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On the chemical distance in critical percolation

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Abstract

We consider two-dimensional critical bond percolation. Conditioned on the existence of an open circuit in an annulus, we show that the ratio of the expected size of the shortest open circuit to the expected size of the innermost circuit tends to zero as the side length of the annulus tends to infinity, the aspect ratio remaining fixed. The same proof yields a similar result for the lowest open crossing of a rectangle. In this last case, we answer a question of Kesten and Zhang by showing in addition that the ratio of the length of the shortest crossing to the length of the lowest tends to zero in probability. This suggests that the chemical distance in critical percolation is given by an exponent strictly smaller than that of the lowest path.

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1 Introduction

The object of this paper is to prove a result concerning the chemical distance inside large open clusters in critical independent bond percolation on $\mathbb{Z}^2$. The chemical distance between two sets A and B is the minimum number of edges in any lattice path of open edges joining A to B.

Distances inside the infinite cluster in supercritical percolation are known to be comparable to the Euclidean distance on $\mathbb{Z}^d$, through the work of G. Grimmett and J. Marstrand [9, Section 5 (g)]. P. Antal and A. Pisztora [2] give exponential bounds for the probability of deviation from this linear behavior.

By contrast, little is known in the critical case. The most complete results are available in high dimensions ($d \geq 19$). Using techniques of G. Kozma and A. Nachmias [19, 20], R. van der Hofstad and A. Sapozhnikov [12, Theorem 1.5] have shown that, conditioned on the existence of an open path to Euclidean distance n, the chemical distance from the origin to the boundary of a Euclidean box of side length n is at least of order ϵn^2 with probability at least $1 - C\sqrt{\epsilon}$. The matching upper bound follows directly

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from the work of Kozma and Nachmias (see also [11, Theorem 2.8] for a more general result, which applies also to long-range percolation). These estimates presumably hold for any dimension above the critical dimension $d = 6$, but the current proofs rely on results derived from the lace expansion. To the best of our knowledge, there is currently no rigorous work addressing the chemical distance in percolation for $2 < d < 19$.

Unlike in the high dimensional case, there has been very little work on the chemical distance in two-dimensional percolation. This is mainly because it is not clear how to analyze it using the usual tools: gluing constructions and arm exponents. Two notable exceptions include the work of J. van den Berg and A. Jarai [31], and G.J. Morrow and Y. Zhang [21], both of which study asymptotics for the lowest crossing. The techniques in our paper are most similar to the ones in these papers, and also particularly the recent work involving arm events in self-destructive percolation by D. Kiss, I. Manolescu and V. Sidoravicius [18]. For extensive background on relevant methods from critical and near-critical percolation, the reader should also see Nolin's survey [23].

As observed by Pisztora [24], the work of M. Aizenman and A. Burchard [1] implies that distances in planar critical percolation are bounded below by a power greater than one of the Euclidean distance, with high probability. Letting $B(n) = [-n, n]^2$, there is an $\epsilon > 0$ such that, for any $\kappa > 0$,

$$\mathbf{P}(\exists \text{ an open crossing of } B(n) \text{ with cardinality } \leq n^{1+\epsilon} \mid \exists \text{ an open crossing}) \leq C_\kappa n^{-\kappa}. \tag{1.1}$$

For definiteness, we consider horizontal crossings of $B(n)$. Pisztora treats the "near-critical" case, when the percolation parameter p is sufficiently close to $p_c = \frac{1}{2}$ and obtains essentially the same result as long as n is below the correlation length for p. H. Kesten and Y. Zhang [16] had previously outlined a proof of an estimate analogous to (1.1) for some fixed κ, for the size of the *lowest* open crossing in $B(n)$.

We know of no explicit estimate for ϵ in (1.1). In principle, such an estimate could be obtained from careful examination of the proof in [1], but the resulting value would be exceedingly small, and it is not likely to correspond to the true typical length of crossings.

In this work we will be concerned with upper, rather than lower bounds for the chemical distance. Conditioned on the existence of a crossing, the obvious approach is to identify a distinguished crossing of $B(n)$ whose size can be estimated. This provides an upper bound for the shortest crossing.

The lowest open crossing of $B(n)$ has a well-known characterization: an edge $e \in B(n)$ lies on the lowest open crossing if and only if it is connected to the left and right sides of $B(n)$ by disjoint open paths, and the dual edge e^* is connected to the bottom side of $B(n)^*$, the dual to $B(n)$. (For precise definitions, see Section 2.) G. J. Morrow and Zhang have used this fact to show that if $\tilde{L}_n$ is the size of the lowest open crossing of $B(n)$, then for each positive integer k,

$$C_{1,k} n^{2k} (\pi_3(n))^k \leq \mathbf{E}\tilde{L}_n^k \leq C_{2,k} n^{2k} (\pi_3(n))^k, \tag{1.2}$$

with $\pi_3(n)$ denoting the "three-arm" probability (see (2.1)). On the triangular lattice, the existence and asymptotic value of the three-arm exponent are known [29], and (1.2) becomes

$$\mathbf{E}\tilde{L}_n^k = n^{4k/3+o(1)}.$$

It is natural to ask whether this is also the correct order of magnitude for the shortest crossing of $B(n)$. This question was asked by Kesten and Zhang in [16]:

Question 1.1 (H. Kesten and Y. Zhang, 1992)**.** Let H_n be the event that there is an open horizontal crossing of $[-n, n]^2$. Let $\tilde{S}_n$ be the number of edges in the crossing of $[-n, n]^2$

of minimal length. Is it the case that

$$\tilde{S}_n/\tilde{L}_n \to 0, \tag{1.3}$$

in probability, conditionally on H_n? From [16, p. 603]: "It is not clear that $\tilde{S}_n/\tilde{L}_n \to 0$ in probability." In this paper, we give a positive answer to this question. (See Corollary 2.5.)

We present our result on the chemical distance in terms of circuits in annuli. The same proof, with minor modifications, applies to the case of horizontal crossings. Let $A(n) = B(3n) \setminus B(n)$. By Russo-Seymour-Welsh (RSW) [27, 30] estimates, the probability that there is an open circuit around $B(n)$ in $A(n)$ is bounded below by a positive number independent of n. Conditioned on the existence of such a circuit, one defines the innermost open circuit γ_n as the circuit with minimal interior surrounding $B(n)$ inside $A(n)$. As in the case of the lowest path, one can show that if L_n is the size of γ_n, then for some $C > 0$

$$(1/C)n^2\pi_3(n) \le \mathbf{E}L_n \le Cn^2\pi_3(n).$$

Let S_n be the number of edges on the shortest open circuit around $B(n)$ in $A(n)$ (defined to be zero when there is no circuit). Our main result, Theorem 2.2, is the following: as $n \to \infty$,

$$\frac{\mathbf{E}S_n}{n^2\pi_3(n)} \to 0. \tag{1.4}$$

This shows that in an averaged sense, S_n is much shorter than the typical size of L_n. The formulation (1.4) in terms of circuits in annuli serves as an illustration of the fractal nature of percolation clusters. If macroscopic open paths were smooth, in the sense that they had no small-scale features, one would not expect the shortest circuit to be much shorter than the innermost, since the latter encloses a smaller area.

Remark 1.2. Between the original submission and final acceptance, we obtained results stronger than that in (1.4). First, in the arXiv note [6], we refined the strategy introduced in this paper to give the bound $\frac{\mathbf{E}S_n}{n^2\pi_3(n)} \le \frac{1}{(\log n)^c}$ for any $c < 1/4$. Further improvements are in preparation.

1.1 Conjectures in the literature

Here we make a few brief remarks and give additional references to the literature on the subject of the chemical distance in critical percolation.

Physicists expect that there exists an exponent d_{min} such that

$$\tilde{S}_n \sim n^{d_{min}}, \tag{1.5}$$

where the precise meaning of the equivalence $\sim$ remains to be specified. O. Schramm included the determination of d_{min} in a list of open problems on conformally invariant scaling limits [28], noting that the question does not "seem accessible to SLE methods." Even the existence claim has so far not been substantiated.

Following Schramm and Kesten-Zhang, we have formulated the problem in terms of crossings of large boxes. More generally, d_{min} is predicted to govern the chemical distance between any two points inside the same critical percolation cluster in the sense that if $x, y \in \mathbb{Z}^2$ are connected by an open path and $\|x - y\|_1 = n$, then

$$\mathrm{dist}_{\text{chemical}}(x, y) \sim n^{d_{min}}. \tag{1.6}$$

It follows from the results of Aizenman and Burchard that if x and y are at Euclidean distance of order n, then with high probability, the chemical distance between x and y is greater than n^η for $\eta > 1$. One might expect, based on (1.2), that the average

point-to-point chemical distance can be bounded by $n^2\pi_3(n)$, but this bound does not follow directly from the method of Morrow and Zhang. Our main result and numerical simulations suggest that a sharp upper bound would involve a quantity smaller than $n^2\pi_3(n)$ by a power of n.

Simulations have yielded the approximation $d_{min} \approx 1.130\ldots$ [8, 10, 32]. In contrast to other critical exponents, there is no agreement on an exact value for d_{min}, and several proposed values seem inconsistent with each other, and with numerical results. See the introduction and bibliography in [25] for a more extensive review of these questions. In that article, the authors use the formula of V. Beffara [3] for the dimension of SLE curves

$$d_{\mathrm{SLE}(\kappa)} = \min\left(1 + \frac{\kappa}{8}, 2\right)$$

along with a conjectured value for d_{min} to compare, based on simulations, the behavior of SLE(κ) with the shortest path accross a domain.

1.2 Outline of the proof

The obvious strategy is to use the following well-known fact: given any circuit L in $A(n)$, the event that the innermost circuit γ_n in $A(n)$ (around 0) coincides with L depends only on the edges in $A(n)$ which also lie in the interior and on γ_n. One would then try to condition on the value of the innermost circuit and use RSW constructions along with independence of the region outside, to construct a large number of short paths off of L. Unfortunately, this strategy does not work. The innermost circuit with high probability has many very irregular sections, so that it is not possible to make useful constructions with probability bounds that are independent of L.

A main contribution of this paper is to replace this approach, which attempts to use independence of the region outside the innermost circuit, with one where we instead condition on the event that an edge far from the boundary belongs to the innermost circuit, and show that with high conditional probability, there is a short path around this edge. In other words, fixing any edge e which is far from the boundary, if $e \in \gamma_n$, we try to find in several concentric annuli around e a "detour": an open arc lying outside γ_n, but with its endpoints on γ_n. Given such an arc, we can form a new open circuit in $A(n)$ by replacing a portion of γ_n by the detouring open arc. Provided the resulting curve still surrounds $B(n)$, we obtain a candidate for a circuit which could be shorter than γ_n.

Given $\epsilon > 0$ fixed but small, the strategy is then to show that not only are there many detours around edges on γ_n, but they can be chosen so that their length is less than ϵ times the length of the portion of γ_n they detour. If most of γ_n can be covered by such short detours in this manner, we can build an open circuit σ_n consisting of sections of the innermost circuit, with many detours attached, and one might hope that $\#\sigma_n \leq (\epsilon + o(1))\#\gamma_n$ with high probability.

In trying to implement this strategy, we are faced with a number of problems:

1. We are forced to construct detours locally (that is, around edges e that are conditioned to be in γ_n), and not globally. Therefore, multiple detours around different edges might intersect. A systematic method is needed to keep track of how much of γ_n we have replaced by detours.
2. We lack prior knowledge about the size of open paths in the critical cluster. It is thus not obvious that one of the many detours around each edge will have length smaller than ϵ times that of the detoured path.
3. The orientation and rough geometry of γ_n could make it difficult to carry out the percolation estimates required to construct detours. Because we do not condition on the value L of γ_n at any point, the global structure of γ_n does not appear in

the detour construction. An essential point of our construction is that given the occurrence of our "detour" events, the local structure of the innermost circuit is controlled.

We address the first point by considering "shielded" detours: short detours which are also covered by a closed dual arc; see Definition 3.2. Two shielded detours are either equal or disjoint, and this allows us to estimate the total contribution of the detours to the circuit σ_n.

To address the second point, we must show that very short shielded detour paths exist with positive probability in every annulus. The only tool that we have to upper bound the length of paths is the result of Morrow and Zhang, which gives asymptotics for the length of the lowest crossing (innermost circuit). We use the fact that the fractal structure of this innermost circuit of an annulus implies that it can be made much smaller than its expected size, by forcing it to lie in a very thin region. This observation, applied to outermost partial circuits within shields, allows us to construct short detours as in Definition 3.2, by constraining them to be in thin annuli. As an illustration of this idea, we give the following proposition:

Proposition 1.3. *Let $\tilde{L}_n$ be the length of the lowest horizontal crossing in $[-n,n]^2$. For any $\epsilon > 0$, there is $C(\epsilon) > 0$ such that*

$$\mathbf{P}(\tilde{L}_n < \epsilon \mathbf{E}\tilde{L}_n \mid \text{there is an open crossing of } [-n,n]^2) \geq C(\epsilon), \tag{1.7}$$

for all n large enough.

Sketch of proof. We only provide an outline of the proof here. For a more detailed argument, see the proof of Lemma 5.8. The size of the lowest open crossing of $[-n,n] \times [-n,-(1-\alpha)n]$ is of order $\alpha n^2 \pi_3(\alpha n)$. Using quasimultiplicativity [23, Proposition 12.2], and the fact that the three arm-exponent is < 1, this is smaller than $\alpha^{1-\eta} n^2 \pi_3(n)$ for some $\eta < 1$. Choosing α small enough, the result follows. □

To show that most of γ_n can be covered by short detours, we show in Section 6 (see equation (6.1)), that the probability of e having no shielded detour around it is small, conditioned on e lying on γ_n:

$$\limsup_{n\to\infty} \mathbf{P}(\text{no detour around } e \mid e \in \gamma_n) = 0, \tag{1.8}$$

for edges e away from the boundary of $A(n)$ and $\epsilon > 0$ arbitrary.

We then estimate S_n by considering separately the contributions to $\#\sigma_n$ of the union Π of all the short detours, and the edges on $\gamma_n \setminus \hat{\Pi}$, where $\hat{\Pi}$ is the union of the "detoured" portions of the innermost circuit:

$$\begin{aligned}
\mathbf{E}S_n &\leq \mathbf{E}\#\Pi + \mathbf{E}\#(\gamma_n \setminus \hat{\Pi}) \\
&\leq \epsilon \mathbf{E}\#\gamma_n + \mathbf{E}\#\{e \in \gamma_n : \text{there is no detour around } e\} \\
&\leq \epsilon \mathbf{E}\#\gamma_n + \sum_e \mathbf{P}(\text{no detour around } e \mid e \in \gamma_n)\mathbf{P}(e \in \gamma_n).
\end{aligned}$$

Using (1.8), this gives

$$\mathbf{E}S_n \leq (\epsilon + o(1)) \cdot \mathbf{E}\#\gamma_n. \tag{1.9}$$

The proof of Corollary 2.3 (see Section 2.2 for the statement) concerning the expected size of the lowest crossing is identical to the argument for the innermost circuit. To obtain the statement of convergence in probability in Corollary 2.5 (also stated in Section 2.2) we need an additional argument. Essentially, it remains to prove that the lowest

crossing of $[-n,n]^2$ cannot be smaller than $o(1)\mathbf{E}\tilde{L}_n$ with positive probability. The basic idea for our proof comes from Kesten's lower bound for the number of pivotals in a box [14, (2.46)], but the requirement to find a large (of order $n^2\pi_3(n)$) points rather than one at each scale introduces substantial new technical difficulties. See Section 7.

For clarity, we have ignored the edges very close to the boundary in this rough sketch of our proof; for such edges, no estimate like (1.8) holds.

To obtain the estimate (1.8), we define a sequence $E_k(e)$, $k \geq 1$ of events which depend on edges inside concentric annuli around e, and whose occurrence implies the existence of a shielded detour (in the sense of Definition 3.2) if $e \in \gamma_n$. The definition and construction of $E_k(e)$ are given in Section 5, where it is also proved that

$$\mathbf{P}(E_k(e)) \geq c_1 \tag{1.10}$$

uniformly in $k \geq k_0$ for some $c_1 > 0$. A schematic representation of the event $E_k(e)$ appears in Figure 1; see also the accompanying description at the beginning of Section 5. We use closed dual circuits with defects to force the lowest crossing to traverse certain regions inside the annulus where $E_k(e)$ is defined, regardless of the "local orientation" of the innermost circuit outside. To connect the innermost circuit, the detour path and its shielding closed dual path, we use five-arm points (see Section 5.2), avoiding any conditioning on the realization of the lowest path.

To pass from (1.10) to (1.8), we show in Section 6.1 that

1. The estimate (1.10) remains true (with a different, but still n-independent constant) when we condition on e lying in the innermost circuit. See (5.24) in Section 5 and Proposition 6.1.
2. Although the $E_k(e)$'s are no longer independent under the conditional measure $\mathbf{P}(\cdot \mid e \in \gamma_n)$, the dependence is weak enough to obtain an estimate on the event that none of the E_k's occur; see Proposition 6.4. Here we use arm separation tools which appeared in [7] (which we state as Lemma 6.6).

2 Notation and results

On the square lattice $(\mathbb{Z}^2, \mathcal{E}^2)$, let $\mathbf{P}$ be the critical bond percolation measure

$$\prod_{e \in \mathcal{E}^2} \frac{1}{2}(\delta_0 + \delta_1)$$

on $\Omega = \{0,1\}^{\mathcal{E}^2}$.

A lattice path is a sequence $v_0, e_1, v_1, \ldots, v_{N-1}, e_N, v_N$ such that for all $k = 1, \ldots, N$, $\|v_{k-1} - v_k\|_1 = 1$ and $e_k = \{v_{k-1}, v_k\}$. A circuit is a path with $v_0 = v_N$. For such paths we denote $\#\gamma = N$, the number of edges in γ. If $V \subset \mathbb{Z}^2$ then we say that $\gamma \subset V$ if $v_k \in V$ for $k = 0, \ldots, N$.

A path γ is said to be (vertex) self-avoiding if $v_i = v_j$ implies $i = j$ and a circuit is (vertex) self-avoiding if $v_i = v_j$ implies $i = j$ whenever $0 \notin \{i, j\}$. Given $\omega \in \Omega$, we say that γ is open in ω if $\omega(e_k) = 1$ for $k = 1, \ldots, N$. Any self-avoiding circuit γ can be viewed as a Jordan curve and therefore has an interior int γ and exterior ext γ (component of the complement that is unbounded). In this way, $\mathbb{Z}^2$ is the disjoint union int $\gamma \cup$ ext $\gamma \cup \gamma$. We say a self-avoiding circuit surrounds a vertex v if $v \in$ int γ.

The dual lattice is written $((\mathbb{Z}^2)^*, (\mathcal{E}^2)^*)$, where $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (1/2)(\mathbf{e}_1 + \mathbf{e}_2)$ with its nearest-neighbor edges. Here, we have denoted by $\mathbf{e}_i$ the coordinate vectors:

$$\mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1).$$

Given $\omega \in \Omega$, we obtain $\omega^* \in \Omega^* = \{0,1\}^{(\mathcal{E}^2)^*}$ by the relation $\omega^*(e^*) = \omega(e)$, where e^* is the dual edge that shares a midpoint with e. We blur the distinction between ω and ω^* and say, for example, that e^* is open in ω. For any $V \subset \mathbb{Z}^2$ we write $V^* \subset (\mathbb{Z}^2)^*$ for $V + (1/2)(\mathbf{e}_1 + \mathbf{e}_2)$. For two subsets X and Y of the plane, we denote by $\mathrm{dist}(X,Y)$ the Euclidean distance from X to Y.

The symbols C, c will denote positive constants whose value may change between occurrences, but is independent of any parameters. Dependence on parameters is indicated by an argument, as in $C(\alpha)$, and we have numbered some recurring constants using subscript for clarity.

2.1 Circuits in annuli

For $n \geq 1$, let $B(n)$ be the box of side-length $2n$,

$$B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty \leq n\},$$

and $A(n)$ the annulus

$$A(n) = B(3n) \setminus B(n) \ .$$

For $n \geq 1$, let $\partial B(n) = \{x \in \mathbb{Z}^2 : \|x\|_\infty = n\}$.

Let $\mathcal{C}(n)$ be the collection of all self-avoiding circuits in $A(n)$ that surround the origin and, given ω, let $\Xi(n) = \Xi(n)(\omega)$ be the sub-collection of $\mathcal{C}(n)$ of open circuits.

We will be interested in the event

$$\Omega_n = \{\Xi(n) \neq \emptyset\} \ ,$$

which we know has $0 < \inf_n \mathbf{P}(\Omega_n) \leq \sup_n \mathbf{P}(\Omega_n) < 1$ by RSW arguments [27, 30]. On Ω_n we may define γ_n, the innermost element of $\Xi(n)$, as the unique $\gamma \in \Xi(n)$ which has $\mathrm{int}\,\gamma \subset \mathrm{int}\,\sigma$ for all $\sigma \in \Xi(n)$. This allows us to define the random variable

$$L_n = L_n(\omega) = \begin{cases} \#\gamma_n & \text{for } \omega \in \Omega_n \\ 0 & \text{for } \omega \notin \Omega_n \end{cases} \ .$$

This is the length of the innermost open circuit.

The expected length of the innermost open circuit can be estimated using arm events. Let $A_3(n)$ be the "three-arm" event that

1. The edge $(0, \mathbf{e}_1)$ is connected to $\partial B(n)$ by two open vertex disjoint paths and
2. $(1/2)(\mathbf{e}_1 - \mathbf{e}_2)$ is connected to $\partial B(n)^*$ by a closed dual path.

In later sections, we use arm events centered at vertices other than the origin. We define them now. For $v \in \mathbb{Z}^2$, $A_3(v,n)$ denotes the event that $A_3(n)$ occurs in the configuration ω shifted by $-v$. For an edge $e = (v_1, v_2) \in \mathcal{E}^2$, $A_3(e,n)$ denotes the event that

1. e is connected to $\partial B(e,n) := \partial B((v_1+v_2)/2, n)$ by two disjoint open paths and
2. The dual edge e^* is connected to $\partial B((v_1+v_2)/2, n)^*$ by a closed dual path.

In item 2, we view the boundary as a subset of $\mathbb{R}^2$ and say that e^* is connected to it if there is a closed dual path from e^* which (when viewed as a subset of $\mathbb{R}^2$, touches it.

Denoting

$$\pi_3(n) = \mathbf{P}(A_3(n)), \tag{2.1}$$

we have the following simple adaptation of the result of Morrow and Zhang:

Proposition 2.1. *There exist $C_1, C_2 > 0$ such that*

$$C_1 n^2 \pi_3(n) \leq \mathbf{E} L_n \leq C_2 n^2 \pi_3(n) \text{ for all } n \geq 1 \ . \tag{2.2}$$

The characterization of the innermost circuit (based on Morrow and Zhang) we will use throughout the paper is as follows. An edge $e \subset A(n)$ is in the innermost circuit if and only if the following occurs: e^* is connected to $B(n)^*$ by a closed dual path, and e is in a self-avoiding open circuit surrounding $B(n)$. One way to say this is that e has three disjoint arms (two open and one closed), with the closed arm connected to $B(n)^*$ and the open ones connecting into a circuit around $B(n)$.

We can further define the length of the shortest open circuit. That is, set

$$S_n = S_n(\omega) = \begin{cases} \min\{\#\gamma : \gamma \in \Xi(n)\} & \text{for } \omega \in \Omega_n \\ 0 & \text{for } \omega \notin \Omega_n \end{cases} .$$

Our main result for circuits is

Theorem 2.2. *As $n \to \infty$,*

$$\mathbf{E}S_n = o(n^2 \pi_3(n)) \ . \tag{2.3}$$

2.2 The lowest crossing and the question of Kesten-Zhang

The proof of Theorem 2.2 applies equally well to the length $\tilde{L}_n$ of the lowest crossing of $B(n)$. Here on H_n^c, where H_n is the event that there is an open horizontal crossing of $B(n)$, we define $\tilde{L}_n = 0$.

Corollary 2.3. *Let $\tilde{S}_n$ be the minimal number of edges in any open horizontal crossing of $B(n)$ ($\tilde{S}_n = 0$ if there is no such crossing). Then*

$$\mathbf{E}\tilde{S}_n = o(n^2 \pi_3(n)) \ . \tag{2.4}$$

To address the question of Kesten and Zhang stated in the introduction and obtain the result (1.4) on convergence in probability, we combine the preceding corollary with (2.2) (the version for $\tilde{L}_n$ in place of L_n) and an auxiliary estimate for the lower tail of $\tilde{L}_n$:

Lemma 2.4. *Let $\tilde{L}_n$ be the number of edges in the lowest crossing of $[-n, n]^2$. Then*

$$\lim_{\epsilon \downarrow 0} \limsup_n \mathbf{P}(0 < \tilde{L}_n < \epsilon n^2 \pi_3(n)) = 0. \tag{2.5}$$

We give a proof sketch of (2.5) in Section 4. The full proof, which is somewhat involved, appears in an arXiv version of the current paper. With Lemma 2.4 in hand, we can answer the question of Kesten and Zhang.

Corollary 2.5. *Conditionally on H_n, we have the convergence in probability:*

$$\tilde{S}_n / \tilde{L}_n \to 0 \ . \tag{2.6}$$

The proof of Corollary 2.5 will be found in Section 4.

3 Short detours

On the event Ω_n, we will find another another circuit $\sigma_n \in \Xi(n)$ such that $\#\sigma_n = o(\#\hat{\gamma}_n)$, where $\hat{\gamma}_n$ is a truncated version of the innermost circuit γ_n (see equation (3.3)) in a slightly thinned version of $A(n)$. To define this annulus, we note the following:

Lemma 3.1. *For some $C_3 > 0$ and $C_4 \in (0, 1)$,*

$$\pi_3(m, n) \geq C_3 (n/m)^{C_4 - 1} \text{ for all } 1 \leq m \leq n \ . \tag{3.1}$$

Here, $\pi_3(m, n)$ is the probability that there are two disjoint open paths connecting $B(m)$ to $\partial B(n)$ and one closed dual path connecting $B(m)^*$ to $\partial B(n)^*$.

Proof. We use the result of Aizenman and Burchard [1].

Consider the event Λ_1 that

1. there is a closed dual crossing of $([-n,n]\times[-n/2,-n/4])^*$, connected to the bottom of $B(n)^*$ by a closed dual crossing,
2. there is a closed dual crossing of $([-n,n]\times[n/4,n/2])^*$,
3. there is an open left-right crossing of $[-n,n]\times[-n/4,n/4]$.

Then, by RSW, $\mathbf{P}(\Lambda_1)\geq c$ for some $c>0$. Note that on Λ_1, the lowest open crossing of $B(n)$ contains an open crossing of $B(n/2)$.

Tile the box $B(n/2)=[-n/2,n/2]^2$ by boxes of size $(1/10)m\times(1/10)m$, and let $L(m,n)$ be the number of these boxes that intersect the lowest crossing of $B(n)$.

$$\begin{aligned}\mathbf{E}L(m,n)&\leq\sum_B\mathbf{P}(B\cap\tilde{L}_n\neq\emptyset)\\&\leq C\left(\frac{n}{m}\right)^2\pi_3(m,n),\end{aligned}$$

where the sum is over boxes B of side-length $m/10$ in the tiling of $B(n/2)$.

Critical percolation in $\frac{1}{n}\mathbb{Z}^2\cap(B(1)=[-1,1]^2)$ satisfies "Hypothesis H2" in that paper. For $\ell>0$ and $\mathcal{C}$ a curve formed by a self-avoiding concatenation of open edges in $B(1)$, let $N(\mathcal{C},\ell)$ be the minimal number of sets of diameter ℓ required to cover $\mathcal{C}$.

By [1, Theorem 1.3] and [1, Equation (1.21)], there exists $C_4>0$ such that for any $\epsilon>0$:

$$\mathbf{P}_{\frac{1}{n}\mathbb{Z}^2\cap B(1)}\left(\inf_{\mathrm{diam}(\mathcal{C})\geq 1/10}N(\mathcal{C},\ell)\leq C(\epsilon)\ell^{-1-C_4}\right)\leq\epsilon, \tag{3.2}$$

uniformly in n sufficiently large and $\ell\geq 1/n$. Choosing $\ell=m/n$, ϵ sufficiently small and letting Λ_2 be the event that there is an open crossing of $B(n/2)$ with fewer than $C(\epsilon)n^{1+C_4}$ edges, we have by (2.2) and (3.2):

$$\left(\frac{n}{m}\right)^2\pi_3(m,n)\geq(1/C)\mathbf{E}L(m,n)\geq(1/C)\mathbf{E}[L(m,n),\Lambda_1\cap\Lambda_2^c]\geq C_3(\epsilon)(n/m)^{1+C_4}\ . \qquad\square$$

Now define the annulus

$$\hat{A}(n)=B\left(\lfloor 3n-n^{C_4/2}\rfloor\right)\setminus B\left(\lceil n+n^{C_4/2}\rceil\right)$$

and the inner portion $\hat{\gamma}_n$ of the innermost open circuit, defined as the union of all edges $e\in\gamma_n$ which lie entirely inside $\hat{A}(n)$:

$$\hat{\gamma}_n=\{e\in\gamma_n : e\subset\hat{A}(n)\}. \tag{3.3}$$

3.1 Definition of shielded detours

In this section, we define the central objects of our construction, the *shielded detour paths* $\pi(e)$, $e\in\hat{\gamma}_n$.

Definition 3.2. *Given $\omega\in\Omega_n$, $\epsilon\in(0,1)$, and any $e\in\hat{\gamma}_n$, we define the set $\mathcal{S}(e)$ of* ϵ-shielded detours *around e as follows. An element of $\mathcal{S}(e)$ is a self-avoiding open path P with vertex set $w_0,w_1,\ldots,w_M$ such that the following hold:*

1. *for $i=1,\ldots,M-1$, $w_i\in(A(n)\cap\text{ext}\,\gamma_n)$,*
2. *the edges $\{w_0,w_0+\mathbf{e}_1\},\{w_0-\mathbf{e}_1,w_0\},\{w_M,w_M+\mathbf{e}_1\}$ and $\{w_M-\mathbf{e}_1,w_M\}$ are in γ_n and $w_1=w_0+\mathbf{e}_2$, $w_{M-1}=w_M+\mathbf{e}_2$.*
3. *writing Q for the subpath of γ_n from w_0 to w_M that contains e, the circuit $Q\cup P$ does not surround the origin,*

4. *the points* $w_0 + (1/2)(-\mathbf{e}_1 + \mathbf{e}_2)$ *and* $w_M + (1/2)(\mathbf{e}_1 + \mathbf{e}_2)$ *are connected by a dual closed self-avoiding path* R *whose first and last edges are vertical (translates of* $\{0, \mathbf{e}_2\}$*) and is such that the curve formed by the union of* R*, the line segments from the endpoints of* R *to* w_1 *and* w_{M-1}*, and* P *does not enclose the origin, and*
5. $\#P \le \epsilon \#Q$.

Now fix a deterministic ordering of all finite lattice paths and define $\pi(e)$ *to be the first element of* $\mathcal{S}(e)$ *in this ordering. If the set* $\mathcal{S}(e)$ *is empty, then we set* $\pi(e) = \emptyset$.

3.2 Properties of the detour paths

We give the properties of the collection of detours $(\pi(e) : e \in \hat{\gamma}_n)$ which we use in the next section to prove Theorem 2.2 and Corollary 2.5. The definition of $\pi(e)$ (Definition 3.2) appeared in Section 3.1.

Let $0 < \epsilon < 1$. Then:

1. Each $\pi(e)$ is open and for distinct $e, e' \in \hat{\gamma}_n$, $\pi(e)$ and $\pi(e')$ are either equal or have no vertices in common.
2. If $e \in \hat{\gamma}_n$ and $\pi(e) \neq \emptyset$, write $\pi(e) = \{w_0, e_0, \dots, e_{M-1}, w_M\}$. Then $w_0, w_M \in \gamma_n$ but $w_i \in (A(n) \cap \text{ext}\, \gamma_n)$ for $i = 1, \dots, M-1$.
3. If $e \in \hat{\gamma}_n$ then the segment $\hat{\pi}(e)$ of γ_n from w_0 to w_M containing e (that is, the "detoured" portion of γ_n) is such that $\hat{\pi}(e) \cup \pi(e)$ is a circuit that does not surround the origin. Furthermore,
$$\#\pi(e) \le \epsilon \#\hat{\pi}(e) \ . \tag{3.4}$$
4. There exists $C_5(\epsilon) > 0$ such that for all $n \ge C_5$ and $e \in \hat{A}(n)$,
$$\mathbf{P}(\pi(e) = \emptyset \mid e \in \hat{\gamma}_n) \le \epsilon^2 \ . \tag{3.5}$$
(The exponent 2 on ϵ on the right in (3.5) will be used in (4.4).)

We must show the above properties follow from Definition 3.2. Most of the work will be in showing item 4, the probability estimate (3.5), which we will do in Section 6. Items 2 and 3 hold by definition. For item 1, we have

Proposition 3.3. *If* $\omega \in \Omega_n$*, then for distinct* $e, e' \in \hat{\gamma}_n$*,* $\pi(e)$ *and* $\pi(e')$ *are either equal or have no vertices in common.*

The proof of Proposition 3.3 will be found in Section 8.

For the rest of this section, we will identify paths with their edge sets. Given such detour paths $\pi(e)$ (which necessarily do not share edges with γ_n, due to (3.4)) and detoured paths $\hat{\pi}(e)$ (subpaths of γ_n), we construct σ_n as follows. First choose a subcollection Π of $\{\pi(e) : e \in \hat{\gamma}_n\}$ that is maximal in the following sense: for all $\pi(e), \pi(e') \in \Pi$ with $e \neq e'$, the paths $\hat{\pi}(e)$ and $\hat{\pi}(e')$ share no vertices and the total length of detoured paths $\sum_{\pi \in \Pi} \#\hat{\pi}$ is maximal. The choice of Π can be arbitrary among all maximal ones.

We put
$$\hat{\Pi} = \{\hat{\pi} : \pi \in \Pi\}.$$
Now define σ_n to be the path with edge set equal to the union of Π and those edges in γ_n that are not in $\hat{\Pi}$.

We must now show that if such a construction can be made, then $\limsup_n \frac{\mathbf{E}\#\sigma_n}{\mathbf{E}L_n} \le \epsilon$ (so Theorem 2.2 follows). To do this we will need two tools:

Lemma 3.4. *For* $\omega \in \Omega_n$*,* σ_n *is an open circuit in* $A(n)$ *surrounding the origin.*

and

Lemma 3.5. *For* $\omega \in \Omega_n$*, if* $e \in \hat{\gamma}_n \setminus \hat{\Pi}$ *then* $\pi(e) = \emptyset$.

The proofs of these two lemmas is detailed in Section 8.

4 Estimate for $\mathbf{E}S_n$

Now we show that if paths $\pi(e)$ can be defined so as to satisfy the properties in Definition 3.2, and if we prove Lemmas 3.4 and 3.5, in conjunction with the other claimed results from the last section, we can then conclude Theorem 2.2 and Corollary 2.3.

Let $\ell(\Pi) = \sum_{\pi\in\Pi} \#\pi$ be the total length of the detours in the collection Π. Assuming Lemmas 3.4 and 3.5, we estimate the length of σ_n:

$$\begin{aligned}\#\sigma_n &= \ell(\Pi) + \#(\gamma_n \setminus \hat{\Pi})\\ &\le \ell(\Pi) + \#\{e \in A(n) : e \cap (A(n) \setminus \hat{A}(n)) \neq \emptyset\} + \#(\hat{\gamma}_n \setminus \hat{\Pi})\\ &\le \ell(\Pi) + 30n^{1+C_4/2} + \#\{e \in \hat{\gamma}_n : \pi(e) = \emptyset\}\ .\end{aligned}$$

We have:

$$\ell(\Pi) = \sum_{\pi\in\Pi} \#\pi \le \epsilon \sum_{\pi\in\Pi} \#\hat{\pi} = \epsilon\#\left(\cup_{\pi\in\Pi}\hat{\pi}\right) \le \epsilon\#\gamma_n\ . \tag{4.1}$$

Furthermore, due to (3.5),

$$\mathbf{E}\#\{e \in \hat{\gamma}_n : \pi(e) = \emptyset\} = \sum_{e\subset\hat{A}(n)} \mathbf{P}(\pi(e) = \emptyset \mid e \in \hat{\gamma}_n)\mathbf{P}(e \in \hat{\gamma}_n) \le \epsilon^2\mathbf{E}L_n\ . \tag{4.2}$$

Therefore

$$\mathbf{E}S_n \le \mathbf{E}\#\sigma_n \le (\epsilon + \epsilon^2)\cdot\mathbf{E}L_n + 30n^{1+C_4/2}\ .$$

Since ϵ is arbitrary and $\mathbf{E}L_n \ge C_3 n^{1+C_4}$ (using (2.2) and Lemma 3.1), (2.2) gives $\mathbf{E}S_n = o(n^2\pi_3(n))$, finishing the proof of Theorem 2.2.

To obtain Corollary 2.5 for crossings of a box, we repeat the construction above to obtain a crossing $\tilde{\sigma}_n$ and a union $\tilde{\Pi}$ of detours from the lowest crossing l_n. Denoting by $\tilde{\mathbf{P}}$ the probability measure conditioned on the existence of an open crossing

$$\tilde{\mathbf{P}} = \mathbf{P}(\ \cdot \mid H_n),$$

write:

$$\begin{aligned}\tilde{\mathbf{P}}(\#\tilde{\sigma}_n > 3\epsilon^{1/2}\tilde{L}_n) \le\ &\tilde{\mathbf{P}}(\ell(\tilde{\Pi}) > \epsilon^{1/2}\tilde{L}_n) + \tilde{\mathbf{P}}(30n^{1+C_4/2} > \epsilon^{1/2}\tilde{L}_n)\\ &+ \tilde{\mathbf{P}}(\#\{e \in \hat{l}_n : \pi(e) = \emptyset\} > \epsilon^{1/2}\#l_n)\end{aligned} \tag{4.3}$$

By an estimate analogous to (4.1), the first probability on the right is zero. We decompose the last term further:

$$\begin{aligned}\tilde{\mathbf{P}}(\#\{e \in \hat{l}_n : \pi(e) = \emptyset\} > \epsilon^{1/2}\tilde{L}_n) \le\ &\tilde{\mathbf{P}}(\#\{e \in \hat{l}_n : \pi(e) = \emptyset\} > \epsilon\mathbf{E}\tilde{L}_n)\\ &+ \tilde{\mathbf{P}}(\tilde{L}_n \le (\epsilon/\epsilon^{1/2})\mathbf{E}\tilde{L}_n).\end{aligned} \tag{4.4}$$

By Markov's inequality and (4.2), the term $\tilde{\mathbf{P}}(\#\{e \in \hat{l}_n : \pi(e) = \emptyset\} > \epsilon\mathbf{E}\tilde{L}_n)$ is bounded by ϵ. (Recall that $0 < \mathbf{P}(H_n) < 1$ uniformly in n, so $\tilde{\mathbf{P}}$ is uniformly absolutely continuous with respect to $\mathbf{P}$.)

It remains to estimate the second term on the right in (4.3) and the last term on the right in (4.4). Using again that $\mathbf{E}\tilde{L}_n \ge C_3 n^{1+C_4}$, we see that it will suffice to show that

$$\limsup_{n\to\infty} \tilde{\mathbf{P}}(0 < \tilde{L}_n \le \epsilon^{1/2}\mathbf{E}\tilde{L}_n) \le \lambda(\epsilon)$$

for $\lambda(\epsilon) \to 0$ as $\epsilon \to 0$. Recalling that $\tilde{\mathbf{P}}$ is supported on the event $\{\tilde{L}_n > 0\}$ and $\mathbf{P}(H_n) > 0$ uniformly in n, this reduces to showing

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbf{P}(0 < \tilde{L}_n \le \epsilon^{1/2}\mathbf{E}\tilde{L}_n) = 0. \tag{4.5}$$

This will be done in the Section 7.

5 The events E_k

In this section, we define events E_k (depending on ϵ and η), which will be used in the proof of the probability estimate (3.5) (see (6.2) in Section 6.1). The formal, precise definition of E_k is quite long and complicated and given in Definition 5.10, equation (5.22). The essential property of E_k is

Proposition 5.1. *Let e be an edge of $\mathbb{Z}^2$ and $\epsilon > 0$, and let $E_k(e) = \tau_{-e}E_k$ be the translation of E_k by the edge e. That is, for any $\omega \in \Omega$,*

$$(\omega_{e'})_{e' \in \mathcal{E}^2} \in \tau_{-e}E_k \iff (\omega_{e'-e})_{e' \in \mathcal{E}^2} \in E_k.$$

There is a constant $K(\epsilon)$ such that the following holds. Let

$$k \in \{K(\epsilon), \dots, \lfloor (C_4/8)\log n \rfloor\}.$$

On $E_k(e) \cap \{e \in \hat{\gamma}_n\}$, there is a ϵ-shielded detour $\pi(e) \in \mathcal{S}(e)$ around e contained in $B(3^{k+1}) \setminus B(3^{k-1})$. That is,

$$\mathcal{S}(e) \neq \emptyset.$$

Proposition 5.1 will be proved in Section 5.6. As for the probability of occurrence of E_k, we have the following lower bound, where $A_3(d)$ was defined in Section 2.1.

Proposition 5.2. *For some $K(\eta) > 0$,*

$$\mathbf{P}(E_k \mid A_3(d)) > C_7 \text{ for all } k \in \{K(\eta), \dots, \lfloor (C_4/8)\log n \rfloor\} \text{ and } d \geq \lfloor (C_4/8)\log n \rfloor \quad (5.1)$$

for some uniform constant C_7 independent of n. E_k will be defined as the conjunction of a large number of crossing events (see equation (5.22)*). These events will be gradually introduced over the next few subsections. Figure 1 illustrates most of the crossing events in E_k.*

The proof of Proposition 5.2 takes up most of the remainder of Section 5, where lower bounds for the subevents of E_k are given. The proof is completed in Section 5.5.

After we prove Propositions 5.1 and 5.2, Lemmas 3.4 and 3.5, and Proposition 3.3, we can conclude Theorem 2.2.

5.1 Sketch of proof of Proposition 5.2

Because the proof of the above proposition requires many constructions, we now give a sketch of the main ideas. Let $\eta \in (1, 11/10)$ be given and $k \geq 1$. The quantity $\delta = \eta - 1 > 0$ should be thought of as small. Define the annuli

$$Ann_i = Ann_i(\eta) = B(\eta 3^{k+i-2}) \setminus B(3^{k+i-2}) \text{ for } i = 1, 2.$$

The event $E_k(e)$ has three main features:

1. Two closed circuits (in green in Figure 1), in Ann_1 and Ann_2, each with two defects, in thin concentric annuli. These serve to isolate the inside of the annulus $[-3^k, 3^k]^2 \setminus [-\eta 3^{k-1}, \eta 3^{k-1}]^2$ from portions of the innermost circuit $\hat{\gamma}_n$ which are not involved in E_k. Their thickness is controlled by the small parameter $\delta \ll 1$. If the origin has 3 arms to a macroscopic distance, the open arms are forced to pass through the defects.

   The existence of these circuits is shown to have probability bounded below independently of n in Section 5.2.

2. An open half-circuit connected to the crossings of the annulus emanating from the origin (in red in Figure 1). This will act as a detour for the portion of $\hat{\gamma}_n$ inside the larger box.

   In Section 5.3, we show that, given the existence of the circuits in the previous item, the open half-circuit contains at most $\epsilon 3^{2k}\pi_3(3^k)$ edges with high probability, where $\epsilon > 0$ is small.
3. Boxes containing order $3^{2k}\pi_3(3^k)$ many three-arm points connected to the arms emanating from e. (The relevant connections appear in blue in Figure 1.) On $E_k(e) \cap \{e \in \hat{\gamma}_n\}$, these points lie on $\hat{\gamma}_n$.

   We give a lower bound of order $C3^{2k}\pi_3(3^k)$ for the number of three arm points on the open arms emanating from the origin. This holds with probability bounded below independently of n, conditionally on the events in the previous items. This is done in Section 5.4.

Given these three features, the proof proceeds as follows. We first show existence of circuits with defects: we show that for any given η close enough to 1, there is a constant $D_1 = D_1(\eta)$ such that for suitable values of k, one has

$$\mathbf{P}(X_1(k,\eta)) \geq D_1(\eta), \tag{5.2}$$

where $X_1(k,\eta)$ is the event that the closed circuits with defects from feature one above exist in the annuli Ann_i, and the defected edges have five arms as in Figure 1. Next, we give an upper bound for the length of a thin detour. Namely, let $X_2(k,\eta,\epsilon)$ be the event that there is an open half-circuit connected to the defects from feature one, staying in Ann_2, and having length at most $\epsilon 3^{2k}\pi_3(3^k)$. We show in Section 5.3 that for any $\epsilon > 0$, there exists $\eta(\epsilon)$ close enough to 1 such that

$$\mathbf{P}(X_2(k,\eta,\epsilon) \mid X_1(k,\eta)) \geq D_2(\epsilon) \tag{5.3}$$

for suitable values of k. Last, we show existence of many edges on paths that will function as the innermost circuit. Let $X_3(k,\eta,c)$ be the event that in boxes in the interior of the annulus, there are at least $c3^{2k}\pi_3(3^k)$ edges with three disjoint arms: two open to the defects and one closed dual path to the bottom of the annulus. In Section 5.4, we prove that there exists $c > 0$ and a constant $D_3 > 0$ such that for all η close to 1 and suitable values of k,

$$\mathbf{P}(X_3(k,\eta,c) \mid X_2(k,\eta(\epsilon),\epsilon), X_1(k,\eta)) \geq D_3. \tag{5.4}$$

The most important thing here is that c has no dependence on η or ϵ, essentially because as $\eta \downarrow 1$, the size of the boxes in which the three arm points lie does not decrease to 0.

To put these pieces together, we first choose c such that (5.4) holds. Next, given $\epsilon > 0$, choose $\eta = \eta(c\epsilon)$ to guarantee (5.3) with $c\epsilon$ in place of ϵ. For this value of η, one also has (5.2). Combining the above three inequalities, and, for the purposes of this proof sketch, putting

$$E_k = X_1(k,\eta) \cap X_2(k,\eta,\epsilon) \cap X_3(k,\eta,c),$$

one has

$$\mathbf{P}(E_k) \geq D_1 D_2 D_3 \tag{5.5}$$

for suitable k. On this intersection, one is guaranteed that the volume of the detour is at most ϵ times the volume of the three-arm points in the boxes connected to the defects. To finish the proof, one notes that E_k implies that there is a three-arm connection across the annulus in which E_k occurs. Thus one uses arm separation to see that if E_k and

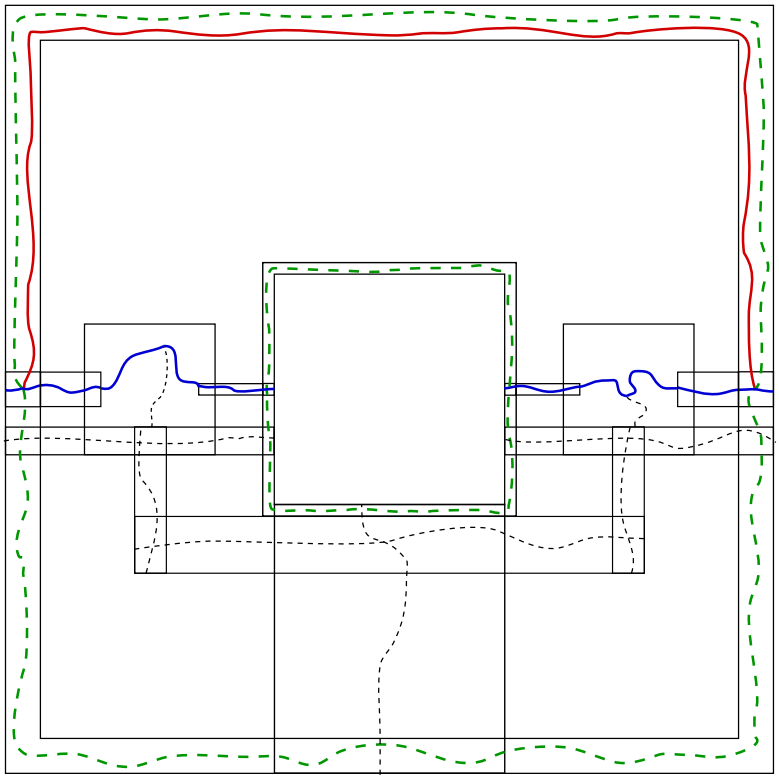

Figure 1: An approximate depiction of the event E_k. Not all connections are shown.

$A_3(d)$ both occur, then one can, with uniformly positive conditional probability, route the arms from $A_3(d)$ to the inner and outer boundaries of the annulus to connect to the arms from E_k. Thus there is a constant D_4 independent of n, η, c and k such that

$$(1/D_4)\mathbf{P}(E_k)\mathbf{P}(A_3(d)) \leq \mathbf{P}(E_k, A_3(d)) \leq D_4\mathbf{P}(E_k)\mathbf{P}(A_3(d)).$$

This step is standard, so we leave the details to the reader. Combining it with (5.5) completes the proof of Proposition 5.2.

5.2 Five-arm events and shielded circuit

In this section we put in place most of the components of the construction of the event E_k. Define the boxes

$$B_1 = [-\eta 3^k, -3^k] \times [-\frac{3^k}{2}(\eta - 1), \frac{3^k}{2}(\eta - 1)],$$
$$B_2 = [-\eta 3^{k-1}, -3^{k-1}] \times [-\frac{3^{k-1}}{2}(\eta - 1), \frac{3^{k-1}}{2}(\eta - 1)],$$
$$B_3 = B_1 + (3^k(\eta - 1), 0), \quad B_4 = B_2 + (-3^{k-1}(\eta - 1), 0),$$

and the "long" rectangles:

$$B_5 = [-\eta 3^k, -\frac{3^k(1+\eta)}{2}] \times [\frac{3^k}{2}(\eta-1), 3^k\eta],$$
$$B_6 = [-\frac{3^k(1+\eta)}{2}, -3^k] \times [\frac{3^k}{2}(\eta-1), \frac{3^k}{2}(1+\eta)],$$
$$B_7 = [-\eta 3^k, -3^k] \times [-\eta 3^k, -\frac{3^k}{2}(\eta-1)],$$
$$B_8 = [-\eta 3^{k-1}, -3^{k-1}] \times [-\eta 3^{k-1}, -\frac{3^{k-1}}{2}(\eta-1)],$$

$$B_9 = [-\eta 3^{k-1}, -3^{k-1}] \times [\frac{3^{k-1}}{2}(\eta-1), \eta 3^{k-1}], \quad B_{10} = [-\eta 3^{k-1}, \eta 3^{k-1}] \times [3^{k-1}, \eta 3^{k-1}],$$
$$B_{11} = [-\eta 3^{k-1}, \eta 3^{k-1}] \times [-\eta 3^{k-1}, -3^{k-1}], \quad B_{12} = [-\eta 3^k, \eta 3^k] \times [\frac{3^k}{2}(1+\eta), \eta 3^k],$$
$$B_{13} = [-\frac{3^k}{2}(1+\eta), \frac{3^k}{2}(1+\eta)] \times [3^k, \frac{3^k}{2}(1+\eta)], \quad B_{14} = [-\eta 3^k, \eta 3^k] \times [-\eta 3^k, -3^k].$$

The relative placement of these boxes inside $S(e, \eta 3^k)$ is shown in Figure 2. From this point on, we will restrict to n and k such that

$$n \geq 1 \text{ and } k \in \{K(\eta), \ldots, \lfloor (C_4)/8 \log n \rfloor\}\ , \tag{5.6}$$

where $K(\eta)$ is chosen so that all boxes involved have lengths at least some constant, say 10. (This includes the above boxes, but also those used in Section 5.4.) If we decrease η, then the range of valid k decreases.

The most important definition of this section is the following

Definition 5.3 (Five-arm event)**.** *$M_1 = M_1(k)$ is the event: there is a* five-arm point *$w \in \mathbb{Z}^2$ in the box B_1. That is,*

1. *The edge $\{w + (1/2)(\mathbf{e}_2 - \mathbf{e}_1), w + (1/2)(3\mathbf{e}_2 - \mathbf{e}_1)\}$ is closed and has a closed arm γ_1 to*
$$I_1 = [-\eta 3^k, -\frac{3^k}{2}(1+\eta)] \times \{\frac{3^k}{2}(\eta-1)\},$$
2. *$\{w, w + \mathbf{e}_2\}$ has an open arm γ_2 to*
$$I_2 = [-\frac{3^k}{2}(1+\eta), -3^k] \times \{\frac{3^k}{2}(\eta-1)\},$$
3. *$\{w, w + \mathbf{e}_1\}$ has an open arm γ_3 to the right side of B_1,*
4. *$\{w - (1/2)(\mathbf{e}_1 + \mathbf{e}_2), w - (1/2)(3\mathbf{e}_2 + \mathbf{e}_1)\}$ has a closed arm γ_4 to the bottom of B_1, and*
5. *$\{w - \mathbf{e}_1, w\}$ has an open arm γ_5 to the left side of B_1.*

The event M_1 is illustrated in Figure 3.

$M_2 = M_2(k)$ is the event that there is a four-arm point z in the box B_2 with a two open arms, one to each horizontal side of B_2, and two closed arms to the top and bottom of B_2.

Our first claim is

Proposition 5.4. *There is a constant C independent of k such that*

$$\min\{\mathbf{P}(M_1), \mathbf{P}(M_2)\} \geq C$$

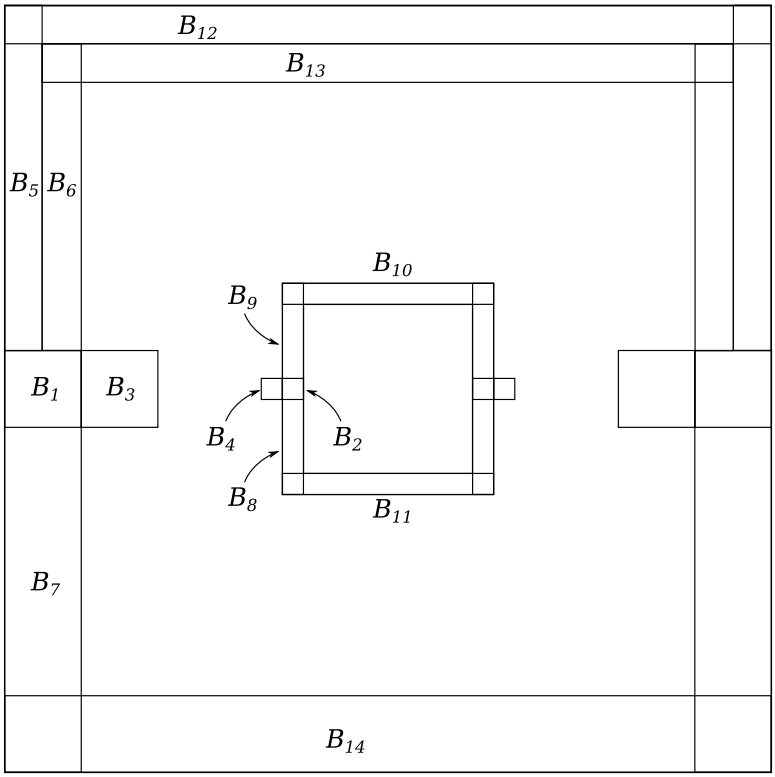

Figure 2: The boxes $B_1, \ldots, B_{14}$. This figure is not to scale; only the relative placement of the boxes is illustrated. In particular, in our application, η is smaller relative to the ratio of the sizes of the inner to outer annuli.

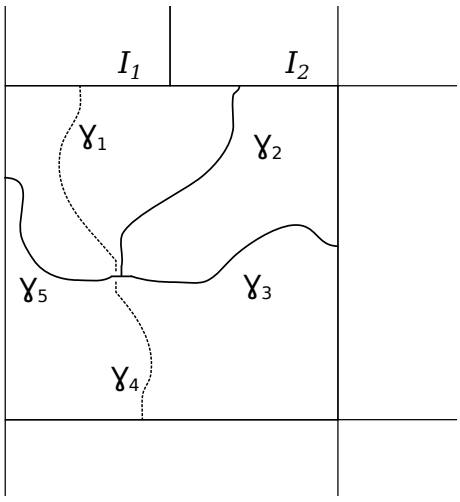

Figure 3: The "five-arm" event $M_1(k)$

Proof. The argument is a standard application of the second moment method. The 5-arm exponent is universal and equal to 2 [17, Lemma 5], [23, Theorem 24, 3.], so the expected number of five arm points in B_2 is bounded away from zero. By planarity, there can be at most one five arm point with the prescribed landing sequences for the endpoints in this region. The result follows by a gluing argument. □

Let $M_3 = M_3(k)$ be the event that there is a closed top-down crossing of B_5, and an open top-down crossing of B_6. $M_4(k)$ is defined to be the event that there are closed top-bottom crossings of B_7, B_8 and B_9 and open left-right and top-down crossings of B_3 and B_4. By Russo-Seymour-Welsh and Harris-FKG, we have

$$\mathbf{P}(M_3), \mathbf{P}(M_4) \geq C(\eta) > 0.$$

We let $G_1(k)$ be the event that

1. M_1, M_2, M_3 and M_4 occur.
2. The closed arm γ_1 is connected to the crossing of B_5.
3. The open arm γ_2 is connected to the crossing of B_6.
4. The open arm γ_3 is connected to the crossings of B_3.
5. The closed arm γ_4 is connected to the crossing of B_7.
6. The four arm-point in B_2 is connected to an open crossing of B_4.

Here and elsewhere in the paper, we will make extensive use of the following generalized FKG inequality [15, Lemma 3] (see also [23, Lemma 13]):

Lemma 5.5 (Generalized FKG inequality)**.** *Let A and D be increasing events, and B and E decreasing events. Assume that A, B D, E depend only on edges in the finite sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$, respectively. If*

$$\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{E} = \mathcal{B} \cap \mathcal{D} = \emptyset,$$

then

$$\mathbf{P}(A \cap B \mid D \cap E) \geq \mathbf{P}(A \cap B).$$

By generalized FKG and standard gluing constructions, we have, using Proposition 5.4,

$$\mathbf{P}(G_1) \geq C_9 \mathbf{P}(M_3)\mathbf{P}(M_4),$$

for some constant C_9 independent of η.

$G_2(k)$ is defined to be the reflection of G_1 about the vertical axis through 0. For a box B_i, $i = 1, \ldots, 9$, we let B_i' be its reflection about the $\mathbf{e}_2$-axis. That is, if $x = (x_1, x_2) \in \mathbb{Z}^2$, then $x \in B_i'$ if and only if $(-x_1, x_2) \in B_i$. The same applies to the "landing zones" I_i. We say that $G_2(k)$ occurs if all the conditions in the definition of $G_1(k)$ occur, replacing each B_i, $i = 1, \ldots 9$, and I_1, I_2 by B_i' and I_1', I_2', respectively. By symmetry and independence:

$$\mathbf{P}(G_1(k) \cap G_2(k)) \geq C_9^2 \mathbf{P}(M_3(k))^2 (\mathbf{P}(M_4(k))^2.$$

We let $R_1(k)$ be the event that there is a closed left-right crossing of B_{12}; $R_2(k)$ is the event that there is an open left-right crossing of B_{13}; $R_3(k)$ is the event that there are closed left-right crossings of B_{10}, B_{11} and B_{14}.

By Russo-Seymour-Welsh and generalized FKG, we have

$$\begin{aligned} &\mathbf{P}(G_1(k) \cap G_2(k) \cap R_1(k) \cap R_2(k) \cap R_3(k)) \\ \geq\ & C_9^2 \mathbf{P}(M_3(k))^2 \mathbf{P}(M_4(k))^2 \mathbf{P}(R_1(k)) \mathbf{P}(R_2(k)) \mathbf{P}(R_3(k)). \end{aligned} \tag{5.7}$$

The occurrence of the intersection

$$Q_1 = Q_1(k,\eta) = G_1(k) \cap G_2(k) \cap R_1(k) \cap R_2(k) \cap R_3(k) \tag{5.8}$$

(whose right side depends on η, although this is not visible in the notation) implies the events:

1. There are five-arm points w and w' in B_1 and B_1', respectively.
2. There is a closed dual circuit α_1 with 2 defects near w and w' in

$$Ann_2 = B(\eta 3^k) \setminus B(3^k).$$

   The arc $\tilde{\alpha}_1$ of α_1 between w and w' is contained in

$$\Gamma(k,\eta) = Ann_2 \cap \left(\mathbb{R} \times [-\frac{3^k}{2}(\eta-1), \infty) \right). \tag{5.9}$$

3. There is an open arc α_2 contained in $\Gamma(k,\eta)$ with endpoints at w and w'. Moreover, α_2 is contained in the interior of the dual circuit α_1.

We note that there exists $C_{10}(\eta) > 0$ depending on η such that

$$\mathbf{P}(Q_1(k,\eta)) \geq C_{10} \text{ for } k \geq K(\eta)\ . \tag{5.10}$$

Definition 5.6 (Outermost open path)**.** *Given the occurrence of $Q_1(k,\eta)$, we can define the* outermost open arc $\tilde{\alpha}_2$ *contained inside the dual circuit α_1. For this, we let $\tilde{\alpha}_1$ denote the portion of α_1 between w and w' in $\Gamma(k,\eta)$. The outermost arc $\tilde{\alpha}_2$ is the open arc in $\Gamma(k,\eta)$ with endpoints at w and w' such that the region enclosed by the Jordan curve $\tilde{\alpha}_1 \cup \tilde{\alpha}_2$ (extended near the five-arm points to be a closed curve) is minimal.*

In the above definition, we may choose α_1 arbitrarily, but this choice uniquely defines α_2. We have the following

Lemma 5.7. *On the event $Q_1(k,\eta)$, any edge e on $\tilde{\alpha}_2$ has 3 disjoint arms: two open and one closed. These arms reach to distance at least $3^k(\eta-1)/2$ from e.*

Proof. If e is on $\tilde{\alpha}_2$, then necessarily, there is a closed path from e^* to $\tilde{\alpha}_1$ contained in the region $\mathrm{int}(\tilde{\alpha}_1 \cup \tilde{\alpha}_2)$. Following this closed path until we reach the closed path $\tilde{\alpha}_1$, we can extend it into a closed path of at least the required length, because the path α_1 has diameter greater than 3^k. On the other hand, the path $\tilde{\alpha}_2$ itself has two ends, one of which necessarily has length at least $3^k/2$. As for the other, it emanates from a five-arm point, itself connected to a crossing of a box of width $3^k(\eta-1)$. □

5.3 Upper bound for the volume of the thin detour

The main estimate of this section is the following:

Lemma 5.8. *Let $\epsilon > 0$, let $Q_1(k,\eta)$ be as in* (5.8)*, and define the event and*

$$Q_2 = Q_2(k,\eta,\epsilon) = \{\#\tilde{\alpha}_2 \leq \epsilon 3^{2k} \pi_3(3^k)\}. \tag{5.11}$$

For $\eta > 1$ sufficiently close to 1, there exists C_{11} such that

$$\mathbf{P}(Q_2(k,\eta,\epsilon) \mid Q_1(k,\eta)) \geq C_{11} > 0, \tag{5.12}$$

for all n and all $k \in \{K(\eta), \ldots, \lfloor (C_4/8)\log n \rfloor\}$.

Proof. The key estimate we need is: for each $e \subset \Gamma(k,\eta)$

$$\mathbf{P}(e \in \tilde{\alpha}_2 \mid Q_1(k,\eta)) \le C\pi_3(3^k(\eta-1)).$$

We rewrite the probability (using (5.7)) as

$$\begin{aligned}&\frac{\mathbf{P}(e \in \tilde{\alpha}_2, Q_1(k,\eta))}{\mathbf{P}(Q_1(k,\eta))}\\ \le\ &\frac{1}{C_9^2}\mathbf{P}(M_3(k))^{-2}\mathbf{P}(M_4(k))^{-2}\mathbf{P}(R_1(k))^{-1}\\ &\times \mathbf{P}(R_2(k))^{-1}\mathbf{P}(R_3(k))^{-1}\mathbf{P}(e \in \tilde{\alpha}_2, Q_1(k,\eta)).\end{aligned}$$

Let

$$B(e,3^k(\eta-1)) = \{v : \|(v_1+v_2)/2 - v\|_\infty \le 3^k(\eta-1)\},$$

where $e = \{v_1, v_2\}$. Also let $A_\eta(e)$ be the event that e has 3 arms to the boundary of $B(e,3^k(\eta-1))$, two open and one closed. Recalling the definition of Q_1 (5.8), we have

$$\mathbf{P}(e \in \tilde{\alpha}_2, Q_1(k,\eta)) \le \mathbf{P}(A_\eta(e), M_3 \cap M_3' \cap M_4 \cap M_4' \cap R_1 \cap R_2 \cap R_3).$$

We now define crossing events in "truncated" regions, which depend on the position of e: $\tilde{M}_3$ is the event

1. There are vertical closed crossings of each component of

$$[-\eta 3^k, -\frac{3^k}{2}(\eta+1)] \times [\frac{3^k}{2}(\eta-1), 3^k(2-\eta)] \setminus B(e,3^k(\eta-1)).$$

2. There are vertical open crossings of each component of

$$[-\frac{3^k}{2}(\eta+1), -3^k] \times [\frac{3^k}{2}(\eta-1), 3^k(2-\eta)] \setminus B(e,3^k(\eta-1)).$$

$\tilde{M}_4$ is the event that there are closed top to bottom crossings of B_8 and B_9, open left-right crossings of B_3 and B_4, and a closed vertical crossing of each component of

$$[-\eta 3^k, -3^k] \times [-3^k, -\frac{3^k}{2}(\eta-1)] \setminus B(e,3^k(\eta-1)).$$

$\tilde{M}_3'$, $\tilde{M}_4'$ are the reflections of the events $\tilde{M}_3$ and $\tilde{M}_4$ about the y-axis. $\tilde{R}_1$ is the event that there are closed left-right crossings of each component of

$$[-3^k(2-\eta), 3^k(2-\eta)] \times [\frac{3^k}{2}(\eta+1), \eta 3^k] \setminus B(e,3^k(\eta-1)),$$

and similarly $\tilde{R}_2$ is the event that there are open left-right crossings of each component of

$$[-3^k(2-\eta), 3^k(2-\eta)] \times [3^k, \frac{3^k}{2}(\eta+1)] \setminus B(e,3^k(\eta-1)).$$

The definition of the events implies that the truncated regions considered are either rectangles, or consist of the union of two disjoint rectangles which also do not abut (see Figure 4 – this is the reason for the choice of slightly smaller rectangles with bound $3^k(2-\eta)$), which implies:

$$\begin{aligned} M_3 &\subset \tilde{M}_3, & M_3' &\subset \tilde{M}_3',\\ M_4 &\subset \tilde{M}_4, & M_4' &\subset \tilde{M}_4',\\ R_1 &\subset \tilde{R}_1, & R_2 &\subset \tilde{R}_2.\end{aligned}$$

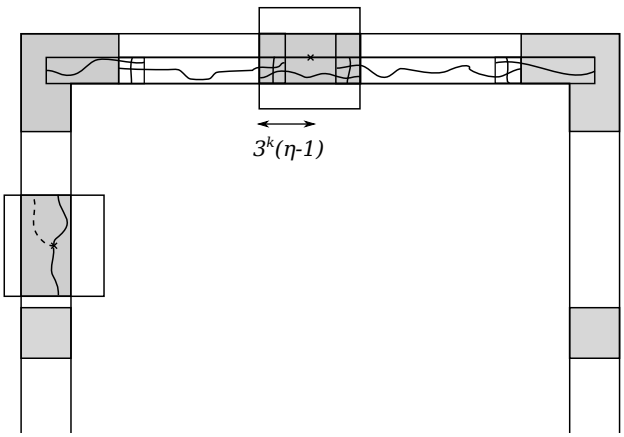

Figure 4: An illustration of the truncated regions for two possible placements of the edge e.

Then:

$$\begin{aligned} & \mathbf{P}(A_\eta(e), M_3 \cap M_3' \cap M_4 \cap M_4' \cap R_1 \cap R_2 \cap R_3) \\ \leq\ & \mathbf{P}(A_\eta(e), \tilde{M}_3 \cap \tilde{M}_3' \cap \tilde{M}_4 \cap \tilde{M}_4' \cap \tilde{R}_1 \cap \tilde{R}_2 \cap R_3) \\ =\ & \mathbf{P}(A_\eta(e))\mathbf{P}(\tilde{M}_2)^2\mathbf{P}(\tilde{M}_3)^2\mathbf{P}(\tilde{R}_1)\mathbf{P}(\tilde{R}_2)\mathbf{P}(R_3). \end{aligned}$$

In the second step we have used independence. Using a gluing construction and FKG, it is easy to "fill in" the truncated regions and show

$$\mathbf{P}(\tilde{M}_4) \leq C\mathbf{P}(M_4), \qquad \mathbf{P}(\tilde{M}_3) \leq C\mathbf{P}(M_3),$$
$$\mathbf{P}(\tilde{R}_1) \leq C\mathbf{P}(R_1), \qquad \mathbf{P}(\tilde{R}_2) \leq C\mathbf{P}(R_2).$$

The point here is that the constants represented by C do not depend on η and k. This is due to the fact that the regions we must fill in have size of order $3^k(\eta - 1)$.

Summarizing all the above, we now have

$$\mathbf{P}(e \in \tilde{\alpha}_2 \mid Q_1(k, \eta)) \leq C\pi_3(3^k(\eta - 1)).$$

Summing over e in $\Gamma(k, \eta)$, this gives

$$\mathbf{E}[\#\tilde{\alpha}_2 \mid Q_1(k, \eta)] \leq C\delta 3^{2k}\pi_3(3^k(\eta - 1)),$$

with $\delta = \eta - 1$. The lemma now follows by Chebyshev's inequality, and the observation that for all $1 \leq m \leq n$, one has $\pi_3(m, n) \geq C_3(n/m)^{-\alpha}$ for some $C_3 > 0$ and $\alpha \in (0, 1)$ (see Lemma 3.1). Combined with quasi-multiplicativity, it gives:

$$\pi_3(3^k(\eta - 1)) \leq (C_3(\eta - 1))^{-\alpha}\pi_3(3^k),$$

and

$$\mathbf{E}[\#\tilde{\alpha}_2 \mid Q_1(k, \eta)] \leq C\delta^{1-\alpha}3^{2k}\pi_3(3^k).$$

Choosing δ sufficiently small completes the proof of Lemma 5.8. □

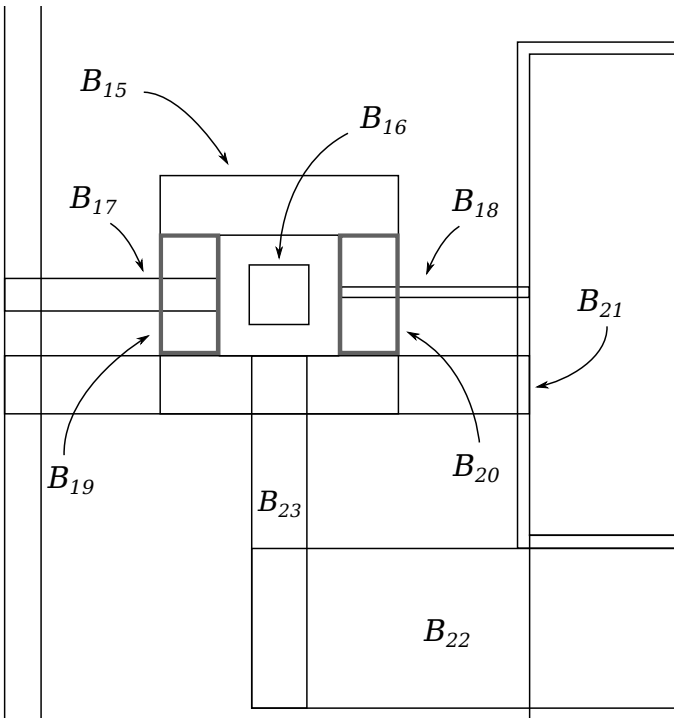

Figure 5: The boxes appearing in the derivation of the lower bound for detoured crossings. Compare to Figure 1, where some of the macroscopic connections are shown.

5.4 Lower bound for the volume of detoured crossings

We define boxes inside the annulus $B(\eta 3^k) \setminus B(3^{k-1})$, and events on which these boxes will be traversed by the open arms emanating from the origin, and contribute on the order of $(3^k)^2 \pi_3(3^k)$ edges. The boxes are centered at the midpoints of $[-3^k, -\eta 3^{k-1}]$ and $[\eta 3^{k-1}, 3^k]$, respectively. Let

$$x_0(k) = \left(-\frac{3^k + 3^{k-1}\eta}{2}, 0\right) = \left(-\frac{3^{k-1}(3+\eta)}{2}, 0\right).$$

The left "interior box" is

$$B_{15} = B\left(x_0(k), \frac{3^{k-1}}{4}(3-\eta)\right).$$

Inside B_{15}, we place a smaller box (a quarter of the size), also centered at $x_0(k)$: it is defined by

$$B_{16} = B\left(x_0(k), \frac{3^{k-1}}{16}(3-\eta)\right).$$

The boxes B_{17} and B_{18} have aspect ratios depending on $\delta = \eta - 1$. Together with B_{19} and B_{20}, they will be used to connect the 3-arm points inside B_{15} to an open crossing of

$Ann_1(k)$:

$$
\begin{aligned}
B_{17} &= [-3^k, -\frac{3^{k-1}}{2}(3+\eta) - \frac{3^{k-1}}{8}(3-\eta)] \\
&\quad \times [-\frac{3^k}{2}(\eta-1), \frac{3^k}{2}(\eta-1)], \\
B_{18} &= [-\frac{3^{k-1}}{2}(3+\eta) + \frac{3^{k-1}}{8}(3-\eta), -\eta 3^{k-1}] \\
&\quad \times [-\frac{3^{k-1}}{2}(\eta-1), \frac{3^{k-1}}{2}(\eta-1)], \\
B_{19} &= [-\frac{3^{k-1}}{2}(3+\eta) - \frac{3^{k-1}}{4}(3-\eta), -\frac{3^{k-1}}{2}(3+\eta) - \frac{3^{k-1}}{8}(3-\eta)] \\
&\quad \times [-\frac{3^{k-1}}{8}(3-\eta), \frac{3^{k-1}}{8}(3-\eta)], \\
B_{20} &= [-\frac{3^{k-1}}{2}(3+\eta) + \frac{3^{k-1}}{8}(3-\eta), -\frac{3^{k-1}}{2}(3+\eta) + \frac{3^{k-1}}{4}(3-\eta)] \\
&\quad \times [-\frac{3^{k-1}}{8}(3-\eta), \frac{3^{k-1}}{8}(3-\eta)].
\end{aligned}
$$

The remaining boxes will serve to define crossing events to connect 3-arm points in B_{16} to a closed crossing of $Ann_1(k)$.

$$
\begin{aligned}
B_{21} &= [-\eta 3^k, -3^{k-1}] \times [-\frac{3^{k-1}}{4}(3-\eta), -\frac{3^{k-1}}{8}(3-\eta)], \\
B_{22} &= [-\frac{3^{k-1}}{2}(3+\eta) - \frac{3^{k-1}}{16}(3-\eta), \frac{3^{k-1}}{2}(3+\eta) + \frac{3^{k-1}}{16}(3-\eta)] \\
&\quad \times [-3^{k-2}(3+\eta), -\eta 3^{k-1}], \\
B_{23} &= [-\frac{3^{k-1}}{2}(3+\eta) - \frac{3^{k-1}}{16}(3-\eta), -\frac{3^{k-1}}{2}(3+\eta) + \frac{3^{k-1}}{16}(3-\eta)] \\
&\quad \times [-3^{k-2}(3+\eta), -\frac{3^{k-1}}{8}(3-\eta)], \\
B_{24} &= [-3^{k-1}, 3^{k-1}] \times [-\eta 3^k, -3^{k-1}], \\
B_{25} &= B\left(x_0(k), \frac{3^{k-1}}{8}(3-\eta)\right).
\end{aligned}
$$

Let $M_5 = M_5(k)$ be the event that there are open left-right crossings of B_{17} and B_{18}. $M_6 = M_6(k)$ is defined as the event that there is a top-down closed dual crossing of B_{23}, and a left-right closed dual crossing of B_{21}. $M_7 = M_7(k)$ is the event that there are open top-down crossings of both B_{19} and B_{20}.

By Russo-Seymour-Welsh and Harris' inequality, there are positive constants $C(\eta)$ and C such that

$$
\begin{aligned}
&\mathbf{P}(M_5) \geq C(\eta), \\
&\mathbf{P}(M_6), \mathbf{P}(M_7) \geq C.
\end{aligned}
$$

We let $R_4(k)$ be the event that there is a left-right dual closed crossing of B_{22} and a top-down dual closed crossings of B_{24}. Again by RSW and Harris, we obtain

$$
\mathbf{P}(R_4(k)) \geq C > 0
$$

for some constant C independent of k and η.

We let

$$G_3(k) = G_3(k,\eta) = M_5(k) \cap M_6(k) \cap M_7(k),$$

and $G_4(k)$ is the reflection of G_3 about the $\mathbf{e}_2$-axis, as previously. The event $Q_3 = Q_3(k,\eta)$ is defined as

$$Q_3(k,\eta) = G_3(k) \cap G_4(k) \cap R_4(k). \tag{5.13}$$

By generalized FKG and independence, we have:

$$\mathbf{P}(Q_3) \geq C\mathbf{P}(M_5)^2.$$

By generalized FKG, we have

$$\mathbf{P}(Q_1 \cap Q_2 \cap Q_3) \geq C(\eta)\mathbf{P}(Q_1 \cap Q_2). \tag{5.14}$$

On the event

$$Q_4 = Q_4(k,\eta) = Q_1 \cap Q_2 \cap Q_3, \tag{5.15}$$

let W_2 be the set of $e \in B_{15}$ such that e has three disjoint arms as follows. Two of the arms are open and connected to the open crossings of B_{17} and B_{18} whose existence is guaranteed by the occurrence of M_5, respectively. One of the arms is closed, and connected to the closed vertical crossing of B_{23} in the definition of M_6. We apply the second moment method (inequality (5.18)), to the number $Z_2 = \#W_2$, conditionally on the event Q_4.

If Q_4 holds and $e \in B_{16}$ has three arms inside the rectangle $B_{17} \cup B_{19} \cup B_{25}$: one open and connected to the horizontal open crossing of B_{17}, another open arm, connected to the horizontal open crossing of B_{18}, and a closed arm connected to the vertical closed dual crossing of B_{23}, then $e \in W_2$. We denote the set of such edges $e \in B_{16}$ by W_2^*, and let $Z_2^* = \#W_2^*$. We have the following:

Proposition 5.9. *Let*

$$Q_5 = Q_5(k,c) = \{Z_2 \geq c3^{2k}\pi_3(3^k)\}. \tag{5.16}$$

There are constants $c, C_{12} > 0$ such that

$$\mathbf{P}(Q_5 \mid Q_4) \geq C_{12}. \tag{5.17}$$

for all $\delta = \eta - 1$, $n \geq 1$ and $k \in \{K(\eta), \ldots, \lfloor (C_4/4)\log n \rfloor\}$.

The important point here is that c does not depend on δ. As $\eta \downarrow 1$ (see Lemma 5.8), the size of the detour shrinks, whereas the lower bound in this proposition does not change.

Proof. Recall the Paley-Zygmund inequality: if $Z \geq 0$ a.s. and $\lambda \leq 1$, then

$$\mathbf{P}(Z \geq \lambda \mathbf{E}Z) \geq (1-\lambda)^2 \frac{(\mathbf{E}Z)^2}{\mathbf{E}Z^2}. \tag{5.18}$$

To apply this with $Z = Z_2^*$ and $\mathbb{P} = \mathbf{P}(\cdot \mid Q_4)$, we give an estimate for the expectation

$$(1/C)3^{2k}\pi_3(3^k) \leq \mathbf{E}[Z_2^* \mid Q_4], \tag{5.19}$$

and an upper bound for the second moment:

$$\mathbf{E}[(Z_2^*)^2 \mid Q_4] \leq C3^{2\cdot 2k}\pi_3(3^k). \tag{5.20}$$

For $e \in B_{16}$, let $A^*(k,e)$ be the event that e has three arms to the boundary of B_{25}: two open arms, one to each vertical side of B_{25}, and a closed arm to the middle third

of the bottom side of B_{25}. By a simple gluing construction with generalized FKG and arms-separation [23, Theorem 11], we have

$$\begin{aligned}\mathbf{P}(e \in W_2^* \mid Q_4) &\geq C\mathbf{P}(A^*(k,e)) \\ &\geq C\mathbf{P}(A_3(e,3^k)),\end{aligned}$$

Summing over $e \in B_{16}$, we obtain

$$\mathbf{E}[Z_2^* \mid Q_4] \geq C3^{2k}\pi_3(3^k).$$

This gives the lower bound in (5.19).

We now estimate the second moment. A similar argument gives the upper bound for the first moment. For simplicity of notation, let

$$m = \frac{3^{k-1}}{16}(3-\eta).$$

$$\begin{aligned}\mathbf{E}[(Z_2^*)^2 \mid Q_4] &= \sum_{e_1,e_2 \in B_{16}} \mathbf{P}(e_1 \in W_2^*, e_2 \in W_2^* \mid Q_4) \\ &\leq \sum_{e_1,e_2 \in B_{16}} \mathbf{P}\left(A_3(e_1,m), A_3(e_2,m)\right),\end{aligned}$$

where in the second step we have used that Q_4 is independent of the status of edges inside B_{25}.

The last double sum is decomposed following an idea of Nguyen [22]:

$$\sum_{e_1}\sum_{d=1}^{2m}\sum_{|e_1-e_2|_\infty=d} \mathbf{P}(A_3(e_1,m), A_3(e_2,m)). \tag{5.21}$$

For $k \leq l$, let $A_3(e,k,l)$ be the probability that there are 3 arms from $\partial B(e,k)$ to $\partial B(e,l)$. Note that

$$\mathbf{P}(A_3(e,k,l)) = \pi_3(k,l),$$

the three-arm probability (open, open, closed) for connection across the annulus $B(l) \setminus B(k)$. For convenience, if $l < k$, we define $A_3(e,k,l)$ to be the entire sample space; that is, $A_3(k,l)$ always holds. Correspondingly, we let $\pi_3(k,l) = 1$ in this case. Then

$$\begin{aligned}\mathbf{P}(A_3(e_1,m), A_3(e_2,m)) &\leq \mathbf{P}(A_3(e_1,d/2), A_3(e_1,3d/2,m), A_3(e_2,d/2)) \\ &\leq \mathbf{P}(A_3(e_1,d/2))\mathbf{P}(A_3(e_1,3d/2,m))\mathbf{P}(A_3(e_2,d/2)) \\ &= \pi_3(d/2)\pi_3(3d/2,m)\pi_3(d/2).\end{aligned}$$

Returning to the sum, we find, for each e_1, the bound

$$\sum_{d=1}^{\lfloor 2m/3 \rfloor} 8d\pi_3(d/2)\pi_3(3d/2,m)\pi_3(d/2) + \sum_{d=\lfloor 2m/3 \rfloor+1}^{2m} 8d\pi_3(d/2)\pi_3(d/2).$$

Now we use RSW theory to rescale some of these quantities by constant factors. First, we have

$$\pi_3(d/2) \leq C\pi_3(d).$$

If $d \geq 2m/3$, we also obtain $\pi_3(d/2) \leq C\pi_3(m)$. By quasimultiplicativity [23, Proposition 12.2]:

$$\pi_3(d/2)\pi_3(3d/2,m) \asymp \pi_3(m).$$

Putting all this back into the sum (5.21), we find the bound

$$m\pi_3(m)\sum_{d=1}^{m}\pi_3(d).$$

This is bounded by $m^2\pi_3(m)^2$. To see why, choose (by Lemma 3.1) $\beta \in (0,1)$ such that $\pi_3(d,m) \geq C(d/m)^\beta$, and use quasimultiplicativity:

$$\sum_{d=1}^{m}\pi_3(d) \asymp \pi_3(m)\sum_{d=1}^{m}\pi_3(d,m)^{-1} \leq C\pi_3(m)m^\beta\sum_{d=1}^{m}d^{-\beta} \leq Cm\pi_3(m)\ .$$

Summing over e_1, we obtain an overall bound

$$\mathbf{E}[(Z_2^*)^2 \mid Q_4] \leq Cm^4(\pi_3(m))^2 \leq C3^{2\cdot 2k}(\pi_3(3^k))^2.$$

A similar, but simpler argument gives the estimate

$$\mathbf{E}[Z_2^* \mid Q_4] \leq Cm^2(\pi_3(m))^2 \leq C3^{2k}\pi(3^k),$$

concluding the proof. □

5.5 Definition of E_k and Proof of Proposition 5.2

We can now give the definition of the events E_k:

Definition 5.10. *Let $\epsilon > 0$ and $\eta = \eta(\epsilon)$ be such that* (5.12) *in Lemma 5.8 is satisfied. Then*

$$E_k = E_k(\eta,\epsilon) = Q_1 \cap Q_2 \cap Q_3 \cap Q_5 \text{ for } k \geq K(\eta), \tag{5.22}$$

where Q_1 is defined in (5.8)*, Q_2 in* (5.11)*, Q_3 in* (5.13)*, Q_5 appears in* (5.16)*, and $K(\eta)$ is defined below* (5.6)*.*

Proof of Proposition 5.2. Combining (5.14), Lemma 5.8, Proposition 5.9 and inequality (5.10), there is a constant $C_{13}(\epsilon)$ such that

$$\mathbf{P}(E_k) \geq C_{13}(\epsilon) > 0 \text{ for } k \geq K(\eta)\ . \tag{5.23}$$

To derive the lower bound

$$\mathbf{P}(E_k \mid A_3(d)) \geq C_7(\epsilon) > 0, \tag{5.24}$$

for $k \geq K(\eta)$ such that $\eta 3^k \leq d$, we use a gluing construction and arms separation [23, Theorem 11], together with the equivalence [23, Proposition 12, 2.]

$$\mathbf{P}(A_3(3^{k-1}))\mathbf{P}(A_3(\eta 3^k, d)) \asymp \mathbf{P}(A_3(d)).$$

Note that the definition of E_k implies $A_3(3^{k-1}, \eta 3^k)$ and that the connections across the annulus are easily extended. □

5.6 Proof of Proposition 5.1

Proof. We will show that if the event

$$E_k(e) \cap \{e \in \hat{\gamma}_n\}$$

occurs for $k \in \{K(\eta), \ldots, \lfloor (C_4/8)\log n\rfloor\}$, then there is a "short" detour around the origin in the sense that

$$\mathcal{S}(e) \neq \emptyset.$$

On $E_k(e)$, there is a closed circuit $\mathcal{C}_2$ with two defects near the five-arm points in B_1 and B_1' inside Ann_2. We denote these (unique) points by x and y. Since e lies on the open, self-avoiding, circuit γ_n, the latter must pass through each of the two five-arm points in the definition of M_1, resp. M_1', exactly once. We denote the portion of γ_n between x and y, and inside $\mathcal{C}_2$, by q. We also let

$$p = \tilde{\alpha}_2,$$

where $\tilde{\alpha}_2$ is from Definition 5.6.

Claim 5.11. *On $E_k(e) \cap \{e \in \hat{\gamma}_n\}$, e^* has a closed connection to the bottom of $B(e, \eta 3^k)$ and to the bottom arc of the closed circuit $\mathcal{C}_2$ with defects in Ann_2.*

Proof. Recall (from (5.9)) that $\Gamma(k,\eta) = Ann_2 \cap \Big(\mathbb{R} \times [-\frac{3^k}{2}(\eta - 1), \infty)\Big)$. The open (detour) arc σ_o in $\Gamma(k,\eta)$ between the two five-arm points and the closed arc σ_c through $Ann_2 \setminus \Gamma(k,\eta)$ form a circuit around e (we can connect them by two line segments of length $1/\sqrt{2}$ to make their union a closed curve). The closed arm emanating from e^* reaches $\partial B(e, \eta 3^k)$, so it must intersect σ_c, which intersects the closed vertical crossing of B_{24} in the definition of the event $R_4(k) \supset Q_2$. This crossing is connected to the bottom side of $B(e, \eta 3^k)$. □

It is important to note that since the closed arm from e^* intersects the bottom of $B(e, \eta 3^k)$ and $e \in \hat{\gamma}_n$, the bottom part of the closed circuit in $Ann_1 \setminus \Gamma(k,\eta)$ must be connected to $B(n)$. This is what forces the "orientation" of the box $B(e,d)$ to be such that p is indeed a detour off the innermost circuit (see Lemma 5.13 below).

Claim 5.12. *The open arc p is disjoint from q except for the five-arm points x and y.*

Proof. This follows from the definition of the five arm events. From this, we obtain the existence of an open crossing α of $B(e, \eta 3^k)$ inside

$$B_{17} \cup B_{15} \cup B_{18} \cup B(e, \eta 3^{k-1}) \cup B_{17}' \cup B_{15}' \cup B_{18}'.$$

whose only intersection with the outermost arc p is x and y. By the previous claim, every dual edge touching q is connected to the bottom of $B(e, \eta 3^k)$ by a closed dual path. This implies that q lies in the region below the Jordan curve α, which separates the box $B(\eta 3^k)$ into two connected components. In particular, q is disjoint from p, except at its endpoints. □

It follows that $p \cup q$ is a Jordan curve lying entirely inside the box $B(e, \eta 3^k)$. This in turn implies

Lemma 5.13. *p lies outside* $\mathrm{int}\gamma_n$.

Proof. The dual edge $\{x - (1/2)(\mathbf{e}_1 + \mathbf{e}_2), x - (1/2)(\mathbf{e}_1 - \mathbf{e}_2)\}$ crosses γ_n and so one endpoint is in each component of the complement of γ_n. The top endpoint can be connected to p and the bottom one can be connected to the bottom arc of $\mathcal{C}_2$, both without crossing γ_n. By Claim 5.11, the bottom is in the interior of γ_n, so p is in the exterior of γ_n. □

We can now prove Proposition 5.1, by setting $K(\epsilon) = K(\eta)$. Letting $P = p$ and $Q = q$, $w_0 = x$, and $w_M = y$, Lemma 5.13 implies that Condition 1. in Definition 3.2 is satisfied. Condition 2. holds by the definition of the five-arm points x and y (Definition 5.3). Condition 3. follows because $P \cup Q = p \cup q$ is contained in the box $B(e, \eta 3^k)$, which does not contain the origin. Condition 4. follows from the existence of the closed dual arc $\tilde{\alpha}_1$, which is implied by the event Q_1 (5.8). Condition 5. holds because of the conjunction of Q_2 (5.11) and Q_5 (5.16) (choose $\epsilon \cdot c$ for ϵ, where c is from the definition of Q_5). □

6 Proof of inequality (3.5)

Our goal in this section is to derive the estimate (3.5). We recall it here:

$$\mathbf{P}(\mathcal{S}(e) = \emptyset \mid e \in \hat{\gamma}_n) < \epsilon^2 \tag{6.1}$$

for some $n \geq C_5$. Once we show this estimate, Lemmas 3.4 and 3.5, and Proposition 3.3, we can conclude Theorem 2.2.

For $k = K(\eta), ..., \lfloor (C_4/8) \log n \rfloor$, we let $E_k(e) = \tau_{-e} E_k$ be the "detour event" inside the annulus $Ann_1(e,k) = B(e, \eta 3^k) \setminus B(e, 3^{k-1})$. (Here η is slightly bigger than 1 and $K = K(\eta)$ is a constant depending on η and which is defined under (5.6).) It is defined precisely in Section 5 (see Definition 5.10). The property we need here is proved in Proposition 5.1: if $E_k(e)$ occurs and $e \in \hat{\gamma}_n$, then $\mathcal{S}(e) \neq \emptyset$. Thus,

$$\mathbf{P}(\mathcal{S}(e) = \emptyset \mid e \in \hat{\gamma}_n) \leq \mathbf{P}((\cup_{k=K}^{\lfloor C_4/8 \log n \rfloor} E_{2k}(e))^c \mid e \in \hat{\gamma}_n). \tag{6.2}$$

6.1 Conditioning on 3-arm event in a box

The next step is to replace the conditioning in (6.2) by conditioning on a "three arm" event:

Proposition 6.1. *There is a constant C such that*

$$\mathbf{P}(\cap_{k=K}^{\lfloor C_4/8 \log n \rfloor} E_{2k}(e)^c \mid e \in \hat{\gamma}_n) \leq C \mathbf{P}(\cap_{k=K}^{\lfloor C_4/8 \log n \rfloor} E_{2k}(e)^c \mid A_3(e, \mathrm{dist}(e, \partial A(n))), \tag{6.3}$$

where $A_3(e,m)$ is the probability that e has three arms, two open and one closed, to distance m from e.

We will omit some details, since most of the arguments are lengthy but standard. To prove (6.3), we use a gluing construction that depends on the position of e inside the annulus $A(n)$, which we split into a number of different regions:

$$A(n) = A \cup B \cup C \cup D \cup E.$$

Region B is the disjoint union of four rectangles:

$$\begin{aligned} B = {} & [-2n, 2n] \times [\frac{5}{2}n, 3n] \cup [-2n, 2n] \times [-3n, -\frac{5}{2}n] \\ & \cup [-3n, -\frac{5}{2}n] \times [-2n, 2n] \cup [\frac{5}{2}n, 3n] \times [-2n, 2n]. \end{aligned}$$

Region A is

$$A = (B(3n) \setminus B(5/2n)) \setminus B.$$

Region C is given by

$$C = B(5n/2) \setminus B(3n/2).$$

Region D is

$$\begin{aligned} D = {} & [-n/2, n/2] \times [n, 3n/2] \cup [-n/2, n/2] \times [-3n/2, -n] \\ & \cup [-3n/2, -n] \times [-n/2, n/2] \cup [n, 3n/2] \times [-n/2, n/2]. \end{aligned}$$

Finally, region E is given by

$$E = (B(3n/2) \setminus B(n)) \setminus D.$$

In each case, we use an adapted gluing construction to connect e to $B(n)$ by a closed dual path inside an open circuit around $B(n)$. Figure 6 depicts the partitioning we will use.

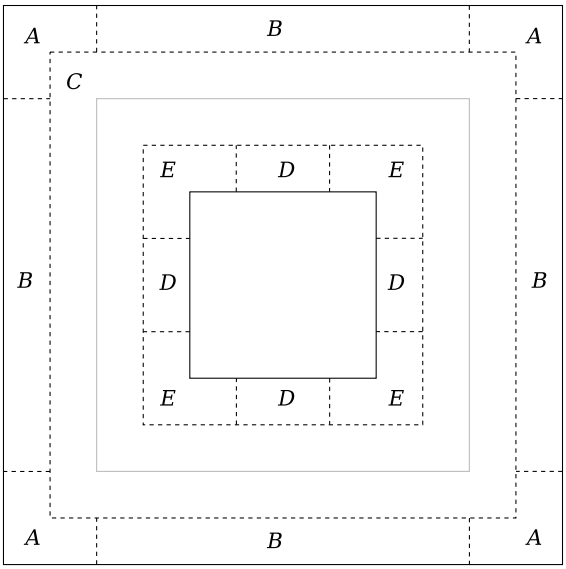

Figure 6: The annulus $A(n)$ is split in a number of regions.

We concentrate on the proof of (6.3) in case $e \in A$. Furthermore, we assume by symmetry that e is in the top-right component of A. We only consider $e \in \hat{\gamma}_n$, so

$$d(e) = \mathrm{dist}(e, \partial A(n)) \geq n^{C_4/2}. \tag{6.4}$$

Since $e \in A$, we have $d(e) = \mathrm{dist}(e, \partial B(3n))$.

Denote by

$$\mathcal{L}(e) \in \{\{3n\} \times [-3n, 3n], [-3n, 3n] \times \{3n\}\},$$

the side of $\partial B(3n)$ such that $d(e) = \mathrm{dist}(e, \mathcal{L}(e))$. If there is more than one possible choice, choose the earliest in the list above.

Let $B(e)$ be the box of side length $2d(e)$ centered at e. We define

$$d'(e) = \mathrm{dist}(e, \partial B(3n) \setminus \mathcal{L}(e)),$$

and

$$\mathcal{H}(e) = B\left(e^L + d'(e) \cdot \frac{e - e_L}{|e - e_L|}, d'(e)\right),$$

where e^L is the projection of e onto $\mathcal{L}(e)$, and let $\mathcal{K}(e)$ be the box

$$\mathcal{K}(e) = [n, 3n] \times [n, 3n],$$

which was chosen based on the assumed placement of e.

We now have

Lemma 6.2. *If $e \in \hat{\gamma}_n$, then the event $F_1(e)$ occurs: There are two open paths and one closed dual path joining $\partial B(e) \setminus \partial B(3n)$ to $\partial \mathcal{H}(e) \setminus \partial B(3n)$ inside $\mathcal{H}(e) \setminus B(e)$, appearing in the order*

$$\textit{open}, \textit{closed}, \textit{open} \tag{6.5}$$

(on the boundary of $\mathcal{H}(e)$). In particular, the closed path is separated from $\partial B(3n)$ by the two open paths.

Similarly, the event $F_2(e)$ occurs: there are two open paths and one closed dual path from $\partial \mathcal{H}(e) \setminus \partial B(3n)$ to $\partial \mathcal{K}(e) \setminus \partial B(3n)$ inside $\mathcal{K}(e) \setminus \mathcal{H}(e)$. These paths also appear in the order (6.5)*, with the closed path separated from $\partial B(3n)$ by the open paths.*

In this lemma, we define $F_1(e)$ to be the sure event (that is, the entire sample space) if $8d(e) > d'(e)$ and we define $F_2(e)$ to be the sure event if $4d'(e) > n$. This is to guarantee that later in the proof, there is enough room between the boxes $B(e)$ and $\mathcal{H}(e)$ (or between $\mathcal{H}(e)$ and $\mathcal{K}(e)$) to do arm separation arguments.

Proof. If $e \in \gamma_n$ then e belongs to an open circuit surrounding $B(n)$ in $A(n)$. Moreover, e^* is connected to $\partial B(n)$ by a closed path contained in the interior $\mathrm{int}\gamma_n$ of the circuit.

Let r_1 be the portion of the open circuit $\hat{\gamma}_n$ obtained by traversing the circuit in one direction from e, until first time it exits $\mathcal{H}(e)$. Call $a(e) \in \partial\mathcal{H}(e)$ the point of exit. Let r_2 be the portion of γ_n obtained by traversing the circuit in the other direction, until it first exits $\mathcal{H}(e)$, at a point $b(e)$. The curve γ_e contained in $\mathcal{H}(e)$ joining $a(e)$ to $b(e)$ separates $\mathcal{H}(e)$ into two regions, each bounded by the curve γ_e and a portion of $\partial\mathcal{H}(e) \setminus \partial B(3n)$. Exactly one of these regions, $R(e)$, say, lies inside the circuit γ_n, and hence contains the portion of the closed dual path from e to $\partial B(n)$ until it first exists $\mathcal{H}(e)$. Following this path from e until this exit point, we obtain a closed dual path whose endpoint $c(e)$ must lie on $\partial\mathcal{H}(e)$, between $a(e)$ and $b(e)$. Traversing r_1 from $a(e)$ and r_2 from $b(e)$ toward e until the first time they enter $B(e)$, we obtain two points $a'(e)$ and $b'(e)$ on $\partial B(e) \setminus \partial B(3n)$. Following the closed path backwards similarly, we find a point $c'(e)$ lying between $a'(e)$ and $b'(e)$ on $\partial B(e) \setminus \partial B(3n)$.

The proof for the paths in $\mathcal{K}(e) \setminus \mathcal{H}(e)$ is similar. □

Returning to the probability in (6.3), write:

$$\begin{aligned}
&\mathbf{P}(\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}(e)^c \mid e \in \hat{\gamma}_n) \\
&= \frac{1}{\mathbf{P}(e \in \hat{\gamma}_n)} \mathbf{P}(\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}(e)^c, e \in \hat{\gamma}_n) \\
&\le \frac{1}{\mathbf{P}(e \in \hat{\gamma}_n)} \mathbf{P}\left(\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}(e)^c, F_1(e), F_2(e), A_3(e, d(e))\right)
\end{aligned}$$

Note that the event

$$\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}^c(e) \tag{6.6}$$

depends only on edges inside $B(e, n^{C_4/4}) \subset B(e, n^{C_4/2})$, so $F_1(e)$ and $F_2(e)$ are independent of (6.6), so we have

$$\begin{aligned}
&\mathbf{P}(\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}(e)^c, F_1(e), F_2(e), A_3(e, d(e))) \\
&\le \mathbf{P}(F_1(e))\mathbf{P}(F_2(e))\mathbf{P}(\cap_{k=K}^{\lfloor C_4/8\log n\rfloor} E_{2k}(e)^c, A_3(e, d(e))).
\end{aligned}$$

Proposition 6.1 now follows from the next estimate:

Lemma 6.3. *There is a constant C independent of n and e such that*

$$\mathbf{P}(e \in \hat{\gamma}_n) \ge \frac{1}{C}\mathbf{P}(F_1(e))\mathbf{P}(F_2(e))\mathbf{P}(A_3(e, d(e))), \tag{6.7}$$

Proof. We first introduce two events that will serve to complete a circuit around $A(n)$, once connected to the open arms coming out of e. Let $\mathcal{C}_1$ be the event that there are open crossings along the long sides of the rectangles

$$\begin{aligned}
[-3n, n] \times [5n/2, 3n],\ [-3n, -5n/2] \times [-3n, 3n], \\
[-3n, 3n] \times [-3n, -5n/2],\ [5n/2, 3n] \times [-3n, n].
\end{aligned}$$

Let $\mathcal{C}_2$ be the event that there are top-down and left-right closed crossings of the rectangle:

$$[3n/4, n] \times [n, 3n/2].$$

By the Russo-Seymour-Welsh theorem, the Harris inequality and independence, there is a positive C_6 independent of n such that

$$\mathbf{P}(\mathcal{C}_1), \mathbf{P}(\mathcal{C}_2) \geq C_6. \tag{6.8}$$

To connect the partial circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ into a circuit containing the edge e, we use a standard arms separation argument (see for example [15, Lemma 4], [23, Theorem 11]), which allows us to specify landing areas on $\partial B(e)$, $\partial\mathcal{H}(e)$ and $\partial\mathcal{K}(e)$ for the arms in events $F_1(e)$ and $F_2(e)$, while not modifying the probability of these events by more than a constant factor. The conclusion (6.7) is then obtained using the generalized FKG inequality.

To define the modified arm events, we need to specify regions ("landing zones") that will contain the endpoints. For this, we divide the left side of $\partial B(e)$ into four vertical segments, which we label from top to bottom: I_1, I_2, I_3, I_4 of equal length $d(e)/2$. The bottom side of $\partial B(e)$ is also divided into three horizontal segments of equal size, which we label according to their position from left to right: I_1',I_2', I_3' and I_4'. We proceed similarly with the left side of $\partial\mathcal{H}(e)$, which we also divide into four parts J_1, J_2, J_3, J_4 of equal size, labeled from top to bottom. The bottom side of $\partial\mathcal{H}(e)$ is also divided into four parts of equal size: J_1', J_2', J_3' and J_4', labeled from left to right. I_4 and I_1' intersect at the lower left corner of $B(e)$; J_4 and J_1' intersect at the lower left corner of $\mathcal{H}(e)$. Note also that

$$J_1 \subset [-3n, 3n] \times [5n/2, 3n]$$
$$J_4' \subset [5n/2, 3n] \times [-3n, 3n].$$

$\tilde{A}_3(e, d(e))$ is the event that $A_3(e, d(e))$ occurs, one of the open arms from e has its other endpoint in I_2, and the other arm has its endpoint in I_2'. The closed arm has its endpoint in I_4^*. $\tilde{F}_1(e)$ is the event that $F_1(e)$ occurs, one of the open arms having its endpoints in I_2 and J_1, respectively, and the other open arm having endpoints in I_2' and J_4'. Moreover, we require the closed arm to have its endpoints in I_4^* and J_4^*. $\tilde{F}_2(e)$ is the event that $F_2(e)$ occurs, one open arm has endpoints in J_1 and $\{n\} \times [5n/2, 3n]$, and the other in J_4' and $[5n/2, 3n] \times \{n\}$. Finally the closed dual arm is required to have one endpoint in J_4^*, and the other in $\{n\} \times [n, 3n/2]$.

On $\tilde{A}_3(e, d(e)) \cap \tilde{F}_1(e) \cap \tilde{F}_2(e)$, standard gluing techniques allow us connect each of the open arms the definition of $\tilde{F}_1(e)$ to one end of the open arc in the event $\mathcal{C}_1$, and the closed arm to the vertical crossing of $[3n/4, n] \times [n, 3n/2]$ appearing in the definition of the event $\mathcal{C}_2$, and to connect the arms in each of the three events to the arms with endpoints in the same region. Combined with the generalized FKG inequality, this gives:

$$\mathbf{P}(e \in_n) \geq C_6^2 C \mathbf{P}(\tilde{F}_1(e)) \mathbf{P}(\tilde{F}_2(e)) \mathbf{P}(\tilde{A}_3(e, d(e))), \tag{6.9}$$

where C is independent of n and C_6 appears in (6.8).

By arms separation, [23, Theorem 11], we have

$$\mathbf{P}(\tilde{A}_3(e, d(e))) \geq C \mathbf{P}(A_3(e, d(e))). \tag{6.10}$$

An argument similar to the proof of [23, Theorem 11] (see also the proof of [15, Lemma 4]) gives the existence of a constant C independent of n, such that:

$$\mathbf{P}(\tilde{F}_1(e)) \geq C \mathbf{P}(F_1(e)) \tag{6.11}$$
$$\mathbf{P}(\tilde{F}_2(e)) \geq C \mathbf{P}(F_2(e)). \tag{6.12}$$

It is important here that the arms in the definition of $F_1(e)$ and $F_2(e)$ appear in a definite order, as guaranteed by Lemma 6.2.

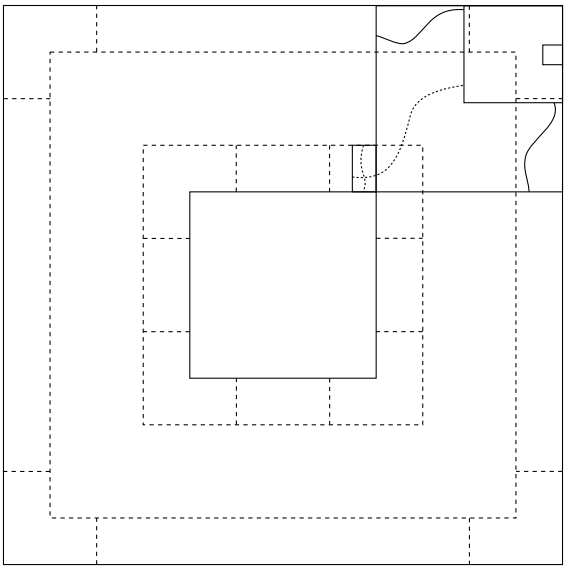

Figure 7: A sketch of the construction inside $\mathcal{K}(e)$ in case e lies in the top right component of A (see Lemma 6.2).

Combining (6.10), (6.11), (6.12) and (6.9), we obtain (6.7) in the case where e lies in the upper right part of the region A. Similar gluing constructions also apply in the other cases. This completes the proof of Lemma 6.3, and thus of Proposition 6.1. □

6.2 Arms separation conditional on A_3

By Proposition 6.1, we have, for any $e \in \hat{A}(n)$:

$$\mathbf{P}(S(e) = \emptyset \mid e \in \hat{\gamma}_n) \leq C\mathbf{P}(\cap_{k=K}^{\lfloor C_4/8 \log n \rfloor} E_{2k}(e)^c \mid A_3(e, d(e))), \tag{6.13}$$

where $d(e)$ is defined in (6.4) and $K = K(\eta)$ is defined below (5.6). The events E_{2k} depend on disjoint sets of edges, and each occurs with probability bounded below independently of k (see (5.23)), so we expect an estimate of the form (6.1).

However, we must ensure that the conditioning on the three arm event $A_3(e, d(e))$ does not affect the probability of occurrence of the E_{2k}'s too drastically. To state our result, let $d = 3^m$ be a (large) integer. We will later take $m = \lfloor \log_3 d(e) \rfloor$. From Proposition 5.2, we have:

$$\mathbf{P}(E_{2k} \mid A_3(d)) \geq C_7 > 0 \text{ for } K(\eta) \leq 2k \leq \lfloor (C_4/8) \log n \rfloor. \tag{6.14}$$

This is the lower bound (5.24). From the definition of E_{2k} (Definition 5.10), we also have

$$E_{2k} \text{ depends only on edges in } B(3^{2k+1}) \setminus B(3^{2k-1}). \tag{6.15}$$

Set $m_n = \lfloor (C_4/8) \log n \rfloor$. Our goal will be to prove that

$$\mathbf{P}(\cap_{k=K}^{m_n} E_{2k}^c \mid A_3(N)) \to 0 \text{ as } n \to \infty \tag{6.16}$$

uniformly in $N \geq n^{C_4/2}$. Given this result, we find by translation invariance

Proposition 6.4. *As $n \to \infty$,*

$$\mathbf{P}(S(e) = \emptyset \mid e \in \hat{\gamma}_n) \to 0$$

uniformly for $e \in \hat{A}(n)$.

We first state and prove the following claim, which will be used later in the proof of Proposition 6.4:

Claim 6.5. *For a sequence of integers*

$$3^{2K+1} < i_1 < i_2 < \ldots < i_k < \ldots$$

let B_k be the event that there exists a closed dual circuit with exactly two defects (that is, two edges that are open) around 0 in $Ann(i_k, i_{k+1})$, and let

$$k_N = \max\{k : i_{k+1} < N\}.$$

Furthermore, let $\hat{B}_k$ be the event that there exists an open circuit with one defect around 0 in $Ann(i_k, i_{k+1})$. Given $\epsilon > 0$, there is a choice of $i_1, i_2, \ldots$ such that $i_{k+1} > 3^6 i_k$ for all k and

$$\mathbf{P}(\hat{B}_1^c \cup (\cup_{k=2}^{k_N} B_k^c) \mid A_3(N)) < \epsilon \tag{6.17}$$

for all N.

Proof. By quasimultiplicativity, we can choose C_{14} such that for all $m_1 < m_2 < N$,

$$\mathbf{P}(A_3(m_1))\mathbf{P}(A_3(m_1, m_2))\mathbf{P}(A_3(m_2, N)) \leq C_{14}\mathbf{P}(A_3(N)).$$

For any sequence $i_1 < i_2 < \ldots$, let

$$\alpha_k = \mathbf{P}(\text{there is an open crossing or a closed dual crossing of } Ann(i_k, i_{k+1})).$$

Choose the sequence $(i_k)_{k\geq 1}$ such that

$$\sum_{k=1}^{\infty} \alpha_k \leq \epsilon / C_{14}.$$

Then, estimate

$$\begin{aligned}\mathbf{P}(A_3(N), \hat{B}_1^c \cup (\cup_{k=2}^{k_N} B_k^c)) &\leq \mathbf{P}(A_3(N), \hat{B}_1^c) + \sum_{k=2}^{k_N} \mathbf{P}(A_3(N), B_k^c) \\ &\leq \mathbf{P}(A_3(i_1))\mathbf{P}(A_3(i_1, i_2), \hat{B}_1^c)\mathbf{P}(A_3(i_2, N)) \\ &\quad + \sum_{k=2}^{k_N} \mathbf{P}(A_3(i_k))\mathbf{P}(A_3(i_k, i_{k+1}), B_k^c)\mathbf{P}(A_3(i_{k+1}, N)). \end{aligned} \tag{6.18}$$

If $A_3(i_k, i_{k+1})$ occurs but B_k does not occur, then there must be an open crossing of $Ann(i_k, i_{k+1})$ that is disjoint from the three crossings from $A_3(i_k, i_{k+1})$. A similar statement holds for $\hat{B}_1$. Therefore, by the van den Berg-Fiebig inequality [4], (6.18) is bounded by

$$\begin{aligned}&\sum_{k=1}^{k_N} \alpha_k \cdot \mathbf{P}(A_3(i_k))\mathbf{P}(A_3(i_k, i_{k+1}))\mathbf{P}(A_3(i_{k+1}, N)) \\ &\leq C_{14}\mathbf{P}(A_3(N)) \sum_{k=1}^{k_N} \alpha_k \leq \epsilon \cdot \mathbf{P}(A_3(N)).\end{aligned}$$

□

Proof of Proposition 6.4. Choose $i_1 < i_2 < \cdots$ from the previous claim corresponding to $\epsilon/2$. For $k_N \geq 3$, consider the event

$$C_N = \hat{B}_1 \cap (\cap_{k=2}^{k_N} B_k).$$

The first step is to show that there is a constant $c_1 > 0$ such that for all N large, all k satisfying $3 \le k \le k_N - 3$, E any event depending on the state of edges in $B(i_k)$, and F any event depending on the state of edges in $B(i_{k+4})^c$, then

$$\mathbf{P}(F \mid A_3(N), C_N, E) \ge c_1 \mathbf{P}(F \mid A_3(N), C_N) \ . \tag{6.19}$$

On the event $C_N \cap A_3(N)$, there is an innermost closed dual circuit with two defects in each annulus $Ann(i_k, i_{k+1})$. (This circuit is also vertex self-avoiding.) For a given dual circuit $\mathcal{C}$ in $Ann(i_k, i_{k+1})$ with edges e_1, e_2 on $\mathcal{C}$, we let $Circ_k(\mathcal{C})$ be the event that $\mathcal{C}$ is the innermost closed dual circuit with defects e_i. Generally, if $A_3(N)$ does not occur, then the event $Circ_k(\mathcal{C})$ means that $\mathcal{C}$ is closed, e_1 and e_2 are open, and there is no closed circuit with two defects around 0 in $Ann(i_k, i_{k+1})$ entirely contained in the union of $\mathcal{C}$ with its interior.

Then we can decompose

$$\mathbf{P}(F, E, A_3(N), C_N) = \sum_{\mathcal{C},\mathcal{D}} \mathbf{P}(F, E, A_3(N), Circ_k(\mathcal{C}), Circ_{k+3}(\mathcal{D}), C_N) \ . \tag{6.20}$$

To decouple, we must introduce three events that build $A_3(N)$. Every closed dual circuit $\mathcal{C}$ or $\mathcal{D}$ above contains two arcs between its defects. Given a deterministic ordering of all arcs and a dual circuit $\mathcal{C}$, let $\mathcal{A}_i(\mathcal{C})$ be the i-th arc of $\mathcal{C}$ in this ordering. For $i = 1, 2$, let $X_-(\mathcal{C}, i)$ be the event that 0 is connected to e_1 and e_2 by disjoint open paths in the interior of $\mathcal{C}$, and to $\mathcal{A}_i(\mathcal{C})$ by one closed dual path in the interior of $\mathcal{C}$. Let C_N^{k-} be the event that $\hat{B}_1 \cap (\cap_{i=2}^{k-1} B_i)$ occurs. For $i, j = 1, 2$, let $X_0(\mathcal{C}, \mathcal{D}, i, j)$ be the event that e_l is connected to f_l (for $l = 1, 2$), where the f_l's are two edges on $\mathcal{D}$, by an open path in the region between $\mathcal{C}$ and $\mathcal{D}$ (not including $\mathcal{C}$) such that these paths are disjoint and $\mathcal{A}_i(\mathcal{C})$ is connected to $\mathcal{A}_j(\mathcal{D})$ by a closed dual path in this same region. Let $X_+(\mathcal{D}, j)$ be the event that f_1 and f_2 are connected to $\partial B(N)$ by disjoint open paths in the exterior of $\mathcal{D}$ and $\mathcal{A}_j(\mathcal{D})$ is connected to $\partial B(N)$ by a closed dual path in the exterior of $\mathcal{D}$. Also let C_N^{k+} be the event that $\cap_{i=k+4}^{k_N} B_i$ occurs.

Then (6.20) becomes by independence,

$$\begin{aligned} &\sum_{\mathcal{C},\mathcal{D}} \sum_{i,j} \mathbf{P}(F, E, X_-(\mathcal{C}, i), X_0(\mathcal{C}, \mathcal{D}, i, j), X_+(\mathcal{D}, j), Circ_k(\mathcal{C}), Circ_{k+3}(\mathcal{D}), C_N) \\ = &\sum_{\mathcal{C},\mathcal{D}} \sum_{i,j} \mathbf{P}(E, X_-(\mathcal{C}, i), Circ_k(\mathcal{C}), C_N^{k-}) \ \mathbf{P}(X_0(\mathcal{C}, \mathcal{D}, i, j), Circ_{k+3}(\mathcal{D}), B_{k+1}, B_{k+2}) \\ &\times \mathbf{P}(X_+(\mathcal{D}, j), F, C_N^{k+}) \ . \end{aligned} \tag{6.21}$$

The effect of this decoupling will be to "reset" the system outside of the outer circuit $\mathcal{D}$, so that the event E no longer significantly affects the occurrence of F. Intuitively speaking, E could affect the system by biasing certain circuits $\mathcal{C}$ to appear in $Ann(i_k, i_{k+1})$, and these could change the conditional probability of F. However, a lemma from [7, Lemma 6.1] below will show that the second circuit $\mathcal{D}$ will mostly remove this possible bias and allow the system to start fresh. We give here a modification of that lemma. We omit the proof, which uses extensions of arm separation techniques developed by Kesten and is essentially the same proof as that found in [7]:

Lemma 6.6. *Consider dual circuits $\mathcal{C}$ in $Ann(i_k, i_{k+1})$, $\mathcal{D}$ in $Ann(i_{k+3}, i_{k+4})$, edges e_1, e_2 on $\mathcal{C}$ and f_1, f_2 on $\mathcal{D}$ respectively. For $i, j = 1, 2$, let $P(\mathcal{C}, \mathcal{D}, i, j)$ be the probability, conditional on the event that all edges in $\mathcal{C} \setminus \{e_1, e_2\}$ are closed and e_1, e_2 are open, that (1) there are disjoint open paths from e_i to f_i in the region between $\mathcal{C}$ and $\mathcal{D}$ (not including $\mathcal{C}$), (2) there is a closed dual path from $\mathcal{A}_i(\mathcal{C})$ to $\mathcal{A}_j(\mathcal{D})$ in the region between*

$\mathcal{C}$ and $\mathcal{D}$ (not including $\mathcal{C}$), (3) $\mathcal{D}$ is the innermost closed dual circuit with defects f_1, f_2 around 0 in $Ann(i_{k+3}, i_{k+4})$, and (4) $B_{i_{k+1}} \cap B_{i_{k+2}}$ occurs. We similarly define $\mathcal{C}', \mathcal{D}', i', j'$, etc. There exists a finite constant C_8 (it does not depend on the particular choice of circuits, defects, or i, i', j, j') such that

$$\frac{P(\mathcal{C}, \mathcal{D}, i, j)P(\mathcal{C}', \mathcal{D}', i', j')}{P(\mathcal{C}, \mathcal{D}', i, j')P(\mathcal{C}', \mathcal{D}, i', j)} < C_8 \ .$$

To apply the lemma, we rewrite the probability in (6.20), (6.21) as:

$$\begin{aligned}
&\mathbf{P}(E, F, A_3(N), C_N) \\
&= \sum_{\mathcal{C},\mathcal{D}} \sum_{i,j} \mathbf{P}(E, X_-(\mathcal{C}, i), Circ_k(\mathcal{C}), C_N^{k-}) P(\mathcal{C}, \mathcal{D}, i, j) \mathbf{P}(X_+(\mathcal{D}, j), F, C_N^{k+}) \ .
\end{aligned}$$

Similarly,

$$\mathbf{P}(A_3(N), C_N) = \sum_{\mathcal{C}',\mathcal{D}'} \sum_{i',j'} \mathbf{P}(X_-(\mathcal{C}', i'), Circ_k(\mathcal{C}'), C_N^{k-}) P(\mathcal{C}', \mathcal{D}', i', j') \mathbf{P}(X_+(\mathcal{D}', j'), C_N^{k+}) \ .$$

Multiplying these and using Lemma 6.6, one obtains

$$\begin{aligned}
&\mathbf{P}(E, F, A_3(N), C_N)\ \mathbf{P}(A_3(N), C_N) \\
&\geq \left(\frac{1}{C_8}\right)^2 \sum_{\mathcal{C},\mathcal{C}',\mathcal{D},\mathcal{D}'} \sum_{i,j,i',j'} \Big[\mathbf{P}(E, X_-(\mathcal{C}, i), Circ_k(\mathcal{C}), C_N^{k-}) P(\mathcal{C}, \mathcal{D}', i, j') \mathbf{P}(X_+(\mathcal{D}', j'), C_N^{k+}) \\
&\quad \times \mathbf{P}(X_-(\mathcal{C}', i'), Circ_k(\mathcal{C}'), C_N^{k-}) P(\mathcal{C}', \mathcal{D}, i', j) \mathbf{P}(X_+(\mathcal{D}, j), F, C_N^{k+})\Big] \\
&= \left(\frac{1}{C_8}\right)^2 \mathbf{P}(E, A_3(N), C_N) \mathbf{P}(A_3(N), F, C_N).
\end{aligned}$$

Dividing gives

$$\mathbf{P}(F \mid E, A_3(N), C_N) \geq (1/C_8)^2 \mathbf{P}(F \mid A_3(N), C_N) \ .$$

This shows (6.19) with $c_1 = (1/C_8)^2$.

To finish the proof of (6.16), we use estimate (6.19) to show that at least one E_{2k} occurs. The idea is to consider a maximal sub collection $F_1, F_2, \ldots$ of the E_{2k}'s such that F_1 depends on the state of edges in $B(i_3)$, F_2 depends on the state of edges in $B(i_8) \setminus B(i_7)$, F_3 depends on the state of edges in $B(i_{13}) \setminus B(i_{12})$, and so on. Write r_n for the largest k such that F_k depends on edges in $B(3^{2m_n+1})$, where m_n was defined below (6.15). Then

$$\begin{aligned}
\mathbf{P}\left(\cap_{k=1}^{r_n} F_k^c \mid A_3(N), C_N\right) &= \prod_{k=1}^{r_n} \mathbf{P}\left(F_k^c \mid A_3(N), C_N, \cap_{l=1}^{k-1} F_l^c\right) \\
&\leq \prod_{k=1}^{r_n} \left(1 - c_1 \mathbf{P}(F_k \mid A_3(N), C_N)\right) \\
&\leq (1 - c_1 \cdot a_{n,N})^{r_n},
\end{aligned}$$

where

$$a_{n,N} = \min_{1 \leq k \leq r_n} \mathbf{P}(F_k \mid A_3(N), C_N).$$

However by the bound (6.14) and Claim 6.5, one has

$$\mathbf{P}(E_{2k} \mid A_3(N), C_N) \geq \mathbf{P}(E_{2k} \mid A_3(N)) - \mathbf{P}(C_N^c \mid A_3(N)) \geq C_7 - \epsilon.$$

So for $\epsilon < C_7$,

$$\mathbf{P}\left(\cap_{k=1}^{r_n} F_k^c \mid A_3(N), C_N\right) \to 0 \text{ in } n \text{ uniformly in } N \geq n^{C_4/2}.$$

Combining this with Claim 6.5, one has

$$\mathbf{P}(\cap_{k=K}^{m_n} E_{2k}^c \mid A_3(N)) \leq \mathbf{P}(\cap_{k=K}^{m_n} E_{2k}^c \mid A_3(N), C_N) + \mathbf{P}(C_N^c \mid A_3(N)) < \epsilon$$

for n large and uniformly in $N \geq n^{C_4/2}$. □

7 The lower tail of $\tilde{L}_n$

In this section we give a proof sketch of the following result:

Lemma 7.1. *Let $\tilde{L}_n$ be the number of edges in the lowest crossing of $[-n,n]^2$. Then*

$$\lim_{\epsilon \downarrow 0} \limsup_n \mathbf{P}(0 < \tilde{L}_n < \epsilon n^2 \pi_3(n)) = 0. \tag{7.1}$$

To limit the length of the current paper, we only provide a sketch of Lemma 7.1. The full proof can be found in the online version of the paper [5].

The idea of the proof is similar to that of Kesten's proof that at criticality, the expected number of edges which are pivotal for a crossing event is at least order $\log n$ [14]. We first restrict to the event that the maximum number of disjoint open left-right crossings of the box is exactly k. Next, we condition successively on the k upper-most disjoint open crossings. Calling T_k the k-th such crossing, we then condition on the leftmost top-down dual closed crossing p connecting T_k to the bottom of the box (there must be such a crossing, since there are no more disjoint left-right open crossings below T_k). Calling e_k the edge at the intersection of p and T_k, we then use independence of the edge variables for edges in the region below and to the right of e_k to build many "three-arm" edges in this region in annuli centered at e_k. The crucial point is that each such edge will have two disjoint open arms to T_k and one closed arm to p, and will therefore be an edge on the lowest crossing of the box. Since there is a lower bound for the probability of many such edges existing in each annulus, we obtain that with high probability, many such edges exist in at least one annulus, and this implies $\tilde{L}_n \geq \epsilon n^2 \pi_3(n)$ with high probability.

The main difficulty in our construction (and it is this point that makes ours more complicated than the one in Kesten's proof) is that we do not want only $\log n$ number of such edges on the lowest crossing, but at least $\epsilon n^2 \pi_3(n)$. This corresponds to the fact that in each annulus, Kesten needs only to produce one pivotal point, whereas we need to build many. For us to do so, the region below and to the right of e_k must have many open spaces. Specifically, we must first show that with probability of order $1 - o_\epsilon(1)$, in each large enough annulus centered at e_k, we can find a box of size at least $\epsilon^\delta n$, for some c and $\delta > 0$, which lies entirely in the region below and to the right of e_k.

7.1 Sketch of the proof of Lemma 7.1

Let $A_n = A_n(\epsilon)$ be the event in the probability in (7.1). Let $M_K = M_K(n)$ be the event that there are at most K disjoint open crossings of the box $[-n,n]^2$. Note that by the BK-Reimer inequality and the RSW theorem,

$$\mathbf{P}(M_K^c) \leq (1-C)^K,$$

uniformly in n.

We further let $D_k = M_k \setminus M_{k+1}$ be the event that the maximal number of disjoint crossings equals k. Then

$$M_K = \cup_{k=1}^K D_k,$$

and the union is disjoint. The main work in deriving Lemma 7.1 is to show that for $\epsilon > 0$ small,

$$\mathbf{P}(A_n, D_k) \leq \left(C' \log \frac{1}{\epsilon}\right)^k \epsilon^{c'} \text{ for all } k \geq 1, \tag{7.2}$$

if n is large. Here, $C', c' > 0$ are independent of ϵ, k and n. It then follows that

$$\begin{aligned}\mathbf{P}(A_n) &\leq \mathbf{P}(M_K^c) + \left(C'' \log \frac{1}{\epsilon}\right)^K \epsilon^{c'} \\ &\leq (1-C)^K + \left(C'' \log \frac{1}{\epsilon}\right)^K \epsilon^{c'}.\end{aligned}$$

Letting $K = \lceil \log \log \frac{1}{\epsilon} \rceil$, we obtain (7.1).

To derive (7.2), we condition successively on top-most paths $T_1, \ldots, T_k$. T_1 is defined as the horizontal open crossing of $B(n)$ such that the region above T_1 is minimal. T_2 is then defined as the highest crossing of the region below T_1, and T_i, $i = 3, \ldots, k$ is defined analogously.

For any k-tuple of paths $t_1, \ldots, t_k$ that is admissible in the sense that

$$\mathbf{P}(T_1 = t_1, \ldots, T_k = t_k) > 0,$$

the event

$$\{T_1 = t_1, \ldots, T_k = t_k\}$$

is independent of the status of edges below t_k (see [13, Prop. 2.3]). Moreover, the event

$$D_k \cap \{T_1 = t_1, \ldots, T_k = t_k\}$$

is equal to

$$E(t_1, \ldots, t_k) = \{T_1 = t_1, \ldots, T_k = t_k\} \cap R(t_k),$$

where $R(t_k)$ is the event that there is a dual closed path from e^*, where e is some edge on the path t_k, to the bottom of the box $[-n, n]^2$. Note in particular that $R(t_k)$ is independent of the status of edges on and above t_k.

On $E(t_1, \ldots, t_k)$, there is a unique left-most closed dual path from t_k to the bottom of $[-n, n]^2$, and we denote it by $P(t_k)$. It is characterized by the following three-arm condition: each dual edge on p has one closed dual arm to the path t_k, a disjoint closed dual arm to the bottom of the box, and an open arm to the left side of the box in the region below t_k. Given a closed dual path p in the region below t_k, the event $E(t_1, \ldots, t_k) \cap \{P(t_k) = p\}$ is independent of the status of edges in the region below t_k and to the right of p. Our goal will be to use this independence to connect at least $\sqrt{\epsilon} n^2 \pi_3(n)$ points to T_k by two disjoint open paths and to $P(T_k)$ by a closed dual path. On D_k, we can uniquely define the edge $e_k = \{x_k, y_k\}$ where $P(T_k)$ meets T_k.

Let $\alpha(T_k)$ and $\beta(T_k)$ be the left and right endpoints, respectively, of T_k. By RSW, we have, for some $c = c(\nu) > 0$, n large, ϵ small, and all k,

$$\begin{aligned}\mathbf{P}(\mathrm{dist}(e_k, \alpha(T_k)) < 2\epsilon^\nu n,\ D_k) &\leq \epsilon^c \\ \mathbf{P}(\mathrm{dist}(e_k, \beta(T_k)) < 2\epsilon^\nu n,\ D_k) &\leq \epsilon^c.\end{aligned}$$

Also by RSW, we can arrange that T_k remains at distance $10\epsilon^\nu n$ from the each of the 4 corners: for n large, ϵ small, and all k,

$$\mathbf{P}(\mathrm{dist}(T_k \cup P(T_k), Corner_i) < 10\epsilon^\nu n,\ D_k) \leq \epsilon^c,$$

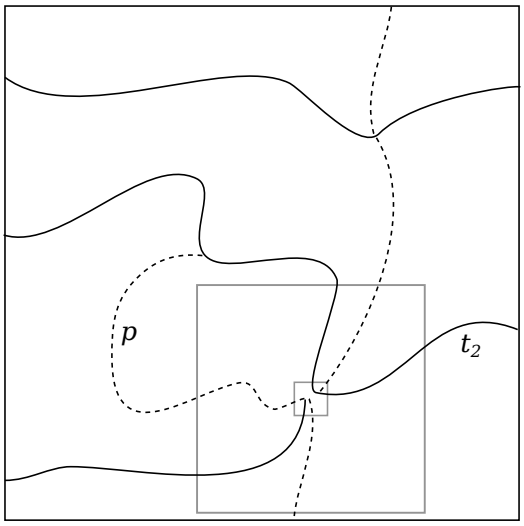

Figure 8: An illustration of the estimate for I_k with $k = 2$. If p and t_k come too close together at some point, a six-arm event occurs in an annulus around that point.

for $i = 1, \dots, 4$ and

$$Corner_1 = (-n, -n) \qquad Corner_2 = (n, -n)$$
$$Corner_3 = (n, n) \qquad Corner_4 = (-n, n).$$

On D_k, let $I_k = I_k(n)$ be the event e_k is close to the endpoints of T_k and the corners or $[-n, n]^2$ and that

$$\mathrm{dist}(e_k, \alpha(T_k)) \geq 2\epsilon^\nu n, \tag{7.3}$$
$$\mathrm{dist}(e_k, \beta(T_k)) \geq 2\epsilon^\nu n, \tag{7.4}$$
$$\mathrm{dist}((T_k \cup P(T_k)), Corner_i) \geq 10\epsilon^\nu n, \quad i = 1, \dots, 4, \tag{7.5}$$

and there is a pair $u, v \in \mathbb{Z}^2$ such that $u \in T_k$, $v \in P(T_k)$,

$$\mathrm{dist}(u, e_k) \geq 10\epsilon^\nu n, \quad \mathrm{dist}(v, e_k) \geq 10\epsilon^\nu n \tag{7.6}$$

and

$$\mathrm{dist}(u, v) < \epsilon^\delta n. \tag{7.7}$$

Using the fact that the event that two points $u \in T_k$ and $v \in P(T_k)$ coming too close would generate a six-arm event (see Figure 8), and the six arm exponent is greater than 2, we show the following:

Lemma 7.2. *There exist $C, \eta' > 0$ and $0 < \delta < \nu < 1/2$ such that for small enough $\epsilon > 0$,*

$$\mathbf{P}(I_k) \leq \left(C \log \frac{1}{\epsilon}\right)^k \epsilon^{\eta'} \text{ for all } k \geq 1$$

if n is large.

It follows that to estimate $\mathbf{P}(A_n, D_k)$, we can write

$$\begin{aligned}\mathbf{P}(A_n, D_k) \leq & 10e^c + \left(C \log \frac{1}{\epsilon}\right)^k \epsilon^c + \mathbf{P}(D_k, I_k) \\ & + \sum_{t_1, \dots, t_k, p} \mathbf{P}(A_n, D_k, T_i = t_i \ \forall i, P(T_k) = p),\end{aligned} \tag{7.8}$$

where the sum is only over t_i and p such that I_k occurs and

1. $\mathrm{dist}((T_k \cup P(T_k)), Corner_i) > 10\epsilon^\nu n$, for all $i = 1, \ldots, 4$ and
2. e_k is at least distance $2\epsilon^\nu n$ from the bottom and right sides of $[-n, n]^2$,

where e_k is the edge where p and t_k meet. Condition 2 follows from an argument similar to the one given to estimate $\mathbf{P}(I_k)$, involving the half-plane 3-arm argument: this is where the term $\left(C \log \frac{1}{\epsilon}\right)^k \epsilon^c$ in (7.8) comes from.

Consider concentric annuli

$$Ann(e_k, 2^l) = B(e_k, 2^l) \setminus B(e_k, 2^{l-1}),$$

for $l = 4 + \lceil \log_2 \epsilon^\delta n \rceil, \ldots, \lceil \log_2 \epsilon^\nu n \rceil$. On $\{D_k, T_i = t_i \ \forall i, P(T_k) = p\}$, if $Ann(e_k, 2^l)$ contains more than $\epsilon n^2 \pi_3(n)$ points connected to t_k by two disjoint open paths, and to p by a closed dual path, then

$$\tilde{L}_n \geq \epsilon n^2 \pi_3(n).$$

We claim that with probability bounded away from 0 independently of l, n, t_k and p, the number of such edges in $Ann(e_k, 2^l)$ is bounded below by

$$C\epsilon^{2\delta} n^2 \pi_3(\epsilon^\delta n). \tag{7.9}$$

Choosing δ appropriately, this is bounded below by $\epsilon n^2 \pi_3(n)$ for ϵ small.

To obtain the lower bound (7.9) with uniformly positive probability, we use the second moment method to find a large number of three arm points in a box inside the region $\mathcal{R}$ below t_k and to the right of p. To this effect, we need to show that it is always possible to find such a box of side-length r at least $\epsilon^\delta n$. In addition, to use RSW and connect the three arm points to p and t_k, we need the box to be at a distance from these crossings that is roughly comparable to r, and the crossings themselves to be separated on this scale. It turns out that a suitable box can always be found, regardless of the choice of $t_1, \ldots, t_k$, p, as long as I_k holds, and the conditions 1. and 2. are satisfied, but showing this requires a somewhat delicate geometric argument. We refer the reader to the full proof in [5] for details. Using this construction in the order $\log \frac{1}{\epsilon}$ annuli $Ann(e_k, 2^l)$, we obtain the estimate

$$\begin{aligned} &\mathbf{P}(A_n, D_k, T_1 = t_1, \ldots, T_k = t_k, P(T_k) = p) \\ &\leq (1 - C)^{c \log_2 \frac{1}{\epsilon}} \mathbf{P}(T_1 = t_1, \ldots, T_k = t_k, P(T_k) = p), \end{aligned}$$

which, together with (7.8) and Lemma 7.2 implies the desired result (7.2).

8 Lemmas of a topological nature

In this section, we complete the detour construction by providing proofs of the proposition and lemmas assumed in Sections 3 and 3.1.

First, recall Proposition 3.3: If $\omega \in C_n$, then for distinct edges $e, f \in \gamma_n$, $\pi(e)$ and $\pi(f)$ are either equal or have no vertices in common.

Regarding σ_n, we establish the following two properties:

1. (*Lemma 3.4*) For $\omega \in C_n$, σ_n is an open circuit in $A(n)$ surrounding the origin.
2. (*Lemma 3.5*) For $\omega \in C_n$, if $e \in \hat{\gamma}_n \setminus \hat{\Pi}$ then $\pi(e) = \emptyset$.

8.1 Proof of Proposition 3.3

Because $\pi(e)$ was defined as the first element of $\mathcal{S}(e)$ in a deterministic ordering, we see that it will suffice to show: given $P(e) \in \mathcal{S}(e)$ and $P(f) \in \mathcal{S}(f)$, if

$$V(P(e)) \cap V(P(f)) \neq \emptyset, \tag{8.1}$$

(their vertex sets intersect) then

$$P(e) \in \mathcal{S}(f) \text{ and } P(f) \in \mathcal{S}(e). \tag{8.2}$$

Now, suppose the intersection of $\pi(e)$ and $\pi(f)$ is non-empty. Then $\pi(e) \in \mathcal{S}(e) \cap \mathcal{S}(f)$, so $\pi(e) = \pi(f)$.

Let $P(e) \in S(e)$. By extending the two open ends of the closed path in condition 4. to meet the midpoints of the edges $\{w_0, w_0 - \mathbf{e}_1\}$ and $\{w_M, w_M + \mathbf{e}_1\}$, we can form a Jordan curve $\theta(e)$ (i.e. a continuous, self-avoiding closed curve) by traversing the closed dual path from $w_0 + (1/2)(-\mathbf{e}_1 + \mathbf{e}_2)$ to $w_M + (1/2)(\mathbf{e}_1 + \mathbf{e}_2)$, traversing the path $Q(e)$ (listed as Q in the definition of $\mathcal{S}(e)$), and returning to $w_0 + (1/2)(-\mathbf{e}_1 + \mathbf{e}_2)$. By the Jordan Curve Theorem, any connected set of which does not intersect $\theta(e)$ must lie completely on either side of $\theta(e)$.

We apply the preceding to $P(e) \setminus \{w_0(e), w_M(e)\}$ and $P(f) \setminus \{w_0(f), w_M(f)\}$. By assumption (Condition 1.) both sets lie in the exterior of γ_n, so neither can intersect $Q(e)$ or $Q(f)$. Since they consist of open edges, they also cannot intersect either the dual portion of $\theta(e)$ or that of $\theta(f)$. Thus, assuming (8.1), we have

Claim 8.1. *Except for the endpoints $w_0(e)$, $w_0(f)$, $w_M(e)$, $w_M(f)$, the paths $P(e)$ and $P(f)$, considered as the union of their edges and vertices, lie in the same connected components of $\mathbb{R}^2 \setminus \theta(e)$ and $\mathbb{R}^2 \setminus \theta(f)$.*

We now assert that this implies:

Claim 8.2. *The endpoints of $P(e)$ and $P(f)$ coincide.*

To see this, we start with the following

Claim 8.3. *Any vertex of $\gamma_n \setminus Q(e)$ lies in the component of $\theta(e)^c$ which does not contain $P(e)$.*

Proof. The vertices $w_1(e)$ and $w_1(e) - \mathbf{e}_1$ lie on opposite sides of $\theta(e)$, since they can be connected by a segment which intersects the curve exactly once. The vertex $w_0(e) - \mathbf{e}_1$ is in the same component as $w_1(e) - \mathbf{e}_1$, so it is in the component which does not contain $P(e)$. However each vertex of $\gamma_n \setminus Q(e)$ can be connected to $w_0(e) - \mathbf{e}_1$ by simply following γ_n (without crossing $\theta(e)$). □

Claim 8.3 implies that $P(f)$, lying as it does in the same component as $P(e)$, must have both of its endpoints in $Q(e)$ (since they must be in γ_n). The argument is symmetric, so that both endpoints of $P(e)$ must lie in $Q(f)$, so that $w_0(e) = w_0(f)$ and $w_M(e) = w_M(f)$. At this point, we have established Claim 8.2.

The coincidence of the endpoints of $P(e)$ and $P(f)$ implies also $Q(e) = Q(f)$, so that $P(e)$, together with the dual path from Condition 4. in the definition of $\mathcal{S}(e)$, satisfies Conditions 1-5 defining $\mathcal{S}(f)$. We have proved Proposition 4.

8.2 Proof of Lemma 3.4

By Proposition 3.3, the set $E(\Pi)$ is a disjoint union of paths $P(e) = \pi(e)$ for a finite collection of edges $e \in \sigma_n$. Our strategy will be to inductively replace each portion $Q(e)$ of γ_n by the detour path $P(e)$ and show that at each stage, we still have a circuit around the origin. In other words, enumerating the paths $P_1, P_2, \ldots$, and $Q_1, Q_2, \ldots$, we replace Q_1 with P_1 to create $\gamma_n^{(1)}$ from the original circuit γ_n. Then we replace a portion of $\gamma_n^{(1)}$ (which is Q_2) with P_2 to create $\gamma_n^{(2)}$, and so on.

Note that at stage k, the path P_{k+1} satisfies the definition of ϵ-shielded detour with γ_n replaced by $\gamma_n^{(k)}$. Indeed, since all paths Q are disjoint for paths in Π, points 2-5 are obvious. Furthermore, assuming that P_{k+1} satisfies point 1 with γ_n equal to $\gamma_n^{(k-1)}$, it must also satisfy it with γ_n equal to $\gamma_n^{(k)}$. If this were not the case, then if we write σ^o for

a path σ excluding its endpoints, we would have $P^o_{k+1} \subset \mathrm{ext}\gamma_n^{(k-1)}$ but $P^o_{k+1} \cap \mathrm{int}\gamma_n^{(k)} \neq \emptyset$. But P^o_{k+1} must also contain a point of $\mathrm{ext}\gamma_n^{(k)}$ since the paths Q_i are disjoint (choose a point near an endpoint of Q_{k+1}), so this implies that P_{k+1} must cross from the interior to the exterior. It cannot cross γ_n, so it must cross one of the P_i's for $i \neq k+1$. This is a contradiction, since the P_i's are disjoint.

Therefore to prove Lemma 3.4, it will suffice to show the following:

Claim 8.4. *Let σ be a self-avoiding circuit surrounding the origin, and let P denote a self-avoiding path with endpoints in σ, such that P satisfies Conditions 1-5 in the definition of $\pi(e)$, with γ_n replaced by σ. Then*

$$(\sigma \setminus Q) \cup P$$

is a circuit in $A(n)$ surrounding the origin. Here Q is defined relative to σ and P as in Condition 3. in the definition of $\pi(e)$.

We first show:

Claim 8.5.

$$Q^o \subset \mathrm{int}(P \cup (\gamma_n \setminus Q)).$$

The connected set Q^o lies either in $\mathrm{int}(P \cup (\gamma_n \setminus Q))$ or in $\mathrm{ext}(P \cup (\gamma_n \setminus Q))$. It will suffice to exclude the latter case. For this, we will use a simple intermediate result.

Lemma 8.6 (The ABC lemma)**.** *Let $A(t)$, $B(t)$ and $C(t)$, $t \in [0,1]$ be three self-avoiding, continuous curves in $\mathbb{R}^2$. Denote their images by $A = A([0,1])$, $B = B([0,1])$, $C = C([0,1])$. Suppose*

$$A(0) = B(0) = C(0) = a$$

and

$$A(1) = B(1) = C(1) = b$$

with $a \neq b$, and

$$A \cap B = A \cap C = B \cap C = \{a, b\}.$$

Form the three Jordan curves $A \cup B$, $A \cup C$ and $B \cup C$, and suppose

$$C \setminus \{a, b\} \subset \mathrm{int}(A \cup B).$$

Then

$$\mathrm{int}(B \cup C) \subset \mathrm{int}(A \cup B).$$

Proof. Any continuous path from a point in $\mathrm{int}(B \cup C)$ to infinity must cross either B or C. If it does not cross B, it must cross C at a point in $\mathrm{int}(A \cup B)$, after which it must cross A, and the lemma follows. □

Let us now return to the proof of Lemma 3.4. We assume

$$Q^o \subset \mathrm{ext}(P \cup (\gamma_n \setminus Q)). \tag{8.3}$$

Because P does not cross γ_n, two possibilities arise: either

$$(\gamma_n \setminus Q)^o \subset \mathrm{int}(P \cup Q), \tag{8.4}$$

or

$$(\gamma_n \setminus Q)^o \subset \mathrm{ext}(P \cup Q). \tag{8.5}$$

In case (8.4) holds, we can apply the "ABC Lemma" 8.6 to find $0 \in \mathrm{int}\gamma_n \subset P \cup Q$, which is not possible by Condition 3 in the definition of P.

Now, assume (8.5) holds. P is contained the boundary of both $\mathrm{int}(P \cup Q)$ and $\mathrm{int}(P \cup (\gamma_n \setminus Q))$. Recalling that we have also (8.3), we see that these two Jordan domains are disjoint. Thus, every point in a sufficiently small neighborhood of a point of P^o lies in exactly one of these two domains (if it is not in P). It follows that any curve joining a point in P^o to infinity, intersecting P^o only at its starting point, must intersect either $\gamma_n \setminus Q$ or Q, so

$$P^o \subset \mathrm{int}\gamma_n,$$

a contradiction to Condition 2. in the definition of $\pi(e)$. At this point, we have established Claim 8.5. Applying Lemma 8.6, find

$$0 \in \mathrm{int}\gamma_n \subset \mathrm{int}(P \cup (\gamma_n \setminus Q)),$$

which is Claim 8.4.

8.3 Proof of Lemma 3.5

By Proposition 3.3, the set $E(\Pi)$ is a disjoint union of sets of the form $\pi(e)$, where e ranges over a finite collection S of edges in $\hat{\Pi}$. If $e \in \hat{\gamma}_n \setminus \hat{\Pi}$ and $\pi(e) \neq \emptyset$ then, again by Proposition 3.3,

$$\pi(f) \cap \pi(e) = \emptyset$$

for all $f \in S$. This contradicts the maximality of Π, provided we show

Claim 8.7. *For all $f \in S$,*

$$\hat{\pi}(e) \cap \hat{\pi}(f) = \emptyset.$$

This is because if the intersection is non-empty, then $\hat{\pi}(e)$ and $\hat{\pi}(f)$ must share a common segment, which forces

$$\pi(e) \cap \pi(f) \neq \emptyset. \tag{8.6}$$

Indeed, if $\hat{\pi}(e)$ and $\hat{\pi}(f)$ coincide, then the two initial and two final vertices of $\pi(e)$ and $\pi(f)$ coincide by Condition 2. in the definition of $\mathcal{S}(e)$, and otherwise one endpoint of $\hat{\pi}(e)$ must be equal to some non-endpoint vertex of $\hat{\pi}(f)$. By the Jordan Curve Theorem, $\pi(e)$ must then intersect $\pi(f)$. Given (8.6), Proposition 3.3 now implies $\pi(e) = \pi(f)$, which contradicts the assumption on e.

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