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Continuity of the time and isoperimetric constants in supercritical percolation*

Olivier Garet† Régine Marchand† Eviatar B. Procaccia‡
Marie Théret§

Abstract

We consider two different objects on supercritical Bernoulli percolation on the edges of $\mathbb{Z}^d$: the time constant for i.i.d. first-passage percolation (for $d \geq 2$) and the isoperimetric constant (for $d = 2$). We prove that both objects are continuous with respect to the law of the environment. More precisely we prove that the isoperimetric constant of supercritical percolation in $\mathbb{Z}^2$ is continuous in the percolation parameter. As a corollary we obtain that normalized sets achieving the isoperimetric constant are continuous with respect to the Hausdorff metric. Concerning first-passage percolation, equivalently we consider the model of i.i.d. first-passage percolation on $\mathbb{Z}^d$ with possibly infinite passage times: we associate with each edge e of the graph a passage time $t(e)$ taking values in $[0, +\infty]$, such that $\mathbb{P}[t(e) < +\infty] > p_c(d)$. We prove the continuity of the time constant with respect to the law of the passage times. This extends the continuity property of the asymptotic shape previously proved by Cox and Kesten [8, 10, 20] for first-passage percolation with finite passage times.

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1 Introduction

We consider supercritical bond percolation on $\mathbb{Z}^d$, with parameter $p > p_c(d)$, the critical parameter for this percolation. Almost surely, there exists a unique infinite cluster $\mathcal{C}_\infty$ – see for instance Grimmett's book [16]. We study the continuity properties of two

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†Université de Lorraine, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France and CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France. E-mail: Olivier.Garet@univ-lorraine.fr,Regine.Marchand@univ-lorraine.fr

‡Texas A&M University, Mailstop 3368, College Station TX 77843, USA. E-mail: procaccia@math.tamu.edu

§LPMA UMR 7599, Université Paris Diderot, Sorbonne Paris Cité, CNRS, F-75013 Paris, France. E-mail: marie.theret@univ-paris-diderot.fr

distinct objects defined on this infinite cluster: the isoperimetric (or Cheeger) constant, and the asymptotic shape (or time constant) for an independent first-passage percolation. In this section, we introduce briefly the studied objects and state the corresponding results: more precise definitions will be given in the next section.

1.1 Isoperimetric constant of the infinite cluster in dimension 2

For a finite graph $\gimel = (V(\gimel), E(\gimel))$, the isoperimetric constant is defined as

$$\varphi_\gimel = \min\left\{ \frac{|\partial_\gimel A|}{|A|} : A \subset V(\gimel), 0 < |A| \leq \frac{|V(\gimel)|}{2} \right\},$$

where $\partial_\gimel A$ is the edge boundary of A in $\gimel$, $\partial_\gimel A = \{e = (x,y) \in E(\gimel) : x \in A, y \notin A\}$, and $|B|$ denotes the cardinal of the finite set B.

We consider the isoperimetric constant $\varphi_n(p)$ of $\mathcal{C}_\infty \cap [-n,n]^d$, the intersection of the infinite component of supercritical percolation of parameter p with the box $[-n,n]^d$:

$$\varphi_n(p) = \min\left\{ \frac{|\partial_{\mathcal{C}_\infty \cap [-n,n]^d} A|}{|A|} : A \subset \mathcal{C}_\infty \cap [-n,n]^d, 0 < |A| \leq \frac{|\mathcal{C}_\infty \cap [-n,n]^d|}{2} \right\}.$$

In several papers (e.g. [2], [23], [24], [3]), it was shown that there exist constants $c, C > 0$ such that $c < n\varphi_n(p) < C$, with probability tending rapidly to 1. This led Benjamini to conjecture the existence of $\lim_{n\to+\infty} n\varphi_n(p)$. In [26], Rosenthal and Procaccia proved that the variance of $n\varphi_n(p)$ is smaller than Cn^{2-d}, which implies $n\varphi_n(p)$ is concentrated around its mean for $d \geq 3$. In [4], Biskup, Louidor, Procaccia and Rosenthal proved the existence of $\lim_{n\to+\infty} n\varphi_n(p)$ for $d = 2$. This constant is called the Cheeger constant. In addition, a shape theorem was obtained: any set yielding the isoperimetric constant converges in the Hausdorff metric to the normalized Wulff shape $\widehat{W}_p$, with respect to a specific norm given in an implicit form, see Proposition 2.4 below. For additional background and a wider introduction on Wulff construction in this context, the reader is referred to [4]. Our first result is the continuity of the Cheeger constant and of the Wulff shape in dimension $d = 2$:

Theorem 1.1. *For $d = 2$, the applications*

$$p \in (p_c(2), 1] \mapsto \lim_{n\to+\infty} n\varphi_n(p) \qquad \text{and} \qquad p \in (p_c(2), 1] \mapsto \widehat{W}_p$$

are continuous, the last one for the Hausdorff distance between non-empty compact sets of $\mathbb{R}^2$.

1.2 First-passage percolation on the infinite cluster in dimension $d \geq 2$

Consider a fixed dimension $d \geq 2$. First-passage percolation on $\mathbb{Z}^d$ was introduced by Hammersley and Welsh [17] as a model for the spread of a fluid in a porous medium. To each edge of the $\mathbb{Z}^d$ lattice is attached a nonnegative random variable $t(e)$ which corresponds to the travel time needed by the fluid to cross the edge. When the passage times are independent identically distributed variables with common distribution G, with suitable moment conditions, the time needed to travel from 0 to nx behaves like $n\mu_G(x)$ for large n, where μ_G is a semi-norm associated to G called the time constant; Cox and Durrett [9] proved this result under necessary and sufficient integrability conditions on the distribution G of the passage times. Kesten in [18] proved that the semi-norm μ_G is a norm if and only if $G(\{0\}) < p_c(d)$. In casual terms, the asymptotic shape theorem (in its geometric form) says that in this case, the random ball of radius n, *i.e.* the set of points that can be reached within time n from the origin, asymptotically looks like $n\mathcal{B}_{\mu_G}$, where $\mathcal{B}_{\mu_G}$ is the unit ball for the norm μ_G. The ball $\mathcal{B}_{\mu_G}$ is thus called the asymptotic shape associated to G.

A natural extension is to replace the $\mathbb{Z}^d$ lattice by a random environment given by the infinite cluster $\mathcal{C}_\infty$ of a supercritical Bernoulli percolation model. This is equivalent to allow $t(e)$ to be equal to $+\infty$. The existence of a time constant in first-passage percolation in this setting was first proved by Garet and Marchand in [12], in the case where $(t(e)\mathbb{1}_{t(e)<+\infty})$ is a stationary integrable ergodic field. Recently, Cerf and Théret [6] focused of the case where $(t(e)\mathbb{1}_{t(e)<+\infty})$ is an independent field, and managed to prove the existence of an appropriate time constant without any integrability assumption. In the following, we adopt the settings of Cerf and Théret: the passage times are independent random variables with common distribution G taking its values in $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$, and we denote by μ_G the corresponding time constant.

Our second result is the continuity of the time constants $\mu_G(x)$ with respect to the distribution G of the passage times, uniformly in the direction. More precisely, let $(G_n)_{n\in\mathbb{N}}$ and G be probability measures on $[0,+\infty]$. We say that G_n converges weakly towards G when n goes to infinity, and we write $G_n \overset{d}{\to} G$, if for any continuous bounded function $f:[0,+\infty]\mapsto[0,+\infty)$ we have

$$\lim_{n\to+\infty}\int_{[0,+\infty]} f\,dG_n \;=\; \int_{[0,+\infty]} f\,dG\,.$$

Equivalently, $G_n \overset{d}{\to} G$ if and only if $\lim_{n\to\infty} G_n([t,+\infty]) = G([t,+\infty])$ for all $t\in[0,+\infty)$ such that $t\mapsto G([t,+\infty])$ is continuous at t. Let $\mathbb{S}^{d-1} = \{x\in\mathbb{R}^d \,:\, \|x\|_2 = 1\}$.

Theorem 1.2. *Let $G,(G_n)_{n\in\mathbb{N}}$ be probability measures on $[0,+\infty]$ such that for every $n\in\mathbb{N}$, $G_n([0,+\infty)) > p_c(d)$ and $G([0,+\infty)) > p_c(d)$. If $G_n \overset{d}{\to} G$, then*

$$\lim_{n\to+\infty}\sup_{x\in\mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| \;=\; 0\,.$$

Remark 1.3. Notice that if $G_n \overset{d}{\to} G$ and $G([0,+\infty)) > p_c(d)$ then $G_n([0,+\infty)) > p_c(d)$ at least for n large enough.

This result extends the continuity of the time constant in classical first-passage percolation proved by Cox and Kesten [8, 10, 20] to first-passage percolation with possibly infinite passage times. As in the classical case, the semi-norm μ_G is a norm if and only if $G(\{0\}) < p_c(d)$ (see proposition 2.7 below). In that case, we denote by $\mathcal{B}_{\mu_G}$ its unit ball and call it the asymptotic shape associated to G. We can quite easily deduce from Theorem 1.2 the following continuity of the asymptotic shapes when they exist:

Corollary 1.4. Let $G,(G_n)_{n\in\mathbb{N}}$ be probability measures on $[0,+\infty]$ such that for every $n\in\mathbb{N}$, $G_n([0,+\infty)) > p_c(d)$, $G([0,+\infty)) > p_c(d)$ and $G(\{0\}) < p_c(d)$. If $G_n \overset{d}{\to} G$, then

$$\lim_{n\to+\infty} d_H(\mathcal{B}_{\mu_{G_n}},\mathcal{B}_{\mu_G}) \;=\; 0\,,$$

where d_H is the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

Particularly, when $G_p = p\delta_1 + (1-p)\delta_{+\infty}$, the norm μ_{G_p} governs the asymptotic distance in the infinite cluster of a supercritical Bernoulli percolation (see [12, 13, 14]). We get the following corollary:

Corollary 1.5. For $p > p_c(d)$, let us denote by $\mathcal{B}_p$ the unit ball for the norm that is associated to the chemical distance in supercritical bond percolation with parameter p. Then,

$$p\in(p_c(d),1]\mapsto\mathcal{B}_p$$

is continuous for the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

As a key step of the proof of Theorem 1.2, we study the effect of truncations of the passage time on the time constant. Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. For every $K > 0$, we set

$$G^K \;=\; \mathbb{1}_{[0,K)} G + G([K,+\infty])\delta_K \,,$$

i.e., G^K is the law of the truncated passage time $t_G^K(e) = \min(t_G(e), K)$. We have the following control on the effect of these truncations on the time constants:

Theorem 1.6. *Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. Then*

$$\forall x \in \mathbb{Z}^d \quad \lim_{K\to\infty} \mu_{G^K}(x) \;=\; \mu_G(x)\,.$$

Remark 1.7. In fact the convergence in Theorem 1.6 (and in its Corollary 1.8) is uniform with respect to direction. This can be obtained as a consequence of Theorem 1.2 applied to $G_n = G^n$ (or by adapting the proof of the uniformity of the convergence in Theorem 1.2). However, Theorem 1.6, as it is written, is a key step in the proof of Theorem 1.2.

As a consequence of these results, we can approximate the time constants for the chemical distance in supercritical percolation on $\mathbb{Z}^d$ by the time constants for some finite passage times:

Corollary 1.8. Let $p > p_c(d)$, and consider $G = p\delta_1 + (1-p)\delta_{+\infty}$. Then $G^K = p\delta_1 + (1-p)\delta_K$ for all $K \geq 1$ and

$$\forall x \in \mathbb{Z}^d \quad \lim_{K\to\infty} \mu_{G^K}(x) \;=\; \mu_G(x)\,.$$

1.3 Idea of the proofs

Obviously, the two main theorems of the paper, Theorems 1.1 and 1.2, state results of the same nature. Beyond this similarity, their proofs share a common structure and a common renormalisation step. The idea of the delicate part of both proofs is inspired by Cox and Kesten's method in [10]. Consider that some edges of $\mathbb{Z}^d$ are "good" (i.e. open, or of passage time smaller than some constant), and the others are "bad", for a given law of the environment (a parameter p for the percolation, or a given law G of passage times), and look at a path of good edges in this setting. Then change a little bit your environment : decrease p to $p-\varepsilon$, or increase the passage times of the edges. Some edges of the chosen path become bad. To recover a path of good edges, you have to bypass these edges. The most intuitive idea is to consider the cluster of bad edges around each one of them, and to bypass the edge by a short path along the boundary of this cluster. This idea works successfully in Cox and Kesten's paper. Unfortunately in our setting the control we have on these boundaries, or on the number of new bad edges we create, is not good enough. This is the reason why we cannot perform our construction of a modified good path at the scale of the edges. Thus we need to use a coarse graining argument to construct the bypasses at the scale of good blocks.

In section 2, we give more precise definitions of the studied objects and state some preliminary results. In Section 3, we present the renormalization process and the construction of modified paths that will be useful to study both the time constant and the isoperimetric constant. Sections 4 and 5 are devoted to the study of first-passage percolation. In Section 4, we use the renormalization argument to study the effect of truncating the passage times on the time constant. We then use it in Section 5 to prove the continuity of the time constant. Finally Section 6 is devoted to the proof of the continuity of the isoperimetric constant, using again the renormalization argument.

2 Definitions and preliminary results

In this section we give a formal definition of the objects we briefly presented in the introduction. We also present the coupling that will be useful in the rest of the paper, and prove the monotonicity of the time constant.

2.1 Lattice and passage times

Let $d \geq 2$. We consider the graph whose vertices are the points of $\mathbb{Z}^d$, and we put an edge between two vertices x and y if and only if the Euclidean distance between x and y is equal to 1. We denote this set of edges by $\mathbb{E}^d$. We denote by 0 the origin of the graph. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $\|x\|_1 = \sum_{i=1}^d |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$ and $\|x\|_\infty = \max\{|x_i| \,:\, i \in \{1, \dots, d\}\}$.

Let $(t(e), e \in \mathbb{E}^d)$ be a family of i.i.d. random variables taking values in $[0, +\infty]$ with common distribution G. We emphasize that $+\infty$ is a possible value for the passage times, on the contrary to what is assumed in classical first-passage percolation. The random variable $t(e)$ is called the *passage time* of e, i.e., it is the time needed to cross the edge e. If x, y are vertices in $\mathbb{Z}^d$, a path from x to y is a sequence $r = (v_0, e_1, \dots, e_n, v_n)$ of vertices $(v_i)_{i=0,\dots,n}$ and edges $(e_i)_{i=1,\dots,n}$ for some $n \in \mathbb{N}$ such that $v_0 = x$, $v_n = y$ and for all $i \in \{1, \dots, n\}$, e_i is the edge of endpoints v_{i-1} and v_i. We define the length $|r|$ of a path r as its number of edges and we define the passage time of r by $T(r) = \sum_{e \in r} t(e)$. We obtain a random pseudo-metric T on $\mathbb{Z}^d$ in the following way (the only possibly missing property is the separation of distinct points):

$$\forall x, y \in \mathbb{Z}^d\,, \quad T(x,y) \,=\, \inf\{T(r) \,:\, r \text{ is a path from } x \text{ to } y\} \in [0, +\infty]\,.$$

Since different laws appear in this article, we put a subscript G on our notations to emphasize the dependence with respect to the probability measure G : $t_G(e)$, $T_G(r)$ and $T_G(x,y)$.

As we are interested in the asymptotic behavior of the pseudo-metric T_G, we will only consider laws G on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$. Here and in the following, $p_c(d)$ denotes the critical parameter for bond Bernoulli percolation on $(\mathbb{Z}^d, \mathbb{E}^d)$. Thus there a.s. exists a unique infinite cluster $\mathcal{C}_{G,\infty}$ in the super-critical percolation $(\mathbb{1}_{\{t_G(e) < \infty\}}, e \in \mathbb{E}^d)$ that only keeps edges with finite passage times. Our generalized first-passage percolation model with time distribution G is then equivalent to standard i.i.d. first-passage percolation (where the passage time of an edge e is the law of $t_G(e)$ conditioned to be finite) on a super-critical Bernoulli percolation performed independently (where the parameter for an edge to be closed is $G(\{+\infty\})$). For instance, if we take $G_p = p\delta_1 + (1-p)\delta_{+\infty}$ with $p > p_c(d)$, the pseudo-distance T_{G_p} is the chemical distance in supercritical bond percolation with parameter p.

To get round the fact that the times T_G can take infinite values, we introduce some regularized times $\widetilde{T}_G^{\mathcal{C}}$, for well chosen sets $\mathcal{C}$. These regularized passage times have better integrability properties. Let $\mathcal{C}$ be a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$. Typically, $\mathcal{C}$ will be the infinite cluster of an embedded supercritical Bernoulli bond percolation. For every $x \in \mathbb{Z}^d$, we define the random vertex $\widetilde{x}^{\mathcal{C}}$ as the vertex of $\mathcal{C}$ which minimizes $\|x - \widetilde{x}^{\mathcal{C}}\|_1$, with a deterministic rule to break ties. We then define the regularized passages times $\widetilde{T}_G^{\mathcal{C}}$ by

$$\forall x, y \in \mathbb{Z}^d\,, \quad \widetilde{T}_G^{\mathcal{C}}(x,y) \,=\, T_G(\widetilde{x}^{\mathcal{C}}, \widetilde{y}^{\mathcal{C}})\,.$$

2.2 Definition of the Cheeger constant in supercritical percolation on $\mathbb{Z}^2$

We collect in this subsection the definitions and properties of the Cheeger constant obtained in [4]. The Cheeger constant can be represented as the solution of a continuous

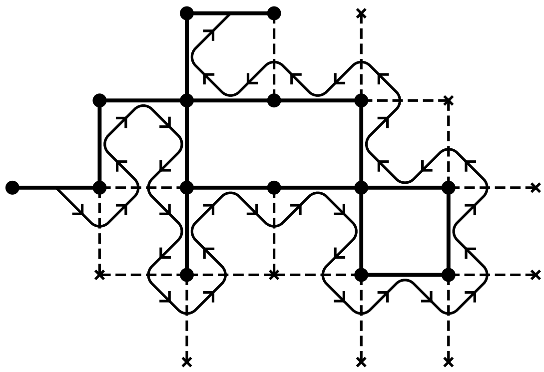

Figure 1: A right most path

isoperimetric problem with respect to some norm. To define this norm, we first require some definitions. We fix $p > p_c(2)$, we denote by $\mathcal{C}_p$ the $\mathbb{P}_p$-a.s. unique infinite cluster $\mathcal{C}_{G_p,\infty}$ and we set $\theta_p = \mathbb{P}_p(0 \in \mathcal{C}_p)$.

For a path $r = (v_0, e_1, \ldots, e_n, v_n)$, and $i \in \{2, \ldots, n-1\}$, an edge $e = (x_i, z)$ is said to be a right-boundary edge if z is a neighbor of x_i between x_{i+1} and x_{i-1} in the clockwise direction. The right boundary $\partial^+ r$ of r is the set of right-boundary edges. A path is called right-most if it uses every edge at most once in every orientation (thus an edge can be used twice, but in different orientations) and it doesn't contain right-boundary edges. See Figure 1; the solid lines represent the path, dashed lines represent the right-boundary edges, and the curly line is a path in the medial graph which shows the orientation (see [4] for a thorough discussion). For $x, y \in \mathbb{Z}^2$, let $\mathcal{R}(x,y)$ be the set of right-most paths from x to y. For a path $r \in \mathcal{R}(x,y)$, define $\mathbf{b}(r) = |\{e \in \partial^+ r : e \text{ is open}\}|$. For $x, y \in \mathcal{C}_p$ we define the right boundary distance, $b(x,y) = \inf\{\mathbf{b}(r) : r \in \mathcal{R}(x,y),\ r \text{ is open}\}$. The next result yields uniform convergence of the right boundary distance to a norm on $\mathbb{R}^2$.

Proposition 2.1 (Definition of the norm, Theorem 2.1 in [4])**.** For any $p > p_c(2)$, there exists a norm β_p on $\mathbb{R}^2$ such that for any $x \in \mathbb{R}^2$,

$$\beta_p(x) := \lim_{n\to\infty} \frac{b(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})}{n} \qquad \mathbb{P}_p - a.s. \text{ and in } L^1(\mathbb{P}_p).$$

Moreover, the convergence is uniform on $\mathbb{S}^1 = \{x \in \mathbb{R}^2 :\ \|x\|_2 = 1\}$.

We will require the following control on the length of right-most paths.

Lemma 2.2 (Proposition 2.9 in [4])**.** There exist $C, C', \alpha > 0$ (depending on p) such that for all n,

$$\mathbb{P}\left[\exists \gamma \in \bigcup_{x\in\mathbb{Z}^2} \mathcal{R}(0,x) \ :\ |\gamma| > n\,,\ \mathbf{b}(\gamma) \le \alpha n\right] \le Ce^{-C'n}.$$

The connection between the Cheeger constant and the norm β_p goes through a continuous isoperimetric problem. For a continuous curve $\lambda : [0,1] \to \mathbb{R}^2$, and a norm ρ, let the ρ-length of λ be

$$\mathrm{len}_\rho(\lambda) = \sup_{N\ge 1} \sup_{0\le t_0 < \ldots < t_N \le 1} \sum_{i=1}^{N} \rho(\lambda(t_i) - \lambda(t_{i-1})).$$

A curve λ is said to be rectifiable if $\mathrm{len}_\rho(\lambda) < \infty$ for any norm ρ. A curve λ is called a Jordan curve if λ is rectifiable, $\lambda(0) = \lambda(1)$ and λ is injective on $[0,1)$. For any Jordan curve λ, we can define its interior $\mathrm{int}(\lambda)$ as the unique finite component of $\mathbb{R}^2 \setminus \lambda([0,1])$. Denote by Leb the Lebesgue measure on $\mathbb{R}^2$. The Cheeger constant can be represented as the solution of the following continuous isoperimetric problem:

Proposition 2.3 (Theorem 1.6 in [4])**.** For every $p > p_c(2)$,

$$\lim_{n\to+\infty} n\varphi_n(p) = (\sqrt{2}\,\theta_p)^{-1} \inf\{\mathrm{len}_{\beta_p}(\lambda) : \lambda \text{ is a Jordan curve}, \mathrm{Leb}(\mathrm{int}(\lambda)) = 1\}.$$

Moreover one obtains a limiting shape for the sets that achieve the minimum in the definition of $\varphi_n(p)$. This limiting shape is given by the Wulff construction [28]. Denote by

$$W_p = \bigcap_{\hat{n}:\|\hat{n}\|_2=1} \{x \in \mathbb{R}^2 : \hat{n}\cdot x \le \beta_p(\hat{n})\} \text{ and } \widehat{W}_p = \frac{W_p}{\sqrt{\mathrm{Leb}(W_p)}}, \tag{2.1}$$

where $\cdot$ denotes the Euclidean inner product. The set $\widehat{W}_p$ is a minimizer for the isoperimetric problem associated with the norm β_p, and it gives the asymptotic shape of the minimizer sets in the definition of $\varphi_n(p)$. Denote by $\mathcal{U}_n(p)$ be the set of minimizers of $\varphi_n(p)$; then

Proposition 2.4 (Shape theorem for the minimizers, [4] Theorem 1.8)**.** For every $p > p_c(2)$, $\mathbb{P}_p$ almost surely,

$$\max_{U\in\mathcal{U}_n(p)} \inf_{\xi\in\mathbb{R}^2} d_{\mathrm{H}}\left(\frac{U}{n}\,,\ \xi + \sqrt{2}\,\widehat{W}_p\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

By Proposition 2.3 and Definition (2.1), Theorem 1.1 will follow from the continuity of $p \mapsto \beta_p$.

At this stage, a few words should be said about the Cheeger constant in higher dimensions. Very recently, Gold extended in [15] the results of Biskup, Louidor, Procaccia and Rosenthal [4] to dimensions 3 and higher. To this purpose, he generalizes the definition of the norm β_p to higher dimensions, see Proposition 3.2 in [15]. However, in a general dimension $d \ge 2$, this norm β_p is defined as the limit of the infimum of the weight (properly rescaled) of a large hypersurface belonging to a certain class of hypersurfaces. Here by hypersurface we mean an object of dimension $d-1$. For $d=2$, we recover an object of dimension 1, i.e., a path, and β_p is defined as the infimum of the rescaled weight of a large path belonging to the class of the right-most paths. The study of the norm β_p in dimension 2 is thus closely related to the study of geodesics in first-passage percolation. However, in higher dimension, the study of the norm β_p is no longer related to the study of geodesics in first-passage percolation, but it is more naturally related to the study of minimal cutsets in first-passage percolation as defined in [21]. This connects the question of the continuity of the Cheeger constant in dimensions 3 or higher to the question of the continuity of the flow constant (rather than the time constant) in first-passage percolation. For this reason, the technics developed in this article, when we deal with modifications of geodesics, cannot be used straightforwardly to prove the continuity of the Cheeger constant in dimensions 3 and higher.

2.3 Definition and properties of the time constant

As announced in the introduction, we follow the approach by Cerf and Théret in [6], which requires no integrability condition on the restriction of G to $[0,+\infty)$. We collect in this subsection the definition and properties of the time constants obtained in their paper.

Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$, and let $M > 0$ be such that $G([0,M]) > p_c(d)$. We denote by $\mathcal{C}_{G,M}$ the a.s. unique infinite cluster of the percolation $(\mathbb{1}_{\{t_G(e)\le M\}}, e \in \mathbb{E}^d)$, *i.e.* the percolation obtained by keeping only edges with passage times less than or equal to M. For any $x, y \in \mathbb{Z}^d$, the (level M) regularized passage time $\widetilde{T}_G^{\mathcal{C}_{G,M}}(x,y)$ is then

$$\widetilde{T}_G^{\mathcal{C}_{G,M}}(x,y) \,=\, T_G(\widetilde{x}^{\mathcal{C}_{G,M}}, \widetilde{y}^{\mathcal{C}_{G,M}})\,.$$

The parameter M only plays a role in the choice of $\widetilde{x}^{\mathcal{C}_{G,M}}$ and $\widetilde{y}^{\mathcal{C}_{G,M}}$. Once these points are chosen, the optimization in $\widetilde{T}_G^{\mathcal{C}_{G,M}}(x,y)$ is on all paths between $\widetilde{x}^{\mathcal{C}_{G,M}}$ and $\widetilde{y}^{\mathcal{C}_{G,M}}$, paths using edges with passage time larger than M included. But as $\widetilde{x}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M}$ and $\widetilde{y}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M}$, we know that exists a path using only edges with passage time less than or equal to M between these two points. To be more precise, we denote by $D^{\mathcal{C}}(x,y)$ the chemical distance (or graph distance) between two vertices x and y on $\mathcal{C}$:

$$\forall x,y \in \mathbb{Z}^d\,, \quad D^{\mathcal{C}}(x,y) \,=\, \inf\{|r| \,:\, r \text{ is a path from } x \text{ to } y\,,\, r\subset \mathcal{C}\}\,,$$

where $\inf \varnothing = +\infty$. The event that the vertices x and y are connected in $\mathcal{C}$ is denoted by $\{x \overset{\mathcal{C}}{\longleftrightarrow} y\}$. Then, for any $x,y \in \mathbb{Z}^d$,

$$\widetilde{T}_G^{\mathcal{C}_{G,M}}(x,y) \,\leq\, M D^{\mathcal{C}_{G,M}}(\widetilde{x}^{\mathcal{C}_{G,M}}, \widetilde{y}^{\mathcal{C}_{G,M}})\,.$$

The regularized passage time $\widetilde{T}_G^{\mathcal{C}_{G,M}}$ enjoys then the same good integrability properties as the chemical distance on a supercritical percolation cluster (see [1]):

Proposition 2.5 (Moments of $\widetilde{T}$, [6])**.** Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. For every $M \in [0,+\infty)$ such that $G([0,M]) > p_c(d)$, there exist positive constants C_1, C_2 and C_3 such that

$$\forall x \in \mathbb{Z}^d\,,\ \forall l \geq C_3\|x\|_1\,, \quad \mathbb{P}\left[\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,x) > l\right] \,\leq\, C_1 e^{-C_2 l}\,.$$

We denote by $\mathcal{C}_{G,\infty}$ the a.s. unique infinite cluster of the percolation obtained by keeping only edges with finite passage time, *i.e.* the percolation $(\mathbb{1}_{\{t_G(e)<\infty\}}, e \in \mathbb{E}^d)$. Proposition 2.5 implies in particular that the times $\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,x)$ are integrable. A classical application of a subadditive ergodic theorem gives the existence of a time constant:

Proposition 2.6 (Convergence to the time constant, [6])**.** Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. There exists a deterministic function $\mu_G : \mathbb{Z}^d \to [0,+\infty)$ such that for every $M \in [0,+\infty)$ satisfying $G([0,M]) > p_c(d)$, we have the following properties:

$$\forall x \in \mathbb{Z}^d \quad \mu_G(x) \,=\, \inf_{n\in\mathbb{N}^*} \frac{\mathbb{E}\left[\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)\right]}{n} \,=\, \lim_{n\to+\infty} \frac{\mathbb{E}\left[\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)\right]}{n}\,, \tag{2.2}$$

$$\forall x \in \mathbb{Z}^d \quad \lim_{n\to\infty} \frac{\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)}{n} \,=\, \mu_G(x) \quad \text{a.s. and in } L^1, \tag{2.3}$$

$$\forall x \in \mathbb{Z}^d \quad \lim_{n\to\infty} \frac{\widetilde{T}_G^{\mathcal{C}_{G,\infty}}(0,nx)}{n} \,=\, \mu_G(x) \quad \text{in probability}\,, \tag{2.4}$$

$$\forall x \in \mathbb{Z}^d \quad \lim_{n\to\infty} \frac{T_G(0,nx)}{n} \,=\, \theta_G^2 \delta_{\mu_G(x)} + (1-\theta_G^2)\delta_{+\infty} \quad \text{in distribution}, \tag{2.5}$$

where $\theta_G = \mathbb{P}[0 \in \mathcal{C}_{G,\infty}]$.

Note that even if the definition (2.2) of the time constants $\mu_G(x)$ requires to introduce a parameter M in the definition of the regularized passage times $\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)$, these time constants $\mu_G(x)$ do not depend on M. Note also that if instead of taking the $\widetilde{x}^{\mathcal{C}_{G,M}}$ in the infinite cluster $\mathcal{C}_{G,M}$ of edges with passage time less than M, we take the $\widetilde{x}^{\mathcal{C}_{G,\infty}}$ in the infinite cluster $\mathcal{C}_{G,\infty}$ of edges with finite passage time, the almost sure convergence is weakened into the convergence in probability (2.4). Without any regularization, the convergence in (2.5) is only in law.

As in the classical first-passage percolation model, the function μ_G can be extended, by homogeneity, into a pseudo-norm on $\mathbb{R}^d$ (the only possibly missing property of μ_G is the strict positivity):

Proposition 2.7 (Positivity of μ_G, [6]). Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. Then either μ_G is identically equal to 0 or $\mu_G(x) > 0$ for all $x \neq 0$, and we know that

$$\mu_G = 0 \iff G(\{0\}) \geq p_c(d)\,.$$

Proposition 2.5 gives strong enough integrability properties of $\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,x)$ to ensure that the convergence to the time constants is uniform in the direction:

Proposition 2.8 (Uniform convergence, [6]). Let G be a probability measure on $[0,+\infty]$ such that $G([0,+\infty)) > p_c(d)$. Then for every $M \in [0,+\infty)$ such that $G([0,M]) > p_c(d)$, we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d,\, \|x\|_1 \geq n} \left| \frac{\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,x) - \mu_G(x)}{\|x\|_1} \right| = 0 \quad \text{a.s.}$$

When $\mu_G > 0$, this uniform convergence is equivalent to the so called *shape theorem*, that we briefly present now. We define $B_{G,t}$ (resp. $\widetilde{B}_{G,t}^{\mathcal{C}_{G,M}}$, $\widetilde{B}_{G,t}^{\mathcal{C}_{G,\infty}}$) as the set of all points reached from the origin within a time t, enlarged by adding a small unit cube around each such point:

$$B_{G,t} \,=\, \{z+u \,:\, z\in\mathbb{Z}^d\,,\, T_G(0,z)\leq t\,,\, u\in[-1/2,1/2]^d\}\,,$$

(resp. $\widetilde{T}_G^{\mathcal{C}_{G,M}}, \widetilde{T}_G^{\mathcal{C}_{G,\infty}}$), and when μ_G is a norm we denote by $\mathcal{B}_{\mu_G}$ its closed unit ball. Roughly speaking, the shape theorem states that the rescaled set $B_{G,t}/t$ (respectively $\widetilde{B}_{G,t}^{\mathcal{C}_{G,M}}/t$, $\widetilde{B}_{G,t}^{\mathcal{C}_{G,\infty}}/t$) converges towards $\mathcal{B}_{\mu_G}$. The convergence holds in a sense that depends on the regularity of times considered (see [6] for more precise results).

2.4 Coupling

To understand how μ_G depends on G, it is useful to consider passage times $(t_G(e))$ with common distribution G, that also have good coupling properties. For any probability measure G on $[0,+\infty]$, we denote by $\mathfrak{G}$ the function

$$\begin{aligned}\mathfrak{G} : [0,+\infty) &\to [0,1]\\ t &\mapsto G([t,+\infty]),\end{aligned}$$

which characterizes G. For two probability measures G_1, G_2 on $[0,+\infty]$, we define the following stochastic domination relation:

$$G_1 \succeq G_2 \quad \Leftrightarrow \quad \forall t\in[0,+\infty) \quad \mathfrak{G}_1(t) \geq \mathfrak{G}_2(t).$$

This is to have this equivalence that we choose to characterize a probability measure G by $\mathfrak{G}$ instead of the more standard distribution function $t\mapsto G([0,t])$.

Given a probability measure G on $[0,+\infty]$, we define the two following pseudo-inverse functions for $\mathfrak{G}$:

$$\begin{aligned}\forall t\in[0,1]\,,\ \hat{\mathfrak{G}}(t) \,&=\, \sup\{s\in[0,+\infty) \,:\, \mathfrak{G}(s)\geq 1-t\} \text{ and}\\ \tilde{\mathfrak{G}}(t) \,&=\, \sup\{s\in[0,+\infty) \,:\, \mathfrak{G}(s) > 1-t\}\,.\end{aligned}$$

These pseudo-inverse functions can be used to simulate random variable with distribution G:

Lemma 2.9. Let U be a random variable with uniform law on $(0,1)$. If G is a probability measure on $[0,+\infty]$, then $\hat{\mathfrak{G}}(U)$ and $\tilde{\mathfrak{G}}(U)$ are random variables taking values in $[0,+\infty]$ with distribution G, and $\tilde{\mathfrak{G}}(U) = \hat{\mathfrak{G}}(U)$ a.s.

Proof. The function $\hat{\mathfrak{G}}$ has the following property

$$\forall t \in [0,1]\,,\ \forall s \in [0,+\infty)\,,\quad \hat{\mathfrak{G}}(t) \geq s \iff \mathfrak{G}(s) \geq 1-t\,. \tag{2.6}$$

Then for all $s \in [0,+\infty)$, we have $\mathbb{P}[\hat{\mathfrak{G}}(U) \geq s] = \mathbb{P}[U \geq 1-\mathfrak{G}(s)] = \mathfrak{G}(s)$, thus $\hat{\mathfrak{G}}(U)$ has distribution G. The function $\tilde{\mathfrak{G}}$ does not satisfy the property (2.6). However, $\hat{\mathfrak{G}}(t) \neq \tilde{\mathfrak{G}}(t)$ only for $t \in [0,1]$ such that $\mathfrak{G}^{-1}(\{1-t\})$ contains an open interval, thus the set $\{t \in [0,1] \,:\, \hat{\mathfrak{G}}(t) \neq \tilde{\mathfrak{G}}(t)\}$ is at most countable. This implies that $\hat{\mathfrak{G}}(U) = \tilde{\mathfrak{G}}(U)$ a.s., thus $\tilde{\mathfrak{G}}(U)$ has the same law as $\hat{\mathfrak{G}}(U)$. □

In the following, we will always build the passage times of the edges with this lemma. Let then $(u(e), e \in \mathbb{E}^d)$ be a family of i.i.d. random variables with uniform law on $(0,1)$. For any given probability measure G on $[0,+\infty]$, the family of i.i.d passage times with distribution G will always be

$$\forall e \in \mathbb{E}^d\,,\quad t_G(e) = \hat{\mathfrak{G}}(u(e))\,. \tag{2.7}$$

Of course the main interest of this construction is to obtain couplings between laws: if G_1 and G_2 are probability measures on $[0,+\infty]$,

$$G_1 \preceq G_2 \quad \Rightarrow \quad t_{G_1}(e) \leq t_{G_2}(e) \text{ for all edges } e.$$

In particular in the case of Bernoulli percolation, if $p \leq q$, $G_q = q\delta_1 + (1-q)\delta_\infty \preceq G_p = p\delta_1 + (1-p)\delta_\infty$ thus $\mathcal{C}_p \subset \mathcal{C}_q$. Moreover, we have the following pleasant property:

Lemma 2.10. Let $G, (G_n)_{n\in\mathbb{N}}$ be probability measures on $[0,+\infty]$. We define the passage times $t_G(e)$ and $t_{G_n}(e)$ as in equation (2.7). If $G_n \xrightarrow{d} G$, then

$$\text{a.s.}\,,\ \forall e \in \mathbb{E}^d\,,\quad \lim_{n\to\infty} t_{G_n}(e) \,=\, t_G(e)\,.$$

Proof. (i) Let us prove that if $G_n \succeq G$ for all n, then

$$\forall t \in [0,1]\ \lim_{n\to\infty} \hat{\mathfrak{G}}_n(t) \,=\, \hat{\mathfrak{G}}(t)\,. \tag{2.8}$$

Consider $t \in [0,1]$, let $x = \hat{\mathfrak{G}}(t)$ and $x_n = \hat{\mathfrak{G}}_n(t)$. Since $G_n \succeq G$, we have $\mathfrak{G}_n \geq \mathfrak{G}$ thus $x_n \geq x$. Suppose that $\overline{\lim}_{n\to+\infty} x_n := \overline{x} > x$. Up to extraction, we suppose that $\lim_{n\to+\infty} x_n = \overline{x}$. Choose $\beta \in (x,\overline{x})$ such that $\mathfrak{G}$ is continuous at β, thus $\lim_{n\to\infty} \mathfrak{G}_n(\beta) = \mathfrak{G}(\beta)$. On one hand, by the definition of $\hat{\mathfrak{G}}$ and the monotonicity of $\mathfrak{G}$, we have $\mathfrak{G}(\beta) < 1-t$. On the other hand, $\beta < x_n$ for all n large enough, thus $\mathfrak{G}_n(\beta) \geq 1-t$ for all n large enough, and we conclude that $\mathfrak{G}(\beta) = \lim_{n\to\infty} \mathfrak{G}_n(\beta) \geq 1-t$, which is a contradiction, and (2.8) is proved.

(ii) Similarly, if $G_n \preceq G$ for all n, then $\forall t \in [0,1]\ \lim_{n\to\infty} \hat{\mathfrak{G}}_n(t) \,=\, \tilde{\mathfrak{G}}(t)\,.$

(iii) We define $\underline{\mathfrak{G}}_n = \min\{\mathfrak{G}, \mathfrak{G}_n\}$ (resp. $\overline{\mathfrak{G}}_n = \max\{\mathfrak{G}, \mathfrak{G}_n\}$), and we denote by $\underline{G}_n$ (resp. $\overline{G}_n$) the corresponding probability measure on $[0,+\infty]$. Notice that $\overline{G}_n \xrightarrow{d} G$ and $\underline{G}_n \xrightarrow{d} G$. Fix an edge e. Then $\overline{G}_n \succeq G$ for all n, and (i) implies that

$$\text{a.s.}\quad \lim_{n\to\infty} t_{\overline{G}_n}(e) \,=\, t_G(e)\,.$$

As $\underline{G}_n \preceq G$ for all n and $t_G(e) = \tilde{\mathfrak{G}}(u(e))$ almost surely, (ii) implies that

$$\text{a.s.} \quad \lim_{n\to\infty} t_{\underline{G}_n}(e) \,=\, t_G(e)\,.$$

Finally, as $\underline{G}_n \preceq G_n \preceq \overline{G}_n$ for all n, we know by coupling that $t_{\underline{G}_n}(e) \leq t_{G_n}(e) \leq t_{\overline{G}_n}(e)$, which gives the desired convergence. □

2.5 Stabilization of the point $\widetilde{x}$ and monotonicity of the time constant

We need to extend the monotonicity of the time constant to first-passage percolation on the infinite cluster of supercritical percolation. Since we work with different probability measures, the fact that, in the regularization process, the point $\widetilde{x}^{\mathcal{C}_{G,M}}$ depends on G may be disturbing. We get round this problem by considering an alternative probability measure H:

Lemma 2.11. Let G and H be probability measures on $[0,+\infty]$ such that $G \preceq H$. For all $M \in [0,+\infty)$ satisfying $H([0,M]) > p_c(d)$, we have

$$\mu_G(x) = \inf_{n\in\mathbb{N}^*} \frac{\mathbb{E}\left[\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)\right]}{n} = \lim_{n\to\infty} \frac{\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)}{n} \text{ a.s. and in } L^1.$$

Proof. Since $G \preceq H$ we get by coupling that $t_G(e) \leq t_H(e)$ for all $e \in \mathbb{E}^d$. Let $M \in [0,+\infty)$ satisfying $H([0,M]) > p_c(d)$, then $G([0,M]) > p_c(d)$ and $\mathcal{C}_{H,M} \subset \mathcal{C}_{G,M}$. The proof of the convergence of $\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)/n$ is a straightforward adaptation of the proof of the convergence of $\widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)/n$: by the subadditive ergodic theorem, there exists a function $\mu'_{G,H} : \mathbb{R}^d \to [0,+\infty)$ such that for all $x \in \mathbb{Z}^d$ we have

$$\mu'_{G,H}(x) \,=\, \inf_{n\in\mathbb{N}^*} \frac{\mathbb{E}\left[\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)\right]}{n} \,=\, \lim_{n\to\infty} \frac{\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)}{n} \quad \text{a.s. and in } L^1\,.$$

It remains to prove that $\mu'_{G,H} = \mu_G$. For any $x \in \mathbb{Z}^d$, for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left[\left|\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx) - \widetilde{T}_G^{\mathcal{C}_{G,M}}(0,nx)\right| > n\varepsilon\right] \\
&\quad\leq\; \mathbb{P}\left[T_G(\widetilde{0}^{\mathcal{C}_{G,M}}, \widetilde{0}^{\mathcal{C}_{H,M}}) + T_G(\widetilde{nx}^{\mathcal{C}_{G,M}}, \widetilde{nx}^{\mathcal{C}_{H,M}}) > n\varepsilon\right] \\
&\quad\leq\; 2\,\mathbb{P}\left[T_G(\widetilde{0}^{\mathcal{C}_{G,M}}, \widetilde{0}^{\mathcal{C}_{H,M}}) > n\varepsilon/2\right]\,. \qquad (2.9)
\end{aligned}$$

Since $\widetilde{0}^{\mathcal{C}_{G,M}} \in \mathcal{C}_{G,M} \subset \mathcal{C}_{G,\infty}$ and $\widetilde{0}^{\mathcal{C}_{H,M}} \in \mathcal{C}_{H,M} \subset \mathcal{C}_{G,M} \subset \mathcal{C}_{G,\infty}$, the time $T_G(\widetilde{0}^{\mathcal{C}_{G,M}}, \widetilde{0}^{\mathcal{C}_{H,M}})$ is finite a.s. thus the right hand side of inequality (2.9) goes to 0 as n goes to infinity. This concludes the proof of Lemma 2.11. □

As a simple consequence of the coupling built in section 2.4 and the stabilization Lemma 2.11, we obtain the monotonicity of the function $G \mapsto \mu_G$:

Lemma 2.12. Let G, H be probability measures on $[0,+\infty]$. we have

$$G \preceq H \Longrightarrow \mu_G \leq \mu_H\,.$$

Proof. By construction of μ_G and μ_H, it suffices to prove that $\mu_G(x) \leq \mu_H(x)$ for all $x \in \mathbb{Z}^d$. By coupling, since $G \preceq H$, we have $t_G(e) \leq t_H(e)$ for every edge e. Using Lemma 2.11 the conclusion is immediate, since we have a.s.

$$\mu_G(x) \,=\, \lim_{n\to\infty} \frac{\widetilde{T}_G^{\mathcal{C}_{H,M}}(0,nx)}{n} \,\leq\, \lim_{n\to\infty} \frac{\widetilde{T}_H^{\mathcal{C}_{H,M}}(0,nx)}{n} \,=\, \mu_H(x)\,.$$ □

2.6 Stabilization of the point $\widetilde{x}$ for the Cheeger constant

Concerning the Cheeger constant, we need a stabilization result similar to Lemma 2.11. For a path $r \in \mathcal{R}(x,y)$, let us define $\mathbf{b}_p(r) = |\{e \in \partial^+ r \,:\, e \text{ is } p\!-\!open\}|$. For $x, y \in \mathcal{C}_p$, we define $b_p(x,y) = \inf\{\mathbf{b}_p(r) \,:\, r \in \mathcal{R}(x,y), r \text{ is } p - \text{open}\}$.

Lemma 2.13. For any p, p_0 such that $p_c(2) < p_0 \le p \le 1$, for any $x \in \mathbb{R}^2$, we have

$$\beta_p(x) = \lim_{n\to\infty} \frac{b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_{p_0}})}{n} \qquad \mathbb{P}_p - a.s. \text{ and in } L^1(\mathbb{P}_p).$$

Proof. Exactly as in the proof of Lemma 2.11, since the convergence of $b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})/n$ follows from a subadditive argument, the proof can be adapted straightforwardly to prove the existence of

$$\beta_{p,p_0}(x) := \lim_{n\to\infty} \frac{b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_{p_0}})}{n} \qquad \mathbb{P}_p - a.s. \text{ and in } L^1(\mathbb{P}_p).$$

The only thing we have to prove is the equality $\beta_{p,p_0}(x) = \beta_p(x)$. By the almost-subbadditivity of b_p (see equation (2.27) in [4]), we have

$$b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p}) \le b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_{p_0}}) + b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{0}^{\mathcal{C}_p}) + b_p(\widetilde{x}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_p}) + 4\,.$$

Since $\mathcal{C}_{p_0} \subset \mathcal{C}_p$, there exists a finite simple (thus also right-most) path γ which is p-open between $\widetilde{0}^{\mathcal{C}_{p_0}}$ and $\widetilde{0}^{\mathcal{C}_p}$, and by [4, Lemma 2.5] we know that $|\partial^+\gamma| < 3|\gamma|$, thus $b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{0}^{\mathcal{C}_p}) \le 3|\gamma| < +\infty$. The same holds for $b_p(\widetilde{nx}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_p})$. As in the proof of Lemma 2.11, this is enough to conclude that $\beta_{p,p_0}(x) = \beta_p(x)$. □

Notice that Lemma 2.13 does not imply the monotonicity of the Cheeger constant. Indeed, consider $p_c(2) < p \le q$, then

- a q-open path γ may not be p-open,
- a p-open path γ is q-open by coupling, but $\mathbf{b}_q(\gamma)$ may be strictly bigger than $\mathbf{b}_p(\gamma)$,

thus no trivial comparison between $b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})$ and $b_q(\widetilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p})$ holds.

3 Renormalization

In this section we present the renormalization process and the construction of modified paths that will be useful to study both the time constant and the isoperimetric constant. We consider coupled bond i.i.d. Bernoulli percolations of different parameters. As we will see in Section 4, the construction of modified paths in the model of first passage percolation reduces to this simplest case.

3.1 Definition of the renormalization process

Consider parameters p_0 and q such that $p_c(d) < p_0 \le q \le 1$. Consider i.i.d. Bernoulii percolation on the edges of $\mathbb{Z}^d$ for these two parameters that are coupled, i.e. any p_0-open edge is q-open. Denote as before by $\mathcal{C}_{p_0}$ the a.s. unique infinite cluster of the supercritical Bernoulli field of parameter p_0. We call this field the p_0-percolation and its clusters the p_0-clusters.

We use a renormalization process in the spirit of the work of Antal and Pisztora [1]. For a large integer N, that will be apropriately chosen later, we chop $\mathbb{Z}^d$ into disjoint N-boxes as follows: we set B_N to be the box $[-N,N]^d \cap \mathbb{Z}^d$ and define the family of N-boxes by setting, for $\mathbf{i} \in \mathbb{Z}^d$,

$$B_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_N),$$

where τ_b stands for the shift in $\mathbb{Z}^d$ with vector $b \in \mathbb{Z}^d$. We will also refer to the box $B_N(\mathbf{i})$ as the N-box with coordinate $\mathbf{i}$. The coordinates of N-boxes will be denoted in bold and considered as macroscopic sites, to distinghish them from the microscopic sites in the initial graph $\mathbb{Z}^d$. We also introduce larger boxes: for $\mathbf{i} \in \mathbb{Z}^d$,

$$B'_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_{3N}).$$

As in [1], we say that a connected cluster C is a crossing cluster for a box B, if for all d directions there is an open path contained in $C \cap B$ joining the two opposite faces of the box B.

Let $\mathcal{C}'_{p_0} = (\mathbb{Z}^d, \{e \in \mathbb{E}^d : e \text{ is } p_0\text{-open}\})$ be the graph whose edges are opened for the Bernoulli percolation of parameter p_0. We recall that $\mathcal{C}_{p_0}$ is the infinite cluster of $\mathcal{C}'_{p_0}$, and we have $D^{\mathcal{C}_{p_0}}(x,y) = D^{\mathcal{C}'_{p_0}}(x,y)$ for every vertices x and y in $\mathcal{C}_{p_0}$, and $D^{\mathcal{C}_{p_0}}(x,y) = +\infty$ if x or y are not in $\mathcal{C}_{p_0}$. Let us recall the following result, obtained by Antal and Pisztora [1, Theorem 1.1], that says that the chemical distance $D^{\mathcal{C}'_{p_0}}$ can't be too large when compared to $\|\cdot\|_1$ or $\|\cdot\|_\infty$ (or any other equivalent norm): there exist positive constants $\hat{A}, \hat{B}, \beta$ such that

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_1 \leq D^{\mathcal{C}'_{p_0}}(0,x) < +\infty) \leq \hat{A}\exp(-\hat{B}\|x\|_1)\,, \tag{3.1}$$

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_\infty \leq D^{\mathcal{C}'_{p_0}}(0,x) < +\infty) \leq \hat{A}\exp(-\hat{B}\|x\|_\infty)\,, \tag{3.2}$$

and

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_2 \leq D^{\mathcal{C}'_{p_0}}(0,x) < +\infty) \leq \hat{A}\exp(-\hat{B}\|x\|_2)\,. \tag{3.3}$$

In fact Antal and Pisztora proved (3.1), but different norms being equivalent in $\mathbb{R}^d$, we can obtain (3.2) and (3.3) by changing the constants.

Definition 3.1. We say that the macroscopic site $\mathbf{i}$ is (p_0,q)-*good* (or that the box $B_N(\mathbf{i})$ is (p_0,q)-good) if the following events occur:

(i) There exists a unique p_0-cluster $\mathcal{C}$ in $B'_N(\mathbf{i})$ which has more than N vertices;

(ii) This p_0-cluster $\mathcal{C}$ is crossing for each of the 3^d N-boxes included in $B'_N(\mathbf{i})$;

(iii) For all $x,y \in B'_N(\mathbf{i})$, if x and y belong to this p_0-cluster $\mathcal{C}$, then $D^{\mathcal{C}'_{p_0}}(x,y) \leq 6\beta N$;

(iv) If π is a q-open path in $B'_N(\mathbf{i})$ such that $|\pi| \geq N$, then π intersects this p_0-cluster $\mathcal{C}$ in $B'_N(\mathbf{i})$, i.e., they share a common vertex.

We call this cluster $\mathcal{C}$ the crossing p_0-cluster of the (p_0,q)-good box $B_N(\mathbf{i})$.

Otherwise, $B_N(\mathbf{i})$ is said to be (p_0,q)-*bad*. For short, we say that B is good or bad if there is no doubt about the choice of (p_0,q).

On the macroscopic grid $\mathbb{Z}^d$, we consider the same standard nearest neighbour graph structure as on the microscopic initial grid $\mathbb{Z}^d$. Moreover we say that two macroscopic sites $\mathbf{i}$ and $\mathbf{j}$ are $*$-neighbors if and only if $\|\mathbf{i}-\mathbf{j}\|_\infty = 1$. If C is a connected set of macroscopic sites, we define its exterior vertex boundary

$$\partial_v C = \left\{ \mathbf{i} \in \mathbb{Z}^d \setminus C : \begin{array}{c} \mathbf{i} \text{ has a neighbour in } C \\ \text{and } \mathbf{i} \text{ is connected to infinity by a } \mathbb{Z}^d\text{-path in } \mathbb{Z}^d \setminus C \end{array} \right\}.$$

For a bad macroscopic site $\mathbf{i}$, denote by $C(\mathbf{i})$ the connected cluster of bad macroscopic sites containing $\mathbf{i}$. If $C(\mathbf{i})$ is finite, the set $\partial_v C(\mathbf{i})$ is then a $*$-connected set of good macroscopic sites, see Lemma 2 in [27]. For a good macroscopic site $\mathbf{i}$, we define $\partial_v C(\mathbf{i})$ to be $\{\mathbf{i}\}$.

3.2 Modification of a path

Let $p_c(d) < p_0 \leq p \leq q$ and N be fixed. Let now γ be a path in $\mathbb{Z}^d$. What we want to do is to remove from γ the edges that are p-closed, and to look for bypasses for these edges using only edges that are p_0-open.

To γ, we associate the connected set $\Gamma \subset \mathbb{Z}^d$ of N-boxes it visits: this is a lattice animal, *i.e.* a connected finite set of $\mathbb{Z}^d$, containing the box that contains the starting point of γ. We decompose γ into two parts, namely $\gamma_a = \{e \in \gamma : e \text{ is } p\text{-open}\}$ and $\gamma_b = \{e \in \gamma : e \text{ is } p\text{-closed}\}$. We denote by Bad the (random) set of bad connected components of the macroscopic percolation given by the states of the N-boxes.

Lemma 3.2. Assume that $y \in \mathcal{C}_{p_0}$, that $z \in \mathcal{C}_{p_0}$, that the N-boxes containing y and z are (p_0, q)-good and belong to an infinite cluster of (p_0, q)-good boxes. Let γ be a path between y and z. Then there exists a p-open path γ' between y and z that has the following properties :

1. $\gamma' \setminus \gamma$ is a collection of disjoint self avoiding p_0-open paths that intersect $\gamma' \cap \gamma$ only at their endpoints;
2. $|\gamma' \setminus \gamma| \leq \rho_d \left(N \sum_{C \in \mathrm{Bad}:\, C \cap \Gamma \neq \varnothing} |C| + N^d |\gamma_b| \right)$, where ρ_d is a constant depending only on the dimension d.

The proof of Lemma 3.2 is in the spirit of Antal and Pisztora's proof in [1]. However, our construction is a bit more involved, since what we do can be seen as a refinement of Antal and Pisztora's arguments, in order to obtain more precise estimates. In fact, we will use Antal and Pisztora's result itself to prove that the property (iii) of good blocks is typical (see Lemma 3.5).

Before proving Lemma 3.2, we need a simpler estimate on the cardinality of a path inside a set of good blocks.

Lemma 3.3. There exists a constant $\hat{\rho}_d$, depending only on d, such that for every fixed N, for every $n \in \mathbb{N}^*$, if $(B_N(\mathbf{i}))_{\mathbf{i} \in \mathcal{I}}$ is a $*$-connected set of n (p_0, q)-good N-blocks, if $x \in B_N(\mathbf{j})$ for $\mathbf{j} \in \mathcal{I}$ and x is in the crossing p_0-cluster of $B_N(\mathbf{j})$, if $y \in B_N(\mathbf{k})$ for $\mathbf{k} \in \mathcal{I}$ and y is in the crossing p_0-cluster of $B_N(\mathbf{k})$, then there exists a p_0-open path from x to y of length at most equal to $\hat{\rho}_d(Nn + N^d)$.

Proof of Lemma 3.3. Since $\mathcal{I}$ is a $*$-connected set of macroscopic sites, there exists a self-avoiding $*$-connected path $(\boldsymbol{\varphi}_i)_{1 \leq i \leq r}$ from $\mathbf{j}$ and $\mathbf{k}$ in $\mathcal{I}$. We notice that $r \leq 3^d |\mathcal{I}| \leq 3^d n$. Since $B_N(\mathbf{i})$ is a good block for all $\mathbf{i} \in \mathcal{I}$, the definition of good boxes ensures that there exists a p_0-cluster $\mathcal{C}$ in $\mathcal{C}'_{p_0} \cap \cup_{\mathbf{i} \in \mathcal{I}} \{e \in B'_N(\mathbf{i})\}$ which is crossing for every N-box included in $\cup_{\mathbf{i} \in \mathcal{I}} B'_N(\mathbf{i})$ (see Proposition 2.1 in Antal and Pisztora [1]), and by hypotheses x and y are in $\mathcal{C}$. We now consider a sequence of points $(z_i)_{1 \leq i \leq r}$ such that for each $i \in \{1, \dots, r\}$, $z_i \in B_N(\boldsymbol{\varphi}_i)$ and z_i belongs to the p_0-crossing cluster of $B_N(\boldsymbol{\varphi}_i)$. For every $i \in \{2, \dots, r\}$, we have $z_i \in B_N(\boldsymbol{\varphi}_i)$, $z_{i-1} \in B_N(\boldsymbol{\varphi}_{i-1}) \subset B'_N(\boldsymbol{\varphi}_i)$, and these two points belong to the crossing p_0-cluster of $B'_N(\boldsymbol{\varphi}_i)$. The fact that $B_N(\boldsymbol{\varphi}_i)$ is good ensures that there exists a p_0-open path from z_{i-1} to z_i of length at most equal to $6\beta N$ (see property (iii) of good boxes). By concatenating these paths, we obtain a p_0-open path from z_1 to z_r of length at most equal to $6\beta N r \leq 6\beta N 3^d n$. Finally, since x and z_1 belong to the crossing p_0-cluster of $B_N(\mathbf{j}) = B_N(\boldsymbol{\varphi}_1)$, there exists a p_0-open path from x to z_1 of length at most $|\{e \in B'_N(\mathbf{j})\}| \leq 2d\, 3^d N^d$. The same holds for z_r and y in $B_N(\mathbf{k}) = B_N(\boldsymbol{\varphi}_r)$. By glueing these paths, we obtain a p_0-open path from x to y of length at most equal to $3\beta N 3^d n + 4d\, 3^d N^d$. ☐

Proof of Lemma 3.2. To the path γ, we associate the sequence $\boldsymbol{\varphi}_0 = (\boldsymbol{\varphi}_0(j))_{1\leq j\leq r_0}$ of N-boxes it visits. Note that $\boldsymbol{\varphi}$ is not necessarily injective, and that the previously defined lattice animal Γ is equal to $\boldsymbol{\varphi}_0(\{1,\dots,r_0\})$.

From the sequence $\boldsymbol{\varphi}_0$, we extract the subsequence $(\boldsymbol{\varphi}_1(j))_{1\leq j\leq r_1}$, with $r_1 \leq r_0$, of N-boxes B such that $\gamma \cap B$ contains at least one edge that is p-closed (more precisely, we keep the indices of the boxes B that contain the smallest extremity, for the lexicographic order, of an edge of γ that is p-closed). Notice that $r_1 \leq |\gamma_b|$. The idea is the following:

(1) If $\boldsymbol{\varphi}_1(j)$ is good, we add to γ all the p_0-open edges in B': there will be enough such edges in the good N-box to find a by-pass for the edge of γ that is p-closed.

(2) If $\boldsymbol{\varphi}_1(j)$ is bad, we will look for such a by-pass in the exterior vertex boundary $\partial_v C(\boldsymbol{\varphi}_1(j)))$ of the connected component of bad boxes of $\boldsymbol{\varphi}_1(j)$.

In the second case, we use Lemma 3.3 to control the length of the by-pass we create. We recall that if $\mathbf{i}$ is good, then $\partial_v C(\mathbf{i}) = \{\mathbf{i}\}$. Note that some $\partial_v C(\boldsymbol{\varphi}_1(j)))$ may coincide or be nested one in another or overlap or be $*$-connected. In order to define properly the modification of our path, we need thus to extract a subsequence once again, see Figure 2. We first consider the $*$-connected components $(S_{\varphi_2(j)})_{1\leq j\leq r_2}$, with $r_2 \leq r_1$, of the

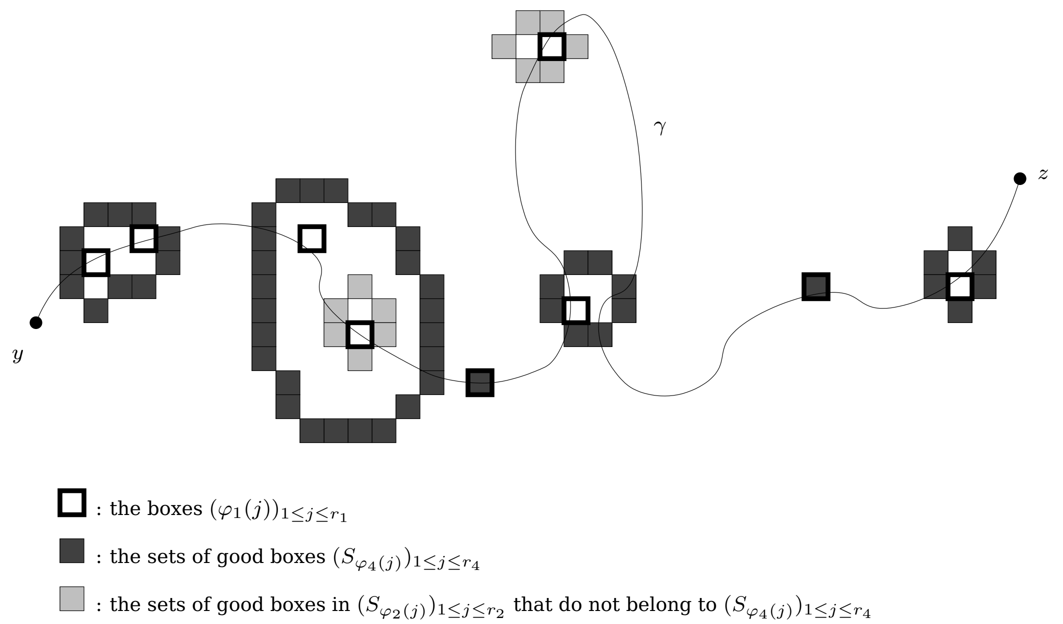

Figure 2: Construction of the path γ' - step 1.

union of the $(\partial_v C(\boldsymbol{\varphi}_1(j)))_{1\leq j\leq r_1}$, by keeping only the smallest index for each connected component. Thus, for all $i \neq j$, $i,j \in \{1,\dots,r_2\}$, $S_{\varphi_2(i)}$ and $S_{\varphi_2(j)}$ cannot be $*$-connected. Next, in case of nesting, we only keep the largest connected component. We denote by $(S_{\varphi_3(j)})_{1\leq j\leq r_3}$, with $r_3 \leq r_2$, the remaining hypersurfaces of good N-boxes. Finally it may happen that γ visits several times the same $S_{\varphi_3(j)}$ for some j: in this situation we can remove the loops that γ makes between its different visits in $S_{\varphi_3(j)}$. Thus by a last extraction we obtain $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$, where $S_{\varphi_4(1)} = S_{\varphi_3(1)}$ and for all $k \geq 1$, $\varphi_4(k+1)$ is the infimum of the indices $(\varphi_3(j))_{1\leq j\leq r_3}$ such that γ visits $S_{\varphi_3(j)}$ after it exits $S_{\varphi_4(k)}$ for the last time (if such a j exists).

Note that the path γ must visit each $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$. We now cut γ in several pieces. Let $\Psi_{in}(1) = \min\{k \geq 1 \,:\, \gamma_k \in \cup_{\mathbf{i}\in S_{\varphi_4(1)}} B_N(\mathbf{i})\}$ and $\mathbf{i}_{in}(1)$ be the macroscopic site such that $\gamma_{\Psi_{in}(1)} \in B_N(\mathbf{i}_{in}(1))$, let $\Psi_{out}(1) = \max\{k \geq \Psi_{in}(1) \,:\, \gamma_k \in \cup_{\mathbf{i}\in S_{\varphi_4(1)}} B_N(\mathbf{i})\}$ and $\mathbf{i}_{out}(1)$

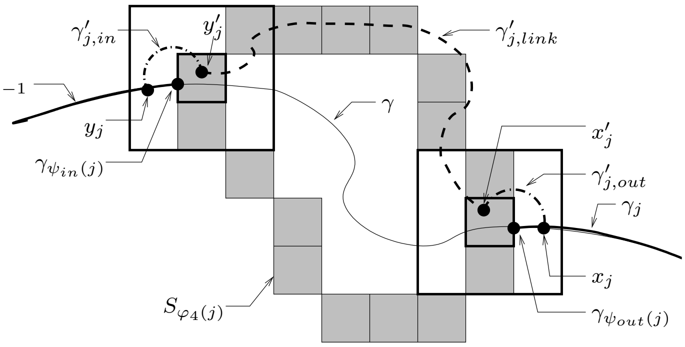

Figure 3: Construction of the path γ' - step 2.

be the macroscopic site such that $\gamma_{\Psi_{out}(1)} \in B_N(\mathbf{i}_{out}(1))$ (see Figure 3). By recurrence, for all $1 \leq j \leq r_4$, we define $\Psi_{in}(j) = \min\{k \geq \Psi_{out}(j-1) \, : \, \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(j)}} B_N(\mathbf{i})\}$, $\gamma_{\Psi_{in}(j)} \in B_N(\mathbf{i}_{in}(j))$, and $\Psi_{out}(j) = \max\{k \geq \Psi_{in}(j) \, : \, \gamma_k \in \cup_{\mathbf{i} \in S_{\varphi_4(j)}} B_N(\mathbf{i})\}$, $\gamma_{\Psi_{out}(j)} \in B_N(\mathbf{i}_{out}(j))$. For all $1 \leq j \leq r_4 - 1$, let γ_j be the part of γ from $\gamma_{\Psi_{out}(j)}$ to $\gamma_{\Psi_{in}(j+1)}$. By construction γ_j contains no p-closed edge, and has at least N vertices in $B'_N(\mathbf{i}_{out}(j))$ (resp. in $B'_N(\mathbf{i}_{in}(j+1))$) since $\mathbf{i}_{out}(j)$ and $\mathbf{i}_{in}(j+1)$ cannot be $*$-connected, thus γ_j intersects the crossing p_0-cluster of $B'_N(\mathbf{i}_{out}(j))$ (resp. $B'_N(\mathbf{i}_{in}(j+1))$) by property (iv) of the good boxes; let us denote by x_j (resp. y_{j+1}) the last (resp. first) intersection of γ_j with the crossing p_0-cluster of $B'_N(\mathbf{i}_{out}(j))$ (resp. $B'_N(\mathbf{i}_{in}(j+1))$). The vertex x_j (resp. y_{j+1}) is not inside $B_N(\mathbf{i}_{out}(j))$ (resp. $B_N(\mathbf{i}_{in}(j+1))$), but it is connected inside the p_0-cluster of $B'_N(\mathbf{i}_{out}(j))$ (resp. $B'_N(\mathbf{i}_{in}(j+1))$) to a vertex x'_j (resp. y'_{j+1}) of $B_N(\mathbf{i}_{out}(j))$ (resp. $B_N(\mathbf{i}_{in}(j+1))$) by a path $\gamma'_{j,out}$ (resp. $\gamma'_{j+1,in}$) of length at most equal to $2d3^dN^d \leq \hat{\rho}_d N^d$.

Let us study more carefully the beginning of the path γ. Since the N-box containing y belongs to an infinite cluster of good boxes, it cannot be in the interior of $S_{\varphi_4(1)}$. Let $\mathbf{i}_0$ be the macroscopic site such that $y \in B_N(\mathbf{i}_0)$. We have to consider three cases.

- If $\mathbf{i}_0$ is in $S_{\varphi_4(1)}$, then $\Psi_{in}(1) = 1$ and $\gamma_{\Psi_{in}(1)} = y$. As $y \in \mathcal{C}_{p_0}$ and $B_N(\mathbf{i}_0)$ is good, y is in the crossing p_0-cluster of $B_N(\mathbf{i}_0)$, thus we can define $y_1 = y'_1 = y$, $\gamma_0 = \varnothing$ and $\gamma'_{1,in} = \varnothing$.
- If $\mathbf{i}_0$ is not in $S_{\varphi_4(1)}$ (thus it is outside $S_{\varphi_4(1)}$), then $\Psi_{in}(1) > 1$ and $\gamma_{\Psi_{in}(1)}$ does not belong to $B_N(\mathbf{i}_0)$. Let $\mathbf{i}_1 = \mathbf{i}_{in}(1)$ be the macroscopic site such that $\gamma_{\Psi_{in}(1)} \in B_N(\mathbf{i}_1)$.
  - If $\mathbf{i}_0$ and $\mathbf{i}_1$ are $*$-connected, then we can choose $y_1 = y$, $y'_1 \in B_N(\mathbf{i}_1)$ such that y'_1 belongs to the crossing p_0-cluster of the good box $B_N(\mathbf{i}_1)$, and we can connect $y = y_1$ to y'_1 inside $B'_N(\mathbf{i}_1)$ by a path $\gamma'_{1,in}$ of length at most equal to $2d3^dN^d \leq \hat{\rho}_d N^d$. We define $\gamma_0 = \emptyset$.
  - If $\mathbf{i}_0$ and $\mathbf{i}_1$ are not $*$-connected, let γ_0 be the part of γ between y and $\gamma_{\Psi_{in}(1)}$, then by construction γ_0 does not contain any p-closed edge and has at least N vertices in $B'_N(\mathbf{i}_1)$, thus by property (iv) γ_0 intersects the crossing p_0-cluster of the good box $B'_N(\mathbf{i}_1)$. We denote by y_1 the first intersection of γ_0 with the crossing p_0-cluster of $B'_N(\mathbf{i}_1)$. As previously, we know that y_1 is connected inside the crossing p_0-cluster of $B'_N(\mathbf{i}_1)$ to a vertex y'_1 of $B_N(\mathbf{i}_1)$ by a path $\gamma'_{1,in}$ of length at most equal to $2d3^dN^d \leq \hat{\rho}_d N^d$.

Similarly, we define $x_{r_4}, x'_{r_4}, \gamma_{r_4}$ and $\gamma'_{r_4,out}$ depending on the fact that the box containing z belongs to $S_{\varphi_4(r_4)}$ or not, and is $*$-connected to the box containing $\gamma_{\Psi_{out}(r_4)}$ or not.

For all $1 \leq j \leq r_4$, we can apply Lemma 3.3 to state that there exists a p_0-open path $\gamma'_{j,link}$ from y'_j to x'_j of length at most $\hat{\rho}_d(N^d + N|S_{\varphi_4(j)}|)$. For all $1 \leq j \leq r_4$, define $\gamma'_j = \gamma'_{j,in} \cup \gamma'_{j,link} \cup \gamma'_{j,out}$. By construction each γ'_j is p_0-open. We can glue together the paths $\gamma_0, \gamma'_1, \gamma_1, \gamma'_2, \dots, \gamma'_{r_4}, \gamma_{r_4}$ in this order to obtain a p-open path γ' from y to z. Up to cutting parts of these paths, we can suppose that each γ'_i is a self-avoiding path, that the γ'_i are disjoint and that each γ'_i intersects only γ_{i-1} and γ_i, and only with its endpoints.

Finally we need an estimate on $|\gamma' \smallsetminus \gamma|$. Obviously $\gamma' \smallsetminus \gamma \subset \cup_{i=1}^{r_4} \gamma'_i$, thus

$$\begin{aligned} |\gamma' \smallsetminus \gamma| &\leq 2r_4 \hat{\rho}_d N^d + \sum_{j=1}^{r_4} \hat{\rho}_d (N^d + N|S_{\varphi_4(j)}|) \\ &\leq 3r_4 \hat{\rho}_d N^d + \hat{\rho}_d N \sum_{j=1}^{r_4} |S_{\varphi_4(j)}| \\ &\leq 4\hat{\rho}_d N^d |\gamma_b| + \hat{\rho}_d N \sum_{C \in \mathrm{Bad}:\, C \cap \Gamma \neq \varnothing} |\partial_v C| \,, \end{aligned}$$

where we have used the fact that

$$\sum_{j=1}^{r_4} |S_{\varphi_4(j)}| \leq |\gamma_b| + \sum_{C \in \mathrm{Bad}:\, C \cap \Gamma \neq \varnothing} |\partial_v C| \,.$$

To conclude, we just have to remark that $|\partial_v C| \leq 2d|C|$. □

3.3 Probabilistic estimates

We want to bound the probability that $|\gamma' \smallsetminus \gamma|$ is big for $q - p_0$ small enough. Lemma 3.2 makes appear the connected set $\Gamma \subset \mathbb{Z}^d$ of N-boxes visited by the path γ. To control $|\gamma' \smallsetminus \gamma|$, we need to have a deterministic control on $|\Gamma|$. This is the purpose of the following Lemma.

Lemma 3.4. There exists a constant $\tilde{C}_d$, depending only on d, such that for every path γ of $\mathbb{Z}^d$, for every $N \in \mathbb{N}^*$, if Γ is the animal of N-blocks that γ visits, then

$$|\Gamma| \leq \tilde{C}_d \left(1 + \frac{|\gamma| + 1}{N}\right) .$$

Proof. Let $\gamma = (\gamma_i)_{i=1,\dots,n}$ be a path of $\mathbb{Z}^d$ for a $n \in \mathbb{N}^*$ (γ_i is the i-th vertex of γ, $n = |\gamma| + 1$), and fix $N \in \mathbb{N}^*$. Let Γ be the animal of N-blocks that γ visits. We will include Γ in a bigger set of blocks whose size can be controlled. Let $p(1) = 1$ and $\mathbf{i}_1$ be the macroscopic site such that $\gamma_1 \in B_N(\mathbf{i}_1)$. If $p(1), \dots, p(k)$ and $\mathbf{i}_1, \dots, \mathbf{i}_k$ are constructed, define $p(k+1) = \inf\{j \in \{p(k), \dots, n\} : \gamma_j \notin B'_N(\mathbf{i}_k)\}$ if this set is not empty and let $\mathbf{i}_{k+1}$ be the macroscopic site such that $\gamma_{p(k+1)} \in B_N(\mathbf{i}_{k+1})$, and stop the process if for every $j \in \{p(k), \dots, n\}$, $\gamma_j \in B'_N(\mathbf{i}_k)$. We obtain two finite sequences $(p(1), \dots, p(r))$ and $(\mathbf{i}_1, \dots, \mathbf{i}_r)$. First notice that

$$\Gamma \subset \bigcup_{k=1}^{r} B'_N(\mathbf{i}_k)$$

by construction, thus $|\Gamma| \leq 3^d r$. Moreover for every $k \in \{1, \dots, r-1\}$, $\|\gamma_{p(k+1)} - \gamma_{p(k)}\|_1 \geq N$, thus $p(k+1) - p(k) \geq N$. This implies that $N(r-1) \leq p(r) - p(1) \leq n$, and we conclude that

$$|\Gamma| \leq 3^d \left(1 + \frac{n}{N}\right) .$$ □

Then we need a control on the probability that a block is good.

Lemma 3.5. (i) For every $q > p_c(d)$, there exists $\delta_0(q) > 0$ such that if $p_0 \in (p_c(d), q]$ satisfy $q - p_0 \leq \delta_0$, then for every $\mathfrak{p} < 1$, there exists an integer $N(p_0, q, \mathfrak{p})$ such that the field $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is } (p_0,\,q)\text{-good}\}})_{\mathbf{i} \in \mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathfrak{p}$.

(ii) For every $p_0 > p_c(d)$, there exists $\delta_1(p_0) > 0$ such that if $q_1 \in [p_0, 1]$ satisfy $q_1 - p_0 \leq \delta_1$, then for every $\mathfrak{p} < 1$ there exists an integer $N'(p_0, q_1, \mathfrak{p})$ such that for any $q \in [p_0, q_1]$ the field $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is } (p_0,\, q)\text{-good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathfrak{p}$.

Proof. Obviously, the states of $(B_N(\mathbf{i}))_{\mathbf{i}\in\mathbb{Z}^d}$ have a finite range of dependance and are identically distributed. Then, by the Liggett–Schonmann–Stacey Theorem [22], it is sufficient to check that $\lim_{N\to+\infty} \mathbb{P}(B_N \text{ good}) = 1$, for $B_N = B_N(\mathbf{0})$.

Consider first the properties (i) and (ii) of the Definition 3.1, that depend only on p_0. For any $p_0 > p_c(d)$, when $d \geq 3$, the fact that

$$\lim_{N\to+\infty} \mathbb{P}(B_N \text{ satisfies (i) and (ii)}) = 1$$

follows from the Pisztora coarse graining argument (see Pisztora [25] or the coarse graining section in Cerf [5]), see also for instance Grimmett [16] Lemma (7.104). When $d = 2$, see Couronné and Messikh [7]. We now study the property (iii) in the Definition 3.1, that also depends only on p_0. Let us define the property (iii') by

(iii') For all $x, y \in B'_N$, if $\|x - y\|_\infty \geq N$ and x and y belong to the crossing p_0-cluster of B_N, then $D^{\mathcal{C}'_{p_0}}(x, y) \leq 3\beta N$.

Notice that if x and y belong to B'_N, there exists z in the crossing p_0-cluster of B'_N such that $\|x - z\|_\infty \geq N$ and $\|y - z\|_\infty \geq N$, thus if B_N satisfies property (iii') it also satisfies property (iii). Using Antal and Pisztora's estimate (3.1), for any fixed $p_0 > p_c(d)$, we have for all N

$$\begin{aligned}
&\mathbb{P}[B_N \text{ does not satisfy (iii)}] \\
&\quad \leq \mathbb{P}[B_N \text{ does not satisfy (iii')}] \\
&\quad \leq \sum_{x\in B'_N} \sum_{y\in B'_N} \mathbb{1}_{\|x-y\|_\infty \geq N} \mathbb{P}\left[x \overset{\mathcal{C}'_{p_0}}{\longleftrightarrow} y \,,\, D^{\mathcal{C}'_{p_0}}(x,y) \geq 3\beta N \right] \\
&\quad \leq \sum_{x\in B'_N} \sum_{y\in B'_N} \mathbb{1}_{\|x-y\|_\infty \geq N} \mathbb{P}\left[x \overset{\mathcal{C}'_{p_0}}{\longleftrightarrow} y \,,\, D^{\mathcal{C}'_{p_0}}(x,y) \geq \beta \|x-y\|_\infty \right] \\
&\quad \leq \sum_{x\in B'_N} \sum_{y\in B'_N} \mathbb{1}_{\|x-y\|_\infty \geq N} \hat{A} e^{-\hat{B}\|x-y\|_\infty} \;\leq\; (3N)^d . (3N)^d \hat{A} e^{-\hat{B}N}
\end{aligned}$$

that goes to 0 when N goes to infinity. The delicate part of the proof is the study of the property (iv) in the Definition 3.1. For $q = p_0$, we are done since property (iv) is implied by the uniqueness of the p_0-crossing cluster in B'_N. We want to deduce from this that property (iv) is asymptotically typical. We follow the proof of Russo's formula, see for instance Theorem 2.25 in [16]. For given parameters $p_c(d) < p_0 < p \leq 1$, we denote by $\mathbb{P}_{p_0,p}$ the probability of the corresponding coupled Bernoulli percolation, and we declare that

- an edge e is in state 0 if e is p-closed,
- an edge e is in state 1 if e is p_0-closed and p-open,
- an edge e is in state 2 if e is p_0-open.

We define A_N as the event that there exists a crossing cluster $\mathcal{C}$ of edges of state 2 in B'_N, and a path $\gamma \subset B'_N$ of edges of state 1 or 2 such that $|\gamma| = N$ and γ does not intersect $\mathcal{C}$. Let us fix p_0. When p vary, the edges of state 2 remain unchanged, we only change the state of edges from 0 to 1 and conversely. For a given p_0, the event A_N is increasing in p.

We denote by $\mathcal{N}(A_N)$ the random number of edges that are $0-1$-pivotal for A_N, i.e., the number of edges e such that if e is in state 1 then A_N occurs, and if e is in state 0 then A_N does not occur. Following the proof of Russo's formula, we obtain that

$$\frac{\partial}{\partial p}\mathbb{P}_{p_0,p}(A_N) \;=\; \frac{1}{p}\mathbb{E}_{p_0,p}[\mathcal{N}(A_N)\,|\,A_N]\,\mathbb{P}_{p_0,p}(A_N)\,.$$

We remark that when A_N occurs, $\mathcal{N}(A_N)\leq N$, the length of the desired path, thus

$$\mathbb{E}_{p_0,p}[\mathcal{N}(A_N)\,|\,A_N]=\mathbb{E}_{p_0,p}[\mathbb{1}_{A_N}\mathcal{N}(A_N)\,|\,A_N]\leq N\,.$$

We obtain that

$$\begin{aligned}\mathbb{P}_{p_0,q}(A_N) \;&=\; \mathbb{P}_{p_0,p_0}(A_N)\exp\left(\int_{p_0}^{q}\frac{1}{p}\mathbb{E}_{p_0,p}[\mathcal{N}(A_N)\,|\,A_N]\,dp\right)\\ &\leq\; \mathbb{P}_{p_0,p_0}(A_N)\exp\left(N\int_{p_0}^{q}\frac{1}{p}\,dp\right)\\ &\leq\; \mathbb{P}_{p_0,p_0}(A_N)\exp\left(N\log\left(\frac{q}{p_0}\right)\right)\\ &\leq\; \mathbb{P}_{p_0,p_0}(A_N)\exp\left(N\log\left(1+\frac{q-p_0}{p_0}\right)\right)\end{aligned}\tag{3.4}$$

It comes from the coarse graining arguments previously cited to study property (i) that $\mathbb{P}_{p_0,p_0}(A_N)$ decays exponentially fast with N: there exists $\kappa_1(p_0),\kappa_2(p_0)$ such that

$$\mathbb{P}_{p_0,p_0}(A_N)\;\leq\;\kappa_1(p_0)e^{-\kappa_2(p_0)N}\,.\tag{3.5}$$

Part (ii) of Lemma 3.5: When p_0 is fixed, combining (3.4) and (3.5) is enough to conclude that there exists $\delta_1(p_0)>0$ such that if $q_1<p_0+\delta_1$, then

$$\lim_{N\to\infty}\mathbb{P}_{p_0,q_1}(A_N)\;=\;0\,.\tag{3.6}$$

We conclude that for every $p_0>p_c(d)$, there exists $\delta_1(p_0)>0$ such that if $q_1\in[p_0,1]$ satisfies $q_1-p_0\leq\delta_1$, then for every $\mathfrak{p}<1$ there exists an integer $N'(p_0,q_1,\mathfrak{p})$ such that for $q=q_1$ the field $(\mathbb{1}_{\{B_N(\mathbf{i})\text{ is }(p_0,\,q_1)\text{-good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathfrak{p}$. The only property of a good block that depends on q is property (iv), and if $p_0\leq q\leq q_1$ then any q-open path is also a q_1-open path, thus if a block is (p_0,q_1)-good then it is (p_0,q)-good for any parameter $q\in[p_0,q_1]$. We conclude that for $N'=N'(p_0,q_1,\mathfrak{p})$, for any $q\in[p_0,q_1]$, the field $(\mathbb{1}_{\{B'_N(\mathbf{i})\text{ is }(p_0,\,q)\text{-good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathfrak{p}$.

Part (i) of Lemma 3.5: If q is fixed, we need to replace (3.5) by a control on $\mathbb{P}_{p_0,p_0}(A_N)$ which is uniform for p_0 in a left neighborhood of q. Let us have a closer look at the proof of (3.5). In dimension $d\geq 3$, we refer to the proof of Lemma 7.104 in Grimmett [16] : the constants $\kappa_1(p_0),\kappa_2(p_0)$ of (3.5) appearing in Grimmett's book are explicit functions of the parameters $\delta(p_0)$ and $L(p_0)$ chosen in Lemma 7.78 in [16]. The probability controlled in Lemma 7.78 in [16] is clearly non decreasing in the parameter p of the percolation, thus the choice of $\delta(p)$ and $L(p)$ made for a given $p>p_c(d)$ can be kept unchanged for any $p'\geq p$. Fixing $p'_0=(q-p_c(d))/2$, we obtain that for any $p_0\in[p'_0,q]$,

$$\mathbb{P}_{p_0,p_0}(A_N)\;\leq\;\kappa_1(p'_0)e^{-\kappa_2(p'_0)N}\,.\tag{3.7}$$

Combining (3.4) and (3.7) we can conclude that in dimension $d\geq 3$, when q is fixed, there exists $\delta_0(q)$ such that if $p_0\in[p'_0,q]$ satisfies $q-p_0\leq\delta_0$, then (3.6) still holds. In dimension

2, (3.5) is obtained by Couronné and Messikh [7], Theorem 9, in a more general setting. The constants appearing in this theorem are explicit functions of the constants appearing in Proposition 6 in [7], and the same remark as in dimension $d \geq 3$ leads to the uniform control (3.7), and the proof is complete. □

We can now use Lemma 3.5 to bound the probability that $\sum_{C\in \mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C|$ is big. Denote by $\mathcal{A}\mathrm{nimals}$ the set of lattice animals containing $\mathbf{0}$, and $\mathcal{A}\mathrm{nimals}_n$ the subset of those having size n.

Lemma 3.6. Let $\varepsilon > 0$. Let $p_c^{\mathrm{site}}(d)$ be the critical parameter for independent Bernoulli site percolation on $\mathbb{Z}^d$. Choose $\alpha = \alpha(\varepsilon) > 0$ and then $\mathfrak{p} = \mathfrak{p}(\varepsilon) \in (p_c^{\mathrm{site}}(d), 1)$, such that

$$7^d \exp(-\alpha\varepsilon) \leq \frac{1}{3}, \tag{3.8}$$

$$\mathfrak{p} + \frac{e^\alpha 7^d(1-\mathfrak{p})}{1-e^\alpha 7^d(1-\mathfrak{p})} \leq \frac{3}{2}. \tag{3.9}$$

For a given $q > p_c(d)$ (resp. $p_0 > p_c(d)$), for a fixed $p_0 \in (p_c(d), q]$ such that $q - p_0 \leq \delta_0(q)$ (resp. $q_1 \geq p_0$ such that $q_1 - p_0 \leq \delta_1(p_0)$ and any $q \in [p_0, q_1]$), let finally $N = N(p_0, q, \mathfrak{p}(\varepsilon))$ (resp. $N = N'(p_0, q_1, \mathfrak{p}(\varepsilon))$) be large enough to have the stochastic comparison of Lemma 3.5 with this parameter $\mathfrak{p}(\varepsilon)$. Then for all $m \in \mathbb{N}$, we have

$$\mathbb{P}\left(\exists\Gamma\in\mathcal{A}\mathrm{nimals},\ |\Gamma| \geq \frac{m}{N},\ \sum_{C\in\mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C| \geq \varepsilon|\Gamma|\right) \leq e^{-\frac{m}{N}+1}.$$

Proof. We have

$$\begin{aligned}
\mathcal{P}(m) &\stackrel{\text{def}}{=} \mathbb{P}\left(\exists\Gamma\in\mathcal{A}\mathrm{nimals},\ |\Gamma| \geq \frac{m}{N},\ \sum_{C\in\mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C| \geq \varepsilon|\Gamma|\right) \\
&\leq \sum_{n\geq\frac{m}{N}}\sum_{\Gamma\in\mathcal{A}\mathrm{nimals}_n}\mathbb{P}\left(\sum_{C\in\mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C| \geq \varepsilon|\Gamma|\right) \\
&\leq \sum_{n\geq\frac{m}{N}}\sum_{\Gamma\in\mathcal{A}\mathrm{nimals}_n}\mathbb{P}_{\mathfrak{p}}\left(\sum_{C\in\mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C| \geq \varepsilon|\Gamma|\right).
\end{aligned}$$

For the last inequality, we use the coupling Lemma 3.5 to replace the locally dependent states of our N-boxes by an independent Bernoulli site percolation with parameter $\mathfrak{p}$ chosen in (3.9); by analogy, we still denote by Bad the random set of closed connected components for this independent Bernoulli site percolation. From now on, we work with this Bernoulli site percolation with parameter $\mathfrak{p}$. Denote by $C(0)$ the connected component of closed sites containing 0 (with the convention that if 0 is open, then $C(0) = \varnothing$). Let $(\tilde{C}(i))_{i\in\mathbb{Z}^d}$ be independent and identically distributed random sets of $\mathbb{Z}^d$ with the same law as $C(0)$. Fix a set $\Gamma = (\Gamma(i))_{1\leq i\leq n}$ of sites; we first prove that, for the independent Bernoulli site percolation, the following stochastic comparison holds:

$$\sum_{C\in\mathrm{Bad}\colon C\cap\Gamma\neq\varnothing}|C| \preceq \sum_{i=1}^n |\tilde{C}(i)|. \tag{3.10}$$

The idea is to build algorithmically the real clusters from the sequence of pre-clusters $(\tilde{C}(i))_{i\in\mathbb{Z}^d}$, as in the work of Fontes and Newman [11], proof of Theorem 4. Note however that in our sum (3.10), each visited cluster is only counted once, while they count each cluster the number of times it is visited, which explains the difference between our stochatic domination and their one. We proceed by induction on $j \in \{1, \dots, n\}$ to build a new family $(\overline{C}(i))_{1 \le i \le n}$ such that

$$A_j \stackrel{def}{=} \bigcup_{C \in \mathrm{Bad}:\ C \cap \{\Gamma(i):\ 1 \le i \le j\} \neq \varnothing} C \stackrel{law}{\subset} \bigcup_{i=1}^{j} \overline{C}(i) \subset \bigcup_{i=1}^{j} (\Gamma(i) + \tilde{C}(i)) ,$$

where for two random sets A and B, $A \stackrel{law}{\subset} B$ means that the field $(\mathbb{1}_A(x), x \in \mathbb{Z}^d)$ is stochastically dominated by the field $(\mathbb{1}_B(x), x \in \mathbb{Z}^d)$. Set $\overline{C}(1) = \Gamma(1) + \tilde{C}(1)$. Assume now that $(\overline{C}(i))_{1 \le i \le j}$ are built for some $j < n$:

- if $\Gamma(j+1) \in A_j$, then $A_{j+1} = A_j$, so we set $\overline{C}(j+1) = \varnothing$;
- if $\Gamma(j+1) \in \partial_v A_j$ (the exterior vertex boundary of A_j), then it is a good site, so we set $\overline{C}(j+1) = \varnothing$;
- otherwise, the conditional distribution of the bad cluster C containing the site $\Gamma(j+1)$, given A_j, is that of the percolation cluster of $\Gamma(j+1)$ in a site percolation model where $\mathbb{Z}^d$ is replaced by $\mathbb{Z}^d \backslash (A_j \cup \partial_v A_j)$; thus, it has the same law as the connected component of $\Gamma(j+1)$ in

$$\overline{C}(j+1) = \left(\Gamma(j+1) + \tilde{C}(j+1) \right) \setminus (A_j \cup \partial_v A_j) ,$$

which ends the construction and proves (3.10). As the number of lattice animals containing 0 with size n is bounded from above by $(7^d)^n$ (see Kesten [19], p 82. or Grimmett [16], p.85), we have, by the Markov inequality,

$$\mathcal{P}(m) \le \sum_{n \ge \frac{m}{N}} (7^d)^n \exp(-\alpha \varepsilon n) \left(\mathbb{E}_{\mathfrak{p}}(\exp(\alpha |C(0)|)) \right)^n .$$

But

$$\begin{aligned} \mathbb{E}_{\mathfrak{p}}(\exp(\alpha |C(0)|)) &= \mathfrak{p} + \sum_{k \ge 1} \exp(\alpha k) \mathbb{P}_{\mathfrak{p}}(|C(0)| = k) \le \mathfrak{p} + \sum_{k \ge 1} \exp(\alpha k) \mathbb{P}_{\mathfrak{p}}(|C(0)| \ge k) \\ &\le \mathfrak{p} + \sum_{k \ge 1} \exp(\alpha k) (7^d)^k (1 - \mathfrak{p})^k = \mathfrak{p} + \frac{e^{\alpha} 7^d (1 - \mathfrak{p})}{1 - e^{\alpha} 7^d (1 - \mathfrak{p})}. \end{aligned}$$

With the choices (3.8) and (3.9) we made for α and $\mathfrak{p}$, this ensures that

$$\mathcal{P}(m) \le \sum_{n \ge \frac{m}{N}} 2^{-n} \le 2^{-\frac{m}{N} + 1}. \qquad \square$$

4 Truncated passage times, proof of Theorem 1.6

Let G be a probability measure on $[0, +\infty]$ such that $q := G([0, +\infty)) > p_c(d)$. Let $\delta_0(q)$ be given by Lemma 3.5. Fix M_0 large enough so that $p_0 := G([0, M_0]) > p_c(d)$ and $q - p_0 \le \delta_0$. For a $K \in [M_0, +\infty)$, define $p = p(K) = G([0, K])$. We define the following i.i.d. Bernoulli bond percolations :

- an edge e is declared p_0-open if and only if $t_G(e) \le M_0$,
- an edge e is declared p-open if and only if $t_G(e) \le K$,
- an edge e is declared q-open if and only if $t_G(e) < \infty$.

These percolations are naturally coupled, thus we can use the modification of paths presented in the previous section. Denote as before by $\mathcal{C}_{G, M_0}$ the a.s. unique infinite cluster of the supercritical Bernoulli field $\{\mathbb{1}_{t_G(e) \le M_0} :\ e \in \mathbb{E}^d\}$. We call this field the M_0-percolation and its clusters the M_0-clusters. They correspond exactly to the p_0-percolation and the p_0-clusters.

4.1 Estimation for the passage time of the modified path

Lemma 4.1. There exists a positive constant ρ'_d (depending only on d and M_0) such that the following holds: Assume that $y \in \mathcal{C}_{G,M_0}$, that $z \in \mathcal{C}_{G,M_0}$, that the N-boxes containing y and z are good and belong to an infinite cluster of good boxes. Then for every $K \geq M_0$,

$$T_G(y,z) \leq T_{G^K}(y,z)\left(1+\frac{\rho'_d N^d}{K}\right) + \rho'_d N \sum_{C\in\mathrm{Bad}:\ C\cap\Gamma\neq\varnothing} |C|,$$

where Γ is the lattice animal of N-boxes visited by an optimal path between y and z for the passage times with distribution G^K.

Remark 4.2. In this Lemma, Γ can be the lattice animal of N-boxes visited by any arbitrary geodesic between y and z for the passage times with distribution G^K if there is more than one.

Proof. As $y \in \mathcal{C}_{G,M_0}$ and $z \in \mathcal{C}_{G,M_0}$, the quantities $T_G(y,z)$ and $T_{G^K}(y,z)$ are bounded by M_0 times the chemical distance in $\mathcal{C}_{G,M_0}$ between y and z, and are thus finite. Let γ be an optimal path between y and z for $T_{G^K}(y,z)$. We can consider the modification γ' given by Lemma 3.2. Since γ' is a path between y and z, and $\gamma' \setminus \gamma$ is p_0-open, we have

$$T_G(y,z) \leq \sum_{e\in\gamma'} t_G(e) = \sum_{e\in\gamma\cap\gamma'} t_G(e) + \sum_{e\in\gamma'\setminus\gamma} t_G(e) \leq \sum_{e\in\gamma_a} t_G(e) + M_0\,|\gamma'\setminus\gamma|\,.$$

On one hand, since γ is an optimal path between y and z for $T_{G^K}(y,z)$, we have

$$\sum_{e\in\gamma_a} t_G(e) = \sum_{e\in\gamma_a} t_{G^K}(e) \leq \sum_{e\in\gamma} t_{G^K}(e) = T_{G^K}(y,z)\,.$$

On the other hand, using the estimate on the cardinality of $\gamma'\setminus\gamma$ given in Lemma 3.2, and noticing that the number of edges in γ_b is less than $T_{G^K}(\gamma)/K$, we obtain

$$|\gamma'\setminus\gamma| \leq \rho_d\left(\frac{N^d T_{G^K}(\gamma)}{K} + N\sum_{C\in\mathrm{Bad}:\ C\cap\Gamma\neq\varnothing}|C|\right). \qquad \square$$

Lemma 4.3. Suppose that $G(\{0\}) < p_c(d)$. For every $\varepsilon > 0$ there exists $p_1(\varepsilon) > 0$ and $A(\varepsilon) > 0$ such that for every $K \geq M_0$, for all x large enough,

$$\mathbb{P}\left(\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,x) \leq \widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,x)\left(1+\frac{A(\varepsilon)}{K}\right) + \varepsilon\|x\|_1\right) \geq p_1(\varepsilon).$$

Proof. Let $\varepsilon > 0$ be fixed. Let $p_c^{\mathrm{site}}(d)$ be the critical parameter for independent Bernoulli site percolation on $\mathbb{Z}^d$. Choose $\alpha = \alpha(\varepsilon) > 0$ and then $\mathfrak{p} = \mathfrak{p}(\varepsilon) \in (p_c^{\mathrm{site}}(d), 1)$, such that

$$7^d \exp(-\alpha\varepsilon) \leq \frac{1}{3}, \tag{4.1}$$

$$\mathfrak{p} + \frac{e^\alpha 7^d(1-\mathfrak{p})}{1-e^\alpha 7^d(1-\mathfrak{p})} \leq \frac{3}{2} \tag{4.2}$$

as in Lemma 3.6. Let $N = N(p_0, q, \mathfrak{p}(\varepsilon))$ be large enough to have the stochastic comparison of Lemma 3.5 with this parameter $\mathfrak{p}(\varepsilon)$. Let $K \geq M_0$. Fix a large x, at least large enough so that $\|x\|_1 \geq 12dN$.

Let F_x be the following good event: the N-boxes containing 0 and x and all the adjacent boxes belong to an infinite cluster of good boxes (in particular they are good). For any y in the same $(3N)$-box as 0, for any z in the same $(3N)$-box as x, let $E_{y,z}$ be the event that $y \in \mathcal{C}_{G,M_0}$, $z \in \mathcal{C}_{G,M_0}$ and the N-boxes containing y and z and belong to an infinite cluster of good boxes (in particular they are good). For any such (y,z), we have $\|y-z\|_1 \le \|x\|_1 + 6dN \le 2\|x\|_1$. For any given β'', we have

$$\begin{aligned}
&\mathbb{P}\left(\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,x) \ge \widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,x)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 4\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|x\|_1\right)\\
&\le \mathbb{P}[F_x^c]\\
&+\mathbb{P}\left(F_x \cap \left\{\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,x) \ge \widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,x)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 4\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|x\|_1\right\}\right)\\
&= \mathbb{P}[F_x^c]\\
&+\sum_{y,z}\mathbb{P}\left(\begin{array}{c} F_x \cap \{\widetilde{0}^{\mathcal{C}_{G,M_0}} = y\,,\, \widetilde{x}^{\mathcal{C}_{G,M_0}} = z\}\\ \cap\left\{T_G(y,z) \ge T_{G^K}(y,z)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 4\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|x\|_1\right\}\end{array}\right)\\
&\le \mathbb{P}[F_x^c]\\
&+\sum_{y,z}\mathbb{P}\left(E_{y,z}\cap\left\{T_G(y,z) \ge T_{G^K}(y,z)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 2\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|y-z\|_1\right\}\right)\,,
\end{aligned} \tag{4.3}$$

where the sum is over every y in the same $(3N)$-box as 0, and every z in the same $(3N)$-box as x - indeed, on the event F_x, we know that $\mathcal{C}_{G,M_0}$ intersects the box of 0 (resp. x) thus $\widetilde{0}^{\mathcal{C}_{G,M_0}}$ (resp. $\widetilde{x}^{\mathcal{C}_{G,M_0}}$) belongs to the same $(3N)$-box as 0 (resp. x). Note that the stochastic comparison and the FKG inequality ensure that

$$\mathbb{P}(F_x) \ge \theta_{\text{site},\mathfrak{p}(\varepsilon)}^{2\cdot 3^d} > 0\,, \tag{4.4}$$

where $\theta_{\text{site},\mathfrak{p}(\varepsilon)}$ denotes the density of the infinite cluster in a supercritical vertex i.i.d. Bernoulli percolation of parameter $\mathfrak{p}(\varepsilon)$.

Consider a couple (y,z) as in (4.3). On the event $E_{y,z}$, we have

$$T_{G^K}(y,z) \le M_0 D^{\mathcal{C}_{G,M_0}}(y,z) < \infty\,.$$

Let $\gamma_{y,z}$ be a geodesic for $T_{G^K}(y,z)$, and let $\Gamma_{y,z}$ be the lattice animal of the N-boxes visited by this geodesic. By Lemma 4.1, we have

$$\begin{aligned}
&\mathbb{P}\left(E_{y,z}\cap\left\{T_G(y,z) \ge T_{G^K}(y,z)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 2\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|y-z\|_1\right\}\right)\\
&\le \mathbb{P}\left(E_{y,z}\cap\left\{\sum_{C\in\mathrm{Bad}:\ C\cap\Gamma_{y,z}\neq\varnothing}|C| \ge \frac{2\,\varepsilon\,\tilde{C}_d\,\beta''\|y-z\|_1}{N(\varepsilon)}\right\}\right)\,.
\end{aligned}$$

Note that by construction, on the event $E_{y,z}$, we have $|\Gamma_{y,z}| \ge \|y-z\|_1/N$. On the other hand Lemma 3.4 implies that $|\Gamma_{y,z}| \le \tilde{C}_d(1+(|\gamma_{y,z}|+1)/N) \le 2\tilde{C}_d|\gamma_{y,z}|/N$ at least for x large enough (remember that $|\gamma_{y,z}| \ge \|y-z\|_1 \ge \|x\|_1 - 6dN \ge \|x\|_1/2$). Thus we obtain

$$\begin{aligned}
&\mathbb{P}\left(E_{y,z}\cap\left\{T_G(y,z) \ge T_{G^K}(y,z)\left(1+\frac{\rho'_d N(\varepsilon)^d}{K}\right) + 2\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|y-z\|_1\right\}\right)\\
&\le \mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}| > \beta''\|y-z\|_1\}\right)\\
&\qquad + \mathbb{P}\left(E_{y,z}\cap\left\{\sum_{C\in\mathrm{Bad}:\ C\cap\Gamma_{y,z}\neq\varnothing}|C| \ge \varepsilon|\Gamma_{y,z}|\right\}\right)\,.
\end{aligned} \tag{4.5}$$

Since $G^K(\{0\}) = G(\{0\}) < p_c(d)$, there exist positive constants A', B', β' such that for all $k\in\mathbb{N}^*$ (see Proposition 5.8 in Kesten [20]):

$$\mathbb{P}\left[\exists r \text{ s.a. path starting at } y \text{ s.t. } |r| \ge k \text{ and } T_{G^K}(r) \le \beta' k\right] \le A'\exp(-B'k). \tag{4.6}$$

Let β be given by Antal and Pisztora's estimate (3.1). By (3.1) we have

$$\begin{aligned}\mathbb{P}(E_{y,z}\cap\{D^{\mathcal{C}_{G,M_0}}(y,z)\geq\beta\|y-z\|_1\}) &\leq \mathbb{P}(\beta\|y-z\|_1\leq D^{\mathcal{C}'_{G,M_0}}(y,z)<+\infty)\\ &\leq \hat{A}\exp(-\hat{B}\|y-z\|_1)\,. \qquad (4.7)\end{aligned}$$

Fix $\beta''=\frac{\beta M_0}{\beta'}>0$. Combining (4.6) and (4.7) we obtain the existence of positive constants A'',B'' such that

$$\begin{aligned}&\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta''\|y-z\|_1\}\right)\\ &\quad\leq \mathbb{P}(E_{y,z}\cap\{D^{\mathcal{C}_{G,M_0}}(y,z)\geq\beta\|y-z\|_1\})\\ &\qquad+\mathbb{P}\left(E_{y,z}\cap\{T_{G^K}(y,z)\leq M_0\beta\|y-z\|_1\}\cap\{|\gamma_{y,z}|>\beta''\|y-z\|_1\}\right)\\ &\quad\leq \hat{A}e^{-\hat{B}\|y-z\|_1}+A'e^{-B'\beta''\|y-z\|_1}\leq A''e^{-B''\|y-z\|_1}\,. \qquad (4.8)\end{aligned}$$

By Lemma 3.6, with the choices (4.1) and (4.2) we made for α and $\mathfrak{p}$, we know that

$$\begin{aligned}&\mathbb{P}\left(E_{y,z}\cap\left\{\sum_{C\in\mathrm{Bad}:\ C\cap\Gamma_{y,z}\neq\varnothing}|C|\geq\varepsilon|\Gamma_{y,z}|\right\}\right)\\ &\quad\leq \mathbb{P}\left(\exists\Gamma\in\mathcal{A}\mathrm{nimals},\ |\Gamma|\geq\frac{\|y-z\|_1}{N(\varepsilon)},\ \sum_{C\in\mathrm{Bad}:\ C\cap\Gamma\neq\varnothing}|C|\geq\varepsilon|\Gamma|\right)=\mathcal{P}(\|y-z\|_1)\\ &\quad\leq 2^{-\frac{\|y-z\|_1}{N(\varepsilon)}+1}. \qquad (4.9)\end{aligned}$$

Combining (4.3), (4.4), (4.5), (4.8) and (4.9), we obtain that

$$\begin{aligned}&\mathbb{P}\left(\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,x)\geq\widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,x)\left(1+\frac{\rho'_dN(\varepsilon)^d}{K}\right)+4\,\varepsilon\,\tilde{C}_d\,\rho'_d\,\beta''\|x\|_1\right)\\ &\quad\leq 1-\theta_{\mathrm{site},\mathfrak{p}(\varepsilon)}^{2\dot{3}^d}+\sum_{y,z}\left(A''e^{-B''\|y-z\|_1}+2^{-\frac{\|y-z\|_1}{N(\varepsilon)}+1}\right)\\ &\quad\leq 1-\theta_{\mathrm{site},\mathfrak{p}(\varepsilon)}^{2\dot{3}^d}+2(3N(\varepsilon))^d\left(A''e^{-B''\|x\|_1/2}+2^{-\frac{\|x\|_1}{2N(\varepsilon)}+1}\right)\\ &\quad\leq 1-p_1(\varepsilon),\end{aligned}$$

for a well-chosen $p_1(\varepsilon)>0$ and every x large enough. □

4.2 Proof of Theorem 1.6

If $G(\{0\})\geq p_c(d)$, then $\mu_{G^K}(x)=\mu_G(x)=0$, so there is nothing to prove. Suppose from now on that $G(\{0\})<p_c(d)$.

For any $\varepsilon>0$, consider $p_1(\varepsilon)$ and $A(\varepsilon)$ as given by Lemma 4.3, and define, for $K\geq M_0$, $\Psi(K)=\inf_{\varepsilon>0}\frac{A(\varepsilon)}{K}+\varepsilon$. It is easy to see that $\lim_{K\to+\infty}\Psi(K)=0$. Fix $\varepsilon>0$, $\delta>0$, $K\geq M_0$ and $x\in\mathbb{Z}^d$.

With the convergence (2) in Proposition 2.6 and Lemma 4.3, we can choose n large enough such that

$$\begin{aligned}\mathbb{P}\left(\mu_G(x)-\delta\leq\frac{\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,nx)}{n}\right)&\geq 1-\frac{p_1(\varepsilon)}{3},\\ \mathbb{P}\left(\frac{\widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,nx)}{n}\leq\mu_{G^K}(x)+\delta\right)&\geq 1-\frac{p_1(\varepsilon)}{3},\\ \mathbb{P}\left(\widetilde{T}_G^{\mathcal{C}_{G,M_0}}(0,nx)\leq\widetilde{T}_{G^K}^{\mathcal{C}_{G,M_0}}(0,nx)\left(1+\frac{A(\varepsilon)}{K}\right)+\varepsilon n\|x\|_1\right)&\geq p_1(\varepsilon).\end{aligned}$$

For every $\varepsilon > 0$, for every $\delta > 0$, on the intersection of these 3 events, that has positive probability, we obtain

$$\forall K \geq M_0, x \in \mathbb{Z}^d \quad \mu_G(x) - \delta \leq (\mu_{G^K}(x) + \delta)\left(1 + \frac{A(\varepsilon)}{K}\right) + \varepsilon \|x\|_1,$$

and by letting δ going to 0 we get

$$\forall \varepsilon > 0, K \geq M_0, x \in \mathbb{Z}^d \quad \mu_G(x) \leq \mu_{G^K}(x)\left(1 + \frac{A(\varepsilon)}{K}\right) + \varepsilon \|x\|_1.$$

It follows that for every $\varepsilon > 0$,

$$0 \leq \mu_{G^K}(x) - \mu_G(x) \leq \mu_{G^K} \frac{A(\varepsilon)}{K} + \varepsilon\|x\|_1 \leq (\mu_G(x) + \|x\|_1)\left(\frac{A(\varepsilon)}{K} + \varepsilon\right),$$

thus, by optimizing ε,

$$0 \leq \mu_{G^K}(x) - \mu_G(x) \leq (\mu_G(x) + \|x\|_1)\Psi(K)\,.$$

Theorem 1.6 is proved by using the fact that $\lim_{K\to+\infty} \Psi(K) = 0$.

5 Continuity of the time constant, proof of Theorem 1.2

We first state two properties:

Lemma 5.1. Suppose that $G, (G_n)_{n\in\mathbb{N}}$ are probability measures on $[0, +\infty]$ such that $G([0, +\infty)) > p_c(d)$ and $G_n([0, +\infty)) > p_c(d)$ for all $n \in \mathbb{N}$. If $G_n \xrightarrow{d} G$ and $G_n \succeq G$ for all n, then

$$\forall x \in \mathbb{Z}^d\,, \qquad \overline{\lim_{n\to+\infty}}\ \mu_{G_n}(x) \ \leq\ \mu_G(x)\,.$$

Lemma 5.2. Suppose that $G, (G_n)_{n\in\mathbb{N}}$ are probability measures on $[0, R]$ for some common and finite $R \in [0, +\infty)$. If $G_n \xrightarrow{d} G$, then

$$\forall x \in \mathbb{Z}^d\,, \quad \lim_{n\to\infty} \mu_{G_n}(x) \ =\ \mu_G(x)\,.$$

To prove Theorem 1.2, we follow the general structure of Cox and Kesten's proof of the continuity of the time constant in first-passage percolation with finite passage times in [10]. We first deduce Theorem 1.2 from Theorem 1.6 and Lemmas 5.1 and 5.2. Lemmas 5.1 and 5.2 will be respectively proved in subsections 5.3 and 5.4.

5.1 Proof of Theorem 1.2

Let $G, (G_n)_{n\in\mathbb{N}}$ be probability measures on $[0, +\infty]$. We first prove that for all fixed $x \in \mathbb{Z}^d$, we have

$$\lim_{n\to\infty} \mu_{G_n}(x) = \mu_G(x)\,. \tag{5.1}$$

We define $\underline{\mathfrak{G}}_n = \min\{\mathfrak{G}, \mathfrak{G}_n\}$ (resp. $\overline{\mathfrak{G}}_n = \max\{\mathfrak{G}, \mathfrak{G}_n\}$), and we denote by $\underline{G}_n$ (resp. $\overline{G}_n$) the corresponding probability measure on $[0, +\infty]$. Then $\underline{\mathfrak{G}}_n \leq \mathfrak{G} \leq \overline{\mathfrak{G}}_n$ (resp. $\underline{G}_n \leq G_n \leq \overline{G}_n$), thus by Lemma 2.12 we have $\mu_{\underline{G}_n}(x) \leq \mu_G(x) \leq \mu_{\overline{G}_n}(x)$. To conclude that (5.1) holds, it is sufficient to prove that

$$(i) \quad \underline{\lim_{n\to\infty}}\ \mu_{\underline{G}_n}(x) \geq \mu_G(x) \quad \text{and} \quad (ii) \quad \overline{\lim_{n\to+\infty}}\ \mu_{\overline{G}_n}(x) \leq \mu_G(x)\,.$$

Notice that $\overline{G}_n \xrightarrow{d} G$ and $\underline{G}_n \xrightarrow{d} G$. Inequality (ii) is obtained by a straightforward application of Lemma 5.1. For any $K \in [0, +\infty)$, we define $G^K = \mathbb{1}_{[0,K)}G + G([K, +\infty])\delta_K$

(resp. $\underline{G}_n^K = \mathbb{1}_{[0,K)}\underline{G}_n + \underline{G}_n([K,+\infty])\delta_K$), the distribution of $t_G^K(e) = \min(t_G(e),K)$ (resp. $t_{\underline{G}_n}^K(e) = \min(t_{\underline{G}_n}(e),K)$). Using Lemmas 2.12 and 5.2, since $\underline{G}_n^K \xrightarrow{d} G^K$, we obtain for all K

$$\varliminf_{n\to\infty} \mu_{\underline{G}_n}(x) \,\geq\, \lim_{n\to\infty} \mu_{\underline{G}_n^K}(x) \,=\, \mu_{G^K}(x)\,,$$

and by Theorem 1.6 we have $\lim_{K\to\infty}\mu_{G^K}(x) = \mu_G(x)$. This concludes the proof of (i), and of (5.1).

By homogeneity, (5.1) also holds for all $x \in \mathbb{Q}^d$. We know that $|\mu_{G_n}(x) - \mu_{G_n}(y)| \leq \mu_{G_n}(e_1)\|x-y\|_1$, where $e_1 = (1,0,\dots,0)$. Moreover $\lim_{n\to\infty}\mu_{G_n}(e_1) = \mu_G(e_1)$, thus for all $n \geq n_0$ large enough we have $|\mu_{G_n}(x) - \mu_{G_n}(y)| \leq 2\mu_G(e_1)\|x-y\|_1$ for all $x,y\in\mathbb{R}^d$. This implies that for any fixed $\varepsilon > 0$, there exists $\eta > 0$ such that for all $x,y \in \mathbb{R}^d$ such that $\|x-y\|_1 \leq \eta$, we have

$$\sup\{|\mu_G(x) - \mu_G(y)|, |\mu_{G_n}(x) - \mu_{G_n}(y)|, n \geq n_0\} \,\leq\, \varepsilon\,.$$

There exists a finite set $(y_1,\dots,y_m)$ of rational points of $\mathbb{R}^d$ such that

$$\mathbb{S}^{d-1} \,\subset\, \bigcup_{i=1}^m \{x\in\mathbb{R}^d \,:\, \|y_i - x\|_1 \leq \eta\}\,.$$

Thus $\varlimsup_{n\to+\infty} \sup_{x\in\mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| \leq 2\varepsilon + \lim_{n\to+\infty}\max_{i=1,\dots,m}|\mu_{G_n}(y_i) - \mu_G(y_i)| = 2\varepsilon$. Since ε was arbitrary, Theorem 1.2 is proved.

5.2 Bound on sequences of probability measures

Lemma 5.3. Suppose that G and $(G_n)_{n\in\mathbb{N}}$ are probability measures on $[0,+\infty]$ such that $G_n \xrightarrow{d} G$.

- (i) If $G([0,+\infty)) > p_c(d)$ and $G_n([0,+\infty)) > p_c(d)$ for all $n\in\mathbb{N}$, then there exists a probability measure H^+ on $[0,+\infty]$ such that $G_n \preceq H^+$ for all n and $H^+([0,+\infty)) > p_c(d)$.
- (ii) If $G(\{0\}) < p_c(d)$ and $G_n(\{0\}) < p_c(d)$ for all $n\in\mathbb{N}$, then there exists a probability measure H^- on $[0,+\infty]$ such that $G_n \succeq H^-$ for all n and $H^-(\{0\}) < p_c(d)$.

Proof. (i) We define $\hat{\mathfrak{H}}^+ = \sup_{n\in\mathbb{N}}\mathfrak{G}_n$, and $\mathfrak{H}^+(x) = \inf\{\hat{\mathfrak{H}}^+(y) \,:\, y < x\}$ for all $x\in[0,+\infty)$. Then $\hat{\mathfrak{H}}^+$ and $\mathfrak{H}^+$ are non-increasing functions defined on $[0,+\infty)$ and they take values in $[0,1]$. By construction $\mathfrak{H}^+$ is left continuous and $\mathfrak{H}^+ \geq \mathfrak{G}_n$, for all $n\in\mathbb{N}$. Moreover we have $\hat{\mathfrak{H}}^+(x) = \mathfrak{H}^+(x) = 1$ for all $x\leq 0$. Thus there exists a probability measure H^+ on $[0,+\infty]$ such that $\mathfrak{H}^+(t) = H^+([t,+\infty])$ for all $t\in[0,+\infty)$. It remains to prove that $H^+([0,+\infty)) > p_c(d)$. Since $G([0,+\infty)) > p_c(d)$, i.e. $\lim_{+\infty}\mathfrak{G} < 1 - p_c(d)$, there exist $A\in[0,+\infty)$ and $\varepsilon > 0$ such that $\mathfrak{G}$ is continuous at A and $\mathfrak{G}(A) \leq 1 - p_c(d) - 2\varepsilon$. Moreover $G_n \xrightarrow{d} G$ and $\mathfrak{G}$ is continuous at A, thus there exists n_0 such that for all $n\geq n_0$ we have $\mathfrak{G}_n(A) \leq \mathfrak{G}(A) + \varepsilon \leq 1 - p_c(d) - \varepsilon$. For any $i\in\{1,\dots,n_0-1\}$, $G_i([0,+\infty)) > p_c(d)$ thus there exists $A_i < +\infty$ such that $\mathfrak{G}_i(A_i) < 1 - p_c(d)$. Fix $A' = \max(A, A_0,\dots,A_{n_0-1}) < +\infty$. We conclude that

$$\begin{aligned}\hat{\mathfrak{H}}^+(A') \,&=\, \max\left(\mathfrak{G}_0(A'),\dots,\mathfrak{G}_{n_0-1}(A'), \sup_{n\geq n_0}\mathfrak{G}_n(A')\right)\\ &\leq\, \max\left(\mathfrak{G}_0(A_0),\dots,\mathfrak{G}_{n_0-1}(A_{n_0-1}), \sup_{n\geq n_0}\mathfrak{G}_n(A)\right) \,<\, 1 - p_c(d)\,,\end{aligned}$$

thus $H^+([0,+\infty)) = 1 - \lim_{+\infty}\mathfrak{H}^+ > p_c(d)$.

(ii) We define $\mathfrak{H}^- = \inf_{n\in\mathbb{N}} \mathfrak{G}_n$. Then $\mathfrak{H}^-$ is non-increasing, defined on $[0,+\infty)$ and it takes values in $[0,1]$. Fix $t_0 \in [0,+\infty)$. Let us prove that $\mathfrak{H}^-$ is left continuous at t_0. By definition of $\mathfrak{H}^-$, for any $\varepsilon > 0$, there exists n_0 such that $\mathfrak{H}^-(t_0) \geq \mathfrak{G}_{n_0}(t_0) - \varepsilon$. Since $\mathfrak{G}_{n_0}$ is left continuous, there exists $\eta > 0$ such that for all $t \in (t_0 - \eta, t_0]$ we have $\mathfrak{G}_{n_0}(t) \leq \mathfrak{G}_{n_0}(t_0) + \varepsilon$. Thus for all $t \in (t_0-\eta, t_0]$, we obtain

$$\mathfrak{H}^-(t) \,\leq\, \mathfrak{G}_{n_0}(t) \,\leq\, \mathfrak{G}_{n_0}(t_0) + \varepsilon \,\leq\, \mathfrak{H}^-(t_0) + 2\varepsilon\,,$$

thus $\mathfrak{H}^-$ is right continuous. By construction $\mathfrak{H}^- \leq \mathfrak{G}_n$, for all $n \in \mathbb{N}$. Moreover $\mathfrak{H}^-(t) = 1$ for all $t \leq 0$. Thus there exists a probability measure H^- on $[0,+\infty]$ such that $\mathfrak{H}^-(t) = H^-([t,+\infty])$ for all $t \in [0,+\infty)$. It remains to prove that $H^-(\{0\}) < p_c(d)$. Since $G(\{0\}) < p_c(d)$, there exists $\eta > 0$ such that $G([0,\eta)) < p_c(d)$, i.e., $\mathfrak{G}(\eta) > 1 - p_c(d)$. Let $\varepsilon > 0$ such that $\mathfrak{G}(\eta) \geq 1 - p_c(d) + 2\varepsilon$. There exists $\delta \in [0,\eta)$ such that $\mathfrak{G}$ is continuous at δ. Then $\lim_{n\to\infty} \mathfrak{G}_n(\delta) = \mathfrak{G}(\delta)$, thus there exists n_0 such that for all $n \geq n_0$, $\mathfrak{G}_n(\delta) \geq \mathfrak{G}(\delta) - \varepsilon \geq 1 - p_c(d) + \varepsilon$. For any $i \in \{1,\dots,n_0-1\}$, there exists $\delta_i > 0$ such that $\mathfrak{G}_i(\delta_i) > 1 - p_c(d)$. Fix $\delta' = \min(\delta, \delta_0, \dots, \delta_{n_0-1}) > 0$. We conclude that

$$\begin{aligned}\mathfrak{H}^-(\delta') \,&=\, \min\left(\mathfrak{G}_0(\delta'),\dots,\mathfrak{G}_{n_0-1}(\delta'), \inf_{n\geq n_0} \mathfrak{G}_n(\delta')\right)\\ &\geq\, \min\left(\mathfrak{G}_0(\delta_0),\dots,\mathfrak{G}_{n_0-1}(\delta_{n_0-1}), \inf_{n\geq n_0} \mathfrak{G}_n(\delta)\right) \,>\, 1 - p_c(d)\,,\end{aligned}$$

and $H^-(\{0\}) \,=\, 1 - \lim_{t\to 0, t>0} \mathfrak{H}^-(t) \,\leq\, 1 - \mathfrak{H}(\delta') < p_c(d)\,.$ ☐

5.3 Proof of Lemma 5.1

We follow the structure of Cox and Kesten's proof of Lemma 1 in [10].

We take H^+ as given in Lemma 5.3 (i), and we fix $M \in [0,+\infty)$ such that $H^+([0,M]) > p_c(d)$. We work with the stabilized points $\widetilde{x}^{\mathcal{C}_{H^+,M}}$. We consider a point $x \in \mathbb{Z}^d$, and $k \in \mathbb{N}^*$. For any path r from $\widetilde{0}^{\mathcal{C}_{H^+,M}}$ to $\widetilde{kx}^{\mathcal{C}_{H^+,M}}$, using Lemma 2.10 we have a.s.

$$T_G(r) \,=\, \sum_{e\in r} t_G(e) \,=\, \lim_{n\to+\infty} \sum_{e\in r} t_{G_n}(e) \,\geq\, \varlimsup_{n\to+\infty} \widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx)\,.$$

Taking the infimum over any such path r, we obtain

$$\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,kx) \,\geq\, \varlimsup_{n\to+\infty} \widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx)\,.$$

Conversely, since $G \preceq G_n$, thanks to the coupling of the laws we get $\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,kx) \leq \widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx)$ for all n, thus

$$\forall k \in \mathbb{N}^*\,,\ \text{a.s.,} \quad \lim_{n\to\infty} \widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx) \,=\, \widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,kx)\,.$$

Since for all n we have $\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx) \leq \widetilde{T}^{\mathcal{C}_{H^+,M}}_{H^+}(0,kx)$ that is integrable by Proposition 2.5, the dominated convergence theorem implies that, for all $k \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \mathbb{E}\left[\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G_n}(0,kx)\right] \,=\, \mathbb{E}\left[\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,kx)\right]\,. \tag{5.2}$$

By Lemma 2.11, we know that $\mu_G(x) = \inf_{k\in\mathbb{N}^*} \mathbb{E}\left[\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,kx)\right]/k$. For any $\varepsilon > 0$, there exists $K(G,\varepsilon)$ such that

$$\mu_G(x) \,\geq\, \frac{\mathbb{E}\left[\widetilde{T}^{\mathcal{C}_{H^+,M}}_{G}(0,Kx)\right]}{K} - \varepsilon\,, \tag{5.3}$$

and using (5.2) we know that there exists $n_0(\varepsilon, K)$ such that for all $n \geq n_0$ we have

$$\frac{\mathbb{E}\left[\widetilde{T}_G^{\mathcal{C}_{H^+,M}}(0,Kx)\right]}{K} \geq \frac{\mathbb{E}\left[\widetilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0,Kx)\right]}{K} - \varepsilon\,. \tag{5.4}$$

Since $\mu_{G_n}(x) = \inf_{k\in\mathbb{N}^*} \mathbb{E}\left[\widetilde{T}_{G_n}^{\mathcal{C}_{H^+,M}}(0,kx)\right]/k$, combining equations (5.3) and (5.4), we obtain that for any $\varepsilon > 0$, for all n large enough,

$$\mu_G(x) \geq \mu_{G_n}(x) - 2\varepsilon\,.$$

This concludes the proof of Lemma 5.1.

Remark 5.4. The domination we use to prove (5.2) is free, since whatever the probability measure H^+ on $[0,+\infty]$ satisfying $H^+([0,+\infty)) > p_c(d)$ we consider, the regularized times $\widetilde{T}_{H^+}^{\mathcal{C}_{H^+,M}}(0,x)$ are always integrable. In [8], Cox considered the (non regularized) times $T_{G_n}(0,x)$ for probability measures G_n on $[0,+\infty)$. By Lemma 5.3 it is easy to obtain $T_{G_n}(0,x) \leq T_H(0,x)$ for some probability measure H on $[0,+\infty)$. However, without further assumption, $T_H(0,x)$ may not be integrable. This is the reason why Cox supposed that the family $(G_n, n\in\mathbb{N})$ was uniformly integrable. In [9], Cox and Kesten circumvent this problem by considering some regularized passage times that are always integrable. There is no straigthtforward generalization of their regularized passage times to the case of possibly infinite passage times, but the $\widetilde{T}$ introduced in [6] plays the same role.

5.4 Proof of Lemma 5.2

Of course, Lemma 5.2 can be seen as a particular case of the continuity result by Cox and Kesten. But, as noted by Kesten in his Saint-Flour course [20], the Cox–Kesten way makes use of former results by Cox in [9] and is not the shortest path to a proof in the compact case. In [20] Kesten also gave a sketch of a shorter proof in the compact case. We thought the reader would be pleased to have a self-contained proof, so we present a short but full proof of Lemma 5.2, quite inspired by Kesten [20].

Let $G, (G_n)_{n\in\mathbb{N}}$ be probability measures on $[0,R]$, and consider $x\in\mathbb{Z}^d$. As in the proof of Theorem 1.2, we have $G_n \preceq \overline{G}_n$, where $\overline{\mathfrak{G}}_n = \max(\mathfrak{G}, \mathfrak{G}_n)$, thus $\mu_{G_n} \leq \mu_{\overline{G}_n}$. Applying Lemma 5.1, we know that

$$\overline{\lim_{n\to+\infty}}\ \mu_{\overline{G}_n}(x) \leq \mu_G(x)\,.$$

If $\mu_G(x) = 0$, then $\lim_{n\to\infty}\mu_{G_n}(x) = \lim_{n\to\infty}\mu_{\overline{G}_n}(x) = \mu_G(x) = 0$ and the proof is complete. We suppose from now on that $\mu_G(x) > 0$, thus $x \neq 0$. Since the passage times $t_G(e)$ are finite, it is well known that $\mu_G(x) > 0$ for $x\neq 0$ if and only if $G(\{0\}) < p_c(d)$ (see Theorem 6.1 in [20], or Proposition 2.7 in a more general setting). We want to prove that $\underline{\lim}_{n\to\infty}\mu_{\underline{G}_n}(x) \geq \mu_G(x)$, where $\underline{\mathfrak{G}}_n = \min(\mathfrak{G},\mathfrak{G}_n)$. Notice that $\widetilde{x}^{\mathcal{C}_{\underline{G}_n,M}} = \widetilde{x}^{\mathcal{C}_{G,M}} = x$ for any $M \geq R$, thus we do not need to introduce regularized times $\widetilde{T}$. In what follows we note *s.a.* for *self avoiding*. Since $\underline{G}_n \xrightarrow{d} G$, we have $\lim_{n\to\infty}\underline{G}_n(\{0\}) \leq G(\{0\}) < p_c(d)$, thus we consider only n large enough so that $\underline{G}_n(\{0\}) < p_c(d)$. Applying Lemma 5.3 (ii) to the sequence of functions $\underline{G}_n$, we obtain the existence of a probability measure H^- on $[0,+\infty]$ (in fact on $[0,R]$) such that $H^- \preceq \underline{G}_n$ for all n and $H^-(\{0\}) < p_c(d)$. Thanks to the coupling, we know that $t_{H^-}(e) \leq t_{\underline{G}_n}(e) \leq t_G(e)$ for every edge e, thus we obtain that for all $A \in \mathbb{N}^*$, for all $C \in [0, +\infty)$,

$$\begin{aligned}
&\mathbb{P}[T_{\underline{G}_n}(0,kx) \leq T_G(0,kx) - \varepsilon k] \\
&\leq \mathbb{P}\left[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq Ak \text{ and } T_{\underline{G}_n}(r) \leq ACk\right] \\
&\quad + \mathbb{P}[T_{\underline{G}_n}(0,kx) > ACk] + \sum_{\substack{r \text{ s.a. path from } 0 \\ \text{s.t. } |r| \leq Ak}} \mathbb{P}\left[\sum_{e \in r} t_G(e) - t_{\underline{G}_n}(e) \geq \varepsilon k\right] \\
&\leq \mathbb{P}\left[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq Ak \text{ and } T_{H^-}(r) \leq ACk\right] \\
&\quad + \mathbb{P}[T_G(0,kx) > ACk] + (2d)^{Ak}\, \mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right],
\end{aligned}$$

where $(e_i, i = 1, \dots, Ak)$ is a collection of distinct edges. Since $H^-(\{0\}) < p_c(d)$, we know that we can choose $C \in (0, +\infty)$ (depending on d and H) such that there exist finite and positive constants D, E (depending also on d and H) satisfying, for all $k \in \mathbb{N}^*$,

$$\mathbb{P}\left[\exists r \text{ s.a. path starting at } 0 \text{ s.t. } |r| \geq k \text{ and } T_{H^-}(r) \leq Ck\right] \leq De^{-Ek}$$

(see Proposition 5.8 in [20]). Since the support of G is included in $[0, R]$ for some finite R, we know that $T_G(0,kx) \leq Rk\|x\|_1$, thus we choose A large enough (depending on F, d and C) so that

$$\mathbb{P}[T_G(0,kx) > ACk] = 0 .$$

If we prove that there exists $n_0(G, (\underline{G}_n), \varepsilon)$ such that for all $n \geq n_0$,

$$\sum_{k>0} (2d)^{Ak}\, \mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right] < +\infty , \tag{5.5}$$

then for all $n \geq n_0$ we have $\sum_k \mathbb{P}[T_{\underline{G}_n}(0,kx) \leq T_G(0,kx) - \varepsilon k] < +\infty$. By Borel-Cantelli's lemma we obtain that for all $n \geq n_0$, a.s., for all $k \geq k_0(n)$ large enough,

$$T_{\underline{G}_n}(0,kx) > T_G(0,kx) - \varepsilon k ,$$

thus for all $n \geq n_0$ we get

$$\mu_{\underline{G}_n}(x) \geq \mu_G(x) - \varepsilon .$$

We conclude that $\underline{\lim}_{n\to\infty} \mu_{\underline{G}_n}(x) \geq \mu_G(x)$. It remains to prove (5.5). For any $\alpha > 0$, by Markov's inequality we have

$$\begin{aligned}
&(2d)^{Ak}\, \mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right] \\
&\qquad\qquad \leq \left(2d \exp\left(\frac{-\alpha\varepsilon}{A}\right) \mathbb{E}\left[\exp\left(\alpha(t_G(e) - t_{\underline{G}_n}(e))\right)\right]\right)^{Ak} .
\end{aligned}$$

By Lemma 2.10 we have $\lim_{n\to\infty} t_{\underline{G}_n}(e) = t_G(e)$ a.s. Since $t_{\underline{G}_n}(e), t_G(e) \leq R$ we obtain by a dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(\alpha(t_G(e) - t_{\underline{G}_n}(e))\right)\right] = 1 .$$

We choose $\alpha(\varepsilon)$ large enough so that

$$2d \leq \exp\left(\frac{\alpha\varepsilon}{4A}\right) ,$$

and then $n_0(G, (\underline{G}_n), \varepsilon)$ large enough so that for all $n \geq n_0$, we have

$$\mathbb{E}\left[\exp\left(\alpha(t_G(e) - t_{\underline{G}_n}(e))\right)\right] \leq \exp\left(\frac{\alpha\varepsilon}{4A}\right).$$

Thus for all $n \geq n_0$, we have

$$(2d)^{Ak}\mathbb{P}\left[\sum_{i=1}^{Ak} t_G(e_i) - t_{\underline{G}_n}(e_i) \geq \varepsilon k\right] \leq \exp\left(-\frac{\alpha\varepsilon}{2A}\right),$$

so (5.5) is proved.

6 Continuity of the Cheeger constant, proof of Theorem 1.1

The definition of the objects used in this section are given in section 2.2. The main step in the proof of Theorem 1.1 is the following lemma:

Lemma 6.1. For every $p > p_c(2)$, $\lim_{p' \to p} \sup_{x \in \mathbb{S}^1} |\beta_{p'}(x) - \beta_p(x)| = 0$.

Proof. Let $x \in \mathbb{S}^1$. Let $p_c(2) < p_0 \leq p \leq q$, and define $\delta = q - p$. We couple the percolations with different parameters in the usual way using uniform variables. We extend the definition of $\tilde{y}^{\mathcal{C}}$ to any $y \in \mathbb{R}^d$. For a path $r \in \mathcal{R}(x, y)$, let us define $\mathbf{b}_p(r) = |\{e \in \partial^+ r : e \text{ is } p-open\}|$. For $x, y \in \mathcal{C}_p$, we define $b_p(x, y) = \inf\{\mathbf{b}_p(r) : r \in \mathcal{R}(x, y), r \text{ is } p-\text{open}\}$.

Step (i). By Lemma 2.2 there exist $C, C', \alpha > 0$ (depending on p_0) such that $\forall p \geq p_0$, $\forall n$,

$$\mathbb{P}\left[\exists \gamma \in \bigcup_{x \in \mathbb{Z}^2} \mathcal{R}(0, x) : |\gamma| > n, \ \mathbf{b}_p(\gamma) \leq \alpha n\right] \leq Ce^{-C'n}. \tag{6.1}$$

Let F_{p_0} be the event $\{0 \in \mathcal{C}_{p_0}\} \cap \{nx \in \mathcal{C}_{p_0}\}$. On the event F_{p_0}, by [4, Lemma 2.5], we have $b_p(0, nx) \leq 3D^{\mathcal{C}_p}(0, nx) \leq 3D^{\mathcal{C}_{p_0}}(0, nx)$, thus using (3.3) we know that there exist positive constants $\hat{A}, \hat{B}, \beta$ (depending only on p_0) such that for all $p > p_0$, for all $x \in \mathbb{S}^1$,

$$\mathbb{P}\left[F_{p_0} \cap \{b_p(0, nx) \geq 3\beta n\}\right] \leq \hat{A}\exp(-\hat{B}n). \tag{6.2}$$

For any p-open path γ, γ is also q-open. However some additional right-boundary edges may be open. To bound the difference between $\mathbf{b}_q(\gamma)$ and $\mathbf{b}_p(\gamma)$, note that if $|\gamma| < \alpha' n$ by [4, Lemma 2.5] $|\partial^+\gamma| < 3\alpha' n$. We can bound $\mathbf{b}_q(\gamma) - \mathbf{b}_p(\gamma)$ by Cramér's theorem. For every fixed path γ such that $|\gamma| < \alpha' n$, for every $\varepsilon > 0$ and $\delta < \varepsilon$,

$$\mathbb{P}\left[\mathbf{b}_q(\gamma) - \mathbf{b}_p(\gamma) > 3\varepsilon\alpha' n\right] \leq e^{-3\alpha' n\left(\varepsilon \log\frac{\varepsilon}{\delta} + (1-\varepsilon)\log\frac{1-\varepsilon}{1-\delta}\right)}. \tag{6.3}$$

Fix $\alpha' = 3\beta/\alpha$. Since there are at most $4^{\alpha' n}$ paths of length smaller than $\alpha' n$ containing 0, we obtain that for all $p_0 \leq p < q$, for all $x \in \mathbb{S}^1$,

$$\begin{aligned}
&\mathbb{P}\left[b_q(\tilde{0}^{\mathcal{C}_q}, \widetilde{nx}^{\mathcal{C}_q}) > b_p(\tilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p}) + 3\varepsilon\alpha' n\right] \\
&\quad \leq \mathbb{P}[F_{p_0}^c] + Q_p[F_{p_0} \cap \{b_p(0, nx) > 3\beta n\}] \\
&\qquad + \mathbb{P}\left[F_{p_0} \cap \{\exists \gamma \in \mathcal{R}(0, nx) : |\gamma| > \alpha' n, \ \mathbf{b}_p(\gamma) \leq 3\beta n\}\right] \\
&\qquad + \mathbb{P}\left[F_{p_0} \cap \{\exists \gamma \in \mathcal{R}(0, nx) : |\gamma| \leq \alpha' n, \ \mathbf{b}_q(\gamma) > \mathbf{b}_p(\gamma) + 3\varepsilon\alpha' n\}\right] \\
&\quad \leq (1 - \theta_{p_0}^2) + \hat{A}e^{-\hat{B}n} + Ce^{-C'\alpha' n} \\
&\qquad + 4^{\alpha' n} e^{-3\alpha' n\left(\varepsilon \log\frac{\varepsilon}{\delta} + (1-\varepsilon)\log\frac{1-\varepsilon}{1-\delta}\right)}.
\end{aligned}$$

For every $p_0 > p_c(d)$, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ and $p_2(\varepsilon) > 0$ such that for every $x \in \mathbb{S}^1$, for every $p_0 \leq p < q$ satisfying $q - p < \delta$, we have

$$\mathbb{P}\left[b_q(\tilde{0}^{\mathcal{C}_q}, \widetilde{nx}^{\mathcal{C}_q}) > b_p(\tilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p}) + 3\varepsilon\alpha' n\right] \leq 1 - p_2(\varepsilon),$$

thus for every $p_0 > p_c(d)$, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $x \in \mathbb{S}^1$, for every $p_0 \leq p < q$ satisfying $q - p < \delta$, we have

$$\beta_q(x) < \beta_p(x) + 3\alpha'\varepsilon.$$

Step (ii). Given a q-open path γ, γ may not be p-open. Thus we use the results of Section 3 to modify the path to a p-open path which does not gain too many extra right-boundary edges. We mimic the proof of Lemma 4.3. Fix $\varepsilon > 0$. Choose $\alpha(\varepsilon)$ and $\mathfrak{p}(\varepsilon)$ as in (3.8) and (3.9).

- If p is fixed and we let q goes to p, we choose $p_0 = p$, $q_1 \in (p_0, p_0 + \delta_1(p))$ as defined in Lemma 3.5, and we consider only values of q such that $q \in [p_0, q_1]$. Then we choose $N = N'(p_0, q_1, \mathfrak{p}(\varepsilon))$ as given in Lemma 3.5.

- If q is fixed and we let p goes to q, we choose $p_0 \in (p_c(d), q]$ such that $q - p_0 \leq \delta_0(q)$ as defined in Lemma 3.5, and we consider only values of p in the interval $[p_0, q]$. Then we choose $N = N(p_0, q, \mathfrak{p}(\varepsilon))$ as given in Lemma 3.5.

Let $x \in \mathbb{S}^1$, we denote by $\lfloor nx \rfloor$ the point y of $\mathbb{Z}^d$ which minimizes $\|nx - y\|_1$ (with a deterministic rule to break ties). Let F' be the following good event: the N-boxes containing 0 and $\lfloor nx \rfloor$ and all the adjacent boxes are good and belong to an infinite cluster of good boxes. By the FKG inequality and the stochastic comparison, we have

$$\mathbb{P}(F') \geq \theta^{18}_{\mathrm{site},\mathfrak{p}(\varepsilon)} . \tag{6.4}$$

Fix $\alpha'' = 6\beta/\alpha = 2\alpha'$ as defined in step (i). We have

$$\begin{aligned} &\mathbb{P}\left[b_p(\tilde{0}^{\mathcal{C}_{p_0}}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_{p_0}}) > b_q(\tilde{0}^{\mathcal{C}_{p_0}}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_{p_0}}) + 12\alpha'' \tilde{C}_d \rho_d \varepsilon n \right] \\ &\quad \leq \mathbb{P}[F^c] + \sum_{y,z} \mathbb{P}\left(E_{y,z} \cap \{ b_p(y,z) > b_q(y,z) + 12\alpha'' \tilde{C}_d \rho_d \varepsilon n \} \right) , \end{aligned} \tag{6.5}$$

where the sum is over every y in the same $(3N)$-box as 0 and z in the same $(3N)$-box as $\lfloor nx \rfloor$, and $E_{y,z}$ is the event that $y \in \mathcal{C}_{p_0}, z \in \mathcal{C}_{p_0}$ and the N-boxes containing y and z are good and belong to an infinite cluster of good boxes. For any such (y,z), on $E_{y,z}$, let $\gamma_{y,z} \in \mathcal{R}(y,z)$ be a q-open right-most path from y to z such that $b_q(y,z) = \mathbf{b}_q(\gamma_{y,z})$, and let $\Gamma_{y,z}$ be the lattice animal of N-boxes it visits. For short, we write γ for $\gamma_{y,z}$. As previously we define

$$\begin{aligned} \gamma_a &= \{ e \in \gamma : e \text{ is } p\text{-open} \} \\ \gamma_b &= \{ e \in \gamma : e \text{ is } p\text{-closed} \} . \end{aligned}$$

By Lemma 3.2, on the event $E_{y,z}$, there exists a path γ' with the following properties:

1. γ' is a path from y to z which is p-open;
2. $\gamma' \setminus \gamma$ is a collection of simple paths (and also right-most) that intersect $\gamma' \cap \gamma$ only at their endpoints thus γ' is a right-most path (see [4, Lemma 2.6]);
3. $|\gamma' \setminus \gamma| \leq \rho_d \left(N^d |\gamma_b| + N \sum_{C \in \mathrm{Bad}:\, C \cap \Gamma \neq \varnothing} |C| \right)$.

Note that

$$\mathbf{b}_p(\gamma') \leq \mathbf{b}_q(\gamma') \leq \mathbf{b}_q(\gamma) + 3|\gamma' \setminus \gamma|.$$

Moreover, since a simple path is also a right-most path, we have for all $y, z \in \mathcal{C}_{p_0}$,

$$b_q(y,z) \leq 3D^{\mathcal{C}_{p_0}}(y,z) . \tag{6.6}$$

Using Equation (6.6), Proposition 2.3, Antal and Pizstora's estimate (3.1), Cramér's theorem again and Lemma 3.6, for all $x \in \mathbb{S}^1$ and for all n large enough (in particular such that $\|y-z\|_1 \le \|\lfloor nx \rfloor\| + 12N \le 2n$ and $\|y-z\|_1 \ge \|\lfloor nx \rfloor\| - 12N \ge n/2$), we have

$$
\begin{aligned}
&\mathbb{P}\left(E_{y,z} \cap \{b_p(y,z) > b_q(y,z) + 12\alpha''\tilde{C}_d\rho_d\varepsilon n\}\right) \\
&\quad \le \mathbb{P}[E_{y,z} \cap \{b_q(y,z) > 6\beta n\}] \\
&\qquad + \mathbb{P}\left[E_{y,z} \cap \{\exists \gamma \in \mathcal{R}(y,z) \,:\, |\gamma| > \alpha'' n\,,\ \mathbf{b}_{p_0}(\gamma) \le \mathbf{b}_q(\gamma) \le 6\beta n\}\right] \\
&\qquad + \mathbb{P}\left[E_{y,z} \cap \{\exists \gamma \in \mathcal{R}(y,z) \,:\, \gamma \text{ is } q\text{-open}, |\gamma| \le \alpha'' n\,,\ |\gamma' \setminus \gamma| > 4\alpha''\tilde{C}_d\rho_d\varepsilon n\}\right] \\
&\quad \le \mathbb{P}\left(\beta\|y-z\|_1 \le 2\beta n \le D^{\mathcal{C}_{p_0}}(y,z) < \infty\right) + Ce^{-C'\alpha'' n} \\
&\qquad + \mathbb{P}\left[\exists \gamma \,:\, \gamma \text{ starts at } y, \gamma \text{ is } q\text{-open}, |\gamma| \le \alpha'' n\,,\ |\gamma_b| > \frac{2\alpha''\tilde{C}_d\varepsilon n}{N^d}\right] \\
&\qquad + \mathbb{P}\left[\exists \Gamma \in \mathcal{A}\text{nimals},\ |\Gamma| \ge \frac{n}{N},\ \sum_{C \in \text{Bad}:\, C \cap \Gamma \ne \varnothing} |C| \ge \varepsilon |\Gamma|\right] \\
&\quad \le \hat{A}e^{-\hat{B}\|y-z\|_1} + Ce^{-C'\alpha'' n} \\
&\qquad + 4^{\alpha n} e^{-3\alpha'' n\left(\frac{2\tilde{C}_d\varepsilon}{N^d}\log\frac{2\tilde{C}_d\varepsilon}{\delta N^d} + (1-\frac{2\tilde{C}_d\varepsilon}{N^d})\log\frac{1-2\tilde{C}_d\varepsilon/N^d}{1-\delta}\right)} + \mathcal{P}(n) \\
&\quad \le \hat{A}e^{-\hat{B}n/2} + Ce^{-C'\alpha'' n} \\
&\qquad + 4^{\alpha n} e^{-3\alpha'' n\left(\frac{2\tilde{C}_d\varepsilon}{N^d}\log\frac{2\tilde{C}_d\varepsilon}{\delta N^d} + (1-\frac{2\tilde{C}_d\varepsilon}{N^d})\log\frac{1-2\tilde{C}_d\varepsilon/N^d}{1-\delta}\right)} + 2^{-\frac{n}{N}+1}\,.
\end{aligned}
\tag{6.7}
$$

Combining Equations (6.4), (6.5) and (6.7), we conclude that for every fixed $\varepsilon > 0$ and every fixed $p > p_c(d)$ (thus p_0, q_1 and N are fixed), there exists $\delta(\varepsilon, p) \in (0, q_1 - p]$ and $p_3(\varepsilon, p) > 0$ such that for every $q > p$ satisfying $q - p < \delta$, for every $x \in \mathbb{S}^1$, for every n large enough, we have

$$\mathbb{P}\left[b_p(\tilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p}) > b_q(\tilde{0}^{\mathcal{C}_q}, \widetilde{nx}^{\mathcal{C}_q}) + 12\alpha''\tilde{C}_d\rho_d\varepsilon n\right] \le 1 - p_3\,,$$

thus for every $\varepsilon > 0$ and for every $p > p_c(d)$, there exists $\delta(\varepsilon, p) > 0$ such that for every $q > p$ satisfying $q - p < \delta$, for every $x \in \mathbb{S}^1$,

$$\beta_p(x) < \beta_q(x) + 12\alpha''\tilde{C}_d\rho_d\varepsilon.$$

Similarly, for every fixed $\varepsilon > 0$ and every fixed $q > p_c(d)$ (thus p_0 and N are fixed), there exists $\delta'(\varepsilon, q) \in (0, q - p_0]$ and $p_4(\varepsilon, q) > 0$ such that for every $p < q$ satisfying $q - p < \delta'$, for every $x \in \mathbb{S}^1$, for every n large enough, we have

$$\mathbb{P}\left[b_p(\tilde{0}^{\mathcal{C}_p}, \widetilde{nx}^{\mathcal{C}_p}) > b_q(\tilde{0}^{\mathcal{C}_q}, \widetilde{nx}^{\mathcal{C}_q}) + 12\alpha''\tilde{C}_d\rho_d\varepsilon n\right] \le 1 - p_4\,,$$

thus for every $\varepsilon > 0$ and for every $q > p_c(d)$, there exists $\delta'(\varepsilon, q) > 0$ such that for every $p < q$ satisfying $q - p < \delta$, for every $x \in \mathbb{S}^1$,

$$\beta_p(x) < \beta_q(x) + 12\alpha''\tilde{C}_d\rho_d\varepsilon.$$

This ends the proof of Lemma 6.1. □

Proof of Theorem 1.1.
(i) Proof of the continuity of the Cheeger constant $\lim_{n\to\infty} n\varphi_n(p)$. Let $p > p_c(2)$. For any rectifiable Jordan curve λ, with $\text{Leb}(int(\lambda)) = 1$,

$$\text{len}_{\beta_p}(\lambda) = \sup_{N \ge 1} \sup_{0 \le t_0 < \ldots < t_N \le 1} \sum_{i=1}^{N} \beta_p\left(\frac{\lambda(t_i) - \lambda(t_{i-1})}{\|\lambda(t_i) - \lambda(t_{i-1})\|_2}\right) \|\lambda(t_i) - \lambda(t_{i-1})\|_2.$$

By Lemma 6.1 for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p-q| < \delta$ we have $\sup_{x\in\mathbb{S}^1} |\beta_q(x) - \beta_p(x)| < \varepsilon$, thus

$$|\text{len}_{\beta_p}(\lambda) - \text{len}_{\beta_q}(\lambda)| \leq \varepsilon \text{len}_{\|\cdot\|_2}(\lambda). \tag{6.8}$$

The infimum in Theorem 2.3 is achieved (by compactess of the set of Lipschitz curves), so let us denote by λ_p (resp. λ_q) a Jordan curve such that $\text{Leb}(int(\lambda_p)) = 1$ and $\text{len}_{\beta_p}(\lambda_p) = \sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p)$ (resp. $\text{Leb}(int(\lambda_q)) = 1$ and $\text{len}_{\beta_q}(\lambda_q) = \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q)$). All norms in $\mathbb{R}^2$ are equivalent thus we know that $\text{len}_{\|\cdot\|_2}(\lambda_p) < \infty$ and $\text{len}_{\|\cdot\|_2}(\lambda_q) < \infty$. From (6.8) we deduce that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|p-q| < \delta$ then

$$\begin{aligned} \sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p) &= \text{len}_{\beta_p}(\lambda_p) \geq \text{len}_{\beta_q}(\lambda_p) - \varepsilon\text{len}_{\|\cdot\|_2}(\lambda_p) \\ &\geq \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q) - \varepsilon\text{len}_{\|\cdot\|_2}(\lambda_p) \end{aligned} \tag{6.9}$$

$$\text{and} \qquad \begin{aligned} \sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p) &\leq \text{len}_{\beta_p}(\lambda_q) \leq \text{len}_{\beta_q}(\lambda_q) + \varepsilon\text{len}_{\|\cdot\|_2}(\lambda_q) \\ &\leq \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q) + \varepsilon\text{len}_{\|\cdot\|_2}(\lambda_q)\,. \end{aligned} \tag{6.10}$$

Let $\beta_q^{\min} = \inf_{x\in\mathbb{S}^1} \beta_q(x)$, for all q. By Lemma 6.1 again we know that for every q satisfying $|p-q| < \delta$ we have $\beta_q^{\min} \geq \beta_p^{\min} - \varepsilon$, which is positive for ε small enough ($\beta_p^{\min}$ is not zero since β_p is a norm), thus

$$\text{len}_{\|\cdot\|_2}(\lambda_q) \leq \frac{\text{len}_{\beta_q}(\lambda_q)}{\beta_q^{\min}} \leq \frac{\text{len}_{\beta_q}(\lambda_q)}{\beta_p^{\min} - \varepsilon}\,.$$

Thanks to Equation (6.10) we obtain

$$\sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p) \leq \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q) \left(1 + \frac{\varepsilon}{\beta_p^{\min} - \varepsilon}\right)\,. \tag{6.11}$$

Combining (6.9) and (6.11) we obtain that

$$\lim_{q\to p} \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q) = \sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p)\,.$$

Since $p \mapsto \theta_p$ is continuous on $(p_c(2), 1]$, this conludes the first part of the proof.

(ii) Proof of the continuity of the Wulff shape. Next we prove that $p \mapsto \widehat{W}_p$ is continuous for the Hausdorff distance. Fix $\eta > 0$ and $p > p_c(2)$ and let $\varepsilon = \varepsilon(\eta, p) > 0$ be small enough such that

$$\varepsilon \leq \frac{\beta_p^{\min}}{2} \min(\eta, 1)\,. \tag{6.12}$$

As previously let $\delta > 0$ satisfy $\sup_{x\in\mathbb{S}^1} |\beta_q(x) - \beta_p(x)| < \varepsilon$ for all $q > p_c(2)$ such that $|p-q| < \delta$. For every $x \in W_q$ we have by definition of W_q that for every $\hat{n} \in \mathbb{S}^1$, $\hat{n} \cdot x \leq \beta_q(\hat{n})$. Thus for all $q > p_c(2)$ such that $|p-q| < \delta$,

$$\hat{n} \cdot x \leq \beta_q(\hat{n}) \leq \beta_p(\hat{n}) + \varepsilon \leq (1+\eta)\beta_p(\hat{n}),$$

where the last inequality comes from (6.12), thus $x \in (1+\eta)W_p$. We obtain that for all $p > p_c(2)$, for all $\eta > 0$, there exists $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p-q| < \delta$,

$$W_q \subset (1+\eta)W_p. \tag{6.13}$$

For every $q > p_c(2)$ satisfying $|p-q| < \delta$, we also have $\beta_q^{\min} \geq \beta_p^{\min} - \varepsilon \geq \beta_p^{\min}/2 \geq \varepsilon/\eta$ by (6.12), thus by the same method we obtain that for every $x \in W_p$, for every $\hat{n} \in \mathbb{S}^1$,

$$\hat{n} \cdot x \leq \beta_p(\hat{n}) \leq \beta_q(\hat{n}) + \varepsilon \leq (1+\eta)\beta_q(\hat{n}),$$

thus

$$W_p \subset (1+\eta) W_q . \tag{6.14}$$

For every $x \in W_p$, $\|x\|_2 = x \cdot x / \|x\|_2 \leq \beta_p(x) \leq \beta_p^{\max}$, where $\beta_p^{\max} = \sup_{x\in\mathbb{S}^1} \beta_p(x) < \infty$, thus $\|(1+\eta)x - x\|_2 \leq \eta \beta_p^{\max}$. Similarly, for all $q > p_c(2)$ satisfying $|p-q| < \delta$, $\|x\|_2 \leq \beta_q^{\max} \leq 2\beta_p^{\max}$ and $\|(1+\eta)x - x\|_2 \leq 2\eta\beta_p^{\max}$. With (6.13) and (6.14), we conclude that for every $p > p_c(2)$, for every $\eta > 0$, there exists $\delta > 0$ such that for every $q > p_c(2)$ satisfying $|p-q| < \delta$,

$$d_H(W_p, W_q) \,\leq 2\eta \beta_p^{\max} \,,$$

thus $\lim_{q\to p} d_H(W_p, W_q) = 0$. This implies that $\lim_{q\to p} \mathrm{Leb}(W_q) = \mathrm{Leb}(W_p)$, and since $\widehat{W}_p = \frac{W_p}{\sqrt{\mathrm{Leb}(W_p)}}$ we deduce from (6.13) and (6.14) that $\lim_{q\to p} d_H(\widehat{W}_p, \widehat{W}_q) = 0$ by a similar argument. This concludes the proof of Theorem 1.1. □

Remark 6.2. To deduce the continuity of the Wulff crystal from Lemma 6.1, we can also consider a more general setting. Consider β_p^* the dual norm of β_p, defined by

$$\forall x \in \mathbb{R}^d \,, \quad \beta_p^*(x) \,=\, \sup\{x \cdot y \,:\, \beta_p(y) \leq 1\} \,.$$

Then β_p^* is a norm, and what we did is equivalent to deduce from Lemma 6.1 the same result concerning β_p^* :

$$\lim_{q\to p} \sup_{x\in\mathbb{S}^1} |\beta_q^*(x) - \beta_p^*(x)| \,=\, 0 \,. \tag{6.15}$$

Notice that W_p, the Wulff crystal associated to β_b, is in fact the unit ball associated to β_b^*, then (6.15) implies the continuity of $p \mapsto W_p$ according to the Hausdorff distance.

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