

Stein approximation for functionals of independent random sequences*

Nicolas Privault[†]

Grzegorz Serafin[‡]

Abstract

We derive Stein approximation bounds for functionals of uniform random variables, using chaos expansions and the Clark-Ocone representation formula combined with derivation and finite difference operators. This approach covers sums and functionals of both continuous and discrete independent random variables. For random variables admitting a continuous density, it recovers classical distance bounds based on absolute third moments, with better and explicit constants. We also apply this method to multiple stochastic integrals that can be used to represent U -statistics, and include linear and quadratic functionals as particular cases.

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1 Introduction

The Stein and Chen-Stein methods have been developed together with the Malliavin calculus to derive bounds on the distances between probability laws on the Wiener and Poisson spaces, cf. [9], [12], [13] and for discrete Bernoulli sequences, cf. [10], [4], [5]. The results of these works rely on covariance representations based on the number (or Ornstein-Uhlenbeck) operator L on multiple Wiener-Poisson stochastic integrals and its inverse L^{-1} . Other covariance representations based on the Clark-Ocone representation formula have been used in [18] on the Wiener and Poisson spaces, and in [19] for Bernoulli processes.

This paper focuses on functionals of a countable number of uniformly distributed random variables, and uses the framework of [14], cf. also [15], [16], to derive covariance representations from chaos expansions in multiple stochastic integrals, based on a

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[†]Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-05-43, 21 Nanyang Link Singapore 637371. E-mail: nprivault@ntu.edu.sg

[‡]Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Ul. Wybrzeże Wyspiańskiego 27, Wrocław, Poland. E-mail: grzegorz.serafin@pwr.edu.pl

version of the Clark-Ocone formula with finite difference or derivation operators. We obtain general bounds on the distance of a random functional to the Gaussian and gamma distributions using Stein kernels, see Propositions 3.1-3.3, and we also derive specific bounds for multiple stochastic integrals, see Corollary 5.2. Other recent approaches to the Stein method for arbitrary univariate distributions using Stein kernels include [7].

When restricted to single stochastic integrals, our framework applies to sums

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k, \quad n \geq 1,$$

of independent centered random variables $(X_k)_{k \geq 1}$ with variance one. This includes the case of discrete random variables and, e.g., sums and polynomials of Bernoulli random variables with variable parameters, as a consequence of Proposition 3.4, see Proposition 4.2. In addition, this approach yields the general bound

$$d_W(Z_n, \mathcal{N}) \leq \frac{2}{n^{3/2}} \sum_{k=1}^n E[|X_k|^3], \quad (1.1)$$

where d_W denotes the Wasserstein distance, see (4.4) below, which recovers classical results such as the bound of Theorem 1.1 in [2], however with an additional factor two.

On the other hand, for random variables which admit a continuous density, as a consequence of Proposition 3.2 we find in Proposition 4.4 that

$$d_W(Z_n, \mathcal{N}) \leq \frac{1}{n^{3/2}} \sqrt{\sum_{k=1}^n \left(\int_{-\infty}^{\infty} \left| \frac{1}{F'_k(y)} \int_{-\infty}^y x dF_k(x) \right|^2 dF_k(y) - 1 \right)}, \quad (1.2)$$

assuming that the cumulative distribution function F_k of X_k admits a non-vanishing density on the support of X_k . This recovers in particular Proposition 3.3 of [18] in the case $n = 1$. For several usual distributions the bound (1.2) improves on (1.1) which is based on absolute third moments. For example in the Gaussian case, (1.2) yields $d_W(Z_n, \mathcal{N}) = 0$ as expected. For the Gamma and Beta distributions it also yields better constants than (1.1). The bound (1.2) may however perform worse than (1.1), or can become infinite if $F'(x)$ becomes too close to 0 on an interval.

Multiple stochastic integrals with respect to a point process with uniform jump times are particularly treated in Proposition 3.4 and 5.1 and Corollaries 5.2 and 5.3, with an application to a combinatorial central limit theorem for general i.i.d. random sequences in Theorem 5.4.

In Section 6 we consider U-statistics, or quadratic functionals of the form

$$Q_n := \sum_{1 \leq k, l \leq n} a_{k,l} X_k X_l,$$

where $(X_k)_{k \geq 1}$ is a sequence of normalized independent identically distributed random variables, such that $\text{Var}[Q_n] = 1$. Corollary 6.2 shows that we have the bound

$$d_{TV}(Q_n, \mathcal{N}) \leq 4\sqrt{n}L_n^2 \left(C + \sqrt{E[X_1^4] + \frac{2}{nL_n^4} \sum_{1 \leq l, p \leq n} \left(\sum_{k=1}^n a_{k,l} a_{k,p} \right)^2} \right), \quad (1.3)$$

where $C = 3E[X_1^4] + (E[X_1^4])^2$ and

$$L_n^2 := \max_{1 \leq k \leq n} \sum_{l=1}^n a_{k,l}^2,$$

which provides a different bound from Theorem 1 in [3], with explicit constants. In case $a_{2k,2k-1} = 1/\sqrt{n}$, the bound (1.3) yields

$$d_{TV}(Q_n, \mathcal{N}) \leq \frac{16E[X_1^4]}{\sqrt{n}},$$

which recovers the known convergence rate in $1/\sqrt{n}$ as on pages 1074-1075 of [3]. Corollary 6.4 provides another bound obtained from derivation operators.

More generally, our approach applies to functionals of uniformly distributed random variables, see Propositions 3.2 and 3.3 which deal respectively with smooth random functionals and with multiple stochastic integrals, cf. Proposition 3.4.

This paper is organized as follows. In Section 2 we recall the framework of [14] for the construction of random functionals of uniform random variables, together with the construction of derivation operators and the associated stochastic integral (Clark-Ocone) decomposition formula. In Section 3 we derive Stein approximation bounds for the distance of the laws of general functionals to the Gaussian and gamma distributions. Section 4 deals with single stochastic integrals which can be used to represent sums of independent random variables. Section 5 treats the general case of multiple stochastic integrals, which can be viewed as U -statistics. Finally, in Section 6, double stochastic integrals are discussed with their applications to quadratic functionals. In the appendix Section 7 we prove a multiplication formula for multiple stochastic integrals.

2 Functionals of uniform random sequences

Stochastic integrals

Consider an i.i.d. sequence $(U_k)_{k \in \mathbb{N}}$ of uniformly distributed random variables on the interval $[-1, 1]$, where $\mathbb{N} := \{0, 1, 2, \dots\}$, and let the jump process $(Y_t)_{t \in \mathbb{R}_+}$ be defined as

$$Y_t := \sum_{k=0}^{\infty} \mathbf{1}_{[2k+1+U_k, \infty)}(t), \quad t \in \mathbb{R}_+.$$

We also denote by $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ the filtration generated by $(Y_t)_{t \in \mathbb{R}_+}$, and let

$$\tilde{\mathcal{F}}_t := \mathcal{F}_{2k}, \quad 2k \leq t < 2k+2, \quad k \in \mathbb{N}.$$

The compensated stochastic integral

$$\int_0^{\infty} u_t d(Y_t - t/2)$$

with respect to the compensated point process $(Y_t - t/2)_{t \in \mathbb{R}_+}$ can be defined for square-integrable $\tilde{\mathcal{F}}_t$ -adapted processes $(u_t)_{t \in \mathbb{R}_+}$ by the isometry relation

$$\begin{aligned} & E \left[\int_0^{\infty} u_t d(Y_t - t/2) \int_0^{\infty} v_t d(Y_t - t/2) \right] \\ &= E \left[\int_0^{\infty} u_t \left(v_t - \sum_{k=0}^{\infty} \mathbf{1}_{(2k, 2k+2]}(t) \int_{2k}^{2k+2} v_r \frac{dr}{2} \right) \frac{dt}{2} \right], \end{aligned} \quad (2.1)$$

see [14], where $(u_t)_{t \in \mathbb{R}_+}$ and $(v_t)_{t \in \mathbb{R}_+}$ are square-integrable $\tilde{\mathcal{F}}_t$ -adapted processes. This also implies the bound

$$E \left[\left(\int_0^{\infty} u_t d(Y_t - t/2) \right)^2 \right] \leq \frac{1}{2} E \left[\int_0^{\infty} |u_t|^2 dt \right],$$

for $(u_t)_{t \in \mathbb{R}_+}$ a square-integrable $\tilde{\mathcal{F}}_t$ -adapted process.

Given $f_1 \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ we define the first order stochastic integral

$$I_1(f_1) := \sum_{k=0}^{\infty} f_1(2k+1+U_k) - \frac{1}{2} \int_0^{\infty} f_1(t) dt = \int_0^{\infty} f_1(t) d(Y_t - t/2).$$

Next, given f_n a function which is square integrable on \mathbb{R}_+^n and belongs to the space $\hat{L}^2(\mathbb{R}_+^n)$ of symmetric functions that vanish outside of

$$\Delta_n := \bigcup_{\substack{k_i \neq k_j \geq 0 \\ 1 \leq i \neq j \leq n}} [2k_1, 2k_1+2] \times \cdots \times [2k_n, 2k_n+2],$$

we define the multiple stochastic integral

$$\begin{aligned} I_n(f_n) &:= \sum_{r=0}^n \frac{(-1)^{n-r}}{2^{n-r}} \binom{n}{r} \\ &\sum_{k_1 \neq \cdots \neq k_r \geq 0} \int_0^{\infty} \cdots \int_0^{\infty} f_n(2k_1+1+U_{k_1}, \dots, 2k_r+1+U_{k_r}, y_1, \dots, y_{n-r}) dy_1 \cdots dy_{n-r} \\ &= n! \int_0^{\infty} \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \dots, t_n) d(Y_{t_1} - t_1/2) \cdots d(Y_{t_n} - t_n/2), \end{aligned}$$

see [16] for a construction using a Wick type product, and [22] for the Poisson point process version. It is easy to notice, see (2.1) above and Propositions 4 and 6 of [14], that $(I_n(f_n))_{n \geq 1}$ forms a family of mutually orthogonal centered random variables which satisfy the bound

$$E[(I_n(f_n))^2] \leq n! \|f_n\|_{L^2(\mathbb{R}_+^n, dx/2)}^2, \quad n \geq 1, \quad (2.2)$$

which allows us to extend the definition of $I_n(f_n)$ to all $f_n \in \hat{L}^2(\mathbb{R}_+^n)$. If in addition we have

$$\int_{2k}^{2k+2} f_n(t, *) dt = 0, \quad k \in \mathbb{N}, \quad (2.3)$$

i.e. the function f_n is canonical [23], then the multiple stochastic integral $I_n(f_n)$ can be written as the U -statistic of order n based on the function f_n , i.e.

$$I_n(f_n) = \sum_{k_1 \neq \cdots \neq k_n \geq 0} f_n(2k_1+1+U_1, \dots, 2k_n+1+U_n), \quad (2.4)$$

with the isometry and orthogonality relation

$$E[I_n(f_n)I_m(f_m)] = \mathbf{1}_{\{n=m\}} n! \langle f_n, f_m \rangle_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}, \quad (2.5)$$

see [14] page 589. Finally, every $X \in L^2(\Omega)$ admits the chaos decomposition

$$X = E[X] + \sum_{n=1}^{\infty} I_n(f_n), \quad (2.6)$$

for some sequence of functions f_n in $\hat{L}^2(\mathbb{R}_+^n)$, $n \geq 1$, cf. Proposition 7 of [14].

Finite difference operator

Consider the finite difference operator ∇ defined on multiple stochastic integrals $X = I_n(f_n)$ as

$$\nabla_t X := X \circ \Psi_t - \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lceil t/2 \rceil + 2} X \circ \Psi_s ds, \quad t \in \mathbb{R}_+, \quad (2.7)$$

where

$$\Psi_t(\omega) := (U_1(\omega), \dots, U_{\lfloor t/2 \rfloor - 1}(\omega), t - 2\lfloor t/2 \rfloor - 1, U_{\lfloor t/2 \rfloor + 1}(\omega), \dots), \quad t \in \mathbb{R}_+,$$

cf. Definition 5 and Proposition 10 of [14]. The operator ∇ does not satisfy the chain rule of derivation, however it possesses a simple form and it can be easily applied to multiple stochastic integrals.

Proposition 2.1. *Given $f_n \in \hat{L}^2(\mathbb{R}_+^n)$, we have*

$$\nabla_t I_n(f_n) = n I_{n-1}(f_n(t, *)) - n \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} I_{n-1}(f_n(s, *)) ds, \quad t \in \mathbb{R}_+. \quad (2.8)$$

Proof. We observe that

$$I_n(f_n) \circ \Psi_t = I_n(f_n) + n I_{n-1}(f_n(t, *)) - n I_{n-1}(f_n(v, *))|_{v=2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}}. \quad (2.9)$$

Consequently we have

$$\begin{aligned} & \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} I_n(f_n) \circ \Psi_s ds \\ &= I_n(f_n) + n \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} I_{n-1}(f_n(s, *)) ds - n I_{n-1}(f_n(v, *))|_{v=2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}}, \quad t \in \mathbb{R}_+, \end{aligned}$$

and applying this to (2.7) we obtain the conclusion. \square

In particular, under the condition (2.3) we have the equality

$$\nabla_t I_n(f_n) = n I_{n-1}(f_n(t, *)), \quad t \in \mathbb{R}_+,$$

as in Proposition 10 of [14]. The operator ∇ also admits an adjoint operator ∇^* given by

$$\nabla^*(I_n(g_{n+1})) := I_{n+1}(\mathbf{1}_{\Delta_{n+1}} \tilde{g}_{n+1}),$$

where \tilde{g}_{n+1} is the symmetrization of $g_{n+1} \in \hat{L}^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+)$ in $n+1$ variables, and ∇ is closable with domain

$$\text{Dom}(\nabla) = \{X \in L^2(\Omega) : E[\|\nabla X\|_{L^2(\mathbb{R}_+)}^2] < \infty\},$$

and we have the duality relation

$$E[\langle \nabla X, u \rangle_{L^2(\mathbb{R}_+, dx/2)}] = E[X \nabla^*(u)], \quad X \in \text{Dom}(\nabla), \quad (2.10)$$

for u in the domain $\text{Dom}(\nabla^*)$ of ∇^* , cf. Proposition 8 of [14]. The operator L defined on linear combinations of multiple stochastic integrals as

$$L I_n(f_n) := -\nabla^* \nabla_t I_n(f_n) = -n I_n(f_n), \quad f_n \in \hat{L}^2(\mathbb{R}_+^n),$$

is called the Ornstein-Uhlenbeck operator. By (2.6) the operator is well-defined, invertible for centered $X \in L^2(\Omega)$, and the inverse operator L^{-1} is given by

$$L^{-1} I_n(f_n) = -\frac{1}{n} I_n(f_n), \quad n \geq 1.$$

Recall that the operator ∇ satisfies the Clark-Ocone formula

$$X = E[X] + \int_0^\infty E[\nabla_t X | \mathcal{F}_t] d(Y_t - t/2), \quad (2.11)$$

for $X \in L^2(\Omega)$, see [14], Theorem 2. This relation is reformulated using the operator Ψ_t in the next proposition.

Proposition 2.2. For all $X \in L^2(\Omega)$ we have

$$X = E[X] + \int_0^\infty E[X \circ \Psi_t | \tilde{\mathcal{F}}_t] d(Y_t - t/2). \quad (2.12)$$

Proof. Since the integral term in the right hand side of (2.7) is constant in t on every interval of the form $[2k, 2k+2)$, $k \in \mathbb{N}$, we get

$$\begin{aligned} \int_0^\infty E[\nabla_t X | \tilde{\mathcal{F}}_t] d(Y_t - t/2) &= \int_0^\infty E \left[X \circ \Psi_t - \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} X \circ \Psi_s ds \middle| \tilde{\mathcal{F}}_t \right] d(Y_t - t/2) \\ &= \int_0^\infty E[X \circ \Psi_t | \tilde{\mathcal{F}}_t] d(Y_t - t/2), \end{aligned}$$

and (2.11) ends the proof. \square

In particular, it follows from the Clark-Ocone formula (2.11) that

$$\int_0^\infty E[I_{n-1}(f_n(t, *) | \tilde{\mathcal{F}}_t)] d(Y_t - t/2) = \frac{1}{n} I_n(f_n), \quad (2.13)$$

since the integral term in the right hand side of (2.8) is constant in t on every interval of the form $[2k, 2k+2)$, $k \in \mathbb{N}$.

Derivation operator

Given X a random variable of the form

$$X = f(U_0, \dots, U_n), \quad f \in \mathcal{C}_b^1([-1, 1]^{n+1}),$$

we consider the gradient D_t defined as

$$D_t X := \sum_{k=1}^n \partial_k f(U_0, \dots, U_n) \left((1 - U_k) \mathbf{1}_{(2k, 2k+1+U_k]}(t) - (1 + U_k) \mathbf{1}_{(2k+1+U_k, 2k+2]}(t) \right),$$

cf. Definition 3 of [14]. By Proposition 5 of [14] the gradient D is closable, and its closed domain is denoted by $\text{Dom}(D)$. For any $X \in \text{Dom}(D)$ and $\phi \in \mathcal{C}_b^1(\mathbb{R})$ we have $\phi(X) \in \text{Dom}(D)$, and the operator D satisfies the chain rule of derivation

$$D_t \phi(X) = \phi'(X) D_t X, \quad X \in \text{Dom}(D), \quad (2.14)$$

for all $\phi \in \mathcal{C}_b^1(\mathbb{R})$. The gradient operator

$$D : \text{Dom}(D) \subset L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$$

with domain $\text{Dom}(D)$, defined by $DX = (D_t X)_{t \in \mathbb{R}_+}$ satisfies the following Clark-Ocone representation formula, see Theorem 2 of [14].

Proposition 2.3. For $X \in L^2(\Omega)$ we have

$$X = E[X] + \int_0^\infty E[D_t X | \tilde{\mathcal{F}}_t] d(Y_t - t/2). \quad (2.15)$$

Covariance identities

From (2.14) the gradient operator D satisfies the following covariance identity, see e.g. Proposition 3.4.1 in [17], p. 121.

Lemma 2.4. Let $X, Y \in \text{Dom}(D)$. We have

$$\text{Cov}(X, Y) = \frac{1}{2} E \left[\int_0^\infty E[D_t X | \tilde{\mathcal{F}}_t] D_t Y dt \right].$$

Proof. By (2.1) and (2.15) we have

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E \left[\int_0^\infty E[D_t X | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \int_0^\infty E[D_t Y | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \right] \\ &= \frac{1}{2} E \left[\int_0^\infty E[D_t Y | \tilde{\mathcal{F}}_t] \left(E[D_t X | \tilde{\mathcal{F}}_t] - \Phi_t(X) \right) dt \right], \end{aligned}$$

where

$$\Phi_t(X) := \frac{1}{2} \sum_{k=0}^\infty \mathbf{1}_{(2k, 2k+2]}(t) \int_{2k}^{2k+2} E[D_r X | \tilde{\mathcal{F}}_r] dr, \quad t \in \mathbb{R}_+.$$

By the independence between $\tilde{\mathcal{F}}_{2k}$ and (U_k, \dots, U_n) we get

$$\begin{aligned} \Phi_t(X) &= \frac{1}{2} \sum_{k=0}^n \mathbf{1}_{(2k, 2k+2]}(t) \int_{2k}^{2k+2} \\ &E[\partial_k f(U_1, \dots, U_n) \left((1 - U_k) \mathbf{1}_{(2k, 2k+1+U_k]}(r) - (1 + U_k) \mathbf{1}_{(2k+1+U_k, 2k+2]}(r) \right) | \tilde{\mathcal{F}}_{2k}] dr \\ &= \frac{1}{2} \sum_{k=0}^n \mathbf{1}_{(2k, 2k+2]}(t) \int_{2k}^{2k+2} E[\partial_k f(y_1, \dots, y_{k-1}, U_k, \dots, U_n) \\ &\quad \times \left((1 - U_k) \mathbf{1}_{(2k, 2k+1+U_k]}(r) - (1 + U_k) \mathbf{1}_{(2k+1+U_k, 2k+2]}(r) \right) |_{(y_1, \dots, y_{k-1})=(U_1, \dots, U_{k-1})}] dr \\ &= \frac{1}{2} \sum_{k=0}^n \mathbf{1}_{(2k, 2k+2]}(t) E[\partial_k f(y_1, \dots, y_{k-1}, U_k, \dots, U_n) \\ &\quad \times \left((1 - U_k)(1 + U_k) - (1 - U_k)(1 + U_k) \right) |_{(y_1, \dots, y_{k-1})=(U_1, \dots, U_{k-1})}] \\ &= 0. \end{aligned}$$

We conclude that

$$\begin{aligned} \text{Cov}(X, Y) &= \frac{1}{2} E \left[\int_0^\infty E[D_t Y | \tilde{\mathcal{F}}_t] E[D_t X | \tilde{\mathcal{F}}_t] dt \right] \\ &= \frac{1}{2} E \left[\int_0^\infty E[E[D_t X | \tilde{\mathcal{F}}_t] D_t Y | \tilde{\mathcal{F}}_t] dt \right] \\ &= \frac{1}{2} E \left[\int_0^\infty E[D_t X | \tilde{\mathcal{F}}_t] D_t Y dt \right]. \quad \square \end{aligned}$$

As a consequence of Lemma 2.4 we have the inequality

$$\frac{1}{2} E[\langle D.X, E[D.X | \tilde{\mathcal{F}}.] \rangle_{L^2(\mathbb{R}_+)}] = \text{Var}[X] \leq \|X\|_{L^2(\Omega)}^2. \quad (2.16)$$

Using the operator ∇ and the Clark-Ocone formula (2.11)-(2.12) we can also obtain the covariance identity

$$\text{Cov}(X, Y) = \frac{1}{2} E \left[\int_0^\infty E[\nabla_t X | \tilde{\mathcal{F}}_t] \nabla_t Y dt \right]$$

from (2.1) and (2.7) as in the proof of Lemma 2.4.

Stein kernel

The next proposition shows that the Stein kernel φ_X defined in (2.17) is a Stein kernel in the sense of Definition (2.1) in [6].

Proposition 2.5. *Let $X \in \text{Dom}(D)$ be such that $E[X] = 0$. The Stein kernel*

$$\varphi_X(z) := \frac{1}{2} E \left[\int_0^\infty D_t X E[D_t X | \tilde{\mathcal{F}}_t] dt \mid X = z \right], \quad z \in \mathbb{R}, \quad (2.17)$$

satisfies

$$\text{Cov}(X, \phi(X)) = E[\phi'(X)\varphi_X(X)], \quad (2.18)$$

for any $\phi \in C_b^1(\mathbb{R})$.

Proof. We note that by Lemma 2.4 and Jensen's inequality we have

$$\|\varphi_X(X)\|_{L^1(\Omega)} \leq \|\varphi_X(X)\|_{L^2(\Omega)} \leq E[X^2] = \sqrt{E \left[\int_0^\infty |D_t X|^2 \frac{dt}{2} \right]} < \infty,$$

and, for any $\phi \in C_b^1(\mathbb{R})$,

$$\begin{aligned} \text{Cov}(X, \phi(X)) &= \frac{1}{2} E \left[\int_0^\infty E[D_t X | \mathcal{F}_t] D_t \phi(X) dt \right] \\ &= \frac{1}{2} E \left[\phi'(X) \int_0^\infty D_t X E[D_t X | \mathcal{F}_t] dt \right] \\ &= \frac{1}{2} E \left[E \left[\phi'(X) \int_0^\infty D_t X E[D_t X | \mathcal{F}_t] dt \mid X \right] \right] \\ &= E[\phi'(X)\varphi_X(X)]. \end{aligned} \quad (2.19)$$

□

In particular, (2.19) shows that we have

$$E[\varphi_X(X)] = \text{Var}[X], \quad X \in \text{Dom}(D).$$

In the sequel we will also use the identity

$$\varphi_{X_k}(y) = -\frac{1}{F'_k(y)} \int_{-\infty}^y x dF_k(x), \quad (2.20)$$

see Relation (3.17) in [11]. Next, we review some examples of Stein kernels.

Gaussian case. The Stein kernel of $X_1 \simeq \mathcal{N}(0, \sigma^2)$ with the Gaussian cumulative distribution function $F(x)$ is given by

$$\varphi_{X_1}(y) = -\frac{1}{F'(y)} \int_{-\infty}^y x dF(x) = \sigma^2, \quad y \in \mathbb{R}.$$

Gamma case. When X_1 has the centered gamma distribution with shape parameter $s > 0$ and density function

$$F'_s(x) = \frac{(x+s)^{s-1}}{\Gamma(s)} e^{-(x+s)} \quad x \in [-s, \infty), \quad k \geq 1,$$

we have $E[|X_1 - s|] = 2s^s e^{-s}$, hence the Stein kernel of X_1 is

$$\varphi_{X_1}(y) = -\frac{1}{F'_s(y)} \int_{-s}^y x dF_s(x) = y + s, \quad y \in \mathbb{R}.$$

Beta case. When X_1 has the centered Beta($\alpha, 1$) distribution, $\alpha > 0$, we have

$$F_\alpha(x) = \left(\frac{\alpha}{\alpha+1} + x \right)^\alpha, \quad x \in \left[-\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1} \right],$$

and the Stein kernel of X_1 is

$$\varphi_{X_1}(y) = -\frac{1}{F'_\alpha(y)} \int_{-\alpha/(\alpha+1)}^y x dF_\alpha(x) = \frac{1}{\alpha+1} \left(\frac{\alpha}{\alpha+1} + y \right) \left(\frac{1}{\alpha+1} - y \right), \quad y \in \mathbb{R}. \quad (2.21)$$

Single stochastic integrals. Such integrals can be used to represent the sum Z_n of independent centered random variables $(X_k)_{k \geq 1}$ as

$$Z_n = \sum_{k=1}^n X_k = I_1(f_1 \mathbf{1}_{[0,2n]}), \quad (2.22)$$

where

$$f_1(t) := \sum_{k=0}^{\infty} F_k^{-1} \left(\frac{t}{2} - k \right) \mathbf{1}_{[2k, 2k+2)}(t), \quad (2.23)$$

satisfies $\int_{2k}^{2k+2} f_1(t) dt = 0$, $k \in \mathbb{N}$, and

$$F_k^{-1}(t) := \inf\{s \in \mathbb{R}_+ : F_X(s) \geq t\}, \quad t \in [0, 1],$$

is the right-continuous inverse of the cumulative distribution function F_k of X_k , $k \geq 1$.

In the sequel we let $C_{\square}^1(\mathbb{R}_+)$ denote the set of functions which are C^1 on every interval of the form $(2k, 2k+2)$, $k \in \mathbb{N}$. The next lemma can be useful when computing the Stein kernel of single stochastic integrals according to (2.17), see Propositions 4.3 and 4.4 below.

Lemma 2.6. Assume that $Z_n = I_1(f_1 \mathbf{1}_{[0,2n]}) = \sum_{k=1}^n X_k$ belongs to $\text{Dom}(D)$, $n \geq 1$. We have

$$\langle D.Z_n, E[D.Z_n | \tilde{\mathcal{F}}] \rangle_{L^2(\mathbb{R}_+)} = -2I_1(\varphi_{X_{1+\lfloor \cdot/2 \rfloor}}(f_1(\cdot))) + E[Z_n^2].$$

Proof. We note that for $f_1 \in C_{\square}^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, we have

$$D_t I_1(f_1) = \sum_{k=0}^{\infty} ((1 - U_k) \mathbf{1}_{(2k, 2k+1+U_k]}(t) - (1 + U_k) \mathbf{1}_{(2k+1+U_k, 2k+2]}(t)) f_1'(2k+1+U_k).$$

Next, by Proposition 10 and Lemma 1 in [14] we get

$$E[D_t I_1(f_1) | \tilde{\mathcal{F}}_t] = E[\nabla_t I_1(f_1) | \tilde{\mathcal{F}}_t] = f_1(t), \quad t \in \mathbb{R}_+,$$

hence by (2.3) we have

$$\begin{aligned} & \langle D.I_1(f_1), E[D.I_1(f_1) | \tilde{\mathcal{F}}] \rangle_{L^2(\mathbb{R}_+)} \\ &= \int_0^\infty \sum_{k=0}^{\infty} ((1 - U_k) \mathbf{1}_{(2k, 2k+1+U_k]}(s) - (1 + U_k) \mathbf{1}_{(2k+1+U_k, 2k+2]}(s)) f_1'(2k+1+U_k) f_1(s) ds \\ &= \int_0^\infty \sum_{k=0}^{\infty} (\mathbf{1}_{(2k, 2k+1+U_k]}(s) - \mathbf{1}_{(2k+1+U_k, 2k+2]}(s)) f_1'(2k+1+U_k) f_1(s) ds \\ &= 2 \int_0^\infty \sum_{k=0}^{\infty} (\mathbf{1}_{(2k, 2k+1+U_k]}(s)) f_1'(2k+1+U_k) f_1(s) ds \end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{k=0}^{\infty} f_1'(2k+1+U_k) \int_0^{2k+1+U_k} f_1(s) ds \\
&= 2 \int_0^{\infty} f_1'(t) \int_0^t f_1(s) ds d(Y_t - t/2) + \int_0^{\infty} f_1'(t) \int_0^t f_1(s) ds dt \\
&= 2 \int_0^{\infty} f_1'(s) \int_0^t f_1(s) ds d(Y_t - t/2) + \int_0^{\infty} |f_1(t)|^2 dt.
\end{aligned}$$

On the other hand, by (2.1) and (2.20), see (3.17) in [11], we have

$$\begin{aligned}
f_1'(x) \int_0^x f_1(t) dt &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{F_k'(F_k^{-1}((x-2k)/2))} \int_{2k}^x F_k^{-1} \left(\frac{t}{2} - k \right) dt \mathbf{1}_{[2k, 2k+2)}(x) \\
&= \sum_{k=0}^{\infty} \mathbf{1}_{[2k, 2k+2)}(x) \frac{1}{F_k'(F_k^{-1}((x-2k)/2))} \int_{-\infty}^{F_k^{-1}((x-2k)/2)} t dF_k(t) \\
&= - \sum_{k=0}^{\infty} \varphi_{X_k}(F_k^{-1}((x-2k)/2)) \mathbf{1}_{[2k, 2k+2)}(x) \\
&= - \sum_{k=0}^{\infty} \varphi_{X_k}(f_1(x)) \mathbf{1}_{[2k, 2k+2)}(x) \\
&= -\varphi_{X_{1+\lfloor x/2 \rfloor}}(f_1(x)) \mathbf{1}_{[2k, 2k+2)}(x),
\end{aligned}$$

where we used the identity (2.20). \square

Density representation and bounds

Working along the lines of the proof of Theorem 3.1 in [11] by replacing (3.15) therein with (2.19) above we can derive the following result, where $\text{Supp}(f)$ denotes the support of the function f .

Proposition 2.7. *Let $X \in \text{Dom}(D)$ be such that $E[X] = 0$. The law of X has a density p_X with respect to the Lebesgue measure if and only if the Stein kernel φ_X defined in (2.17) satisfies $\varphi_X(X) > 0$ a.s. In this case $\text{Supp}(p_X)$ is a closed interval of \mathbb{R} containing 0 and we have*

$$p_X(z) = \frac{E[|X|]}{2\varphi_X(z)} \exp\left(-\int_0^z \frac{u}{\varphi_X(u)} du\right), \quad \text{a.e. } z \in \text{Supp}(p_X).$$

As a consequence of Proposition 2.7 we get the following result on density bounds as in Corollary 3.5 of [11].

Proposition 2.8. *Let $X \in \text{Dom}(D)$ be a centered random variable such that*

$$0 < c \leq \int_0^{\infty} D_s X E[D_s X | \mathcal{F}_s] ds \leq C \quad \text{a.s.},$$

where $C, c > 0$ are positive constants. Then the density p_X satisfies

$$\frac{E[|X|]}{2C} \exp\left(-\frac{z^2}{2c}\right) \leq p_X(z) \leq \frac{E[|X|]}{2c} \exp\left(-\frac{z^2}{2C}\right), \quad \text{a.e. } z \in \mathbb{R},$$

and the tail probabilities satisfy

$$P(X \geq x) \leq \exp\left(-\frac{x^2}{2C}\right) \quad \text{and} \quad P(X \leq -x) \leq \exp\left(-\frac{x^2}{2C}\right), \quad x > 0.$$

3 Stein approximation bounds

The total variation distance between two real-valued random variables X and Y is defined by

$$d_{TV}(X, Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|,$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel subsets of \mathbb{R} . The Wasserstein distance between the laws of X and Y is defined by

$$d_W(X, Y) := \sup_{h \in \text{Lip}(1)} |E[h(X)] - E[h(Y)]|,$$

where $\text{Lip}(1)$ is the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1.

In the following propositions we derive bounds for the Wasserstein and total variation distances between the normal distribution and the distribution of a given random variable $X \in \text{Dom}(D)$. Recall that by Stein's lemma, cf. [21], [8], for any continuous function $h : \mathbb{R} \rightarrow [0, 1]$ the Stein equation

$$h(x) - E[h(X)] = f'_h(x) - xf_h(x),$$

where $X \sim \mathcal{N}$, admits a solution $f_h(x)$ that satisfies the bound $|f'_h(x)| \leq 2$. In the sequel we denote by

$$\mathcal{T} := \{h \in \mathcal{C}_b^2(\mathbb{R}) : \|h'\|_\infty \leq 1, \|h''\|_\infty \leq 2\}$$

the space of twice differentiable functions whose first derivative is bounded by 1 and whose second derivative is bounded by 2. For the gamma approximation we will use the distance

$$d_{\mathcal{H}}(X, Y) := \sup_{h \in \mathcal{H}} |E[h(X)] - E[h(Y)]|,$$

where

$$\mathcal{H} := \{h \in \mathcal{C}_b^2(\mathbb{R}) : \max\{\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty\} \leq 1\}.$$

Derivation operator bounds

In the next Proposition 3.1 we derive a Stein bound using the Stein kernel $\varphi_X(z)$ defined in (2.17), see also Proposition 3.3 of [18] for a bound using a different probabilistic representation of the Stein kernel. Here we denote by $\Gamma(\nu/2)$ a random variable distributed according to the gamma law with parameters $(\nu/2, 1)$, $\nu > 0$. We also let $\langle \cdot, \cdot \rangle$ denote the usual inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$.

Proposition 3.1. *For any $X \in \text{Dom}(D)$ such that $E[X] = 0$, we have*

$$d_W(X, \mathcal{N}) \leq E[|1 - \varphi_X(X)|] \leq |1 - E[X^2]| + \|\varphi_X(X) - E[\varphi_X(X)]\|_{L^2(\Omega)},$$

where the Stein kernel φ_X is defined in (2.17), and

$$d_{TV}(X, \mathcal{N}) \leq 2E[|1 - \varphi_X(X)|] \leq 2|1 - E[X^2]| + 2\|\varphi_X(X) - E[\varphi_X(X)]\|_{L^2(\Omega)}.$$

If moreover X is a.s. $(-\nu, \infty)$ -valued then we have

$$d_{\mathcal{H}}(X, \Gamma_\nu) \leq E[|2(X+\nu) - \varphi_X(X)|] \leq \|2(X+\nu) - E[X^2]\|_{L^2(\Omega)} + \|\varphi_X(X) - E[\varphi_X(X)]\|_{L^2(\Omega)}.$$

Proof. We focus on the first inequalities, as the second inequalities follow from the triangle inequality and Jensen's inequality, and the identity $E[\varphi_X(X)] = E[X^2]$ that follows from Lemma 2.4.

(i) By Lemma 2.4 we have

$$\begin{aligned} E[Xf_h(X)] &= \frac{1}{2}E\left[\int_0^\infty E[D_tX | \tilde{\mathcal{F}}_t]D_t f_h(X) dt\right] \\ &= \frac{1}{2}E\left[f'_h(X)\int_0^\infty E[D_tX | \tilde{\mathcal{F}}_t]D_tX dt\right]. \end{aligned} \quad (3.1)$$

Hence, using the bound (2.33) in [12] and (3.1), we get

$$\begin{aligned} d_W(X, \mathcal{N}) &\leq \sup_{\phi \in \mathcal{T}} |E[\phi'(X) - X\phi(X)]| \\ &= \sup_{\phi \in \mathcal{T}} \left| E\left[\phi'(X)\left(1 - \frac{1}{2}\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\right)\right] \right| \\ &= \sup_{\phi \in \mathcal{T}} |E[\phi'(X)(1 - \varphi_X(X))]| \\ &\leq E[|1 - \varphi_X(X)|]. \end{aligned} \quad (3.2)$$

(ii) By the covariance identity (3.1) we have

$$\begin{aligned} |E[h(X)] - E[h(\mathcal{N})]| &= \left| E\left[f'_h(X)\left(1 - \frac{1}{2}\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\right)\right] \right| \\ &= |E[f'_h(X)(1 - \varphi_X(X))]| \\ &\leq 2E[|1 - \varphi_X(X)|], \end{aligned}$$

and this bound can be extended to $h = \mathbf{1}_C$ for any $C \in \mathcal{B}_b(\mathbb{R})$ by the same approximation argument as in the proof of e.g. Theorem 2.1 of [18].

(iii) Given $h \in \mathcal{H}$ a twice differentiable function bounded above by 1 we choose $c > 0$ and $a < 1/2$ such that

$$|h(x)| \leq ce^{ax}, \quad x > -\nu.$$

By e.g. Lemma 1.3-(ii) of [9], letting $\Gamma_\nu := 2\Gamma(\nu/2) - \nu$, the functional equation

$$2(x + \nu)f'(x) = xf(x) + h(x) - E[h(\Gamma_\nu)], \quad x > -\nu,$$

has a solution f_h which is bounded and differentiable on $(-\nu, \infty)$, and such that

$$\|f_h\|_\infty \leq 2\|h'\|_\infty \quad \text{and} \quad \|f'_h\|_\infty \leq \|h''\|_\infty.$$

By the covariance identity (3.1) on $\mathcal{C}_b^1(\mathbb{R})$ for the centered random variable X we have

$$\begin{aligned} |E[h(X)] - E[h(\Gamma_\nu)]| &= |E[(2(X + \nu)f'_h(X) - Xf_h(X))]| \\ &= \left| E\left[f'_h(X)\left(2(X + \nu) - \langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\right)\right] \right| \\ &= |E[f'_h(X)(2(X + \nu) - \varphi_X(X))]| \\ &\leq \|h''\|_\infty E[|2(X + \nu) - \varphi_X(X)|]. \end{aligned}$$

The claim follows by taking the supremum over all functions $h \in \mathcal{H}$. \square

As a consequence of Proposition 3.1, for any $X \in \text{Dom}(D)$ such that $E[X] = 0$, we have

$$\begin{aligned} d_W(X, \mathcal{N}) &\leq |1 - E[X^2]| + \sqrt{E[(\varphi_X(X) - E[X^2])^2]} \\ &= |1 - E[X^2]| + \sqrt{E[(\varphi_X(X))^2 - 2\varphi_X(X)E[X^2] + (E[X^2])^2]} \\ &= |1 - E[X^2]| + \sqrt{E[(\varphi_X(X))^2] - (E[X^2])^2}, \end{aligned} \quad (3.3)$$

and

$$d_{TV}(X, \mathcal{N}) \leq 2|1 - E[X^2]| + 2\sqrt{E[(\varphi_X(X))^2] - (E[X^2])^2}.$$

Similarly, Proposition 3.1 implies the following corollary which applies in particular to smooth functionals $X \in \text{Dom}(D)$.

Proposition 3.2. *For any $X \in \text{Dom}(D)$ such that $E[X] = 0$, we have*

$$\begin{aligned} d_W(X, \mathcal{N}) &\leq \frac{1}{2} \|2 - \langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\|_{L^2(\Omega)} \\ &\leq |1 - E[X^2]| + \frac{1}{2} \|\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle - E[\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle]\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} d_{TV}(X, \mathcal{N}) &\leq \|2 - \langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\|_{L^2(\Omega)} \\ &\leq 2|1 - E[X^2]| + \|\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle - E[\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle]\|_{L^2(\Omega)}. \end{aligned}$$

For any a.s. $(-\nu, \infty)$ -valued $X \in \text{Dom}(D)$ such that $E[X] = 0$, we have

$$\begin{aligned} d_{\mathcal{H}}(X, \Gamma_\nu) &\leq \|2(X + \nu) - \langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle\|_{L^2(\Omega)} \\ &\leq \|2(X + \nu) - \|X\|_{L^2(\Omega)}^2\|_{L^2(\Omega)} + \|\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle - E[\langle D.X, E[D.X | \tilde{\mathcal{F}}] \rangle]\|_{L^2(\Omega)}. \end{aligned}$$

Finite difference operator bound

Using the finite difference operator ∇ we obtain the following bound which applies in particular to multiple stochastic integrals, see Proposition 3.4 below.

Proposition 3.3. *Let $X \in \text{Dom}(\nabla)$ be such that $E[X] = 0$. We have*

$$\begin{aligned} d_W(X, \mathcal{N}) &\leq E \left[\left| 1 - \frac{1}{2} \langle \nabla.X, -\nabla.L^{-1}X \rangle \right| \right] \\ &\quad + \frac{1}{2} E \left[\int_0^\infty |\nabla_t L^{-1}X| |\nabla_t X|^2 dt \right] + \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1}X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |\nabla_s X|^2 ds dt \right]. \end{aligned} \quad (3.4)$$

Proof. By (2.7), for every function $f \in C^2(\mathbb{R})$, the finite difference operator ∇ satisfies

$$\begin{aligned} \nabla_t f(X) &= \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} (f(X \circ \Psi_t) - f(X \circ \Psi_s)) ds \\ &= \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} (f'(X \circ \Psi_s)(X \circ \Psi_t - X \circ \Psi_s) + R_f(X \circ \Psi_t - X \circ \Psi_s)) ds \\ &= \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} f'(X \circ \Psi_s)(X \circ \Psi_t - X \circ \Psi_s) ds + \frac{1}{2} \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} R_f(X \circ \Psi_t - X \circ \Psi_s) ds, \end{aligned}$$

$t \in \mathbb{R}_+$, where the function R_f is such that $|R_f(y)| \leq y^2 \|f''\|_\infty / 2$, $y \in \mathbb{R}$. Hence for any $f \in \mathcal{T}$, by the duality relation (2.10) we have

$$\begin{aligned} E[f'(X) - Xf(X)] &= E[f'(X) - XLL^{-1}f(X)] \\ &= E \left[f'(X) - \frac{1}{2} \langle \nabla f(X), -\nabla L^{-1}X \rangle \right] \\ &= E \left[f'(X) - \frac{1}{2} \int_0^\infty \nabla_t f(X) (-\nabla_t L^{-1}X) dt \right] \\ &= E \left[f'(X) - \frac{1}{4} \int_0^\infty \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} f'(X \circ \Psi_s)(X \circ \Psi_t - X \circ \Psi_s) ds (-\nabla_t L^{-1}X) dt \right] \end{aligned}$$

$$-\frac{1}{4}E \left[\int_0^\infty \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} R_f(X \circ \Psi_t - X \circ \Psi_s) ds (-\nabla_t L^{-1} X) dt \right]. \quad (3.5)$$

Regarding the first term, we note that for any two square-integrable random variables F and G , by (2.7) we have the relations

$$E[(F \circ \Psi_t)G] = \frac{1}{2}E \left[(F \circ \Psi_t) \int_{2k}^{2k+2} G \circ \Psi_s ds \right] \quad (3.6)$$

and

$$E[(\nabla_t F)G] = \frac{1}{2}E \left[\nabla_t F \int_{2k}^{2k+2} G \circ \Psi_s ds \right], \quad (3.7)$$

$t \in [2k, 2k+2]$, $k \in \mathbb{N}$, hence

$$\begin{aligned} & \left| E \left[f'(X) - \frac{1}{4} \int_0^\infty \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} f'(X \circ \Psi_s) (X \circ \Psi_t - X \circ \Psi_s) ds (-\nabla_t L^{-1} X) dt \right] \right| \\ &= \left| E \left[f'(X) - \frac{1}{2} \int_0^\infty f'(X) (X \circ \Psi_t - X) (-\nabla_t L^{-1} X) dt \right] \right| \\ &= \left| E \left[f'(X) \left(1 - \frac{1}{2} \int_0^\infty (X \circ \Psi_t - X) (-\nabla_t L^{-1} X) dt \right) \right] \right| \\ &= \left| E \left[f'(X) \left(1 - \frac{1}{2} \int_0^\infty \nabla_t X (-\nabla_t L^{-1} X) dt \right) \right] \right| \\ &\leq E \left[\left| 1 - \frac{1}{2} \int_0^\infty \nabla_t X (-\nabla_t L^{-1} X) dt \right| \right], \end{aligned}$$

because $\|f'\|_\infty \leq 1$. Next, given that $\|f''\|_\infty \leq 2$, the term (3.5) can be bounded as

$$\begin{aligned} & \frac{1}{4} \left| E \left[\int_0^\infty \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} R_f(X \circ \Psi_t - X \circ \Psi_s) ds (-\nabla_t L^{-1} X) dt \right] \right| \\ &\leq \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |X \circ \Psi_t - X \circ \Psi_s|^2 ds dt \right] \\ &= \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |\nabla_t X - \nabla_s X|^2 ds dt \right] \\ &= \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} (|\nabla_t X|^2 + |\nabla_s X|^2 - 2\nabla_s X \nabla_t X) ds dt \right] \\ &= \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} (|\nabla_t X|^2 + |\nabla_s X|^2) ds dt \right] \\ &= \frac{1}{2} E \left[\int_0^\infty |\nabla_t L^{-1} X| |\nabla_t X|^2 dt \right] + \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |\nabla_s X|^2 ds dt \right], \end{aligned}$$

where we used the relations

$$E \left[(F \circ \Psi_t) \int_{2k}^{2k+2} \nabla_s G ds \right] = 0 \quad \text{and} \quad E \left[\nabla_t F \int_{2k}^{2k+2} \nabla_s G ds \right] = 0,$$

$t \in [2k, 2k+2]$, $k \in \mathbb{N}$, that hold similarly to (3.6)-(3.7). We conclude to (3.4) by the inequality (3.2), which is the bound (2.33) in [12]. \square

The second term in (3.4) can also be written as

$$\begin{aligned} & \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |X \circ \Psi_t - X \circ \Psi_s|^2 ds dt \right] \\ &= \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |X \circ \Psi_t - X|^2 \circ \Psi_s ds dt \right] \\ &= \frac{1}{4} E \left[\int_0^\infty |\nabla_t L^{-1} X| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |X \circ \Psi_t - X|^2 ds dt \right] \\ &= \frac{1}{2} E \left[\int_0^\infty |\nabla_t L^{-1} X| |X \circ \Psi_t - X|^2 dt \right]. \end{aligned}$$

Taking $X = I_n(f_n)$ in Proposition 3.3, we get the following result.

Proposition 3.4. *Let $f_n \in \hat{L}^2(\mathbb{R}_+^n)$. The following estimate holds:*

$$\begin{aligned} d_W(I_n(f_n), \mathcal{N}) &\leq \sqrt{E \left[\left(1 - \frac{1}{n} \|\nabla I_n(f_n)\|_{L^2(\mathbb{R}_+, dx/2)}^2 \right)^2 \right]} \\ &+ \frac{1}{2n} E \left[\int_0^\infty |\nabla_t I_n(f_n)|^3 dt \right] + \frac{1}{4n} E \left[\int_0^\infty |\nabla_t I_n(f_n)| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |\nabla_s I_n(f_n)|^2 ds dt \right]. \end{aligned}$$

4 Single stochastic integrals

For single stochastic integrals, Proposition 3.4 shows the following.

Proposition 4.1. *For $f_1 \in L^2(\mathbb{R}_+)$ such that $\int_{2k}^{2k+2} f_1(t) dt = 0$, $k \in \mathbb{N}$, we have*

$$\begin{aligned} d_W(I_1(f_1), \mathcal{N}) &\leq \left| 1 - \frac{1}{2} \int_0^\infty |f_1(t)|^2 dt \right| + \frac{1}{2} \int_0^\infty |f_1(t)|^3 dt \\ &+ \frac{1}{4} \int_0^\infty |f_1(t)| \int_{2\lfloor t/2 \rfloor}^{2\lfloor t/2 \rfloor + 2} |f_1(s)|^2 ds dt \\ &\leq \left| 1 - \frac{1}{2} \int_0^\infty |f_1(t)|^2 dt \right| + \int_0^\infty |f_1(t)|^3 dt. \end{aligned} \quad (4.1)$$

Consider now a sum Z_n of independent centered random variables $(X_k)_{k \geq 1}$ written, as in (2.22), as

$$Z_n := I_1(f_1 \mathbf{1}_{[0, 2n]}) = \sum_{k=1}^n X_k, \quad (4.2)$$

with $f_1 \in C_0^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ given by (2.23) from the respective cumulative distribution functions $(F_k)_{k \geq 1}$. In this case, Proposition 4.1 can be rewritten as follows.

Proposition 4.2. *Given $(Z_n)_{n \geq 1}$ written as in (4.2) we have*

$$d_W(Z_n, \mathcal{N}) \leq |1 - E[Z_n^2]| + \sum_{k=1}^n E[|X_k|^3] + \sum_{k=0}^\infty E[|X_k|] E[|X_k|^2]. \quad (4.3)$$

Proof. We note that $f_1(2k+1+U_k) = F_k^{-1}((U_k+1)/2)$ has same distribution as X_k , $k \geq 1$, hence (4.1) can be rewritten as

$$d_W(Z_n, \mathcal{N}) \leq |1 - E[Z_n^2]| + \frac{1}{2} \int_0^{2n} |f_1(t)|^3 dt + \frac{1}{4} \sum_{k=0}^\infty \int_{2k}^{2k+2} |f_1(t)| dt \int_{2k}^{2k+2} |f_1(s)|^2 ds$$

$$\begin{aligned}
 &= |1 - E[Z_n^2]| + \frac{1}{2} \sum_{k=1}^n \int_0^2 |F_k^{-1}(t/2)|^3 dt + \frac{1}{4} \sum_{k=0}^{\infty} \int_0^2 |F_k^{-1}(t/2)| dt \int_0^2 |F_k^{-1}(t/2)|^2 ds \\
 &= |1 - E[Z_n^2]| + \sum_{k=1}^n \int_{-\infty}^{\infty} |x|^3 dF_k(x) + \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} |x| dF_k(x) \int_{-\infty}^{\infty} |y|^2 dF_k(y) \\
 &= |1 - E[Z_n^2]| + \sum_{k=1}^n E[|X_k|^3] + \sum_{k=0}^{\infty} E[|X_k|] E[|X_k|^2]. \quad \square
 \end{aligned}$$

Using Hölder’s inequality, Proposition 4.2 shows that

$$d_W(\tilde{Z}_n, \mathcal{N}) \leq \frac{2}{(E[(Z_n)^2])^{3/2}} \sum_{k=1}^n E[|X_k|^3], \quad n \geq 1, \tag{4.4}$$

for the normalized sum $\tilde{Z}_n := (E[(Z_n)^2])^{-1/2} \sum_{k=1}^n X_k$, which recovers the bound (1.1) of [2], with however a worse constant.

Bernoulli random variables

Given $(p_k)_{k \geq 1}$ a sequence in $(0, 1)$, letting

$$f(t) := \sum_{k=1}^{\infty} \frac{\alpha_k}{\sqrt{p_k(1-p_k)}} (\mathbf{1}_{[2k-2, 2k-2+2p_k]}(t) - p_k), \quad t \in \mathbb{R}_+,$$

the single integral $I_1(f_1 \mathbf{1}_{[0, 2n]})$ becomes a weighted sum

$$I_1(f_1 \mathbf{1}_{[0, 2n]}) = \sum_{k=1}^n \alpha_k X_k$$

of centered and normalized Bernoulli random variables $(X_k)_{k \geq 1}$ with parameters $(p_k)_{k \geq 1}$, and (4.3) shows that

$$d_W(I_1(f_1), \mathcal{N}) \leq \left| 1 - \sum_{k=1}^{\infty} |\alpha_k|^2 \right| + 2 \sum_{k=1}^{\infty} |\alpha_k|^3 \frac{1 - 2p_k(1-p_k)}{\sqrt{p_k(1-p_k)}},$$

which provides a simple distance bound for the sum of non-symmetric Bernoulli random variables, cf. Corollary 3.3 of [10], Corollary 4.1 of [19] and Theorem 4.1 of [4] for other versions.

By Proposition 3.2 we have the following result that uses the derivation operator D .

Proposition 4.3. For $f_1 \in C_{\square}^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ such that $\int_{2k}^{2k+2} f_1(t) dt = 0, k \in \mathbb{N}$, we have

$$d_W(I_1(f_1), \mathcal{N}) \tag{4.5}$$

$$\leq \left| 1 - \frac{1}{2} \int_0^{\infty} |f_1(t)|^2 dt \right| + \frac{1}{2} \sqrt{2 \int_0^{\infty} \left| f_1'(x) \int_0^x f_1(t) dt \right|^2 dx - \sum_{k=0}^{\infty} \left(\int_{2k}^{2k+2} |f_1(t)|^2 dt \right)^2}.$$

Proof. We note that by (2.1) and Lemma 2.6 we have

$$\begin{aligned}
 &E \left[\left(\langle D.Z_n, E[D.Z_n | \tilde{\mathcal{F}}] \rangle - E[\langle D.Z_n, E[D.Z_n | \tilde{\mathcal{F}}] \rangle] \right)^2 \right] \\
 &= \frac{1}{2} \int_0^{2n} \left| f_1'(x) \int_0^x f_1(t) dt \right|^2 dx - \frac{1}{4} \sum_{k=1}^n \left(\int_{2k-2}^{2k} |f_1(t)|^2 dt \right)^2, \quad \square
 \end{aligned}$$

Proposition 4.3 can be rewritten as follows using sums Z_n of random variables $(X_k)_{k \geq 1}$.

Proposition 4.4. *Assume that $(X_k)_{k \geq 1}$ is a sequence of independent centered random variables having non-vanishing continuous densities. Then the sum*

$$Z_n := \sum_{k=1}^n X_k, \quad n \geq 1,$$

satisfies the bound

$$d_W(Z_n, \mathcal{N}) \leq |1 - E[Z_n^2]| + \sqrt{\sum_{k=1}^n (E[(\varphi_{X_k}(X_k))^2] - (E[(X_k)^2])^2)}. \quad (4.6)$$

Proof. By Lemma 2.6, (2.1) and (2.20), see (3.17) in [11], we have

$$\begin{aligned} & \frac{1}{2} \int_0^{2n} \left| f_1'(x) \int_0^x f_1(t) dt \right|^2 dx - \frac{1}{4} \sum_{k=1}^n \left(\int_{2k-2}^{2k} |f_1(t)|^2 dt \right)^2 \\ &= \sum_{k=1}^n \left(\int_0^1 \left| \frac{1}{F_k'(F_k^{-1}(x))} \int_0^x F_k^{-1}(t) dt \right|^2 dx - (E[(X_k)^2])^2 \right) \\ &= \sum_{k=1}^n \left(\int_{-\infty}^{\infty} \left| \frac{1}{F_k'(y)} \int_{-\infty}^y x dF_k(x) \right|^2 dF_k(y) - (E[(X_k)^2])^2 \right) \\ &= \sum_{k=1}^n (E[(\varphi_{X_k}(X_k))^2] - (E[(X_k)^2])^2), \quad \square \end{aligned}$$

Next, we consider some particular cases.

Gaussian case. The Stein kernel of X_k centered Gaussian is given by

$$\varphi_{X_k}(y) = -\frac{1}{F_k'(y)} \int_{-\infty}^y x dF_k(x) = E[X_k^2], \quad y \in \mathbb{R}, \quad k \geq 1,$$

and the bound (4.6) recovers $d_W(Z_n, \mathcal{N}) \leq |1 - E[Z_n^2]|$ as expected.

Gamma case. The Stein kernel of X_k a centered gamma random variable is $\varphi_{Z_n}(y) = y + E[Z_n^2]$, $y \in \mathbb{R}$, hence

$$E[(\varphi_{Z_n}(Z_n))^2] = E[(Z_n + E[Z_n^2])^2] = E[Z_n^2](1 + E[Z_n^2]),$$

and the bound (4.6) shows that the sum Z_n satisfies

$$d_W(Z_n, \mathcal{N}) \leq |1 - E[Z_n^2]| + \sqrt{E[Z_n^2]}, \quad n \geq 1.$$

By the scaling relation

$$\varphi_{aZ_n}(y) = a^2 \varphi_{Z_n}(y/a) = ay + a^2 E[Z_n^2], \quad y \in \mathbb{R},$$

we find that the normalized sum $\tilde{Z}_n := Z_n / \sqrt{E[Z_n^2]}$ satisfies

$$d_W(\tilde{Z}_n, \mathcal{N}) \leq \frac{1}{\sqrt{E[Z_n^2]}}, \quad n \geq 1.$$

In particular, in the i.i.d. case we have

$$d_W(\tilde{Z}_n, \mathcal{N}) \leq \frac{1}{\sqrt{nE[X_1^2]}}, \quad n \geq 1,$$

which systematically improves on (4.4) and on the bound (1.1) of [2], i.e.

$$\begin{aligned} d_W(Z_n, \mathcal{N}) &\leq n \frac{E[|X_1|^3]}{(E[Z_n^2])^{3/2}} \\ &= \frac{2}{\sqrt{nE[X_1^2]}} \left(\frac{2\Gamma(3 + E[X_1^2], E[X_1^2]) + 2(E[X_1^2])^{2+E[X_1^2]}e^{-E[X_1^2]}(1 + E[X_1^2])}{\Gamma(3 + E[X_1^2])} - 1 \right), \end{aligned}$$

where $\Gamma(3 + s, s)$ denotes the upper incomplete gamma function. Indeed, the ratio

$$2 \left(\frac{2\Gamma(3 + s, s) + 2s^{2+s}e^{-s}(1 + s)}{\Gamma(3 + s)} - 1 \right),$$

of the two bounds tends to infinity as s tends to infinity, and has smallest value 2 as s tends to 0.

Beta case. When X_k has the centered Beta($\alpha, 1$) distribution, $\alpha > 0$, $k \geq 1$, we have

$$F(x) = \left(\frac{\alpha}{\alpha + 1} + x \right)^\alpha, \quad x \in \left[-\frac{\alpha}{\alpha + 1}, \frac{1}{\alpha + 1} \right],$$

and $E[X_k^2] = \alpha/((\alpha + 1)^2(\alpha + 2))$, hence by (2.21) we have

$$\begin{aligned} E[(\varphi_{X_k}(X_k))^2] &= \frac{1}{(\alpha + 1)^2} E \left[\left(\frac{\alpha}{\alpha + 1} + X_k \right)^2 \left(\frac{1}{\alpha + 1} - X_k \right)^2 \right] \\ &= \frac{2\alpha}{(\alpha + 4)(\alpha + 3)(\alpha + 2)(\alpha + 1)^2}, \end{aligned} \quad (4.7)$$

and by Proposition 4.4, the normalized sum

$$\tilde{Z}_n := \frac{1}{\sqrt{nE[X_1^2]}} \sum_{k=1}^n X_k, \quad n \geq 1,$$

satisfies

$$d_W(\tilde{Z}_n, \mathcal{N}) \leq \frac{1}{\sqrt{n}} \sqrt{\frac{4 + \alpha(\alpha^2 + \alpha - 2)}{\alpha(\alpha + 3)(\alpha + 4)}},$$

which systematically improves on (4.4) and on the bound (1.1) of [2], i.e.

$$d_W(Z_n, \mathcal{N}) \leq \frac{1}{\sqrt{n}} E[|X_1|^3] = \frac{2}{\sqrt{n}} \sqrt{\frac{\alpha + 2}{\alpha}} \left(\frac{6\alpha(\alpha/(\alpha + 1))^{\alpha+1} + 1 - \alpha}{\alpha + 3} \right),$$

as can be checked from Figure 1.

For example in the uniform case with $\alpha = 1$ we have $X_k = U_k$, $k \in \mathbb{N}$, and $F(x) = (x + 1)/2$, $x \in [-1, 1]$, and

$$f_1(t) = \sqrt{3} \sum_{k=0}^{\infty} (t - 2k - 1) \mathbf{1}_{[2k, 2k+2)}(t),$$

hence (4.1) shows that the sequence $Z_n := \sqrt{3/n} \sum_{k=1}^n U_k$, satisfies

$$d_W(Z_n, \mathcal{N}) \leq \frac{3^{3/2}}{\sqrt{n}} E[|X_1|^3] = \frac{3}{4} \sqrt{\frac{3}{n}},$$

whereas (4.5) yields $d_W(Z_n, \mathcal{N}) \leq 1/\sqrt{5n}$.

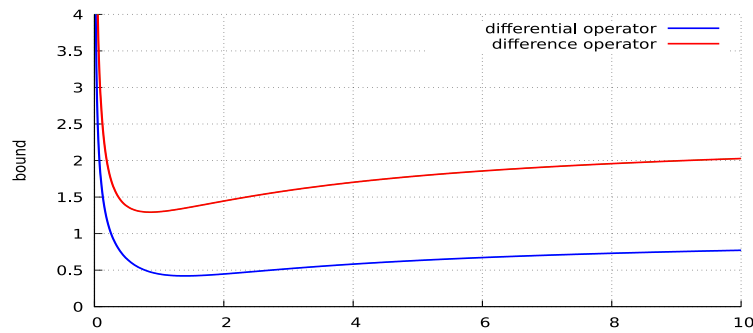


Figure 1: Comparison of bounds.

5 Multiple stochastic integrals

In this section we apply the multiplication formula given in the appendix Section 7 in order to obtain bounds on the distance between multiple stochastic integrals and the normal distribution \mathcal{N} . In the sequel for $0 \leq i \leq k \leq n \wedge m$ we define

$$\begin{aligned}
 & f_n \star_k^i g_m(\gamma_1, \dots, \gamma_{k-i}, t_1, \dots, t_{n-k}, s_1, \dots, s_{m-k}) \\
 & := \frac{1}{2^i} \int_{[0, \infty)^i} f_n(z_1, \dots, z_i, \gamma_1, \dots, \gamma_{k-i}, t_1, \dots, t_{n-k}) \\
 & \quad \times g_m(z_1, \dots, z_i, \gamma_1, \dots, \gamma_{k-i}, s_1, \dots, s_{m-k}) dz_1 \cdots dz_i,
 \end{aligned} \tag{5.1}$$

and we denote by $f_n \tilde{\star}_k^i g_m$ the symmetrisation of $f_n \star_k^i g_m$, i.e.

$$\begin{aligned}
 & f_n \tilde{\star}_k^i g_m(x_1, \dots, x_{m+n-k-i}) \\
 & := \frac{1}{(m+n-k-i)!} \sum_{\sigma \in S_{m+n-k-i}} f_n \star_k^i g_m(x_{\sigma(1)}, \dots, x_{\sigma(m+n-k-i)}),
 \end{aligned}$$

where $S_{m+n-k-i}$ stands for the group of all permutations of the set $\{1, \dots, m+n-k-i\}$. Note that $f_n \star_k^i g_m$ may not satisfy (2.3), even if f_n and g_m satisfy (2.3). The multiplication formula of Theorem 7.1 below can be given in many different forms, one of which is presented in the next Proposition 5.1.

Proposition 5.1. *Let f_n, g_m satisfy (2.3) and $f_n \star_k^i g_m \in L^2(\mathbb{R}_+^{m+n-k-i})$ for every $0 \leq i \leq k \leq m \wedge n$. We have*

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{2(m \wedge n)} I_k(h_k),$$

where

$$h_k = \sum_{r=0}^{n \wedge m} \sum_{l=0}^r \mathbf{1}_{\{2n-r-l=k\}} r! \binom{n}{r} \binom{m}{r} \binom{r}{l} f_n \tilde{\star}_r^l g_m.$$

Bounds obtained from the finite difference operator ∇

To obtain a more explicit bound than in Proposition 3.4 we have to employ the multiplication formula. Precisely, by virtue of Proposition 5.1 we may express $(I_n(f_n))^2$ as follows:

$$(I_n(f_n))^2 = \sum_{k=0}^{2n} I_k(G_k^n f_n), \tag{5.2}$$

where

$$G_k^n f_n(z_1, \dots, z_k) = \mathbf{1}_{\Delta_k}(z_1, \dots, z_k) \sum_{r=0}^n \sum_{l=0}^r \mathbf{1}_{\{2n-r-l=k\}} r! \binom{n}{r}^2 \binom{r}{l} f_n \tilde{\star}_r^l f_n(z_1, \dots, z_k).$$

Corollary 5.2. Let $f_n \in L^2(\mathbb{R}_+^n)$ be a symmetric function satisfying (2.3). Assume that

$$\widehat{G}_k^n f_n(*) := \frac{1}{2} \int_0^\infty G_k^{n-1} f_n(t, *) dt$$

belongs to $L^2(\mathbb{R}_+^k)$ for all $1 \leq k \leq 2n - 2$. We have

$$d_W(I_n(f), \mathcal{N}) \leq \sqrt{\left(1 - n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2\right)^2 + n^2 \sum_{k=1}^{2n-2} k! \|\widehat{G}_k^n f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2} \\ + n^2 \sqrt{2(n-1)!} \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}} \sqrt{\sum_{k=0}^{2n-2} k! \int_0^\infty \|G_k^{n-1} f_n(t, \cdot)\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2 dt}.$$

Proof. We are going to estimate both components appearing in Proposition 3.4. The formula (5.2) lets us write

$$(\nabla_t I_n(f_n))^2 = nn! \|f_n(t, \cdot)\|_{L^2(\mathbb{R}_+, dx/2)^{\circ(n-1)}}^2 + n^2 \sum_{k=1}^{2n-2} I_k(G_k^{n-1} f_n(t, \cdot)).$$

Hence we have

$$\frac{1}{n} \|\nabla \cdot I_n(f_n)\|_{L^2(\mathbb{R}_+, dx/2)}^2 = n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2 + \frac{n}{2} \sum_{k=1}^{2n-2} \int_0^\infty I_k(G_k^{n-1} f_n(t, \cdot)) dt.$$

Since multiple integrals of different orders are orthogonal, we get

$$E \left[\left(1 - \frac{1}{n} \|\nabla \cdot I_n(f_n)\|_{L^2(\mathbb{R}_+, dx/2)}\right)^2 \right] \\ = \left(1 - n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2\right)^2 + n^2 \sum_{k=1}^{2n-2} E \left[\left(\int_0^\infty I_k(G_k^{n-1} f_n(t, \cdot)) dt/2\right)^2 \right].$$

Finally, by (2.5), we obtain

$$E \left[\left|\int_0^\infty I_k(G_k^{n-1} f_n(t, \cdot)) dt/2\right|^2 \right] = E \left[\left(I_k(\widehat{G}_k^n f_n)\right)^2 \right] \leq k! \|\widehat{G}_k^n f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2,$$

which implies

$$E \left[\left|1 - \frac{1}{n} \|\nabla \cdot I_n(f_n)\|_{L^2(\mathbb{R}_+, dx/2)}\right|^2 \right] \leq \left|1 - n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2\right|^2 \\ + n^2 \sum_{k=1}^{2n-2} k! \|\widehat{G}_k^n f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2.$$

To get the second component of the estimates in the thesis we use Cauchy-Schwarz inequality in the following way:

$$\frac{1}{2n} \int_0^\infty E [|\nabla_t I_n(f_n)|^3] dt \leq \frac{1}{2} \sqrt{\int_0^\infty E [(I_{n-1}(f_n(t, *)))^2] dt} \sqrt{\int_0^\infty E [|\nabla_t I_n(f_n)|^4] dt} \\ \leq n^2 \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}} \sqrt{\frac{(n-1)!}{2} \int_0^\infty E [(I_{n-1}(f_n(t, *)))^4] dt}.$$

Since

$$(I_{n-1}(f_n(t, *)))^2 = \sum_{k=0}^{2n-2} I_k(G_k^{n-1} f_n(t, \cdot)),$$

and by orthogonality of multiple integrals, we have

$$\int_0^\infty E[(I_{n-1}(f_n(t, *)))^4] dt \leq \sum_{k=0}^{2n-2} k! \int_0^\infty \|G_k^{n-1} f_n(t, \cdot)\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2 dt,$$

which ends the proof. \square

As noted above, $I_n(f_n)$ can be used to represent various U-statistics, including polynomials of Bernoulli random variables, in which case Corollary 5.2 provides an alternative to the results of [10], [5], [19] for Bernoulli processes.

Bounds obtained from the derivation operator D

Here we let $C_{\square}^1(\mathbb{R}_+^n)$ denote the set of functions which are C^1 on every set of the form

$$(2k_1, 2k_1 + 2) \times \cdots \times (2k_n, 2k_n + 2), \quad k_1, \dots, k_n \in \mathbb{N}.$$

Given $f_n \in C_{\square}^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n)$, we define

$$H_k(s, z_2, \dots, z_{k+1}) := \sum_{r=0}^{n-1} \sum_{l=0}^r \mathbf{1}_{\{2n-2-r-l=k\}} r! \binom{n-1}{r} \binom{r}{l} \left(\partial_1 f_n(s, *) \tilde{\star}_r^l \int_0^s f_n(t, *) \mathbf{1}_{\{*\leq s\}} dt \right) (z_2, \dots, z_{k+1}),$$

and

$$J_k(s, z_1, \dots, z_k) := \sum_{r=0}^{n-1} \sum_{l=0}^r \mathbf{1}_{\{2n-2-r-l=k\}} r! \binom{n-1}{r} \binom{r}{l} \left(f_n(s, *) \tilde{\star}_r^l f_n(s, *) \mathbf{1}_{\{*\leq s\}} \right) (z_1, \dots, z_k),$$

$1 \leq k \leq 2n - 2$, where

$$\{*\leq u\} := \{x \in \mathbb{R}^{n-1} : x_i < u, i = 1, \dots, n-1\},$$

and assume that $H_k, J_k \in L^2(\mathbb{R}_+^{k+1})$. Additionally, we denote

$$\widehat{J}_k(z_1, \dots, z_k) = \frac{1}{2} \int_0^\infty J_k(s, z_1, \dots, z_k) ds.$$

Next is a consequence of Proposition 3.2.

Corollary 5.3. *Let $f_n \in C_{\square}^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n)$ and satisfy (2.3). We have*

$$\begin{aligned} d_W(I_n(f), \mathcal{N}) &\leq \left| 1 - n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2 \right| \\ &+ n^2 \left(\frac{1}{2} \sum_{k=0}^{2n-2} \int_0^\infty E[|I_k(H_k(s, *)))|^2] ds + \sum_{k=1}^{2n-2} E[|I_k(\widehat{J}_k)|^2] \right. \\ &\left. - \frac{1}{4} \sum_{i=0}^\infty \sum_{k=0}^{2n-2} E \left[\left| I_k \left(\int_{2i}^{2i+2} J_k(s, *) ds \right) \right|^2 \right] \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq \left| 1 - n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2 \right| \\ &\quad + n^2 \left(\frac{1}{2} \sum_{k=0}^{2n-2} k! \int_0^\infty \|H_k(s, *)\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2 ds + \sum_{k=1}^{2n-2} k! \left\| \widehat{J}_k \right\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}}^2 \right. \\ &\quad \left. + ((n-1)!)^2 \sum_{i=0}^\infty \left| \int_{2^i}^{2^{i+2}} \|f_n(s, *) \mathbf{1}_{\{s < s\}}\|_{L^2(\mathbb{R}_+, dx/2)^{\circ(n-1)}}^2 ds \right|^2 \right)^{1/2}. \end{aligned}$$

The bounds for $d_{TV}(I_n(f), \mathcal{N})$ are equal to those for $d_W(I_n(f), \mathcal{N})$ multiplied by 2.

Proof. By Lemma 2.4 and formula (2.5) we get

$$E[\langle D.I_n(f_n), E[D.I_n(f_n) | \tilde{\mathcal{F}}] \rangle] = 2E[(I_n(f_n))^2] = 2n! \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^2.$$

Next, we are going to provide an explicit form for the expression $\langle D.I_n(f_n), E[D.I_n(f_n) | \tilde{\mathcal{F}}] \rangle$. We have

$$\begin{aligned} &D_t I_n(f_n) \\ &= n \sum_{0 \leq k_1 \neq \dots \neq k_{n-1}} \\ &\quad \left((1 - U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor, 2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor})}(t) - (1 + U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, 2\lfloor t/2 \rfloor + 2)}(t) \right) \\ &\quad \times \partial_1 f_n(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, 2k_1 + 1 + U_{k_1}, \dots, 2k_{n-1} + 1 + U_{k_{n-1}}) \\ &= n \left((1 - U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor, 2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor})}(t) - (1 + U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, 2\lfloor t/2 \rfloor + 2)}(t) \right) \\ &\quad \times I_{n-1}(\partial_1 f_n(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, *)). \end{aligned}$$

By Proposition 10 and Lemma 1 in [14] we get

$$E[D_t I_n(f_n) | \tilde{\mathcal{F}}_t] = E[\nabla_t I_n(f_n) | \tilde{\mathcal{F}}_t] = n I_{n-1}(f_n(t, *) \mathbf{1}_{\{s < 2\lfloor t/2 \rfloor\}}).$$

Consequently, using the assumption (2.3) twice, we arrive at

$$\begin{aligned} \langle D.I_n(f_n), E[D.I_n(f_n) | \tilde{\mathcal{F}}] \rangle &= n^2 \int_0^\infty \left((1 - U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor, 2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor})}(t) \right. \\ &\quad \left. - (1 + U_{\lfloor t/2 \rfloor}) \mathbf{1}_{(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, 2\lfloor t/2 \rfloor + 2)}(t) \right) \\ &\quad \times I_{n-1}(\partial_1 f_n(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, *)) I_{n-1}(f_n(t, *) \mathbf{1}_{\{s < 2\lfloor t/2 \rfloor\}}) dt \\ &= n^2 \int_0^\infty \left(\mathbf{1}_{(2\lfloor t/2 \rfloor, 2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor})}(t) - \mathbf{1}_{(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, 2\lfloor t/2 \rfloor + 2)}(t) \right) \\ &\quad \times I_{n-1}(\partial_1 f_n(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, *)) I_{n-1}(f_n(t, *) \mathbf{1}_{\{s < 2\lfloor t/2 \rfloor\}}) dt \\ &= 2n^2 \int_0^\infty \left(\mathbf{1}_{(2\lfloor t/2 \rfloor, 2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor})}(t) \right. \\ &\quad \left. \times I_{n-1}(\partial_1 f_n(2\lfloor t/2 \rfloor + 1 + U_{\lfloor t/2 \rfloor}, *)) I_{n-1}(f_n(t, *) \mathbf{1}_{\{s < 2\lfloor t/2 \rfloor\}}) \right) dt \\ &= 2n^2 \sum_{k=0}^\infty I_{n-1}(\partial_1 f_n(2k + 1 + U_k, *)) I_{n-1} \left(\int_0^{2k+1+U_k} f_n(t, *) \mathbf{1}_{\{s < 2k\}} dt \right) \\ &= 2n^2 \int_0^\infty h(s) d(Y_s - s/2) + n^2 \int_0^\infty h(s) ds, \end{aligned}$$

where

$$h(s) := I_{n-1}(\partial_1 f_n(s, *)) I_{n-1} \left(\int_0^s f_n(t, *) \mathbf{1}_{\{s < s\}} dt \right), \quad s \in \mathbb{R}_+,$$

is a random process when $n \geq 2$. Note that by integration by parts we have

$$\int_0^\infty h(s)ds = \int_0^\infty I_{n-1}(f_n(s, *)) I_{n-1}(f_n(s, *) \mathbf{1}_{\{s < * \}}) ds.$$

By the Fubini theorem we may express $\int_0^\infty h(s)d(Y_s - s/2)$ and $\int_0^\infty h(s)ds$ as $2n - 1$ and $2n - 2$ integrals with respect to $d(Y_s - s/2)$, respectively. Then, applying (2.1) $2n - 2$ times together with (2.3), we obtain

$$\begin{aligned} E \left[\int_0^\infty h(s)d(Y_s - s/2) \int_0^\infty h(s)ds \right] &= \\ \int_{\mathbb{R}_+^{2n-1}} E \left[\int_0^\infty \partial_1 f_n(u, x_1, \dots, x_{n-1}) \int_0^u f_n(t, x_n, \dots, x_{2n-2}) dt \mathbf{1}_{\{(x_n, \dots, x_{2n-2}) < u\}} d\left(Y_u - \frac{u}{2}\right) \right] \\ &\times f_n(s, x_1, \dots, x_{n-1}) f_n(s, x_n, \dots, x_{2n-2}) \mathbf{1}_{\{(x_n, \dots, x_{2n-2}) < s\}} dx_1 \cdots dx_{2n-2} ds \\ &= 0, \end{aligned}$$

and consequently

$$\begin{aligned} E \left[\langle D.I_n(f_n), E[D.I_n(f_n) | \tilde{\mathcal{F}}] \rangle^2 \right] \\ = 4n^4 E \left[\left(\int_0^\infty h(s)d(Y_s - s/2) \right)^2 \right] + n^4 E \left[\left(\int_0^\infty h(s)ds \right)^2 \right]. \end{aligned}$$

Using the orthogonality of multiple integrals of different orders and the relation

$$I_{n-1}(f_n(s, *)) I_{n-1}(f_n(s, *) \mathbf{1}_{\{s < * \}}) = \sum_{k=0}^{2n-2} I_k(J_k(s, *)),$$

we rewrite the latter component as follows:

$$\begin{aligned} E \left[\left(\int_0^\infty h(s)ds \right)^2 \right] &= 4E \left[\left(\sum_{k=0}^{n-1} I_k(\hat{J}_k) \right)^2 \right] \\ &= 4 \sum_{k=1}^{n-1} E \left[\left(I_k(\hat{J}_k) \right)^2 \right] + \left((n-1)! \int_0^\infty \|f_n(s, *) \mathbf{1}_{\{s < * \}}\|_{L^2(\mathbb{R}_+, dx/2)^{\circ(n-1)}}^2 ds \right)^2 \\ &= 4 \sum_{k=1}^{n-1} E \left[\left(I_k(\hat{J}_k) \right)^2 \right] + \frac{4}{n^2} \|f_n\|_{L^2(\mathbb{R}_+, dx/2)^{\circ n}}^4. \end{aligned}$$

Furthermore, by Proposition 5.1 we have

$$I_{n-1}(\partial_1 f_n(s, *)) I_{n-1} \left(\int_0^s f_n(t, *) \mathbf{1}_{\{t < * \}} dt \right) = \sum_{k=0}^{2n-2} I_k(H_k(s, *)),$$

hence (2.1) gives us

$$\begin{aligned} E \left[\left(\int_0^\infty h(s)d(Y_s - s/2) \right)^2 \right] &= \frac{1}{2} E \left[\int_0^\infty |h(s)|^2 ds \right] - \frac{1}{4} \sum_{i=0}^\infty E \left[\left(\int_{2i}^{2i+2} h(s)ds \right)^2 \right] \\ &= \frac{1}{2} E \left[\int_0^\infty \left(\sum_{k=0}^{2n-2} I_k(H_k(s, *)) \right)^2 ds \right] - \sum_{i=0}^\infty E \left[\left(\sum_{k=0}^{n-1} I_k \left(\int_{2i}^{2i+2} J_k(s, *) ds/2 \right) \right)^2 \right] \\ &= \frac{1}{2} \sum_{k=0}^{2n-2} \int_0^\infty E \left[\left(I_k(H_k(s, *)) \right)^2 \right] ds - \sum_{i=0}^\infty \sum_{k=0}^{n-1} E \left[\left(I_k \left(\int_{2i}^{2i+2} J_k(s, *) ds/2 \right) \right)^2 \right]. \end{aligned}$$

We apply this to Proposition 3.2 and get the first inequality in the assertion of the theorem. In order to derive the other one we use (2.2) and the estimate

$$\begin{aligned} \sum_{k=0}^{n-1} E \left[\left(I_k \left(\int_{2^i}^{2^{i+2}} J_k(s, *) ds/2 \right) \right)^2 \right] &\geq E \left[\left(I_0 \left(\int_{2^i}^{2^{i+2}} J_0(s, *) ds/2 \right) \right)^2 \right] \\ &= ((n-1)!)^2 \left(\int_{2^i}^{2^{i+2}} \|f(s, *) \mathbf{1}_{\{s < s\}}\|_{L^2(\mathbb{R}_+, dx/2)^{\circ(n-1)}} ds \right)^2. \quad \square \end{aligned}$$

A combinatorial central limit theorem

In this section, we show that the bounds of [4] for the Rademacher combinatorial central limit theorem of [1] can be extended to our setting of random sequences.

Given K a symmetric subset of $\Delta_q := \{a \in \mathbb{N}^q : a_i \neq a_j \text{ if } i \neq j\}$, $(b_k)_{k \geq 0}$ a sequence of real numbers, and $(X_k)_{k \geq 0}$ an i.i.d. sequence of random variables such that $E[X_1] = 0$ and $E[X_1^2] < \infty$, define

$$S^{(b)}(K) := \frac{1}{(q! \mu_b^{\otimes q}(K)(E[X_1^2])^q)^{1/2}} \sum_{(i_1, \dots, i_q) \in K} b_{i_1} \cdots b_{i_q} X_{i_1} \cdots X_{i_q}.$$

Following § 6.3 of [4], we let K_j^* denote the collection of all $(i_1, \dots, i_q) \in K$ such that $i_k = j$ for some $k \in \{1, \dots, q\}$, and we define $K^\# \subset K \times K$ by stating that a pair $(i_1, \dots, i_q), (j_1, \dots, j_q)$ belongs to $K^\#$ if $\{i_1, \dots, i_q\} \cap \{j_1, \dots, j_q\} = \emptyset$ and there are $(k_1, \dots, k_q), (l_1, \dots, l_q) \in K$ such that $\{k_1, \dots, k_q, l_1, \dots, l_q\} = \{i_1, \dots, i_q, j_1, \dots, j_q\}$ and (k_1, \dots, k_q) does not coincide with (i_1, \dots, i_q) or (j_1, \dots, j_q) .

Theorem 5.4. *There exists a constant $C = C(q)$ such that*

$$d_{W/TV} \left(S^{(b)}(K), \mathcal{N} \right) \leq C (E[X_1^4])^q \left[\frac{(\mu_b^{\otimes(2q)}(K^\#))^{1/2}}{\mu_b^{\otimes q}(K)} + \left(\sup_{j \geq 1} \frac{\mu_b^{\otimes q}(K_j^*)}{\mu_b^{\otimes q}(K)} \right)^{1/4} \right].$$

Proof. Let F be the distribution function of X_1 with generalised inverse function F^{-1} . Then we have $S^{(b)}(K) \stackrel{d}{=} I_q(f_q)$, where

$$\begin{aligned} &f_q(t_1, \dots, t_q) \\ &= \frac{\mathbf{1}_K(\lfloor t_1/2 \rfloor, \dots, \lfloor t_q/2 \rfloor)}{(q! \mu_b^{\otimes q}(K)(E[X_1^2])^q)^{1/2}} b_{\lfloor t_1/2 \rfloor} \cdots b_{\lfloor t_q/2 \rfloor} F^{-1} \left(\frac{t_1 - 2\lfloor t_1/2 \rfloor}{2} \right) \cdots F^{-1} \left(\frac{t_q - 2\lfloor t_q/2 \rfloor}{2} \right). \end{aligned}$$

By Theorem 5.2, there exist constants C_1, C_2 depending only on q , such that

$$\begin{aligned} &d_W(I_q(f_q), \mathcal{N}) \\ &\leq C_1 \left(\sum_{k=1}^{2q-2} \left\| \widehat{G}_k^q f_q \right\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}} + \sum_{k=0}^{2q-2} \int_0^\infty \left\| G_k^{q-1} f_q(t, \cdot) \right\|_{L^2(\mathbb{R}_+, dx/2)^{\circ k}} dt \right) \quad (5.3) \end{aligned}$$

$$\leq C_2 \left(\sum_{k=1}^{q-1} \sum_{r=1}^k \left\| (f_q \star_k^r f_q) \mathbf{1}_{\Delta_{2q-k-r}} \right\|_{L^2(\mathbb{R}_+)^{\circ(2q-k-r)}} + \sum_{k=1}^q \sum_{r=0}^{k-1} \left\| f_q \star_k^r f_q \right\|_{L^2(\mathbb{R}_+)^{\circ(2q-k-r)}} \right). \quad (5.4)$$

Note that for $r \leq k$ we have

$$\begin{aligned} &\left\| (f_q \star_k^r f_q) \mathbf{1}_{\Delta_{2q-k-r}} \right\|_{L^2(\mathbb{R}_+)^{\circ(2q-k-r)}} \\ &= 2^{q+r} \left(\int_0^2 (F^{-1}(s))^2 ds \right)^{2r+2q-2k} \left(\int_0^2 (F^{-1}(s))^4 ds \right)^{k-r} \end{aligned}$$

$$\begin{aligned} & \times \sum_{y,z \in \mathbb{N}^{q-k}} \sum_{x \in \mathbb{N}^{k-r}} \mathbf{1}_{\tilde{\Delta}_{2q-k-r}}(x, y, z) \\ & \times \left(\sum_{\substack{(w_1, \dots, w_r, x_1, \dots, x_{k-r}, y_1, \dots, y_{q-k}) \in K \\ \text{left}(w_1, \dots, w_r, x_1, \dots, x_{k-r}, z_1, \dots, z_{q-k}) \in K}} \tilde{f}_q(w, x, y) \tilde{f}_q(w, x, z) \right)^2 \\ & \leq 2^{2q} (E[X_1^4])^q \left\| \left(\tilde{f}_q \tilde{\star}_k^r \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-k-r}} \right\|_{l^2(\mathbb{N}^{\circ(2q-k-r)}} \end{aligned}$$

where, as in [4] or [10], the notation $\tilde{\star}$ is here the discrete version of the product defined in (5.1), and

$$\tilde{f}_q(i_1, \dots, i_q) := \frac{\mathbf{1}_K(i_1, \dots, i_q)}{(q! \mu_b^{\otimes q}(K) (E[X^2])^q)^{1/2}} b_{i_1} \cdots b_{i_q}.$$

Furthermore, for fixed $y, z \in \mathbb{N}^{q-k}$ we get

$$\begin{aligned} & \left\| \left(\tilde{f}_q \tilde{\star}_k^r \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-k-r}} \right\|_{l^2(\mathbb{N}^{\circ(2q-k-r)})} \\ & = \sum_{y,z \in \mathbb{N}^{q-k}} \sum_{x \in \mathbb{N}^{k-r}} \mathbf{1}_{\tilde{\Delta}_{2q-k-r}}(x, y, z) \\ & \times \left(\sum_{\substack{(w_1, \dots, w_r, x_1, \dots, x_{k-r}, y_1, \dots, y_{q-k}) \in K \\ \text{left}(w_1, \dots, w_r, x_1, \dots, x_{k-r}, z_1, \dots, z_{q-k}) \in K}} \tilde{f}_q(w, x, y) \tilde{f}_q(w, x, z) \right)^2 \\ & \leq \sum_{y,z \in \mathbb{N}^{q-k}} \mathbf{1}_{\tilde{\Delta}_{2q-2k}}(y, z) \\ & \times \left(\sum_{\substack{x \in \mathbb{N}^{k-r} \\ (w_1, \dots, w_r, x_1, \dots, x_{k-r}, y_1, \dots, y_{q-k}) \in K \\ \text{left}(w_1, \dots, w_r, x_1, \dots, x_{k-r}, z_1, \dots, z_{q-k}) \in K}} \tilde{f}_q(w, x, y) \tilde{f}_q(w, x, z) \right)^2 \\ & \leq \left\| \left(\tilde{f}_q \tilde{\star}_k^k \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-2k}} \right\|_{l^2(\mathbb{N}^{\circ(2q-2k)})} \end{aligned}$$

where the first inequality follows from the general inequality $\sum_{i \in I} a_i^2 \leq (\sum_{i \in I} a_i)^2$, $a_i \geq 0$, and the fact that $\text{sgn}(\tilde{f}_q(w, x, y) \tilde{f}_q(w, x, z))$ is constant for fixed $y, z \in \mathbb{N}^{q-k}$. Thus, we get

$$\left\| \left(\tilde{f}_q \tilde{\star}_k^r \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-k-r}} \right\|_{L^2(\mathbb{R}_+)^{\circ(2q-k-r)}} \leq 2^{2q} (E[X_1^4])^q \left\| \left(\tilde{f}_q \tilde{\star}_k^k \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-2k}} \right\|_{l^2(\mathbb{N}^{\circ(2q-2k)})}.$$

Analogously, for $r < k$ we obtain

$$\left\| \tilde{f}_q \tilde{\star}_k^r \tilde{f}_q \right\|_{L^2(\mathbb{R}_+)^{\circ(2q-k-r)}} \leq 2^{2q} (E[X_1^4])^q \left\| \left(\tilde{f}_q \tilde{\star}_k^{k-1} \tilde{f}_q \right) \right\|_{l^2(\mathbb{N}^{\circ(2q-2k)})}.$$

Finally, applying this to (5.4), we may write

$$\begin{aligned} & d_W(I_q(f_q)) \\ & \leq C (E[X_1^4])^q \\ & \times \left(\max_{k \in \{1, \dots, q-1\}} \left\| \left(\tilde{f}_q \tilde{\star}_k^k \tilde{f}_q \right) \mathbf{1}_{\tilde{\Delta}_{2q-2k}} \right\|_{l^2(\mathbb{N}^{\circ(2q-2k)})} + \max_{k \in \{1, \dots, q\}} \left\| \left(\tilde{f}_q \tilde{\star}_k^{k-1} \tilde{f}_q \right) \right\|_{l^2(\mathbb{N}^{\circ(2q-2k)})} \right), \end{aligned}$$

for some $C = C(q)$, and both maxima can be calculated as in Theorem 6.2 of [4]. \square

Theorem 5.4 extends the standard Berry-Esseen bound of Corollary 6.2 in [4] to general independent random sequences, in particular when K takes the form $K = \{1, \dots, n\}^q \cap \hat{\Delta}_q$. Note also that the general result on random sequences in Proposition 6.8 of [10] does not apply to the total variation or Wasserstein distances.

6 Quadratic functionals

This section is devoted to double stochastic integrals, which are a special case of the multiple integrals discussed in Section 5. We study them in a separate section because of many applications i.e. to quadratic functionals. Taking $n = 2$ in Corollary 5.2 of Section 5, we get the following result.

Corollary 6.1. *Let $f_2 \in L^2(\mathbb{R}_+^2)$ be a symmetric function satisfying (2.3). Assume that the functions*

$$\hat{G}_1^2 f_2(y) = \frac{1}{2} \int_0^\infty |f_2(x, y)|^2 dx \quad \text{and} \quad \hat{G}_2^2 f_2(y, z) = \frac{1}{2} \int_0^\infty f_2(x, y) f_2(x, z) dx$$

belong to $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+^2)$, respectively. Then we have

$$\begin{aligned} d_W(Z_n, \mathcal{N}) &\leq \left(\left(1 - \frac{1}{4} \|f_2\|_{L^2(\mathbb{R}_+^2)}^2 \right)^2 + \int_0^\infty \left(\int_0^\infty |f_2(x, y)|^2 dx \right)^2 \frac{dy}{2} \right. \\ &\quad \left. + \int_0^\infty \int_0^\infty \left(\int_0^\infty f_2(x, y) f_2(x, z) dx \right)^2 \frac{dy}{2} dz \right)^{1/2} \\ &\quad + \|f_2\|_{L^2(\mathbb{R}_+^2)} \left(\frac{1}{2} \int_0^\infty \left(\int_0^\infty |f_2(x, y)|^2 dx \right)^2 dy + \|f_2\|_{L^4(\mathbb{R}_+^2)}^4 \right. \\ &\quad \left. + \int_0^\infty \int_0^\infty \int_0^\infty |f_2(x, y) f_2(x, z)|^2 dx dy dz \right)^{1/2}. \end{aligned}$$

For example, when $f_2 \in C_\square^1(\mathbb{R}_+^2) \cap L^2(\mathbb{R}_+^2)$ is given by

$$f_2(s, t) := \sum_{1 \leq k, l \leq n} a_{k,l} f_1(s) f_1(t) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(s, t), \quad s, t \in \mathbb{R}_+, \quad (6.1)$$

where $A = (a_{k,l})_{1 \leq k, l \leq n}$ is a symmetric matrix with vanishing diagonal and such that $\sum_{1 \leq k, l \leq n} a_{k,l}^2 = 1$, Corollary 6.1 yields the following result, when f_1 is given by (2.23).

Corollary 6.2. *Given $(X_k)_{k \geq 1}$ a sequence of independent identically distributed random variables such that $E[X_k] = 0$ and $E[X_k^2] = 1$, $k \geq 1$, let Q_n denote the normalized quadratic form*

$$Q_n := \sum_{1 \leq k, l \leq n} a_{k,l} X_k X_l, \quad (6.2)$$

with $E[Q_n] = 0$ and $E[Q_n^2] = 1$, $n \geq 2$. We have

$$\begin{aligned} d_W(Q_n, \mathcal{N}) &\leq 2 \sqrt{E[X_1^4] \sum_{l=1}^n \left(\sum_{k=1}^n a_{k,l}^2 \right)^2 + 2 \sum_{1 \leq l, p \leq n} \left(\sum_{k=1}^n a_{k,l} a_{k,p} \right)^2} \\ &\quad + 4 \sqrt{(3E[X_1^4] + (E[X_1^4])^2) \sum_{k=1}^n \left(\sum_{l=1}^n a_{k,l}^2 \right)^2}. \end{aligned} \quad (6.3)$$

The bound for $d_{TV}(Q_n, \mathcal{N})$ is twice as large as (6.3).

Proof. Writing Q_n as

$$Q_n := I_2(f_2 \mathbf{1}_{[0,2n] \times [0,2n]}) = \sum_{1 \leq k, l \leq n} a_{k,l} I_1(f_1 \mathbf{1}_{(2k-2, 2k]}) I_1(f_1 \mathbf{1}_{(2l-2, 2l]}),$$

$n \geq 2$, we have

$$\begin{aligned} d_W(Q_n, \mathcal{N}) &\leq \frac{1}{\sqrt{2}} \left(\int_0^{2n} \left(\int_0^{2n} |f_2(x, y)|^2 dx \right)^2 dy \right. \\ &\quad \left. + \int_0^{2n} \int_0^{2n} \left(\int_0^{2n} f_2(x, y) f_2(x, z) dx \right)^2 dy dz \right)^{1/2} \\ &\quad + 2 \left(\frac{1}{2} \int_0^{2n} \left| \int_0^{2n} |f_2(x, y)|^2 dx \right|^2 dy + \int_0^{2n} \int_0^{2n} |f_2(x, y)|^4 dx dy \right. \\ &\quad \left. + \int_0^{2n} \int_0^{2n} \int_0^{2n} |f_2(x, y) f_2(x, z)|^2 dx dy dz \right)^{1/2} \\ &= \frac{1}{\sqrt{2}} \int_0^2 |f_1(x)|^2 dx \\ &\quad \times \sqrt{\int_0^2 |f_1(y)|^4 dy \sum_{l=1}^n \left(\sum_{k=1}^n a_{k,l}^2 \right)^2 + \left(\int_0^2 |f_1(x)|^2 dx \right)^2 \sum_{1 \leq l, p \leq n} \left(\sum_{k=1}^n a_{k,l} a_{k,p} \right)^2} \\ &\quad + 2 \left(\left| \int_0^2 |f_1(x)|^2 dx \right|^2 \int_0^2 |f_1(y)|^4 dy \left(\frac{1}{2} \sum_{l=1}^n \left(\sum_{k=1}^n a_{k,l}^2 \right)^2 + \sum_{1 \leq l, p \leq n} \sum_{k=1}^n a_{k,l}^2 a_{k,p}^2 \right) \right. \\ &\quad \left. + \left| \int_0^2 |f_1(x)|^4 dx \right|^2 \sum_{1 \leq k, l \leq n} a_{k,l}^4 \right)^{1/2} \\ &= 2E[X_1^2] \sqrt{E[X_1^4] \sum_{l=1}^n \left(\sum_{k=1}^n a_{k,l}^2 \right)^2 + 2(E[X_1^2])^2 \sum_{1 \leq l, p \leq n} \left(\sum_{k=1}^n a_{k,l} a_{k,p} \right)^2} \\ &\quad + 4 \sqrt{3(E[X_1^2])^2 E[X_1^4] \sum_{l=1}^n \left(\sum_{k=1}^n a_{k,l}^2 \right)^2 + (E[X_1^4])^2 \sum_{k=1}^n \left(\sum_{l=1}^n a_{k,l}^2 \right)^2}, \end{aligned}$$

where we used the relation

$$\int_0^2 |f_1(x)|^4 dx = 2 \int_0^1 |F^{-1}(y)|^4 dy = 2E[X_1^4]. \quad \square$$

Bounds of that type have been already studied in the literature, see e.g. [20] and [3]. They are usually presented by means of the expression

$$L_n^2 := \max_{1 \leq k \leq n} \sum_{l=1}^n a_{k,l}^2.$$

Following this convention we can apply the bound of Corollary 6.2 to obtain

$$\begin{aligned} d_{TV}(Q_n, \mathcal{N}) &\leq 4\sqrt{n} L_n^2 \left(\sqrt{E[X_1^4] + \frac{2}{nL_n^4} \sum_{1 \leq l, p \leq n} \left(\sum_{k=1}^n a_{k,l} a_{k,p} \right)^2} + 2\sqrt{3E[X_1^4] + (E[X_1^4])^2} \right). \quad (6.4) \end{aligned}$$

Note that the constants in the above bound are explicit. For example, when

$$Q_n = \frac{2}{\sqrt{n}} \sum_{k=1}^n X_{2k-1} X_{2k},$$

we have $L_n^2 = 1/n$ and

$$\sum_{1 \leq l, p \leq 2n} \left(\sum_{k=1}^{2n} a_{k,l} a_{k,p} \right)^2 = \frac{1}{n},$$

hence (6.4) recovers the known convergence rate

$$d_{TV}(Q_n, \mathcal{N}) \leq 2\sqrt{\frac{2}{n}} \left(\sqrt{1 + E[X_1^4]} + 2\sqrt{3E[X_1^4] + (E[X_1^4])^2} \right) \leq \frac{16E[X_1^4]}{\sqrt{n}},$$

cf. pages 1074-1075 of [3], with an explicit constant depending on $E[X_1^4]$ instead of $\sqrt{E[|X_1|^3]}$. On the other hand, Corollary 5.3 applied with $n = 2$ gives the following result.

Corollary 6.3. For any $f_2 \in C_{\square}^1(\mathbb{R}_+^2) \cap L^2(\mathbb{R}_+^2)$ satisfying (2.3), we have

$$\begin{aligned} d_W(I_2(f_2), \mathcal{N}) &\leq \left| 1 - \frac{1}{4} \|f_2\|_{L^2(\mathbb{R}_+^2)}^2 \right| + \left(2 \int_0^\infty \left(\int_0^x \partial_1 f_2(x, y) \int_0^x f_2(t, y) dt dy \right)^2 dx \right. \\ &\quad + 4 \int_0^\infty \int_0^x \left(\partial_1 f_2(x, y) \int_0^x f_2(t, y) dt \right)^2 dy dx \\ &\quad + 4 \int_0^\infty \int_0^\infty \int_0^x \left(\partial_1 f_2(x, y) \int_0^x f_2(t, z) dt \right)^2 dz dx dy \\ &\quad + 2 \int_0^\infty \left(\int_y^\infty |f_2(x, y)|^2 dx \right)^2 dy + 2 \int_0^\infty \int_0^\infty \left(\int_z^\infty f_2(x, y) f_2(x, z) dx \right)^2 dy dz \\ &\quad \left. - 4 \sum_{i=0}^\infty \left(\int_{2^i}^{2^{i+2}} \int_0^x |f_2(x, y)|^2 dy dx \right)^2 \right)^{1/2}. \end{aligned}$$

Proof. We apply Corollary 5.3 with

$$H_0(x) = \frac{1}{2} \int_0^x \partial_1 f_2(x, y) \int_0^x f_2(t, y) dt dy,$$

$$H_1(x, y) = \mathbf{1}_{\{y < x\}} \partial_1 f_2(x, y) \int_0^x f_2(t, y) dt,$$

$$H_2(x, y, z) = \mathbf{1}_{\{z < x\}} \frac{1}{2} \partial_1 f_2(x, y) \int_0^x f_2(t, z) dt + \mathbf{1}_{\{y < x\}} \frac{1}{2} \partial_1 f_2(x, z) \int_0^x f_2(t, y) dt,$$

$x, y, z \in \mathbb{R}_+$, and

$$J_1(s, y) = |f_2(s, y)|^2 \mathbf{1}_{\{y < s\}},$$

$$J_2(s, y, z) = \frac{1}{2} f_2(s, y) f_2(s, z) \mathbf{1}_{\{z < s\}} + \frac{1}{2} f_2(s, y) f_2(s, z) \mathbf{1}_{\{y < s\}},$$

$$\hat{J}_1(z_1) = \frac{1}{2} \int_{z_1}^\infty (f_2(s, z_1))^2 ds,$$

$$\hat{J}_2(z_1, z_2) = \frac{1}{4} \int_{z_2}^\infty f_2(s, z_1) f_2(s, z_2) ds + \frac{1}{4} \int_{z_1}^\infty f_2(s, z_1) f_2(s, z_2) ds. \quad \square$$

When $f_2 \in C_{\square}^1(\mathbb{R}_+^2) \cap L^2(\mathbb{R}_+^2)$ is given by (6.1), Corollary 6.3 shows the following bound on quadratic functionals.

Corollary 6.4. Given $(X_k)_{k \geq 1}$ a sequence of independent identically distributed random variables such that $E[X_k] = 0$ and $E[X_k^2] = 1$, $k \geq 1$, the normalized quadratic form Q_n defined in (6.2) satisfies

$$d_W(Q_n, \mathcal{N}) \leq 4 \sqrt{E[(\varphi_{X_k}(X_k))^2](2 + E[X_1^4])L_n^2 + 2 \sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k,q} a_{k,l} \right)^2 - \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k,l}^2 \right)^2}. \quad (6.5)$$

Proof. By Corollary 6.3, we have

$$d_W(I_2(f_2), \mathcal{N}) \leq \sqrt{2I_1 + 4I_2 + 4I_3 + 2I_4 + 2I_5 - 4I_6},$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \left(\int_0^x \sum_{1 \leq k, l, p, q \leq n} a_{k,l} f_1'(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\ &\quad \left. \times \int_0^x a_{p,q} f_1(t) f_1(y) \mathbf{1}_{(2p-2, 2p] \times (2q-2, 2q]}(t, y) dt dy \right)^2 dx \\ &= \int_0^\infty \sum_{k=1}^n \mathbf{1}_{(2k-2, 2k]}(x) \left(\sum_{1 \leq p, l \leq k} a_{k,l} a_{p,l} \int_0^x |f_1(y)|^2 \mathbf{1}_{(2l-2, 2l]}(y) dy f_1'(x) \right. \\ &\quad \left. \times \int_0^x f_1(t) \mathbf{1}_{(2p-2, 2p]}(t) dt \right)^2 dx \\ &= \int_0^\infty \sum_{k=1}^n \mathbf{1}_{(2k-2, 2k]}(x) \left(\sum_{1 \leq l \leq k} (a_{k,l})^2 \int_0^x |f_1(y)|^2 \mathbf{1}_{(2l-2, 2l]}(y) dy f_1'(x) \right. \\ &\quad \left. \times \int_0^x f_1(t) \mathbf{1}_{(2k-2, 2k]}(t) dt \right)^2 dx \\ &= \left(\int_0^2 |f_1(y)|^2 dy \right)^2 \int_0^2 \left(f_1'(x) \int_0^x f_1(t) dt \right)^2 dx \sum_{k=1}^n \left(\sum_{l=1}^{k-1} (a_{k,l})^2 \right)^2 \\ &\leq 8(E[X_1^2])^2 E[(\varphi_{X_k}(X_k))^2] \sum_{k=1}^n \left(\sum_{l=1}^{k-1} (a_{k,l})^2 \right)^2, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^\infty \int_0^x \left(\sum_{1 \leq k, l \leq n} a_{k,l} f_1'(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\ &\quad \left. \times \int_0^x \sum_{1 \leq p, q \leq n} a_{p,q} f_1(t) f_1(y) \mathbf{1}_{(2p-2, 2p] \times (2q-2, 2q]}(t, y) dt \right)^2 dy dx \\ &= \int_0^\infty \int_0^x \left(\sum_{1 \leq k, l \leq n} a_{k,l} f_1'(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\ &\quad \left. \times \int_0^x a_{k,l} f_1(t) f_1(y) \mathbf{1}_{(2k-2, 2k]}(t) dt \right)^2 dy dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq k, l \leq n} a_{k,l}^4 \int_0^\infty \mathbf{1}_{(2k-2, 2k]}(x) \left(f_1'(x) \int_0^x f_1(t) \mathbf{1}_{(2k-2, 2k]}(t) dt \right)^2 \\
&\quad \times \int_0^x |f_1(y)|^4 \mathbf{1}_{(2l-2, 2l]}(y) dy dx \\
&\leq \sum_{1 \leq k, l \leq n} a_{k,l}^4 \int_0^2 \left(f_1'(x) \int_0^x f_1(t) dt \right)^2 dx \int_0^2 |f_1(y)|^4 dy \\
&\leq 4E[X_1^4] E[(\varphi_{X_k}(X_k))^2] \sum_{1 \leq k, l \leq n} a_{k,l}^4, \\
I_3 &= \int_0^\infty \int_0^\infty \int_0^x \left(\sum_{1 \leq k, l \leq n} a_{k,l} f_1'(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\
&\quad \left. \times \int_0^x \sum_{1 \leq p, q \leq n} a_{p,q} f_1(t) f_1(z) \mathbf{1}_{(2p-2, 2p] \times (2q-2, 2q]}(t, z) dt \right)^2 dz dx dy \\
&= \int_0^\infty \int_0^\infty \int_0^x \left(\sum_{1 \leq k, l \leq n} a_{k,l} f_1'(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\
&\quad \left. \times \int_0^x \sum_{1 \leq q \leq n} a_{k,q} f_1(t) f_1(z) \mathbf{1}_{(2k-2, 2k] \times (2q-2, 2q]}(t, z) dt \right)^2 dz dx dy \\
&= \sum_{1 \leq k, l, q \leq n} \int_0^\infty \int_0^\infty \int_0^x \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \mathbf{1}_{(2q-2, 2q]}(z) |f_1(y)|^2 |f_1(z)|^2 a_{k,l}^2 \\
&\quad \times \left(a_{k,q} f_1'(x) \int_0^x f_1(t) \mathbf{1}_{(2p-2, 2p]}(t) dt \right)^2 dz dx dy \\
&\leq \sum_{1 \leq k, l, q \leq n} \left(\int_0^2 |f_1(y)|^2 dy \right)^2 \left(\int_0^2 f_1'(x) \int_0^x f_1(y) dy \right)^2 a_{k,l}^2 a_{k,q}^2 \\
&\leq 4(E[X_1^2])^2 E[(\varphi_{X_k}(X_k))^2] \sum_{1 \leq k, l, q \leq n} a_{k,l}^2 a_{k,q}^2,
\end{aligned}$$

hence

$$\begin{aligned}
&2I_1 + 4I_2 + 4I_3 \\
&\leq 16E[(\varphi_{X_k}(X_k))^2] \left(\sum_{k=1}^n \left(\sum_{l=1}^{k-1} (a_{k,l})^2 \right) \right)^2 + E[X_1^4] \sum_{1 \leq k, l \leq n} a_{k,l}^4 + \sum_{1 \leq k, l, q \leq n} a_{k,l}^2 a_{k,q}^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
I_4 &= \frac{1}{2} \int_0^\infty \left(\int_y^\infty \left| \sum_{1 \leq k, l \leq n} a_{k,l} f_1(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right|^2 dx \right)^2 dy \\
&= \frac{1}{2} \sum_{1 \leq k, l \leq n} a_{k,l}^4 \int_0^\infty \mathbf{1}_{(2l-2, 2l]}(y) |f_1(y)|^4 \left(\int_y^\infty |f_1(x)|^2 \mathbf{1}_{(2k-2, 2k]}(x) dx \right)^2 dy \\
&\leq \frac{1}{2} \sum_{1 \leq k, l \leq n} a_{k,l}^4 \left(\int_0^2 |f_1(y)|^2 dy \right)^4 \leq 8(E[X_1^2])^4 \sum_{1 \leq k, l \leq n} a_{k,l}^4,
\end{aligned}$$

$$\begin{aligned}
I_5 &= \int_0^\infty \int_0^\infty \left(\int_z^\infty \sum_{1 \leq k, l, p, q \leq n} a_{k, l} f_1(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right. \\
&\quad \left. \times a_{p, q} f_1(x) f_1(z) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, z) dx \right)^2 dy dz \\
&= \int_0^\infty \int_0^\infty |f_1(y)|^2 |f_1(z)|^2 \sum_{1 \leq q, l \leq n} \mathbf{1}_{(2l-2, 2l]}(y) \mathbf{1}_{(2q-2, 2q]}(z) \\
&\quad \times \left(\sum_{k=1}^n a_{k, q} a_{k, l} \int_z^\infty |f_1(x)|^2 \mathbf{1}_{(2k-2, 2k]}(x) dx \right)^2 dy dz \\
&\leq \left(\int_0^2 |f_1(y)|^2 dy \right)^4 \sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k, q} a_{k, l} \right)^2 \leq 16 (E[X_1^2])^4 \sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k, q} a_{k, l} \right)^2.
\end{aligned}$$

and

$$\begin{aligned}
I_6 &= \sum_{i=1}^n \left(\int_{2^{i-2}}^{2^i} \int_0^x \left| \sum_{1 \leq k, l \leq n} a_{k, l} f_1(x) f_1(y) \mathbf{1}_{(2k-2, 2k] \times (2l-2, 2l]}(x, y) \right|^2 dy dx \right)^2 \\
&= \sum_{k=1}^n \left(\sum_{l=1}^n a_{k, l}^2 \int_{2^{k-2}}^{2^k} |f_1(x)|^2 \int_0^x |f_1(y)|^2 \mathbf{1}_{(2l-2, 2l]}(y) dy dx \right)^2 \\
&= \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k, l}^2 \int_{2^{k-2}}^{2^k} |f_1(x)|^2 \int_{2^{l-2}}^{2^l} |f_1(y)|^2 dy dx \right)^2 \\
&= 16 (E[X_1^2])^4 \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k, l}^2 \right)^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
2I_4 + 2I_5 - 4I_6 &\leq 32 \left(\sum_{1 \leq k, l \leq n} a_{k, l}^4 + \sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k, q} a_{k, l} \right)^2 - 2 \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k, l}^2 \right)^2 \right) \\
&\leq 32 \left(\sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k, q} a_{k, l} \right)^2 - \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k, l}^2 \right)^2 \right),
\end{aligned}$$

and combining the above bounds gives us (6.5). \square

When $(X_k)_{k \geq 1}$ is a sequence of independent gamma identically distributed normalized random variables we have $E[(\varphi_{X_k}(X_k))^2] = 2$, and (6.5) yields

$$d_W(Q_n, \mathcal{N}) \leq 4 \sqrt{2(2 + E[X_1^4])L_n^2 + 2 \sum_{1 \leq q, l \leq n} \left(\sum_{k=1}^n a_{k, q} a_{k, l} \right)^2 - \sum_{k=1}^n \left(\sum_{l=1}^{k-1} a_{k, l}^2 \right)^2}.$$

A similar expression can be obtained from (4.7) in the beta case.

7 Appendix - multiplication formula

We now formulate and prove the multiplication formula which is used in the proof of Corollary 5.2.

Theorem 7.1. (Multiplication formula). Let f_n, g_m satisfy (2.3) and

$$f_n \star_k^i g_m \in L^2(\mathbb{R}_+^{m+n-k-i}) \text{ for all } 0 \leq i \leq k \leq m \wedge n.$$

Then we have

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-k-i}(f_n \star_k^i g_m).$$

Proof. Without loss of generality we consider only $n \geq m$. We use mathematical induction with respect to m , if $m < n$, and with respect to n , if $m = n$. The formula is clearly valid for $n \geq 0$ and $m = 0$. Let us assume that the formula is valid for the following pairs of indices: $(n, m-1)$, $(n-1, m)$ and $(n-1, m-1)$. By (2.9) we get

$$(I_n(f_n)I_m(g_m)) \circ \Psi_t = S_1(t) + S_2(t) + S_3(t),$$

where

$$\begin{cases} S_1(t) \\ = mnI_{n-1}(f_n(t, *))I_{m-1}(g_m(t, *)) + mI_n(f_n)I_{m-1}(g_m(t, *)) + nI_{n-1}(f_n(t, *))I_m(g_m), \\ S_2(t) \\ = -(nI_{n-1}(f_n(v, *))I_m(g_m) \circ \Psi_t + mI_n(f_n) \circ \Psi_t I_{m-1}(g_m(v, *)))|_{v=2\lfloor t/2\rfloor+1+U_{\lfloor t/2\rfloor}}, \\ S_3(t) = I_n(f_n)I_m(g_m) - mnI_{n-1}(f_n(v, *))I_{m-1}(g_m(v, *))|_{v=2\lfloor t/2\rfloor+1+U_{\lfloor t/2\rfloor}}. \end{cases}$$

We note that by (2.3) we have $E[S_2(t) | \tilde{\mathcal{F}}_t] = 0$. Additionally, the function $s \mapsto E[S_3(t) | \tilde{\mathcal{F}}_s]$ is constant for $s \in [2\lfloor t/2\rfloor, 2\lfloor t/2\rfloor + 2)$ which, combined with (2.12), implies

$$\begin{aligned} & \int_0^\infty E[\nabla_t(I_n(f_n)I_m(g_m)) | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \\ &= \int_0^\infty E[(I_n(f_n)I_m(g_m)) \circ \Psi_t | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \\ &= \int_0^\infty E[S_1(t) | \tilde{\mathcal{F}}_t] d(Y_t - t/2). \end{aligned}$$

Then, by the induction hypothesis and renumeration in the first sum below, we get

$$\begin{aligned} & \int_0^\infty E[\nabla_t(I_n(f_n)I_m(g_m)) | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \\ &= \int_0^\infty E \left[nm \sum_{k=0}^{(m \wedge n)-1} k! \binom{m-1}{k} \binom{n-1}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-2-k-i}(f_n(t, *) \star_k^i g_m(t, *)) \right. \\ & \quad + m \sum_{k=0}^{(m-1) \wedge n} k! \binom{m-1}{k} \binom{n}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-1-k-i}(f_n \star_k^i g_m(t, *)) \\ & \quad \left. + n \sum_{k=0}^{m \wedge (n-1)} k! \binom{m}{k} \binom{n-1}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-1-k-i}(f_n(t, *) \star_k^i g_m) \Big| \tilde{\mathcal{F}}_t \right] d(Y_t - t/2) \\ &= \int_0^\infty E \left[\sum_{k=1}^{(m \wedge n)} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^{k-1} \binom{k}{i} I_{m+n-1-k-i}(f_n(t, *) \star_{k-1}^i g_m(t, *)) \right. \\ & \quad \left. + \sum_{k=0}^{(m-1) \wedge n} k! (m-k) \binom{m}{k} \binom{n}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-1-k-i}(f_n \star_k^i g_m(t, *)) \right] d(Y_t - t/2) \end{aligned}$$

$$+ \sum_{k=0}^{m \wedge (n-1)} k!(n-k) \binom{m}{k} \binom{n}{k} \sum_{i=0}^k \binom{k}{i} I_{m+n-1-k-i} (f_n(t, *) \star_k^i g_m) \Big| \tilde{\mathcal{F}}_t \Big] d(Y_t - t/2).$$

Thus, the formula (2.13) gives us for $m \neq n$

$$\begin{aligned} & \int_0^\infty E[\nabla_t(I_n(f_n)I_m(f_m)) | \tilde{\mathcal{F}}_t] d(Y_t - t/2) \\ &= \sum_{k=1}^{(m \wedge n)} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^{k-1} \frac{k-i}{m+n-k-i} \binom{k}{i} I_{m+n-k-i} (f_n \star_k^i g_m) \\ &+ \sum_{k=0}^{(m-1) \wedge n} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^k \frac{m-k}{m+n-k-i} \binom{k}{i} I_{m+n-k-i} (f_n \star_k^i g_m) \\ &+ \sum_{k=0}^{m \wedge (n-1)} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^k \frac{n-k}{m+n-k-i} \binom{k}{i} I_{m+n-k-i} (f_n \star_k^i g_m). \end{aligned}$$

If $m = n$, then the term $I_0(f_n \star_n^n g_n)$ does not appear in the above sums. Nevertheless, since $I_0(f_n \star_n^n g_n) = \langle f_n, g_n \rangle_{L^2(\mathbb{R}_+^n, dx/2)}$, the assertion of the theorem follows from (2.5) and Clark-Ocone formula (2.11). \square

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