$P_0[\overline{\lim}_{n\to\infty} X_n \cdot l = \infty] = 1. \qquad \square$$

We can now prove the following stronger version of Lemma 2.2 of [5].

Lemma 6.2. *Let $M > 6d - 3$, $\mathfrak{c} > 0$, $\alpha > 0$ and $\phi \in \Phi$. Assume that $(CM)_{\alpha,\phi}|l$ and $(UE)|l$ are satisfied. Then there exists a positive $\mathcal{L}_0 \in |u|_1\mathbb{N}$, such that for all $\mathcal{L} \geq \mathcal{L}_0$ with $\mathcal{L} \in |u|_1\mathbb{N}$ one has that P_0-a.s.*

$$\tau^{(\mathcal{L})} < \infty. \tag{6.3}$$

Proof. Following the arguments in the proof of Lemma 2.2. of [5], we know that for all $\mathcal{L} \in |u|_1\mathbb{N}$ it is true that

$$\bar{P}_0[R_k < \infty] \leq (\phi(\mathcal{L}) + P_0[D' < \infty])^k. \tag{6.4}$$

From the assumption $(CM)_{\alpha,\phi}|l$, we have $\phi(\mathcal{L}) \to 0$ as $\mathcal{L} \to \infty$. On the other hand, by Lemma 5.1,

$$P_0[D' < \infty] < 1.$$

Therefore, we can find an $\mathcal{L}_0$ with the property

$$\phi(\mathcal{L}) + P_0[D' < \infty] < 1,$$

for all $\mathcal{L} \geq \mathcal{L}_0,\ \mathcal{L} \in \mathbb{N}|u|_1$. Then, by Borel-Cantelli Lemma, one has that $\bar{P}_0$-a.s.

$$\inf\{n \geq 1 :\ R_n = \infty\} < \infty. \tag{6.5}$$

Now, observe that $\bar{P}_0$-a.s.

$$\inf\{n \geq 1 :\ R_n = \infty\} = \inf\{n \geq 1 : R_{n-1} < \infty\ R_n = \infty\}. \tag{6.6}$$

In turn, using (6.1), which is satisfied in view of Lemma 6.1, and also by (6.5) and (6.6), we have that

$$\inf\{n \geq 1 : S_n < \infty\ R_n = \infty\} = K < \infty$$

$\bar{P}_0$-a.s., which finishes the proof of (6.3). □

Finally, we can state the following proposition, which gives a control on the second moment of the position of the random walk at the first approximate regeneration time. Define for $x \in \mathbb{Z}^d$ and $\mathcal{L} > 0$ the σ-algebra

$$\mathfrak{F}_{x,\mathcal{L}} := \sigma\left\{\omega(y,\cdot) :\ y \cdot u \leq x \cdot u - \frac{\mathcal{L}}{|u|_1}|u|_2\right\}. \tag{6.7}$$

Proposition 6.3. *Let $\mathfrak{c} > 0$, $l \in \mathbb{S}_q^{d-1}$, $M > 0$, $\phi \in \Phi$ and $0 < \alpha \leq \min\{\frac{1}{9}, \frac{1}{3\mathfrak{c}}\}$. Assume that $(CM)_{\alpha,\phi}|l$, $(UE)|l$ and $(UWP)_{M,\mathfrak{c}}|l$ hold. Then, there exists a constant $c_9 = c_9(d,\kappa,l) > 0$, such that for all $\mathcal{L} \in \mathbb{N}|u|_1$ we have that*

$$\bar{E}_0[(\kappa^{\mathcal{L}} X_{\tau^{(\mathcal{L})}} \cdot l)^2 | \mathfrak{F}_{0,\mathcal{L}}] \leq c_9. \tag{6.8}$$

In the next subsection we will prove Proposition 6.3.

6.2 Proof of Proposition 6.3

Before we prove Proposition 6.3, we will need to state three lemmas. In order to make the reading of the proof of Proposition 6.3 more direct, the proof of these lemmas is postponed to Appendix A.

Lemma 6.4. *Let $\alpha > 0$ and $\phi \in \Phi$. Assume that $(CM)_{\alpha,\phi}|l$ holds. Then, for each $x \in \mathbb{Z}^d$ one has that for all $\mathcal{L} \in \mathbb{N}|u|_1$, $\mathbb{P}$-a.s.*

$$|\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}] - P_0[D' = \infty]| \leq \phi(\mathcal{L}).$$

We will now state the second lemma that will be needed to prove Proposition 6.3. To state it define

$$\mathfrak{M} := \sup_{0 \leq n \leq D'} (X_n - X_0) \cdot u.$$

Lemma 6.5. *Let $M > 4d + 1$ and*

$$3\mathfrak{c} \leq \frac{1}{\alpha}. \tag{6.9}$$

Assume that $(UWP)_{M,\mathfrak{c}}|l$ is satisfied. Then, there exists $c_{10} = c_{10}(d) > 0$ such that $\mathbb{P}$-a.s. one has that

$$E_0[\mathfrak{M}^2, D' < \infty | \mathfrak{F}_{0,\mathcal{L}}] \leq c_{10}.$$

Finally, to state the third lemma, we define

$$b = b(\mathcal{L}) := P_0[D' < \infty] + \phi(\mathcal{L}),$$
$$b' = b'(\mathcal{L}) := P_0[D' = \infty] + \phi(\mathcal{L})$$

and $E_{\mathbb{P}\otimes Q} := \mathbb{E}E_Q$. Note that b and b' are uniformly bounded in $\mathcal{L}$. Furthermore, it will be necessary to define for each $j \geq 0$ and $n \geq \mathcal{L} + j$ the events

$$D_{j,n} := \{\varepsilon \in W^{\mathbb{N}} : (\varepsilon_m, \dots, \varepsilon_{m+\mathcal{L}-1}) \neq \bar{\varepsilon}^{(\mathcal{L})} \text{ for all } j \leq m \leq j + n - \mathcal{L} + 1\}.$$

Lemma 6.6. *There exists a constant c_{11} such that for all $n \geq \mathcal{L}^2$ one has that*

$$Q[D_{0,n}] \leq (1 - c_{11}\mathcal{L}^2\kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]}.$$

We now present the proof of Proposition 6.3, divided in several steps. For the sake of simplicity, we will write τ instead of $\tau^{(\mathcal{L})}$.

Step 0. We first note that

$$\begin{aligned} &\bar{E}_0[(X_\tau \cdot u)^2 \mid \mathfrak{F}_{0,\mathcal{L}}] \\ =&\sum_{k=1}^{\infty}\sum_{k'=0}^{k-1} \bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,\mathcal{L}}]. \end{aligned} \tag{6.10}$$

Throughout the subsequent steps of the proof we will estimate the right-hand side of (6.10).

Step 1. Here we will prove the following estimate valid for all $k \geq 1$ and $0 \leq k' < k$.

$$\begin{aligned} &\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\leq b'b^{k-k'-1}\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k'+1} < \infty \mid \mathfrak{F}_{0,\mathcal{L}}]. \end{aligned} \tag{6.11}$$

Define the set

$$H^{\mathcal{L}} := \left\{ y \in \mathbb{Z}^d : y \cdot u \geq \mathcal{L}\frac{|u|_2}{|u|_1} \right\}.$$

Then, for each $0 \leq k' < k$, one has that

$$\begin{aligned} &\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ =& \sum_{n\geq 1, x \in H^{\mathcal{L}}} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k = n, \\ &\qquad X_{S_k} = x, D' \circ \theta_n = \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ =& \sum_{n\geq 1, x \in H^{\mathcal{L}}} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k = n, X_n = x] \\ &\qquad \times P_{\vartheta_x\omega,\ \theta_n\varepsilon}[D' = \infty] \mid \mathfrak{F}_{0,\mathcal{L}}] \\ =& \sum_{x \in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, X_{S_k} = x] \\ &\qquad \times P_{x,\omega}[D' = \infty] \mid \mathfrak{F}_{0,\mathcal{L}}], \end{aligned} \tag{6.12}$$

where here for each $x \in \mathbb{Z}^d$, ϑ_x is the canonical space-shift in Ω defined in (2.2), while for each $n \geq 0$, θ_n denotes the canonical time-shift in the space $W^{\mathbb{N}}$ so that $(\theta_n\varepsilon)_m = \varepsilon_{n+m}$, and where in the first equality we have used the fact that the value of $X_{S_k} \cdot u \geq X_{S_1} \cdot u$, in the second equality the Markov property and in the last equality we have used the independence of the coordinates of ε and the fact that the law of the random walk is the same under $P_{x,\omega}$ as under $E_Q P_{\vartheta_x\omega,\theta_n\varepsilon}$.

Moreover, by the fact that the first factor inside the expectation of the right-most expression of (6.12) is $\mathfrak{F}_{x,\mathcal{L}}$-measurable, it is equal to

$$\sum_{x\in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty, X_{S_k}=x] \times \mathbb{E}[P_{x,\omega}[D'=\infty]\mid \mathfrak{F}_{x,\mathcal{L}}]\mid \mathfrak{F}_{0,\mathcal{L}}]. \tag{6.13}$$

Applying next Lemma 6.4 to (6.13), we see that

$$\begin{aligned}&\sum_{x\in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty, X_{S_k}=x]\times \mathbb{E}[P_{x,\omega}[D'=\infty]\mid \mathfrak{F}_{x,\mathcal{L}}]\mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\quad\le b'\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty\mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.14}$$

Next, observe that for $k'<k$ one has that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty\mid \mathfrak{F}_{0,\mathcal{L}}]\\ =&\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, R_{k-1}<\infty\mid \mathfrak{F}_{0,\mathcal{L}}]\\ =&\sum_{x\in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_{k-1}<\infty, X_{S_{k-1}}=x, D'\circ\theta_{S_{k-1}}<\infty]\mid \mathfrak{F}_{0,\mathcal{L}}]\\ =&\sum_{x\in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_{k-1}<\infty, X_{S_{k-1}}=x]P_{x,\omega}[D'<\infty]\mid \mathfrak{F}_{0,\mathcal{L}}]\\ =&\sum_{x\in H^{\mathcal{L}}} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_{k-1}<\infty, X_{S_{k-1}}=x]\\ &\quad\times \mathbb{E}[P_{x,\omega}[D'<\infty]\mid \mathfrak{F}_{x,\mathcal{L}}]\mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.15}$$

Again, by Lemma 6.4, we have that $\mathbb{E}[P_{x,\omega}[D'<\infty]\mid \mathfrak{F}_{x,\mathcal{L}}]\le b=P_0[D'<\infty]+\phi(\mathcal{L})$. Using this inequality to estimate the last factor inside the conditional expectation of the right-most hand side of (6.15), we see that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty\mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le b\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_{k-1}<\infty\mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned}$$

By induction on k we get that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_k<\infty\mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le b^{k-k'-1}\bar{E}_0[(X_{S_{k'+1}}\cdot u)^2-(X_{S_{k'}}\cdot u)^2, S_{k'+1}<\infty\mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.16}$$

Combining (6.16) with (6.14) we obtain (6.11).

Step 2. For $k\ge 1$ we define

$$M_k:=\sup_{0\le n\le R_k} X_n\cdot u. \tag{6.17}$$

Define also the sets parametrized by k and $n\ge 0$

$$A_{n,k}:=\left\{\varepsilon\in W^{\mathbb{N}}:\left(\varepsilon_{t_k^{(n)}},\varepsilon_{t_k^{(n)}+1},\ldots,\varepsilon_{t_k^{(n)}+\mathcal{L}-1}\right)=\overline{\varepsilon}^{(\mathcal{L})}\right\} \tag{6.18}$$

and

$$B_{n,k}:=\left\{\varepsilon\in W^{\mathbb{N}}:\left(\varepsilon_{t_k^{(j)}},\varepsilon_{t_k^{(j)}+1},\ldots,\varepsilon_{t_k^{(j)}+\mathcal{L}-1}\right)\neq\overline{\varepsilon}^{(\mathcal{L})}\text{ for all } 0\le j\le n-1\right\}, \tag{6.19}$$

where we define the sequence of stopping times [cf. (2.4)] parameterized by k and recursively on $n \geq 0$ by

$$t_k^{(0)} := \bar{T}^l_{M_k}$$

and the successive times where a record value of the projection of the random walk on l is achieved by

$$t_k^{(n+1)} := \bar{T}^l_{X_{t_k^{(n)}} \cdot u}.$$

In this step we will show that for all $k \geq 0$ one has that

$$\begin{aligned}
&\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, S_{k+1} < \infty | \mathfrak{F}_{0,\mathcal{L}}] \\
&\leq \sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&+ \sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}],
\end{aligned} \tag{6.20}$$

Indeed, note that on the event $A_{n,k} \cap B_{n,k}$ one has that

$$S_{k+1} = t_k^{(n)} + \mathcal{L}.$$

Thus, as a consequence of the definition of S_{k+1}, one has that $\bar{P}_0$-a.s.

$$\{S_{k+1} < \infty\} \subset \bigcup_{n \geq 0} \{t_k^{(n)} < \infty, B_{n,k}, A_{n,k}\}. \tag{6.21}$$

Display (6.20) now follows directly from (6.21).

Step 3. Here we will derive an upper bound for the two sums appearing in the right-hand side in (6.20). In fact, we will prove that there is a constant c_{12} such that for all $k \geq 0$ one has that

$$\begin{aligned}
&\sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\leq c_{12} \kappa^{\mathcal{L}} \left(\mathcal{L}^4 b^{k-1} + \mathcal{L}^2 \bar{E}_0[X_{S_k} \cdot u, S_k < \infty | \mathfrak{F}_{0,\mathcal{L}}] \right)
\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}
&\sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n.k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\leq c_{12} \sum_{n=\mathcal{L}^2}^{\infty} \kappa^{\mathcal{L}} (1 - c_{11} \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} \left((n+\mathcal{L})^2 b^{k-1} + (n+\mathcal{L}) \bar{E}_0[X_{S_k} \cdot u, S_k < \infty | \mathfrak{F}_{0,\mathcal{L}}] \right).
\end{aligned} \tag{6.23}$$

Note that for all $n \geq 0$ one has that

$$X_{t_k^{(n+1)}} \cdot u \leq X_{t_k^{(n)}} \cdot u + |u|_\infty,$$

and hence by induction on n we get that

$$X_{t_k^{(n)}} \cdot u \leq M_k + (n+1)|u|_\infty.$$

Therefore, if we set

$$\mathcal{L}' := \frac{\mathcal{L}|u|_2}{|u|_1} + |u|_\infty \leq c_{13} \mathcal{L}, \tag{6.24}$$

where c_{13} is a constant depending only l and d, we can see that P_0-a.s on the event $\{t_k^{(n)} < \infty,\ A_{n,k}\}$ one has that

$$X_{S_{k+1}} \cdot u \leq N_{k,n} := M_k + n|u|_\infty + \mathcal{L}'. \tag{6.25}$$

Therefore, for all $0 \le n \le \mathcal{L}^2 - 1$ one has that (with the convention that $R_0 := 0$)

$$\begin{aligned} &\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &= \sum_{j=0}^{\infty} \sum_{x \in \mathbb{Z}^d} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[N_{k,n}^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} = j, X_j = x] \mathbb{1}_{\{(\varepsilon_j,\dots,\varepsilon_{j+\mathcal{L}-1}) = \bar{\varepsilon}^{(\mathcal{L})}\}} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \kappa^{\mathcal{L}} \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}], \end{aligned} \tag{6.26}$$

where in the first inequality we have used (6.25), in the equality we have applied the Markov property and in the second inequality the fact that Q is a product measure and that $R_k \le t_k^{(n)}$. Similarly for all $n \ge \mathcal{L}^2$ one has that

$$\begin{aligned} &\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \sum_{j=0}^{\infty} \sum_{j'=j+n}^{\infty} \sum_{y \in \mathbb{Z}^d} E_{\mathbb{P}\otimes Q}\ [E_{\omega,\varepsilon}[N_{k,n}^2 - (X_{S_k} \cdot u)^2, X_{t_k^{(0)}} = y, \\ &\quad t_k^{(0)} = j] P_{\theta_y \omega, \theta_j \varepsilon}[D_{j,n}, t_k^{(n)} = j'] \mathbb{1}_{\{(\varepsilon_{j'},\dots,\varepsilon_{j'+\mathcal{L}-1}) = \bar{\varepsilon}^{(\mathcal{L})}\}}] \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \kappa^{\mathcal{L}} Q[D_{0,n}] \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le \kappa^{\mathcal{L}} (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}}\right]} \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}], \end{aligned} \tag{6.27}$$

where in the first inequality we have used again (6.25), in the second one the Markov property, in the third one the fact that $R_k \le t_k^{(0)}$ and in the last one Lemma 6.6.

Now, by displays (6.26) and (6.27), to finish the proof of inequalities (6.22) and (6.23) it is enough to prove that there is a constant c_{14} such that

$$\begin{aligned} &\bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le c_{14} \left((n+\mathcal{L})^2 b^{k-1} + (n+\mathcal{L}) \bar{E}_0[X_{S_k} \cdot u, S_k < \infty | \mathfrak{F}_{0,\mathcal{L}}] \right), \end{aligned} \tag{6.28}$$

using the fact that $n \le \mathcal{L}^2 - 1$ in the left-hand side of inequality (6.22). To prove (6.28), the following identity will be useful

$$\begin{aligned} N_{k,n}^2 - (X_{S_k} \cdot u)^2 &= (M_k - X_{S_k} \cdot u)^2 + 2(n|u|_\infty + \mathcal{L}')(M_k - X_{S_k} \cdot u) \\ &+ 2(n|u|_\infty + \mathcal{L}') X_{S_k} \cdot u + 2(M_k - X_{S_k} \cdot u) X_{S_k} \cdot u + (n|u|_\infty + \mathcal{L}')^2. \end{aligned} \tag{6.29}$$

We will now insert this decomposition in the left-hand side of (6.28) and bound the corresponding expectations of each term. Let us begin with the expectation of the last term. Note that by an argument similar to the one developed in *Step 1* we know that there is some constant c_{15} such that

$$\bar{E}_0[(n|u|_\infty + \mathcal{L}')^2, R_k < \infty | \mathfrak{F}_{0,\mathcal{L}}] \le c_{15} (n+\mathcal{L})^2 b^k. \tag{6.30}$$

Similarly, the expectation of the first term of the right-hand side of display (6.29) can be bounded using Lemma 6.5, so that

$$\begin{aligned} &\bar{E}_0[(M_k - X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &= \sum_{x \in H^{\mathcal{L}}} \mathbb{E}[\bar{P}_{0,\omega}[S_k < \infty, X_{S_k} = x] E_x[\mathfrak{M}^2, D' < \infty \mid \mathfrak{F}_{x,\mathcal{L}}] \mid \mathfrak{F}_{0,\mathcal{L}}] \le c_{10} b^{k-1}. \end{aligned} \tag{6.31}$$

Again, for the expectation of the second term of the right-hand side of display (6.29), we have that

$$\bar{E}_0[2(n|u|_\infty + \mathcal{L}')(M_k - X_{S_k} \cdot u), R_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \le c_{16} b^{k-1} (n+\mathcal{L}), \tag{6.32}$$

for some suitable positive constant c_{16}. For the expectation of the fourth term of the right-hand side of (6.29), we see by Lemma 6.5 that

$$\bar{E}_0[2(M_k - X_{S_k}\cdot u)X_{S_k}\cdot u, R_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}] \le 2\sqrt{c_{10}}\bar{E}_0[X_{S_k}\cdot u, S_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}]. \tag{6.33}$$

Finally, for the expectation of the third term of the right-hand side of (6.29) we have that

$$\begin{aligned}&\bar{E}_0[2(n|u|_\infty+\mathcal{L}')X_{S_k}\cdot u, R_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le c_{16}b(n+\mathcal{L})\bar{E}_0[X_{S_k}\cdot u, S_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.34}$$

Using the bounds (6.34), (6.33), (6.32), (6.31) and (6.30) we obtain inequality (6.28).

Step 4. Here we will derive for all $k\ge 1$ the inequality

$$\begin{aligned}&\bar{E}_0[X_{S_k}\cdot u, S_k<\infty|\mathfrak{F}_{0,\mathcal{L}}]\\ &\le \sum_{k'=0}^{k-1} b^{k-k'-1}\left(\sum_{n=0}^{\mathcal{L}^2-1}\bar{E}_0[N_{k',n}-X_{S_{k'}}\cdot u, t_{k'}^{(n)}<\infty, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}]\right.\\ &\left.+\sum_{n=\mathcal{L}^2}^{\infty}\bar{E}_0[N_{k',n}-X_{S_{k'}}\cdot u, t_{k'}^{(n)}<\infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}]\right).\end{aligned} \tag{6.35}$$

Note that

$$\bar{E}_0[X_{S_k}\cdot u, S_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}] = \sum_{k'=0}^{k-1}\bar{E}_0[(X_{S_{k'+1}}-X_{S_{k'}})\cdot u, S_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}]. \tag{6.36}$$

By an argument similar to the one used in *Step 1* we see that for $k'<k$ one has that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}}-X_{S'_k})\cdot u, S_k<\infty \mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le b^{k-k'-1}\bar{E}_0[(X_{S_{k'+1}}-X_{S'_k})\cdot u, S_{k'+1}<\infty \mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.37}$$

Now, we can use inclusion (6.21) of *Step 2* in order to get that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}}-X_{S'_k})\cdot u, S_{k'+1}<\infty \mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le \sum_{n=0}^{\mathcal{L}^2-1}\bar{E}_0[(X_{S_{k'+1}}-X_{S'_k})\cdot u, t_{k'}^{(n)}<\infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}]\\ &+\sum_{n=\mathcal{L}^2}^{\infty}\bar{E}_0[(X_{S_{k'+1}}-X_{S'_k})\cdot u, t_{k'}^{(n)}<\infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}],\end{aligned} \tag{6.38}$$

where the events $A_{n,k'}$ and $B_{n,k'}$ are defined in (6.18) and (6.19). Using the fact that on the event $\{t_{k'}^{(n)}<\infty, B_{n,k'}, A_{n,k'}\}$ one has that P_0-a.s.

$$(X_{S_{k'+1}}-X_{S_{k'}})\cdot u \le N_{k',n}-X_{S'_k}\cdot u,$$

and the inequalities (6.36), (6.37) and (6.38) we finish the proof of (6.35).

Step 5. Here we will obtain an upper bound for first summation inside the parenthesis in (6.35). Indeed, note that on $R_{k'}\le t_{k'}^{(n)}$, by an argument similar to the one used to derive inequality (6.26), we have that for all $0\le n\le \mathcal{L}^2$ and $0\le k'\le k-1$

$$\bar{E}_0[N_{k',n}-X_{S_{k'}}\cdot u, t_{k'}^{(n)}<\infty, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \le \kappa^{\mathcal{L}}\bar{E}_0[N_{k',n}-X_{S_{k'}}\cdot u, R_{k'}<\infty \mid \mathfrak{F}_{0,\mathcal{L}}].$$

Step 6. Here we will obtain an upper bound for the second summation inside the parenthesis in (6.35), showing that for all $n\ge\mathcal{L}^2$ and $0\le k'\le k-1$,

$$\begin{aligned}&E_0[N_{k',n}-X_{S_{k'}}\cdot u, t_{k'}^{(n)}<\infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}]\\ &\le \kappa^{\mathcal{L}}\left(1-c_{11}\mathcal{L}^2\kappa^{\mathcal{L}}\right)^{\left[\frac{n}{\mathcal{L}}\right]}\bar{E}_0[N_{k',n}-X_{S_{k'}}\cdot u, R_{k'}<\infty \mid \mathfrak{F}_{0,\mathcal{L}}].\end{aligned} \tag{6.39}$$

Inequality (6.39) follows from the fact that

$$
\begin{aligned}
&E_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\le \sum_{j=0}^{\infty} \sum_{j' \ge j+n} \sum_{y \in \mathbb{Z}^d} E_{\mathbb{P}\otimes Q}\, [E_{\omega,\varepsilon}[N_{k',n} - X_{S_{k'}} \cdot u, X_{t_{k'}^{(0)}} = y, t_{k'}^{(0)} = j] \\
&\quad \times P_{\theta_y \omega, \theta_j \varepsilon}[D_{j,n}, t_{k'}^{(n)} = j'] \mathbb{1}_{\{(\varepsilon_{j'}, \ldots, \varepsilon_{j'+\mathcal{L}-1}) = \bar{\varepsilon}^{(\mathcal{L})}\}}] \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&= \kappa^{\mathcal{L}} Q[D_{0,n}] \mathbb{E}[\bar{E}_{0,\omega}[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(0)} < \infty] \mid \mathfrak{F}_{0,\mathcal{L}}],
\end{aligned}
$$

and then Lemma 6.6 to estimate $Q[D_{0,n}]$ in the right-most hand side of this development.

Step 7. Here we will show that there exist constants c_{17} and c_{18} such that

$$
\sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \le c_{17} \kappa^{\mathcal{L}} \mathcal{L}^4 b^{k'-1} \tag{6.40}
$$

and

$$
\sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'}, B_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \le 4 c_{18} \kappa^{-\mathcal{L}} b^{k'-1}. \tag{6.41}
$$

Let us first note that by an argument similar to the one used to derive the bound in *Step 1* (through Lemmas 6.4 and 6.5), we have that

$$
\bar{E}_0[N_{k',n} - X_{S'_k} \cdot u, R_{k'} < \infty] \le (n|u|_\infty + \mathcal{L}' + c_{19}) b^{k'-1}, \tag{6.42}
$$

where $c_{19} := \sqrt{c_{10}}$. Let us now prove (6.40). Note that by *Step 5* and (6.42) we have that

$$
\begin{aligned}
&\sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\le \kappa^{\mathcal{L}} \sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\le c_{20}\, \mathcal{L}^4\, \kappa^{\mathcal{L}} b^{k'-1},
\end{aligned} \tag{6.43}
$$

for some suitable constant c_{20}. Let us now prove (6.41). First note that

$$
\begin{aligned}
&\sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'}, B_{n,k'} \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\le \sum_{n=\mathcal{L}^2}^{\infty} \kappa^{\mathcal{L}} (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}}\right]} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\
&\le b^{k'-1} \sum_{n=\mathcal{L}^2}^{\infty} \kappa^{\mathcal{L}} (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}}\right]} (n|u|_\infty + \mathcal{L}' + c_{19}) \\
&\le c_{21} b^{k'-1} \sum_{n=\mathcal{L}^2}^{\infty} n \kappa^{\mathcal{L}} (1 - c_{33} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]},
\end{aligned} \tag{6.44}
$$

for some constant c_{21}, where in the first inequality we have used *Step 6* and in the second we have used inequality (6.42). Finally notice that using the fact that for $n \ge \mathcal{L}^2$ one has that $n \le 2\mathcal{L}^2 \left[\frac{n}{\mathcal{L}^2}\right]$, we get that

$$
\begin{aligned}
\textstyle\sum_{n=\mathcal{L}^2}^{\infty} n \kappa^{\mathcal{L}} (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} &\le 2\kappa^{\mathcal{L}} \mathcal{L}^2 \textstyle\sum_{n=\mathcal{L}^2}^{\infty} \left[\frac{n}{\mathcal{L}^2}\right] (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} \\
&= 2\mathcal{L}^4 \kappa^{\mathcal{L}} \textstyle\sum_{m=1}^{\infty} m (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^m \le \frac{2}{(c_{11})^2} \kappa^{-\mathcal{L}}.
\end{aligned}
$$

Using this estimate in (6.44) we obtain (6.41).

Step 8. Here we finish the proof of Proposition 6.3 combining the previous steps we have already developed. Using inequality (6.35) proved in *Step 4* with inequalities (6.40) and (6.41) proved in *Step 7*, we see that there is a constant c_{22} such that for all $k \geq 0$,

$$\bar{E}_0[X_{S_k} \cdot u, S_k < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \leq c_{22} k b^{k-2} \kappa^{-\mathcal{L}}. \tag{6.45}$$

Thus, by inequality (6.22) proved in *Step 3*, for all $k \geq 0$ we have that

$$\sum_{n=0}^{\mathcal{L}^2-1} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \leq c_{23} \mathcal{L}^4 (k+1) b^{k-2}, \tag{6.46}$$

for certain constant $c_{23} > 0$. On the other hand, combining inequality (6.23) proved in *Step 3* with (6.45), we see that there exists a constant c_{24} such that

$$\begin{aligned} &\sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\leq c_{24} \sum_{n=\mathcal{L}^2}^{\infty} \kappa^{\mathcal{L}} (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} \left((n+\mathcal{L})^2 b^{k-1} + (n+\mathcal{L}) k b^{k-2} \kappa^{-\mathcal{L}}\right). \end{aligned} \tag{6.47}$$

Now, note that for some constant c_{25} one has that

$$\textstyle\sum_{n=\mathcal{L}^2}^{\infty} (n+\mathcal{L})^2 (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} \leq c_{25}\ \kappa^{-3\mathcal{L}} \tag{6.48}$$

and

$$\textstyle\sum_{n=\mathcal{L}^2}^{\infty} (n+\mathcal{L}) (1 - c_{11} \mathcal{L}^2 \kappa^{\mathcal{L}})^{\left[\frac{n}{\mathcal{L}^2}\right]} \leq c_{25}\ \kappa^{-2\mathcal{L}}. \tag{6.49}$$

Substituting (6.48) and (6.49) into (6.47) we see that

$$\sum_{n=\mathcal{L}^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,\mathcal{L}}] \leq c_{26} \kappa^{-2\mathcal{L}} b^{k-2} (k+1), \tag{6.50}$$

for some suitable positive constant c_{26}. Substituting (6.47) and (6.50) into inequality (6.20) of *Step 2*, we then conclude that there is a constant c_{27} such that

$$\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, S_{k+1} < \infty | \mathfrak{F}_{0,\mathcal{L}}] \leq c_{27} \kappa^{-2\mathcal{L}} b^{k-2} (k+1). \tag{6.51}$$

Substituting (6.51) into (6.11) of *Step 1*, we get that

$$\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \leq b' b^{k+1} (k'+1). \tag{6.52}$$

From the fact that $\sum_{k=1}^{\infty} \sum_{k'=0}^{k-1} b^{k+1} k'$ is convergent and bounded by a constant that does not depend on $\mathcal{L}$ (see the definition of b and b' in (6.10)), together with (6.52) and (6.10) of *Step 0*, we conclude that

$$\bar{E}_0[(X_\tau \cdot u)^2 | \mathfrak{F}_{0,\mathcal{L}}] \leq c_{28} \kappa^{-2\mathcal{L}},$$

for some constant $c_{28} > 0$, which proves the proposition.

7 Proof of Theorem 2.1

In this section we will prove Theorem 2.1 using Proposition 6.3 of Section 6. First in Subsection 7.1, we will define an approximate sequence of regeneration times. In Subsection 7.2, we will see how we can use this approximate regeneration time sequence, to prove the existence of an approximate asymptotic direction. In Subsection 7.3, we will use the approximate asymptotic direction to prove Theorem 2.1.

7.1 Approximate regeneration time sequence

As in [5], we define approximate regeneration times recursively by $\tau_1^{(\mathcal{L})} := \tau$ [cf. (6.2)] and for $i \geq 2$,

$$\tau_i^{(\mathcal{L})} := \tau_1^{(\mathcal{L})} \circ \theta_{\tau_{i-1}^{(\mathcal{L})}} + \tau_{i-1}^{(\mathcal{L})}.$$

Whenever there is no risk of confusion, we will drop the dependence of $\mathcal{L}$ on $\tau_1^{(\mathcal{L})}$, using the notation τ_i instead $\tau_i^{(\mathcal{L})}$. Let us define σ-algebras corresponding to the information of the random walk and the ε process up to the first approximate regeneration time and of the environment ω at a distance of order $\mathcal{L}$ in the direction l (recall that τ_1 [cf. (6.2)] and hence the sequence of approximate regeneration times depend on the fixed direction l) of the position of the random walk at this approximate regeneration time as

$$\mathcal{H}_1 := \sigma\{\tau_1^{(\mathcal{L})}, X_0, \varepsilon_0, \dots, \varepsilon_{\tau_1^{(\mathcal{L})}-1}, X_{\tau_1^{(\mathcal{L})}}, \{\omega(y,\cdot) : y \cdot u < u \cdot X_{\tau_1^{(\mathcal{L})}} - \mathcal{L}|u|/|u|_1\}\}.$$

Similarly define for $k \geq 2$

$$\mathcal{H}_k := \sigma\{\tau_1^{(\mathcal{L})}, \dots, \tau_k^{(\mathcal{L})}, X_0, \varepsilon_0, \dots, \varepsilon_{\tau_k^{(\mathcal{L})}-1}, X_{\tau_k^{(\mathcal{L})}}, \{\omega(y,\cdot) : y \cdot u < u \cdot X_{\tau_k^{(\mathcal{L})}} - \mathcal{L}|u|/|u|_1\}\}. \tag{7.1}$$

Let us now recall Lemma 2.3 of [5], stated here under the condition $P_0[D' = \infty] > 0$ [cf. (5.1)] instead of Kalikow condition.

Lemma 7.1. *Let $l \in \mathbb{S}_q^{d-1}$, $\alpha > 0$ and $\phi \in \Phi$. Consider a random walk in a random environment satisfying the cone-mixing condition $(CM)_{\alpha,\phi}|l$ and the uniform ellipticity condition $(UE)|l$. Assume that $\mathcal{L}$ is such that*

$$\phi(\mathcal{L}) < P_0[D' = \infty].$$

Then, $\mathbb{P}$-a.s. one has that

$$\left|\bar{P}_0[\{X_{\tau_k+\cdot} - X_{\tau_k}\} \in A \mid \mathcal{H}_k] - \bar{P}_0[\{X_\cdot\} \in A | D' = \infty]\right| \leq \phi'(\mathcal{L}),$$

for all measurable sets $A \subset (\mathbb{Z}^d)^{\mathbb{N}}$, where

$$\phi'(\mathcal{L}) := \frac{2\phi(\mathcal{L})}{(P_0[D' = \infty] - \phi(\mathcal{L}))}.$$

Proof. The argument given in page 890 in ([5, Lemma 2.3]) is still valid here, so we omit it. □

7.2 Approximate asymptotic direction

We will show that a random walk satisfying the cone-mixing, uniform ellipticity condition and the uniform non-effective polynomial condition with high enough degree has an approximate asymptotic direction. The exact statement is given below. It will also be shown that the order with which the random variable X_{τ_1} grows as a function of $\mathcal{L}$ is $\kappa^{-\mathcal{L}}$.

Proposition 7.2. *Let $l \in \mathbb{S}_q^{d-1}$, $\phi \in \Phi$, $\mathfrak{c} > 0$, $M > 6d$ and $0 < \alpha \leq \min\{\frac{1}{9}, \frac{1}{3\mathfrak{c}}\}$. Consider a random walk in a random environment satisfying $(CM)_{\alpha,\phi}|l$, $(UE)|l$, and $(UWP)_{M,\mathfrak{c}}|l$. Then, there exists a sequence $\eta_{\mathcal{L}}$ such that*

$$\lim_{\mathcal{L}\to\infty} \eta_{\mathcal{L}} = 0 \tag{7.2}$$

and $\bar{P}_0$-a.s.

$$\overline{\lim}_{n\to\infty} \left|\frac{\kappa^{\mathcal{L}} X_{\tau_n}}{n} - \lambda_{\mathcal{L}}\right| < \eta_{\mathcal{L}}, \tag{7.3}$$

where for all $\mathcal{L} \geq 1$,

$$\lambda_{\mathcal{L}} := \bar{E}_0[\kappa^{\mathcal{L}} X_{\tau_1} \mid D' = \infty]. \tag{7.4}$$

Furthermore, there is a constant $c_{29} = c_{29}(\kappa, l, d) > 0$ *such that*

$$|\lambda_{\mathcal{L}}|_2 \geq c_{29}. \tag{7.5}$$

We first prove inequality (7.3) of Proposition 7.2. We will follow the argument presented in the proof of Lemma 3.3 of [5]. For each integer $i \geq 1$ define the sequence

$$\overline{X}_i := \kappa^{\mathcal{L}}(X_{\tau_i} - X_{\tau_{i-1}}),$$

with the convention $\tau_0 = 0$. Using Lemma 7.1 and Lemma 3.2 of [5], we can enlarge the probability space where the sequence $\{\bar{X}_i : i \geq 1\}$ is defined so that there we have the following properties:

(1) There exist an i.i.d. sequence $\{(\widetilde{X}_i, \Delta_i) : i \geq 2\}$ of random vectors with values in $(\kappa^{\mathcal{L}}\mathbb{Z}^d, \{0,1\})$, such that $\widetilde{X}_2$ has the same distribution as $\overline{X}_1$ under the measure $\bar{P}_0[\cdot|D' = \infty]$ while Δ_2 has a Bernoulli distribution on $\{0,1\}$ with $\bar{P}_0[\Delta_2 = 1] = \phi'(\mathcal{L})$.

(2) There exists a sequence $\{Z_i : i \geq 2\}$ of random variables such that for all $i \geq 2$ one has that

$$\overline{X}_i = (1 - \Delta_i)\widetilde{X}_i + \Delta_i Z_i \tag{7.6}$$

and Δ_i is independent of Z_i and of

$$\mathcal{G}_i := \sigma\{\overline{X}_j : j \leq i-1\}.$$

We will call P the common probability distribution of the sequences $\{\overline{X}_i : i \geq 2\}$, $\{\widetilde{X}_i : i \geq 2\}$, $\{Z_i : i \geq 2\}$ and $\{\Delta_i : i \geq 2\}$, and E the corresponding expectation. From (7.6) note that

$$\frac{1}{n}\sum_{i=1}^{n} \overline{X}_i = \frac{\overline{X}_1}{n} + \frac{1}{n}\sum_{i=2}^{n} \widetilde{X}_i - \frac{1}{n}\sum_{i=2}^{n} \Delta_i \widetilde{X}_i + \frac{1}{n}\sum_{i=1}^{n} \Delta_i Z_i. \tag{7.7}$$

Let us now examine the behavior as $n \to \infty$ of each of the four terms in the right-hand side of (7.7). Clearly, the first term tends to 0 as $n \to \infty$. For the second term, note that on the event $\{D' = \infty\}$, one has that $|\ \overline{X}_1\ |_2^2 \leq c_{30}(\overline{X}_1 \cdot l)^2$ for some constant c_{30}. Therefore, by Proposition 6.3, and the fact that $\widetilde{X}_2$ has the same distribution as $\overline{X}_1$ under $\bar{P}_0[\cdot|D' = \infty]$, we see that

$$E[|\widetilde{X}_2|_2^2] = \bar{E}_0[|\overline{X}_1|_2^2 | D' = \infty] \leq c_{30}\bar{E}_0[(\overline{X}_1 \cdot l)^2 | D' = \infty] < c_{31}, \tag{7.8}$$

for a suitable constant c_{31}. Hence, by the strong law of large numbers, we actually have that P-a.s.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n} \widetilde{X}_i = \lambda_{\mathcal{L}}. \tag{7.9}$$

For the third term in the right-hand side of (7.7) we have by Cauchy-Schwarz inequality that

$$\left|\frac{1}{n}\sum_{i=2}^{n} \Delta_i \widetilde{X}_i\right|_2 \leq \left(\frac{1}{n}\sum_{i=2}^{n} |\widetilde{X}_i|^2\right)^{\frac{1}{2}} \left(\frac{1}{n}\sum_{i=2}^{n} \Delta_i\right)^{\frac{1}{2}}. \tag{7.10}$$

Again by (7.8) and Proposition 6.3, we know that there is a constant c_{32} such that P-a.s.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n} |\widetilde{X}_i|_2^2 = \bar{E}_0[|\overline{X}_1|_2^2 | D' = \infty] \leq c_{32}.$$

As a result, from (7.10) we see that

$$\overline{\lim}_{n\to\infty}\left|\frac{1}{n}\sum_{i=2}^{n}\Delta_i\widetilde{X}_i\right|_2 \leq \sqrt{c_{32}\phi'(\mathcal{L})}. \tag{7.11}$$

For the fourth term of the right-hand side of (7.7), we note setting $\overline{Z}_i^{(\mathcal{L})} := E[Z_i \mid \mathcal{G}_i]$ that

$$M_n^j := \sum_{i=2}^{n}\frac{\Delta_i(Z_i-\overline{Z}_i)\cdot e_j}{i} \quad \text{for } n\geq 2,\ j\in\{1,2,\ldots,n\}$$

is a martingale with mean zero with respect to the filtration $\{\mathcal{G}_i : i\geq 1\}$. Thus, from the Burkholder-Gundy inequality [22, page 151, Chapter 14], we know that there is a constant c_{33} such that for all $j\in\{1,2,\ldots,d\}$

$$E\left[\left(\sup_n M_n^j\right)^2\right] \leq c_{33}E\left[\sum_{i=2}^{\infty}\frac{|\Delta_i(Z_i-\overline{Z}_i)|_2^2}{i^2}\right]. \tag{7.12}$$

Now, from (7.6), note that for all $i\geq 2$, $|\Delta_i Z_i|\leq|\bar{X}_i|$. It follows that there exists a constant c_{34} such that

$$E[|Z_i|_2^2|\mathcal{G}_i] \leq \frac{1}{\phi'(\mathcal{L})}E_0[|\overline{X}_1|_2^2, D'=\infty|\mathfrak{F}_{0,\mathcal{L}}] \leq \frac{1}{\phi'(\mathcal{L})}c_{34}, \tag{7.13}$$

where we have used Proposition 6.3 and Lemma 6.4 in the second inequality. So that by (7.12) we see that the martingale $\{M_n^j : n\geq 1\}$ converges P-a.s. to a random variable for any $j\in\{1,2,\ldots,d\}$. Thus, by Kronecker's lemma applied to each component $j\in\{1,2,\ldots,d\}$, we conclude that P-a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\Delta_i(Z_i-\overline{Z}_i)=0. \tag{7.14}$$

Now, note from (7.13) that there is a constant c_{35} such that

$$|\overline{Z}_i|_2 \leq E[|Z_i|_2^2 \mid \mathcal{G}_i]^{\frac{1}{2}} \leq c_{35}\phi'(\mathcal{L})^{-\frac{1}{2}}. \tag{7.15}$$

Therefore, P-a.s. we have that

$$\overline{\lim}_{n\to\infty}\left|\frac{1}{n}\sum_{i=2}^{n}\Delta_i\overline{Z}_i\right|_2 \leq c_{35}\phi'(\mathcal{L})^{-\frac{1}{2}}\overline{\lim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Delta_i \leq c_{35}\phi'(\mathcal{L})^{\frac{1}{2}}. \tag{7.16}$$

Substituting the right-most hand side of (7.16), (7.11) and (7.9) into (7.7), we conclude the proof of inequality (7.3) provided we set $\eta_{\mathcal{L}}=c_{36}\phi'(\mathcal{L})^{\frac{1}{2}}$ for some constant c_{36}.

Let us now prove the inequality (7.5). By an argument similar to the one presented in [5, page 892] to show that the random variable τ_1 has a lower bound of order $\kappa^{-\mathcal{L}}$, we can show that $X_{\tau_1}\cdot l$ is bounded from below by a sum of i.i.d. random variables $\sum_{i=1}^{N}U_i$, where $\{U_i : i\geq 1\}$ take values in $\{1,2,\ldots\}$ with law $P[U_i=n]=(1-\kappa)\kappa^n$ for $1\leq n<\infty$, while $N:=\min\{i\geq 1 : U_i=\mathcal{L}\}$. We then have that

$$|E_0[X_{\tau_1}, D'=\infty]|_2 \geq E_0[X_{\tau_1}\cdot l, D'=\infty] \geq c_{37}E[N] = c_{37}\kappa^{-\mathcal{L}},$$

for some constant c_{37}.

7.3 Proof of Theorem 2.1

It will be enough to prove that there is a constant c_{38} such that for all $\mathcal{L}\geq 1$ one has that

$$\overline{\lim}_{n\to\infty}\left|\frac{X_n}{|X_n|_2}-\frac{\lambda_{\mathcal{L}}}{|\lambda_{\mathcal{L}}|_2}\right|_2 < c_{38}\frac{\eta_{\mathcal{L}}}{|\lambda_{\mathcal{L}}|_2}. \tag{7.17}$$

Indeed, let us assume for the moment that (7.17) holds. By compactness, we know that we can choose a sequence $\{\mathcal{L}_m, m \geq 1\}$ such that

$$\lim_{m\to\infty} \frac{\lambda_{\mathcal{L}_m}}{|\lambda_{\mathcal{L}_m}|_2} = \hat{v}, \tag{7.18}$$

exists. On the other hand, by (7.2) and the inequality (7.5) of Proposition 7.2, we know that $\lim_{m\to\infty} \frac{\eta_{\mathcal{L}_m}}{|\lambda_{\mathcal{L}_m}|_2} = 0$. Now note that by the triangle inequality and (7.17), for every $m \geq 1$ one has that

$$\overline{\lim}_{n\to\infty} \left| \frac{X_n}{|X_n|_2} - \hat{v} \right|_2 \leq c_{38} \frac{\eta_{\mathcal{L}_m}}{|\lambda_{\mathcal{L}_m}|_2} + \left| \frac{\lambda_{\mathcal{L}_m}}{|\lambda_{\mathcal{L}_m}|_2} - \hat{v} \right|_2. \tag{7.19}$$

Taking the limit $m \to \infty$ in (7.19) using (7.18) we prove Theorem 2.1.

Let us hence prove inequality (7.17). Choose a nondecreasing sequence $\{k_n : n \geq 1\}$, P-a.s. tending to $+\infty$ so that for all $n \geq 1$ one has that

$$\tau_{k_n} \leq n < \tau_{k_n+1}.$$

Notice that

$$\frac{X_n}{|X_n|_2} = \left(\frac{X_n - X_{\tau_{k_n}}}{|X_n|_2} \right) + \left(\frac{X_{\tau_{k_n}}}{k_n} \frac{k_n}{|X_n|_2} \right). \tag{7.20}$$

On the other hand, we assume for the time being, that for large enough $\mathcal{L}$ we have proved that

$$\overline{\lim}_{n\to\infty} \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} = 0. \tag{7.21}$$

Note first that (7.21) implies that

$$\overline{\lim}_{n\to\infty} \frac{|X_n - X_{\tau_{k_n}}|_2}{|X_n|_2} = 0. \tag{7.22}$$

Indeed, note that $|X_n|_2 \geq X_n \cdot l \geq X_{\tau_{k_n}} \cdot l \geq k_n \mathcal{L} \frac{|l|_2}{|l|_1}$, which in combination with (7.21) implies (7.22). Also, from (7.21) and the fact that

$$\frac{|X_{\tau_{k_n}}|_2}{k_n} - \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} \leq \frac{|X_n|_2}{k_n} \leq \frac{|X_{\tau_{k_n}}|_2}{k_n} + \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n}, \tag{7.23}$$

we see that

$$\overline{\lim}_{n\to\infty} \left| \frac{\kappa^{\mathcal{L}} |X_n|_2}{k_n} - |\lambda_{\mathcal{L}}|_2 \right|_2 \leq \eta_{\mathcal{L}}. \tag{7.24}$$

Combining (7.22) and (7.24) with (7.20) we get (7.17). Thus, it is enough to prove the claim in (7.21). To this end, note that

$$\frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} \leq \sup_{j\geq 0} \frac{|X_{(\tau_{k_n}+j)\wedge\tau_{k_n+1}} - X_{\tau_{k_n}}|_2}{k_n}. \tag{7.25}$$

We now consider the sequence

$$\{\hat{X} : k \geq 1\} := \left\{ \kappa^{\mathcal{L}} \sup_{j\geq 0} |X_{(\tau_k+j)\wedge(\tau_{k+1})} - X_{\tau_k}| : k \geq 1 \right\}.$$

A coupling decomposition as in the proof of Proposition 7.2, enables us to define these random variables in an enlarged probability space with a probability measure P, where there exist two i.i.d. sequences $(\mathrm{X}_k)_{k\geq 1}$, $(\Delta_k)_{k\geq 1}$ and a sequence $(Y_k)_{k\geq 1}$, such that the following is satisfied:

- For $k \geq 1$, the common law of X_k is the same as the law of $\hat{X}_1$ under $\bar{P}[\cdot \mid D' = \infty]$, and one has that Δ_k has a Bernoulli law with values in the set $\{0, 1\}$ independent of $\mathcal{G}_k$ and $\mathrm{P}[\Delta_k = 1] = \phi'(\mathcal{L})$.

- P-a.s. for $k \geq 1$, we have that

$$\hat{X}_k = (1 - \Delta_k)\mathrm{X}_k + \Delta_k Y_k.$$

Furthermore, quite similar arguments as the ones given in the proof of Proposition 7.2 allow us to conclude that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{|\mathrm{X}_j|}{n} = \mathrm{E}[|\hat{X}_1| \mid D' = \infty] < \infty,$$
$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\Delta_j(Y_j - \widetilde{Y}_j)}{n} = 0$$

and for all $n \geq 0$ that

$$\sum_{j=1}^{n} \frac{|\Delta_j \widetilde{Y}_j|}{n} \leq c_{39}\phi'(\mathcal{L})^{\frac{1}{2}}, \tag{7.26}$$

for some constant c_{39}, where $\widetilde{Y}_j := \mathrm{E}[Y_j \mid \mathcal{G}_j]$. Therefore, using the equality

$$\frac{\hat{X}_k}{k} = \frac{\mathrm{X}_k}{k} + \frac{\Delta_k(Y_k - \widetilde{Y}_k)}{k} + \frac{\Delta_k \widetilde{Y}_k}{k}, \tag{7.27}$$

we see that

$$\lim_{k\to\infty} \frac{\mathrm{X}_k}{k} = 0, \tag{7.28}$$

which finishes the proof.

A Proof of the auxiliary lemmas

A.1 Proof of Lemma 6.4

For each $A \in \mathfrak{F}_{x,\mathcal{L}}$ [cf. (6.7)], we define

$$\nu[A] := \mathbb{E}[P_{x,\omega}[D' = \infty]\mathbb{1}_A] \tag{A.1}$$

and

$$\mu[A] := (P_0[D' = \infty] + \phi(\mathcal{L}))\,\mathbb{P}[A] - \nu[A]. \tag{A.2}$$

Clearly (A.1) defines a measure on $(\Omega, \mathfrak{F}_{x,\mathcal{L}})$. We will show that (A.2) does too. Indeed, take an $A \in \mathfrak{F}_{x,\mathcal{L}}$ and note that $P_{x,\omega}[D = \infty]$ is $\sigma\{\omega(y) : y \in C(x, l, \alpha)\}$-measurable. Therefore, by the cone-mixing condition $(CM)_{\alpha,\phi}|l$ one has that

$$\nu[A] \leq (P_0[D' = \infty] + \phi(\mathcal{L}))\mathbb{P}[A],$$

and hence (A.2) defines a measure μ on $(\Omega, \mathfrak{F}_{x,\mathcal{L}})$. Consider the increasing sequence $\{A_n : n \geq 1\}$ of $\mathfrak{F}_{x,\mathcal{L}}$-measurable sets defined by

$$A_n := \left\{\omega \in \Omega : \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}] > P_0[D' = \infty] + \phi(\mathcal{L}) + \frac{1}{n}\right\}$$

and define

$$A := \bigcup_{n\geq 1} A_n.$$

Observe that for each $n \geq 1$ we have that

$$0 \leq \mu(A_n) = (P_0[D = \infty] + \phi(\mathcal{L}))\mathbb{P}[A_n] - \mathbb{E}[\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}]\mathbb{1}_{A_n}] \leq -\frac{1}{n}\mathbb{P}[A_n].$$

Therefore, one has that for each $n \geq 1$, $\mathbb{P}[A_n] = 0$ and consequently $\mathbb{P}[A] = 0$. Observing that

$$A = \{\omega \in \Omega : \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}] > P_0[D' = \infty] + \phi(\mathcal{L})\},$$

we see that

$$\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}] - P_0[D' = \infty] \le \phi(\mathcal{L}). \tag{A.3}$$

Following the same argument used to show (A.3), but changing the event $\{D' = \infty\}$ by $\{D' < \infty\}$, one can prove that

$$-\phi(\mathcal{L}) \le \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,\mathcal{L}}] - P_0[D' = \infty],$$

which finished the proof.

A.2 Proof of Lemma 6.5

To simplify the proof, we will show that the second moment of

$$\mathfrak{M}' := \sup_{0 \le n \le D'} (X_n - X_0) \cdot l$$

is bounded from above. Note that

$$\begin{aligned} E_0[\mathfrak{M}'^2, D' < \infty|\mathfrak{F}_{0,\mathcal{L}}] &\le P_0[D' < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &+ \sum_{m \ge 0} 2^{2(m+1)} P_0[2^m \le \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,\mathcal{L}}]. \end{aligned} \tag{A.4}$$

Therefore, it is enough to obtain an appropriate upper bound of the probability

$$P_0[2^m \le \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,\mathcal{L}}]$$

when m is large. Defining

$$D'(0) := \inf\{n \ge 0 : \; X_n \notin C(0, l, \alpha)\},$$

note that,

$$\begin{aligned} &P_0[2^m \le \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le P_0[T^l_{2^m} < D' < \infty, T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &\le P_0[X_{T^l_{2^m}} \notin \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0), T^l_{2^m} < D' < \infty \mid \mathfrak{F}_{0,\mathcal{L}}] \\ &+ P_0[X_{T^l_{2^m}} \in \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0), T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,\mathcal{L}}]. \end{aligned} \tag{A.5}$$

From $(UWP)_{M,\mathfrak{c}}|l$, we know that there is a constant c_{40} such that the first term of the right-most hand side in (A.5) is bounded by

$$P_0[X_{T_{B_{2^m,\mathfrak{c}2^m,l}(0)}} \notin \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0), T_{B_{2^m,\mathfrak{c}2^m,l}(0)} < \hat{T}^l_0 | \mathcal{H}_{0,l}] \le c_{40} 2^{-Mm}, \tag{A.6}$$

where $\hat{T}^l_0$ is defined in (2.5). As for the second term in the right-most hand side of (A.5), it will be useful to introduce the set

$$F_m := \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0).$$

By the strong Markov property we have the bound

$$\begin{aligned} P_0[X_{T_{B_{2^m,\mathfrak{c}2^m,l}(0)}} \in \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0), T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ \le \sum_{y \in F_m} P_y[T^l_{2^{m+1}} > D'(0) \mid \mathfrak{F}_{0,\mathcal{L}}]. \end{aligned} \tag{A.7}$$

In order to estimate this last conditional probability, we will obtain a lower bound for its complement. To simplify the computations which follow, for each $x \in \mathbb{Z}^d$ we introduce the notation

$$B_x := B_{2^{m-1},\mathfrak{c}2^{m-1},l}(x).$$

Now, note that under the assumption (6.9) we have that

$$\mathfrak{c}\left(2^m + 2^{m-1}\right) \le \frac{1}{\alpha} 2^{m-1},$$

which implies that the boxes B_y and B_z, for all $y \in F_m$ and $z \in \partial^+ B_y$, are inside the cone $C(0,l,\alpha)$ (see Figure 3).

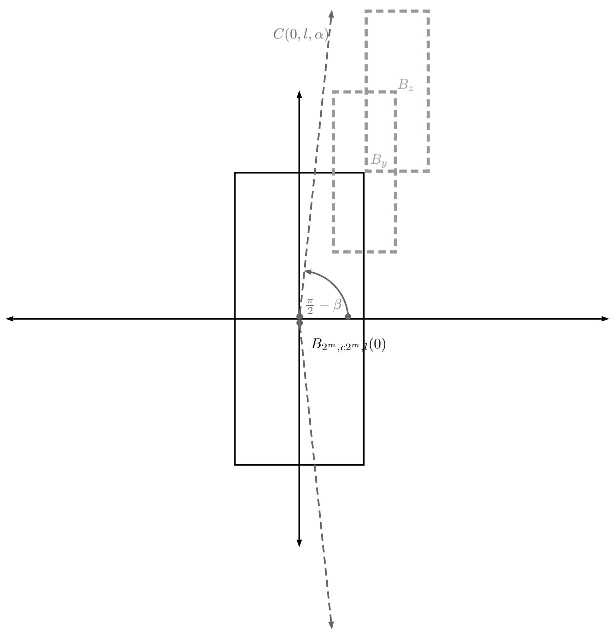

Figure 3: The boxes B_y and B_z are inside of $C(0,l,\alpha)$.

Therefore, fixing $y \in F_m$, it follows that

$$P_y[T^l_{2^{m+1}} < D'(0) \mid \mathfrak{F}_{0,\mathcal{L}}] \geq \sum_{z \in \partial^+ B_y} \mathbb{E}[P_{y,\omega}[X_{T_{B_y}} \in \partial^+ B_y, X_{T_{B_y}} = z, (X_{T_{B_z}} \in \partial^+ B_z) \circ \theta_{T_{B_y}}] | \mathfrak{F}_{0,\mathcal{L}}]. \tag{A.8}$$

To estimate the right-hand side of the above inequality, it will be convenient to introduce the set

$$\bar{F}_m := \partial[\cup_{y \in F_m} B_y] \cap \{R([2^{m-1} + 2^m, \infty) \times \mathbb{R}^{d-1})\},$$

and the event

$$G_{\bar{F}_m} := \{\omega \in \Omega : \ P_{z,\omega}[X_{T_{B_z}} \in \partial^+ B_z] > 1 - 2^{-\frac{M(m-1)}{2}}, \text{ for all } z \in \bar{F}_m\}.$$

Using the strong Markov property, we can now bound from below the right-hand side of inequality (A.8) by

$$(1 - 2^{-\frac{M(m-1)}{2}}) \left(P_y[X_{T_{B_y}} \in \partial^+ B_y | \mathfrak{F}_{0,\mathcal{L}}] - P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,\mathcal{L}}] \right). \tag{A.9}$$

In turn, by means of the polynomial condition and the fact that the boxes B_y and B_z are inside the cone $C(0,l,\alpha)$ we see that (A.9) is greater than or equal to

$$(1 - 2^{-\frac{M(m-1)}{2}}) \left(1 - c_{40} 2^{-M(m-1)} - P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,\mathcal{L}}] \right). \tag{A.10}$$

Now, note that

$$\begin{aligned} P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,\mathcal{L}}] &\leq \sum_{x \in \bar{F}_m} 2^{\frac{M(m-1)}{2}} P_x[X_{T_{B_x}} \notin \partial^+ B_x | \mathfrak{F}_{0,\mathcal{L}}] \\ &\leq c_{40} |\bar{F}_m| 2^{-\frac{M(m-1)}{2}} \leq c_{40} (4\mathfrak{c})^{d-1} 2^{m(d-1)} 2^{-\frac{M(m-1)}{2}}, \end{aligned} \tag{A.11}$$

where in the first inequality we have used Chebyshev's inequality, in the second one the assumption that $(UWP)_{M,\mathfrak{c}}|l$ is satisfied and in the third one the bound $|\bar{F}_{2m}| \leq (4\mathfrak{c})^{d-1}2^{m(d-1)}$.

Consequently, inserting the estimates (A.11) into (A.10) and combining this with inequality (A.8) we conclude that

$$P_y[T^l_{2^{m+1}} \leq D'(0) \mid \mathfrak{F}_{0,\mathcal{L}}] \geq (1-2^{-\frac{M(m-1)}{2}})(1-c_{40}2^{-\frac{M(m-1)}{2}} - c_{35}(4\mathfrak{c})^{d-1}2^{m(d-1)}2^{-\frac{M(m-1)}{2}}) \\ \geq 1 - c_{40}3(4\mathfrak{c})^{d-1}2^{m(d-1)}2^{-\frac{M(m-1)}{2}}. \tag{A.12}$$

Using the bound (A.12) in (A.7), together with the estimate $|F_m| \leq (2\mathfrak{c})^{d-1}2^{m(d-1)}$, we see that

$$P_0[X_{T_{B_{2^m,\mathfrak{c}2^m,l}(0)}} \in \partial^+ B_{2^m,\mathfrak{c}2^m,l}(0), T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,\mathcal{L}}] \\ \leq 3c_{40}(4\mathfrak{c})^{2(d-1)}2^{2m(d-1)}2^{-\frac{M(m-1)}{2}}. \tag{A.13}$$

Combining the estimates (A.13), (A.6), (A.5) with (A.4) we conclude that

$$\begin{aligned} &E_0[\mathfrak{M}'^2, D' < \infty|\mathfrak{F}_{0,\mathcal{L}}] \\ &\quad \leq 1 + 4c_{40}(4\mathfrak{c})^{2(d-1)} \sum_{m\geq 0} 2^{2(m+1)}2^{2m(d-1)}2^{-\frac{M(m-1)}{2}} \\ &\quad \leq 1 + 4c_{40}(4\mathfrak{c})^{2(d-1)} \sum_{m\geq 0} 2^{-m} \leq c_{41}, \end{aligned}$$

where in the second to last inequality we have used the fact that $M > 4d+1$ and c_{41} is a constant that does not depend on $\mathcal{L}$. This completes the proof of the lemma.

A.3 Proof of Lemma 6.6

Here we will prove Lemma 6.6. Let us first remark that it will be enough to show that there exists a constant $c_{42} > 0$ such that for all $\mathcal{L} \in |u|_1\mathbb{N}$

$$Q[D_{0,\mathcal{L}^2}] \leq 1 - c_{42}\mathcal{L}^2\kappa^{\mathcal{L}}. \tag{A.14}$$

Indeed, using this inequality and the product structure of Q, for all $n \geq \mathcal{L}^2$ one has that

$$Q[D_{0,n}] \leq (1 - c_{11}\mathcal{L}^2\kappa^{\mathcal{L}})^{[\frac{n}{\mathcal{L}^2}]}.$$

In order to prove (A.14), for $j = \mathcal{L}^2 - \mathcal{L}$ and $i = 0, 1, \ldots, j$ consider the events

$$A_i = \{\varepsilon : (\varepsilon_i, \ldots, \varepsilon_{i+\mathcal{L}-1}) = \bar{\varepsilon}^{(\mathcal{L})}\}.$$

Then, by the inclusion-exclusion principle we have that

$$Q[(D_{0,\mathcal{L}^2})^c] \geq \sum_{0\leq j_1\leq j} Q[A_{j_1}] - \sum_{0\leq j_1<j_2\leq j} Q[A_{j_1} \cap A_{j_2}]. \tag{A.15}$$

Now, note that

$$\begin{aligned} &\sum_{0\leq j_1<j_2\leq j} Q[A_{j_1} \cap A_{j_2}] \leq j\kappa^{\mathcal{L}+1} + (j-1)\kappa^{\mathcal{L}+2} + \ldots \\ &\ldots + (j-\mathcal{L}+1)\kappa^{2\mathcal{L}} + (j-\mathcal{L})\kappa^{2\mathcal{L}} + \ldots + (j-(j-1))\kappa^{2\mathcal{L}} \\ &\leq j\kappa^{\mathcal{L}}\sum_{n=1}^{\mathcal{L}}\kappa^n + \kappa^{2\mathcal{L}}(j-\mathcal{L})^2 \leq \mathcal{L}^2\kappa^{\mathcal{L}}\frac{1-\kappa^{\mathcal{L}+1}}{1-\kappa} + \mathcal{L}^4\kappa^{2\mathcal{L}} \\ &\leq c_{43}\mathcal{L}^2\kappa^{\mathcal{L}}, \end{aligned} \tag{A.16}$$

for some constant c_{43}. Since $Q[A_i] = \kappa^{\mathcal{L}}$ for all $1 \le i \le j$, we conclude from (A.15) and the bound (A.16) that there is a constant c_{44} such that

$$Q[D_{0,\mathcal{L}^2}] = 1 - Q[(D_{0,\mathcal{L}^2})^c] \le 1 - c_{44}\mathcal{L}^2\kappa^{\mathcal{L}}.$$

This finishes the proof.

B Cone-mixing and ergodicity

The main objective of this appendix is to establish that any stationary probability measure $\mathbb{P}$ defined on the canonical σ-algebra $\mathfrak{F}$, which satisfies property $(CM)_{\alpha,\phi}|l$ [cf. Subsection2.2] is ergodic with respect to space-shifts. We do not claim any originality about such an implication, but we have decided to included the proof of it here for completeness.

Let us recall that a set $E \in \mathfrak{F}$ is an invariant set if

$$\vartheta_x^{-1}E := E$$

for all $x \in \mathbb{Z}^d$ [cf. (2.2)].

Theorem B.1. *Let $\alpha > 0$ and $\phi \in \Phi$. Consider the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and assume that $\mathbb{P}$ is stationary and that it satisfies $(CM)_{\alpha,\phi}|l$. Then the probability measure $\mathbb{P}$ is ergodic, i.e. for any invariant set $E \in \mathfrak{F}$ we have that*

$$\mathbb{P}[E] \in \{0, 1\}.$$

Proof. Let $E \in \mathfrak{F}$ be an invariant set. Note that for each $\epsilon > 0$ there exists a cylinder measurable set $A \in \mathfrak{F}$ such that

$$\mathbb{P}[A \triangle E] < \epsilon.$$

Since A is a cylinder measurable set, there exists a finite subset $F \subset \mathbb{Z}^d$ such that

$$A = \{\omega \in \Omega : (\omega(x) : x \in F) \in \mathcal{B}(\mathcal{P}_d^F)\}, \tag{B.1}$$

where $\mathcal{B}(\mathcal{P}_d^F)$ stands for the Borel σ-algebra on the subset $\mathcal{P}_d^F$. Therefore, we can find an $L > 0$ and $y \in \mathbb{Z}^d$ such that

$$A \in \sigma\{\omega(z, \cdot) : \ z \cdot l \le y \cdot l - L\}$$

along with

$$\vartheta_x A \in \sigma\{\omega(z, \cdot) : \ z \in C(y, l, \alpha)\}.$$

Without loss of generality we will also assume that

$$\phi(L) < \epsilon.$$

We can suppose that $\mathbb{P}[E] > 0$, otherwise there is nothing to prove. So as to complete the proof we have to show that $\mathbb{P}[E] = 1$. Therefore taking ϵ small enough we can suppose that $\mathbb{P}[A] > 0$. Thus, using the cone-mixing property, we get that

$$-\mathbb{P}[A]\phi(L) \le \mathbb{P}[A \cap (\vartheta_x A)^c] - \mathbb{P}[A]\mathbb{P}[A^c] \le \mathbb{P}[A]\phi(L). \tag{B.2}$$

On the other hand, since E is an invariant set, we see that for every $x \in \mathbb{Z}^d$ we have

$$\mathbb{P}[\vartheta_x A \triangle E] = \mathbb{P}[\vartheta_x A \triangle \vartheta_x E] = \mathbb{P}[A \triangle E] < \epsilon, \tag{B.3}$$

which implies

$$\mathbb{P}[A \triangle \vartheta_x A] \le \mathbb{P}[(A \triangle E) \cup (\vartheta_x A \triangle \vartheta_x E)] < 2\epsilon. \tag{B.4}$$

In turn, from inequality (B.4), it is clear that $\mathbb{P}[A\cap(\vartheta_x A)^c] < 2\epsilon$. Now, using the inequality (B.2) one has that

$$\mathbb{P}[A]\mathbb{P}[\Omega - A] \leq 2\epsilon + \mathbb{P}[A]\phi(L).$$

As a result, we see that

$$\mathbb{P}[E]\mathbb{P}[E^c] < (\mathbb{P}[A]+\epsilon)(\mathbb{P}[A^c]+\epsilon) = \mathbb{P}[A]\mathbb{P}[A^c]+\epsilon+\epsilon^2 < 4\epsilon+\phi(L) \leq 5\epsilon.$$

Hence, since $\epsilon > 0$ is arbitrary we conclude that $\mathbb{P}[E]\mathbb{P}[E^c] = 0$. Therefore if $\mathbb{P}[E] > 0$, this implies $\mathbb{P}[E] = 1$. □

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