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Approximation of smooth convex bodies by random polytopes

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Abstract

Let K be a convex body in $\mathbb{R}^n$ and $f : \partial K \to \mathbb{R}_+$ a continuous, strictly positive function with $\int_{\partial K} f(x)\mathrm{d}\mu_{\partial K}(x) = 1$. We give an upper bound for the approximation of K in the symmetric difference metric by an arbitrarily positioned polytope P_f in $\mathbb{R}^n$ having a fixed number of vertices. This generalizes a result by Ludwig, Schütt and Werner [36]. The polytope P_f is obtained by a random construction via a probability measure with density f. In our result, the dependence on the number of vertices is optimal. With the optimal density f, the dependence on K in our result is also optimal.

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1 Introduction and main result

Much research has been devoted to the subject of approximation of convex bodies by polytopes. This is due to the fact that it is fundamental in convex geometry and has applications in stochastic geometry, complexity, geometric algorithms and many more. We refer to, e.g., [13, 15, 17, 18, 19, 24, 25, 45, 47] for further details.

Random constructions are especially advantageous: when one considers random approximation, many metrics, namely metrics related to quermassintegrals of a convex body (see, e.g., [17, 50]), carry essentially the same amount of information as for best approximation.

Volume, surface area and also the mean width are such quermassintegrals. Metrics related to those quermassintegrals are the symmetric difference metric and the surface deviation which reflect the volume deviation, respectively the surface deviation, of the approximating and approximated objects. But many other metrics have been considered

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as well. In particular, approximation of a convex body K by inscribed or circumscribed polytopes with respect to these metrics has been studied extensively and many of the major questions have been resolved. The surveys and books by Gruber [23, 26, 27] and the references cited therein and, e.g., [2, 7, 9, 10, 22, 20, 33, 46, 49, 51, 55] are excellent sources in this context.

Typically, approximation by polytopes involves side conditions, like a prescribed number of vertices, or, more generally, k-dimensional faces [7]. Most often in the literature, it is required that the body contains the approximating polytope or vice versa. In this paper, we drop this condition and investigate the approximation of the body by arbitrarily positioned polytopes. Results in this direction for the Euclidean ball were obtained in [36] for the symmetric difference metric and in [30] for the surface deviation. These results show that one gains by a factor of dimension if one drops the restriction.

Here, we discuss the symmetric difference metric, which for convex bodies K and L in $\mathbb{R}^n$ is defined as

$$\mathrm{vol}_n(K\Delta L) := \mathrm{vol}_n\left(K \cup L\right) - \mathrm{vol}_n\left(K \cap L\right). \tag{1.1}$$

In the case when K is the n-dimensional unit ball $\mathbb{B}^n$ it was proved by Ludwig, Schütt and Werner [36] that there exists an arbitrarily positioned polytope P having N vertices such that for all sufficiently large N,

$$\mathrm{vol}_n(\mathbb{B}^n\Delta P) \leq a\, N^{-\frac{2}{n-1}}\, \kappa_n. \tag{1.2}$$

Here, $a \in (0,\infty)$ is an absolute constant and $\kappa_n = \mathrm{vol}_n\left(\mathbb{B}^n\right)$ is the volume of the n-dimensional unit ball. The corresponding result for convex bodies K in $\mathbb{R}^n$ that are C^2_+, i.e., have twice continuously differentiable boundary ∂K with strictly positive Gaussian curvature $\kappa_K(x)$, $x \in \partial K$, then follows from (1.2), together with a result by Ludwig [33]. Indeed, it was proved in [33] that there exists a constant $\mathrm{ldel}_n \in (0,\infty)$ only depending on the dimension n such that

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K\Delta P) : f_0(P) \leq N\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\,\mathrm{ldel}_n\,\mathrm{as}(K)^{\frac{n+1}{n-1}}, \tag{1.3}$$

where $f_0(P)$ denotes the number of vertices of P and

$$\mathrm{as}(K) = \mathrm{as}_1(K) := \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, \mathrm{d}\mu_{\partial K}(x)$$

is the affine surface area of K. Here, $\mu_{\partial K}$ is the surface measure on ∂K. The affine surface area [6, 32, 37, 54] is an important affine invariant from convex and differential geometry with applications in, e.g., approximation theory [8, 48, 55], the theory of valuations [3, 28, 29, 35, 34, 52], affine curvature flows [1, 58, 59], and has recently been extended to spherical and hyperbolic space [4, 5]. Its related affine isoperimetric inequality, which is equivalent to the famous Blaschke Santaló inequality (see, e.g., [17, 50]), says that

$$\left(\frac{\mathrm{as}(K)}{\mathrm{as}(\mathbb{B}^n)}\right)^{\frac{n+1}{n-1}} \leq \frac{\mathrm{vol}_n\left(K\right)}{\mathrm{vol}_n\left(\mathbb{B}^n\right)}$$

with equality if and only if K is an ellipsoid. Thus, applying the affine isoperimetric inequality, it follows from (1.2) and (1.3) that for sufficiently large N,

$$\mathrm{vol}_n(K\Delta P) \leq a\, N^{-\frac{2}{n-1}}\ \mathrm{vol}_n(K). \tag{1.4}$$

However, it is desirable to have a direct proof available for the approximation of K without having to pass through the results of [33, 36]. Here, we do exactly that and construct a well approximating polytope P_f that is obtained via a random construction. It is the convex hull of randomly chosen points with respect to a probability measure with density f. In fact, it is only via our new approach that different densities can be considered, not just the uniform distribution as in [36]. The optimal density is related to the affine surface area (see below). With this optimal density, the dependence on K in our result is optimal. Our result also gives the optimal dependence on the number of vertices.

Our main theorem reads as follows.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$, $n \geq 2$, that is C^2_+. Let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function with*

$$\int_{\partial K} f(x) \mathrm{d}\mu_{\partial K}(x) = 1.$$

Then, for sufficiently large N there exists a polytope P_f in $\mathbb{R}^n$ having N vertices such that

$$\mathrm{vol}_n(K \Delta P_f) \leq a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x), \tag{1.5}$$

where $a \in (0, \infty)$ is an absolute constant.

Remark 1.2. In particular, (1.4) yields that

$$\mathrm{ldel}_n \leq C, \tag{1.6}$$

where $C \in (0, \infty)$ is an absolute constant. On the other hand, it was shown by Böröczky [7] that

$$\mathrm{vol}_n(\mathbb{B}^n \Delta P) \geq \frac{1}{67 e^2 \pi} \frac{1}{n} N^{-\frac{2}{n-1}}\, \kappa_n$$

for all arbitrarily positioned polytopes P with at most N vertices. Combining the latter estimate with (1.6), one deduces that there are absolute constants $C_1, C_2 \in (0, \infty)$, such that

$$C_1 \frac{1}{n} \leq \mathrm{ldel}_n \leq C_2.$$

Thus, there is a gap between upper and lower bound by a factor of dimension. Adopting the methods used in the proof of Theorem 1.1 below, it might be possible to show that also the corresponding lower bound holds true, that is,

$$\mathrm{vol}_n(K \Delta P_f) \geq a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x).$$

This would imply that with a *random* method one can not achieve a smaller lower bound than the upper one given in (1.5), an indication that the upper bound may be of the right order.

Simultaneously, it might be even possible to get rid of the factor $1/n$ in Böröczky's result without using a random approximation, that is, one would need to find a way to get an effective approximation by some *deterministic* polytope.

We leave both issues to further research.

Before we turn to the proof of the theorem, we want to discuss the expression on the right hand side of (1.5) for various densities.

It has been shown in [55] that the minimum on the right hand side of (1.5) is attained for the normalized affine surface area measure with density

$$f_{as}(x) := \frac{\kappa_K(x)^{\frac{1}{n+1}}}{\int\limits_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)}.$$

In this case, the theorem yields that

$$\mathrm{vol}_n(K\Delta P_f) \leq a\, N^{-\frac{2}{n-1}}\, \mathrm{as}(K)^{\frac{n+1}{n-1}}. \tag{1.7}$$

This shows that choosing the vertices of the random polytope according to curvature is optimal.

Let now K be a convex body in $\mathbb{R}^n$ such that its centroid $\int\limits_K x\mathrm{d}x/\,\mathrm{vol}_n(K)$ is at the origin. For $x \in \partial K$, we denote the corresponding outer unit normal by $N_K(x)$. Put

$$f_{\alpha,\beta}(x) := \frac{\langle x, N_K(x)\rangle^\alpha \kappa_K(x)^\beta}{\int\limits_{\partial K} \langle x, N_K(x)\rangle^\alpha \kappa_K(x)^\beta \mathrm{d}\mu_{\partial K}(x)},$$

where $\alpha, \beta \in \mathbb{R}$. For $p \in [-\infty, \infty]$, $p \neq -n$, let

$$\mathrm{as}_p(K) := \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} \mathrm{d}\mu_{\partial K}(x)$$

be the p-affine surface area of K. Then, the application of Theorem 1.1 gives

$$\begin{aligned}\mathrm{vol}_n(K\Delta P_f) \leq a\, N^{-\frac{2}{n-1}} &\int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1-2\beta}{n-1}}}{\langle x, N_K(x)\rangle^{\frac{2\alpha}{n-1}}} \mathrm{d}\mu_{\partial K}(x) \\ &\times \left(\int\limits_{\partial K} \langle x, N_K(x)\rangle^\alpha \kappa_K(x)^\beta \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{2}{n-1}}.\end{aligned}$$

The second integral is a p-affine surface area if and only if

$$\alpha = -\frac{n(p-1)}{n+p} \quad \text{and} \quad \beta = \frac{p}{n+p}.$$

In this case, we obtain

$$\mathrm{vol}_n(K\Delta P_f) \leq a\, N^{-\frac{2}{n-1}}\, \mathrm{as}_p(K)^{\frac{2}{n-1}}\, \mathrm{as}_q(K),$$

where $q = \frac{n-p}{n+p-2}$. The p-affine surface area, an extension of the classical affine surface area, was introduced by Lutwak for $p > 1$ in [38] and has been extended to all other p in [56]. It is central to the rapidly developing L_p Brunn Minkowski theory, e.g., [12, 14, 31, 42, 57, 61].

The third measure of interest is the surface measure itself given by the constant density

$$f_{sm}(x) := \frac{1}{\mathrm{vol}_{n-1}(\partial K)},$$

where $\mathrm{vol}_{n-1}(\cdot)$ describes the $(n-1)$-dimensional Hausdorff measure of the argument set. For this measure, Theorem 1.1 yields

$$\mathrm{vol}_n(K\Delta P_f) \leq a\, N^{-\frac{2}{n-1}}\ \mathrm{vol}_{n-1}(\partial K)^{\frac{2}{n-1}} \int\limits_{\partial K} \kappa_K(x)^{\frac{1}{n-1}}\, \mathrm{d}\mu_{\partial K}(x).$$

The remaining paper is organized as follows. In Section 2 we present results that are needed for the proof of the main theorem. Its proof is given in Section 3.

2 Preliminaries

We start this section by introducing some more necessary notation. For $u \in \mathbb{S}^{n-1}$ and $h \geq 0$, let $H := H(u,h)$ be the hyperplane orthogonal to u and at distance h from the origin. Let $\mathbb{P}_f$ be the probability measure on ∂K given by

$$\mathrm{d}\mathbb{P}_f = f(x)\mathrm{d}\mu_{\partial K}(x).$$

Let $H \cap K \neq \emptyset$. Then, $\mathbb{P}_{f_{\partial K \cap H}}$ is the probability measure on $\partial K \cap H$ given by

$$\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}} = \frac{f(x)\mathrm{d}\mu_{\partial K\cap H}}{\int\limits_{\partial K\cap H} f(x)\mathrm{d}\mu_{\partial K\cap H}}.$$

For points $x_1,\dots,x_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we denote by $[x_1,\dots,x_i]$ the convex hull of the underlying point set. Moreover, let ω_n denote the $(n-1)$-dimensional Hausdorff measure of the unit sphere $\mathbb{S}^{n-1}$.

In order to prove the main theorem, we need the following results. The first two were proved in [55].

Theorem 2.1 ([55])**.** *Denote by $\mathbb{E}[f,N]$ the expected volume of the convex hull of N points chosen randomly on ∂K with respect to $\mathbb{P}_f$. Then,*

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}[f,N]}{N^{-\frac{2}{n-1}}} = \frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\,\omega_{n-1}^{\frac{2}{n-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\mathrm{d}\mu_{\partial K}(x).$$

Lemma 2.2 ([55])**.** *Let $\sigma = (\sigma_i)_{1\leq i\leq n}$ be a sequence of signs, that is $\sigma_i \in \{-1,1\}$, $1 \leq i \leq n$. We set*

$$K^\sigma := \{x = (x_1,\dots,x_n) \in K : \mathrm{sign}(x_i) = \sigma_i, 1 \leq i \leq n\}.$$

Then,

$$\mathbb{P}_f^N(\{0 \notin [x_1,\dots,x_N]\}) \leq 2^n \left(1 - \min_\sigma \int\limits_{\partial K^\sigma} f(x)\mathrm{d}\mu_{\partial K}\right)^N,$$

where $\mathbb{P}_f^N$ indicates that we choose N points on ∂K with respect to $\mathbb{P}_f$.

We also need the following 'Blaschke-Petkanchin-type' formula that appears as a special case of a result in [62]; an alternative and simpler proof for this version is also given in [48].

Theorem 2.3 ([62])**.** *Let $g(x_1,\dots,x_n)$ be a continuous, non-negative function. Then,*

$$\begin{aligned}&\int_{\partial K}\cdots\int_{\partial K} g(x_1,\dots,x_n)\mathrm{d}\mathbb{P}_f(x_1)\cdots\mathrm{d}\mathbb{P}_f(x_n)\\ &\quad=(n-1)!\int_{\mathbb{S}^{n-1}}\int_0^\infty\int_{\partial K\cap H}\cdots\int_{\partial K\cap H} g(x_1,\dots,x_n)\,\mathrm{vol}_{n-1}([x_1,\dots,x_n])\\ &\qquad\times\prod_{j=1}^n l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\mathrm{d}h\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\end{aligned}$$

with

$$l_H(x_j):=\left\|\mathrm{proj}_H N_K(x_j)\right\|^{-1},$$

where proj_H is the orthogonal projection onto the hyperplane H.

The next result is a special case of [43, equation (29)].

Lemma 2.4 ([43])**.** *It holds that*

$$\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}(\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2\mathrm{d}\mu_{\mathbb{S}^{n-2}}(x_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(x_n)=\frac{n\,\omega_{n-1}^n}{(n-1)!\,(n-1)^{n-1}}\,.$$

3 Proof of the main result

We now turn to the proof of our main theorem. As in [36] we will obtain the approximating polytope in a probabilistic way. To be more precise, we consider a convex body that is slightly bigger than the body K and then choose N points randomly on the boundary of the bigger body and take the convex hull of these points. Such a random polytope exists with high probability.

Without loss of generality we can assume that the origin, denoted by 0, is in the interior of K, namely at the center of gravity of K. Since our density functions live on the boundary of K, we will choose the random points on ∂K and approximate a slightly smaller body, say $(1-c)K$, where $c:=c_{n,N}$ depends on the dimension n and the number of points N and has to be carefully chosen. In fact, we choose c such that

$$\mathbb{E}[f,N]=\mathrm{vol}_n((1-c)K)=(1-c)^n\,\mathrm{vol}_n(K). \tag{3.1}$$

By Theorem 2.1, we get for sufficiently large N,

$$\mathrm{vol}_n(K)-\mathbb{E}[f,N]\;\sim\;N^{-\frac{2}{n-1}}\,\frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\,\omega_{n-1}^{\frac{2}{n-1}}}\int_{\partial K}\frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\mathrm{d}\mu_{\partial K}(x),$$

where for two functions $g_1(x)$ and $g_2(x)$ the relation $g_1(x)\sim g_2(x)$ means that

$$\lim_{x\to\infty}\frac{g_1(x)}{g_2(x)}=1.$$

Hence, with the choice (3.1) of c,

$$\mathrm{vol}_n(K)-(1-c)^n\,\mathrm{vol}_n(K)\;\sim\;N^{-\frac{2}{n-1}}\,\frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\,\omega_{n-1}^{\frac{2}{n-1}}}\int_{\partial K}\frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\mathrm{d}\mu_{\partial K}(x),$$

as $N \to \infty$. This leads to

$$c \sim N^{-\frac{2}{n-1}} \frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\,\omega_{n-1}^{\frac{2}{n-1}}} \frac{1}{n \operatorname{vol}_n(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x), \tag{3.2}$$

as $N \to \infty$. In particular, for sufficiently large N we get that

$$c \geq \left(1-\frac{1}{n}\right) N^{-\frac{2}{n-1}} \frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!\,\omega_{n-1}^{\frac{2}{n-1}}} \frac{1}{n \operatorname{vol}_n(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x). \tag{3.3}$$

In what follows, we calculate the expected volume difference $\mathbb{E}[\operatorname{vol}_n((1-c)K \Delta P_N)]$ between $(1-c)K$ and a random polytope $P_N := [x_1, \dots, x_N]$ whose vertices are randomly chosen from the boundary of K according to the probability measure $\mathbb{P}_f$. Please note that random polytopes are simplicial with probability 1. We split the proof of the main theorem into several lemmas.

Let us recall that for fixed $u \in \mathbb{S}^{n-1}$ and $h \geq 0$, we denote by $H := H(u,h)$ the unique hyperplane orthogonal to u and at distance h from the origin. Let H^+ be the corresponding half space containing the origin and put

$$\mathbb{P}_f(\partial K \cap H^+) := \int\limits_{\partial K \cap H^+} f(x) \mathrm{d}\mu_{\partial K}. \tag{3.4}$$

For fixed $u \in \mathbb{S}^{n-1}$ and sufficiently large N, let $\epsilon > 0$ be such that $ch_K(u) < \epsilon \leq h_K(u)/n$, where $h_K(u) := \max\limits_{x \in K} \langle x, u \rangle$ is the support function of K in direction u.

In what follows, $a \in (0, \infty)$ will always be an absolute constant that may changes from line to line.

Lemma 3.1. *For sufficiently large N, for all $\epsilon > ch_K(u)$ sufficiently small,*

$$\begin{aligned}
&\mathbb{E}[\operatorname{vol}_n((1-c)K \Delta P_N)] \\
&\quad \leq a \binom{N}{n} (n-1)! \int\limits_{\mathbb{S}^{n-1}} \int\limits_{h_K(u)-\epsilon}^{h_K(u)} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n} \max\{0, ((1-c)h_K(u) - h)\} \\
&\qquad \times \int\limits_{\partial K \cap H} \cdots \int\limits_{\partial K \cap H} (\operatorname{vol}_{n-1}([x_1, \dots, x_n]))^2 \prod_{j=1}^{n} l_H(x_j) \\
&\qquad \times \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_1) \cdots \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_n) \mathrm{d}h \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u).
\end{aligned}$$

Proof of Lemma 3.1. With the above choice of the parameter c in (3.1) we obtain for sufficiently large N,

$$\operatorname{vol}_n(K \setminus (1-c)K) = \int\limits_{\partial K} \cdots \int\limits_{\partial K} \operatorname{vol}_n(K \setminus P_N)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_N).$$

We combine this observation with the relation

$$\operatorname{vol}_n((1-c)K \Delta P_N) = \operatorname{vol}_n(K \setminus (1-c)K) - \operatorname{vol}_n(K \setminus P_N) + 2 \operatorname{vol}_n((1-c)K \cap P_N^c)$$

and Lemma 2.2 and obtain that for sufficiently large N,

$$\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n((1-c)K\Delta P_N)]\\
&\quad= \int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\Delta P_N)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= \mathrm{vol}_n(K\setminus(1-c)K) - \int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n(K\setminus P_N)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\qquad+2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= 2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= 2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathbb{1}_{\{0\in P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\qquad+2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathbb{1}_{\{0\notin P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad\le 2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathbb{1}_{\{0\in P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\qquad+2\,\mathrm{vol}_n(K)\,\mathbb{P}_f^N(\{0\notin[x_1,\dots,x_N]\})\\
&\quad\le 2\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K\cap P_N^c)\mathbb{1}_{\{0\in P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\qquad+2\,\mathrm{vol}_n(K)\,2^n\left(1-\min_\sigma\int_{\partial K^\sigma} f(x)\mathrm{d}\mu_{\partial K}\right)^N.
\end{aligned}$$

The density f is strictly positive everywhere and since the origin is in the interior of K, the second summand is essentially of order b^{-N}, where $b>1$. Later we will see that the first summand is of order $N^{-\frac{2}{n-1}}$ and thus it is enough to consider the first one in what follows.

Next, we introduce the function $\Phi_{j_1,\dots,j_n}:\partial K\times\dots\times\partial K\to\mathbb{R}$ as

$$\begin{aligned}
&\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)\\
&:=\begin{cases}0 & :[x_{j_1},\dots,x_{j_n}]\notin\mathcal{F}_{n-1}(P_N)\text{ or }0\notin P_N\\ \mathrm{vol}_n((1-c)K\cap P_N^c\cap\mathrm{cone}(x_{j_1},\dots,x_{j_n}))\mathbb{1}_{\{0\in P_N\}} & :[x_{j_1},\dots,x_{j_n}]\in\mathcal{F}_{n-1}(P_N)\text{ and }0\in P_N,\end{cases}
\end{aligned}$$

where $\mathcal{F}_{n-1}(P_N)$ denotes the set of facets of P_N and

$$\mathrm{cone}(x_1,\dots,x_n):=\left\{\sum_{i=1}^n a_i\,x_i : a_i\ge 0, 1\le i\le n\right\}.$$

For all random polytopes P_N that contain the origin as an interior point, we have that

$$\mathbb{R}^n=\bigcup_{[x_{j_1},\dots,x_{j_n}]\in\mathcal{F}_{n-1}(P_N)}\mathrm{cone}(x_{j_1},\dots,x_{j_n}).$$

Moreover,

$$\mathbb{P}_f^{N-n}(\{(x_{n+1},\dots,x_N) : [x_1,\dots,x_n] \in \mathcal{F}_{n-1}(P_N) \text{ and } 0 \in P_N\}) = \left(\int_{\partial K \cap H^+} f(x)\mathrm{d}\mu_{\partial K}\right)^{N-n} = \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n},$$

where H is the hyperplane spanned by the points $x_1,\dots,x_n$ and we recall the definition of $\mathbb{P}_f(\partial K \cap H^+)$ given in (3.4). Now, note that a random polytope is simplicial with probability 1. Therefore, and since the set where H is not well defined has measure zero and all N points are identically distributed, we arrive at

$$\begin{aligned}
&\int_{\partial K}\cdots\int_{\partial K} \mathrm{vol}_n((1-c)K \cap P_N^c)\mathbb{1}_{\{0\in P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= \int_{\partial K}\cdots\int_{\partial K} \sum_{\{j_1,\dots,j_n\}\subseteq\{1,\dots,N\}} \Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= \binom{N}{n}\int_{\partial K}\cdots\int_{\partial K} \Phi_{1,\dots,n}(x_1,\dots,x_N)\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_N)\\
&\quad= \binom{N}{n}\int_{\partial K}\cdots\int_{\partial K} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n}\\
&\qquad\times \mathrm{vol}_n((1-c)K \cap H^- \cap \mathrm{cone}(x_1,\dots,x_n))\mathbb{1}_{\{0\in P_N\}}\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_n)\\
&\quad\le \binom{N}{n}\int_{\partial K}\cdots\int_{\partial K} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n}\\
&\qquad\times \mathrm{vol}_n((1-c)K \cap H^- \cap \mathrm{cone}(x_1,\dots,x_n))\mathrm{d}\mathbb{P}_f(x_1)\dots\mathrm{d}\mathbb{P}_f(x_n).
\end{aligned}$$

Theorem 2.3 now yields that for sufficiently large N,

$$\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n((1-c)K\Delta P_N)]\\
&\quad\le a\binom{N}{n}(n-1)!\int_{\mathbb{S}^{n-1}}\int_0^\infty\int_{\partial K\cap H}\cdots\int_{\partial K\cap H}\left(\mathbb{P}_f(\partial K\cap H^+)\right)^{N-n}\mathrm{vol}_{n-1}([x_1,\dots,x_n])\\
&\qquad\times \mathrm{vol}_n((1-c)K\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n))\prod_{j=1}^n l_H(x_j)\\
&\qquad\times \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\mathrm{d}h\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u).
\end{aligned}$$

Notice that $h \in [0, h_K(u)]$, where we recall that $h_K(u)$ is the support function of K in direction u. The same arguments as in [36, page 9] and [48, page 2255] show that it is enough to bound the range of integration for h from below by $h_K(u)-\epsilon$, where $\epsilon>0$ is sufficiently small, since the remaining expression decreases exponentially fast. In particular, for sufficiently large N we can choose ϵ such that $ch_K(u) < \epsilon \le h_K(u)/n$. Similar to [36, page 9],

$$\begin{aligned}
&\mathrm{vol}_n((1-c)K\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n))\\
&\qquad\le \frac{h}{n}\,\mathrm{vol}_{n-1}([x_1,\dots,x_n])\cdot\max\left\{0,\left(\frac{(1-c)h_K(u)}{h}\right)^n-1\right\}.
\end{aligned}$$

Since c is of the order $N^{-\frac{2}{n+1}}$ and $\epsilon \le h_K(u)/n$, for sufficiently large N,

$$\frac{(1-c)h_K(u)-h}{h} \le \frac{(1-c)h_K(u)-h_K(u)+\epsilon}{h_K(u)-\epsilon} \le \frac{\frac{1}{n}-c}{1-\frac{1}{n}} \le \frac{1}{n-1}.$$

Thus,

$$\begin{aligned}
&\frac{1}{n}\left[\left(\frac{(1-c)h_K(u)}{h}\right)^n - 1\right] \\
&\quad= \frac{1}{n}\left[\left(\frac{h+(1-c)h_K(u)-h}{h}\right)^n - 1\right] \\
&\quad= \frac{1}{n}\left[\left(1+\frac{(1-c)h_K(u)-h}{h}\right)^n - 1\right] \\
&\quad= \frac{1}{n}\left[n\,\frac{(1-c)h_K(u)-h}{h} + \frac{n(n-1)}{2}\left(\frac{(1-c)h_K(u)-h}{h}\right)^2 + \cdots\right] \\
&\quad\le \frac{(1-c)h_K(u)-h}{h}\cdot\sum_{k=0}^{\infty}\frac{n^k}{k!}\left(\frac{(1-c)h_K(u)-h}{h}\right)^k \\
&\quad\le \exp\left(\frac{n}{n-1}\right)\frac{(1-c)h_K(u)-h}{h} \\
&\quad\le a\,\frac{(1-c)h_K(u)-h}{h}.
\end{aligned}$$

Therefore, for sufficiently large N,

$$\begin{aligned}
&\mathrm{vol}_n((1-c)K\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n)) \\
&\qquad\le a\ \mathrm{vol}_{n-1}([x_1,\dots,x_n])\cdot\max\{0,((1-c)h_K(u)-h)\}.
\end{aligned}$$

This proves the lemma. □

To evaluate the innermost integral in the expression of the foregoing lemma, we first recall the necessary notation and results we need from [48].

Let $x(u)$ be the point on ∂K with fixed outer unit normal vector $u\in\mathbb{S}^{n-1}$. Since K has a twice differentiable boundary, there is a paraboloid $Q_2^{(x(u))}$ given by a quadratic form $b_2:=b_2^{(x(u),x(u))}$ that osculates ∂K at $x(u)$. An explicit construction can be found in, e.g., [48, page 2265].

Let $\mathbb{R}^n=(\mathbb{R}_+\times\mathbb{S}^{n-2})\times\mathbb{R}$, and denote by (rv,z) a point in $\mathbb{R}^n$, where $v\in\mathbb{S}^{n-2}$, $r\in\mathbb{R}_+$ and $z\in\mathbb{R}$. We identify the support plane of ∂K at $x(u)$ with the plane $z=0$ and $x(u)$ with the origin, so that K is contained in the half space corresponding to $z\ge 0$. Please note also that $h=h_K(u)-z$ by construction. The following lemma that summarizes results from [48, pages 2265 and 2271] will be crucial.

Lemma 3.2 ([48])**.** *Let $\delta>0$ be sufficiently small. Then, there exists $\lambda>0$, only depending on δ and K, such that for each boundary point $x(u)\in\partial K$ the λ-neighbourhood U^λ of $x(u)$ in ∂K defined by $\mathrm{proj}_{\mathbb{R}^{n-1}}U^\lambda=\lambda\mathbb{B}^{n-1}$ can be represented by a convex function $z:=g(rv):=g^{(x(u))}(rv)$ that satisfies*

$$(1+\delta)^{-\frac{1}{2}}\,b_2(v)^{-\frac{1}{2}}\,z^{\frac{1}{2}}\le r\le(1+\delta)^{\frac{1}{2}}\,b_2(v)^{-\frac{1}{2}}\,z^{\frac{1}{2}} \tag{3.5}$$

and

$$(1+\delta)^{-\frac{3}{2}}\,2^{-1}\,b_2(v)^{-\frac{1}{2}}\,z^{-\frac{1}{2}}\le\frac{l_H(rv)}{\langle v,N_{K\cap H}(rv)\rangle}\le(1+\delta)^{\frac{3}{2}}\,2^{-1}\,b_2(v)^{-\frac{1}{2}}\,z^{-\frac{1}{2}} \tag{3.6}$$

in this neighborhood U^λ. Here, for fixed rv, H is the hyperplane that contains $(rv,g(rv))$ and is parallel to $\mathbb{R}^{n-1}$ and $N_{K\cap H}(rv)$ is the outer unit normal vector to $\partial K\cap H$ at this point.

Moreover, for the density f, for all $p\in U^\lambda$, it holds that

$$(1+\delta)^{-1}\,f(x(u))\le f(p)\le(1+\delta)f(x(u)). \tag{3.7}$$

We next estimate the innermost integral of Lemma 3.1.

Lemma 3.3. *Let $x(u)$ be the point on ∂K with fixed outer unit normal vector $u \in \mathbb{S}^{n-1}$ and denote by z the distance from H to the support plane of ∂K at $x(u)$, i.e., $z = h_K(u) - h$. Then, for all sufficiently small $\delta > 0$,*

$$\begin{aligned}\int_{\partial K\cap H} \cdots \int_{\partial K\cap H} &(\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2 \prod_{j=1}^{n} l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\\ &\le (1+\delta)^{\frac{n(n+3)}{2}}\, 2^{\frac{n^2-n-2}{2}}\, z^{\frac{n^2-n-2}{2}}\, \frac{n\,\omega_{n-1}^n}{(n-1)!\,(n-1)^{n-1}}\, f(x(u))^n\, \kappa_K(x(u))^{-\frac{n}{2}-1}\\ &\quad + \delta O(z^{\frac{n^2-n-2}{2}}),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ.

Proof of Lemma 3.3. The proof follows closely the arguments given in [48]. We first replace the random points x_i, $i \in \{1,\dots,n\}$, chosen on $\partial K \cap H$ by random points chosen on the intersection of H with the paraboloid $Q_2^{(x(u))}$. In order to do this, we write each point x_i, $i \in \{1,\dots,n\}$, as $x_i = r(v_i)v_i$, where $r(v_i)$ is the radial function of $K \cap H$ with estimates given above in (3.5). The result presented in [48, equation (68)] then implies that

$$|\mathrm{vol}_{n-1}([x_1,\dots,x_n]) - \mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n])| \le \delta O(z^{\frac{n-1}{2}}),$$

where $r_2(v) := b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}}$, $\delta > 0$ is arbitrarily small and the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ. We therefore obtain that

$$\begin{aligned}\int_{\partial K\cap H} \cdots \int_{\partial K\cap H} &(\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2 \prod_{j=1}^{n} l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\\ &= \int_{\partial K\cap H} \cdots \int_{\partial K\cap H} \left[(\mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n]))^2 + \delta O(z^{n-1})\right]\\ &\quad \times \prod_{j=1}^{n} l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ.

We first evaluate the integral involving the $O(\cdot)$ term. Notice that the density f is uniformly bounded and that by (3.6), integration concerning $l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_j)$ results in terms of order $O(z^{-\frac{1}{2}})\,\mathrm{vol}_{n-2}(\partial K\cap H)$. Since, in view of (3.5), $\mathrm{vol}_{n-2}(\partial K\cap H) = O(z^{\frac{n-2}{2}})$,

$$\begin{aligned}\int_{\partial K\cap H} \cdots \int_{\partial K\cap H} &\delta O(z^{n-1}) \prod_{j=1}^{n} l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\\ &= \delta O(z^{n-1}) O(z^{-\frac{n}{2}})(\mathrm{vol}_{n-2}(\partial K\cap H))^n\\ &= \delta O(z^{n-1-\frac{n}{2}+n\frac{n-2}{2}})\\ &= \delta O(z^{\frac{n^2-n-2}{2}}),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ.

Now, we turn to the first summand. Rewriting the integral over $\mathbb{S}^{n-2}$ and using (3.5),

(3.6) and (3.7), we get as in [48, page 2274] that

$$\begin{aligned}
&\int_{\partial K\cap H}\cdots\int_{\partial K\cap H}(\mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n]))^2\prod_{j=1}^n l_H(x_j)\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_n)\\
&\quad=\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}(\mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n]))^2\\
&\qquad\times\prod_{j=1}^n f(r(v_j)v_j)\frac{l_H(r(v_j)v_j)\,r(v_j)^{n-2}}{\langle v_j,N_{K\cap H}(r(v_j)v_j)\rangle}\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_n)\\
&\quad\le(1+\delta)^{\frac{n(n+3)}{2}}\,2^{-n}\,z^{-n}\,f(x(u))^n\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}(\mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n]))^2\\
&\qquad\times\prod_{j=1}^n r_2(v_j)^{n-1}\,\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_n),
\end{aligned}$$

where again $r_2(v)=b_2(v)^{-\frac{1}{2}}z^{\frac{1}{2}}$. Define the ellipsoid E as the $(n-1)$-dimensional convex body with radial function $b_2(v)^{-\frac{1}{2}}$, i.e., E is the intersection of $Q_2^{(x(u))}$ with the hyperplane $z=1$. Thus, since vol_{n-1} is homogeneous, the integral appearing in the latter expression can be rewritten as an integral where the random points are chosen in the interior of E according to the uniform distribution. That is,

$$\begin{aligned}
&\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}(\mathrm{vol}_{n-1}([r_2(v_1)v_1,\dots,r_2(v_n)v_n]))^2\prod_{j=1}^n r_2(v_j)^{n-1}\,\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(v_n)\\
&\quad=z^{\frac{n(n-1)}{2}}\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}\int_0^{b_2(u_1)^{-\frac{1}{2}}}\cdots\int_0^{b_2(u_n)^{-\frac{1}{2}}}(\mathrm{vol}_{n-1}([b_2(u_1)^{-\frac{1}{2}}z^{\frac{1}{2}}u_1,\dots,b_2(u_n)^{-\frac{1}{2}}z^{\frac{1}{2}}u_n]))^2\\
&\qquad\times\prod_{j=1}^n((n-1)t_j^{n-2})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\mathrm{d}\mu_{\mathbb{S}^{n-2}}(u_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(u_n)\\
&\quad=z^{\frac{n(n-1)}{2}+n-1}\int_{\mathbb{S}^{n-2}}\cdots\int_{\mathbb{S}^{n-2}}\int_0^{b_2(u_1)^{-\frac{1}{2}}}\cdots\int_0^{b_2(u_n)^{-\frac{1}{2}}}(\mathrm{vol}_{n-1}([b_2(u_1)^{-\frac{1}{2}}u_1,\dots,b_2(u_n)^{-\frac{1}{2}}u_n]))^2\\
&\qquad\times\prod_{j=1}^n((n-1)t_j^{n-2})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\mathrm{d}\mu_{\mathbb{S}^{n-2}}(u_1)\cdots\mathrm{d}\mu_{\mathbb{S}^{n-2}}(u_n)\\
&\quad=z^{\frac{n^2+n-2}{2}}\,(n-1)^n\int_E\cdots\int_E(\mathrm{vol}_{n-1}([\tilde{x}_1,\dots,\tilde{x}_n]))^2\mathrm{d}x_1\cdots\mathrm{d}x_n,
\end{aligned}$$

where $\tilde{x}_i$ is the projection of the point x_i onto the boundary of E, i.e.,

$$\tilde{x}_i=\frac{x_i}{\|x_i\|}\,r_E\left(\frac{x_i}{\|x_i\|}\right).$$

Here, r_E is the radial function of E and $\|\cdot\|$ is the Euclidean norm, with 0 at the center of E. The random elements $\mathrm{d}x_i$ as well as vol_{n-1} are homogeneous and invariant with respect to volume preserving affinities acting in the affine subspace $\{z=1\}$. Moreover, since the volume of E equals $2^{\frac{n-1}{2}}\kappa_K(x(u))^{-\frac{1}{2}}\kappa_{n-1}$, we get by first transforming the ellipsoid E into the Euclidean ball $\mathbb{B}^{n-1}$ (using a suitable affinity), then again rewriting

the integral as an integral over the sphere $\mathbb{S}^{n-2}$ and finally using Lemma 2.4,

$$\begin{aligned}
&z^{\frac{n^2+n-2}{2}}\,(n-1)^n \int_E \cdots \int_E (\mathrm{vol}_{n-1}([\tilde{x}_1,\dots,\tilde{x}_n]))^2 \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&\quad = z^{\frac{n^2+n-2}{2}} \left(2^{\frac{n-1}{2}} \kappa_K(x(u))^{-\frac{1}{2}}\right)^2 \left(2^{\frac{n-1}{2}} \kappa_K(x(u))^{-\frac{1}{2}}\right)^n \\
&\qquad \times \int_{\mathbb{S}^{n-2}} \cdots \int_{\mathbb{S}^{n-2}} (\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2 \mathrm{d}\mu_{\mathbb{S}^{n-2}}(x_1) \cdots \mathrm{d}\mu_{\mathbb{S}^{n-2}}(x_n) \\
&\quad = z^{\frac{n^2+n-2}{2}}\, 2^{\frac{n^2+n-2}{2}}\, \kappa_K(x(u))^{-\frac{n}{2}-1}\, \frac{n\,\omega_{n-1}^n}{(n-1)!\,(n-1)^{n-1}}\,.
\end{aligned}$$

Combining the above calculations yields that for all sufficiently small $\delta > 0$ it holds that

$$\begin{aligned}
&\int_{\partial K \cap H} \cdots \int_{\partial K \cap H} (\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2 \prod_{j=1}^n l_H(x_j) \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_1) \cdots \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_n) \\
&\quad \le (1+\delta)^{\frac{n(n+3)}{2}}\, 2^{\frac{n^2-n-2}{2}}\, z^{\frac{n^2-n-2}{2}}\, \frac{n\,\omega_{n-1}^n}{(n-1)!\,(n-1)^{n-1}}\, f(x(u))^n\, \kappa_K(x(u))^{-\frac{n}{2}-1} \\
&\qquad + \delta O(z^{\frac{n^2-n-2}{2}}),
\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ. This proves the lemma. □

Now, we further analyze the expression appearing in Lemma 3.1. In order to do this, we put $s := \mathbb{P}_f(\partial K \cap H^-)$, i.e., $\mathbb{P}_f(\partial K \cap H^+) = 1 - s$. The results stated in [48, equation (71)] also imply the following estimates.

Lemma 3.4 ([48])**.** *For all sufficiently small $\delta > 0$, it holds that*

$$\begin{aligned}
&(1+\delta)^{-n}\, 2^{\frac{n-1}{2}}\, f(x(u))\, \kappa_K(x(u))^{-\frac{1}{2}}\, \kappa_{n-1}\, z^{\frac{n-1}{2}} \\
&\qquad \le s \le (1+\delta)^{n+1}\, 2^{\frac{n-1}{2}}\, f(x(u))\, \kappa_K(x(u))^{-\frac{1}{2}}\, \kappa_{n-1}\, z^{\frac{n-1}{2}}
\end{aligned} \tag{3.8}$$

Therefore,

$$z \le (1+\delta)^{\frac{2n}{n-1}}\, \frac{\kappa_K(x(u))^{\frac{1}{n-1}}\, (n-1)^{\frac{2}{n-1}}}{2\, f(x(u))^{\frac{2}{n-1}}\, \omega_{n-1}^{\frac{2}{n-1}}}\, s^{\frac{2}{n-1}} \tag{3.9}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}s} \le (1+\delta)^n\, \frac{\kappa_K(x(u))^{\frac{1}{2}}\, 2^{-\frac{n-3}{2}}}{f(x(u))\, \omega_{n-1}}\, z^{-\frac{n-3}{2}}. \tag{3.10}$$

Using once more results from [48], we continue the proof of the main theorem as follows.

Lemma 3.5. *For sufficiently large N and sufficiently small $\delta > 0$, we have*

$$\mathbb{E}[\mathrm{vol}_n((1-c)K \Delta P_N)] \le I + II,$$

where

$$I := (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1} \int_0^1 (1-s)^{N-n}\, s^{n-1}\, (z - c h_K(u))\, \mathrm{d}s \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)$$

and

$$II := (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1} \times \int\limits_0^{s(ch_K(u))} (1-s)^{N-n}\, s^{n-1}\, (ch_K(u)-z)\, \mathrm{d}s \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u).$$

Here, $z = z(s)$ *and*

$$s(ch_K(u)) := \int\limits_{\partial K \cap H^-} f(x) \mathrm{d}\mu_{\partial K},$$

where H *is the unique hyperplane with unit normal vector* $u \in \mathbb{S}^{n-1}$ *and distance* $(1-c)h_K(u)$ *from the origin and* H^- *the corresponding half space that does not contain the origin.*

Proof of Lemma 3.5. Observe first that

$$\max\{0, ((1-c)h_K(u) - h)\} = 0 \quad \text{if} \quad h > (1-c)h_K(u).$$

This, Lemma 3.1, Lemma 3.3 and the substitution $z = h_K(u) - h$ then yield that

$$\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n((1-c)K \Delta P_N)] \\
&\leq (1+\delta)^{\frac{n(n+3)}{2}}\, a\, 2^{\frac{n^2-n-2}{2}} \binom{N}{n} \frac{n\,\omega_{n-1}^n}{(n-1)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f(x(u))^n\, \kappa_K(x(u))^{-\frac{n}{2}-1} \\
&\quad \times \int\limits_{h_K(u)-\epsilon}^{(1-c)h_K(u)} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n} z^{\frac{n^2-n-2}{2}} ((1-c)h_K(u) - h) \mathrm{d}h \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \\
&\quad + \delta \binom{N}{n} (n-1)! \int\limits_{\mathbb{S}^{n-1}} \int\limits_{h_K(u)-\epsilon}^{(1-c)h_K(u)} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n} O(z^{\frac{n^2-n-2}{2}}) \\
&\quad \times ((1-c)h_K(u) - h) \mathrm{d}h \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \\
&= (1+\delta)^{\frac{n(n+3)}{2}}\, a\, 2^{\frac{n^2-n-2}{2}} \binom{N}{n} \frac{n\,\omega_{n-1}^n}{(n-1)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f(x(u))^n\, \kappa_K(x(u))^{-\frac{n}{2}-1} \\
&\quad \times \int\limits_{ch_K(u)}^{\epsilon} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n} z^{\frac{n^2-n-2}{2}} (z - ch_K(u)) \mathrm{d}z \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \\
&\quad + \delta \binom{N}{n} (n-1)! \int\limits_{\mathbb{S}^{n-1}} \int\limits_{ch_K(u)}^{\epsilon} \left(\mathbb{P}_f(\partial K \cap H^+)\right)^{N-n} O(z^{\frac{n^2-n-2}{2}}) (z - ch_K(u)) \mathrm{d}z \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u).
\end{aligned}$$

As the later calculations will show, the order of both summands is $N^{-\frac{2}{n-1}}$. Since δ is arbitrarily small, it is enough to consider the first summand in what follows.

We use (3.10) and then (3.9) to change from $z^{\frac{(n-1)^2}{2}}$ to s^{n-1} and obtain that, for

sufficiently large N,

$$
\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n((1-c)K\Delta P_N)]\\
&\quad\le (1+\delta)^{\frac{n(n+3)}{2}+n}\, a\, 2^{\frac{n^2-n-2}{2}}\, 2^{-\frac{n-3}{2}} \binom{N}{n} \frac{n\,\omega_{n-1}^n}{(n-1)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f(x(u))^{n-1}\, \kappa_K(x(u))^{-\frac{n}{2}-\frac{1}{2}}\\
&\qquad\times \int\limits_{s(ch_K(u))}^{1} (1-s)^{N-n}\, z^{\frac{n^2-n-2-n+3}{2}} (z-ch_K(u))\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\\
&\quad\le (1+\delta)^{\frac{n^2+5n}{2}}\, a\, 2^{\frac{n^2-2n+1}{2}} \binom{N}{n} \frac{n\,\omega_{n-1}^{n-1}}{(n-1)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f(x(u))^{n-1}\, \kappa_K(x(u))^{-\frac{n}{2}-\frac{1}{2}}\\
&\qquad\times \int\limits_{s(ch_K(u))}^{1} (1-s)^{N-n}\, z^{\frac{(n-1)^2}{2}} (z-ch_K(u))\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\\
&\quad\le (1+\delta)^{\frac{n^2+5n}{2}+n(n-1)}\, a\, 2^{\frac{(n-1)^2}{2}}\, 2^{-\frac{(n-1)^2}{2}} \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1}\\
&\qquad\times \int\limits_{s(ch_K(u))}^{1} (1-s)^{N-n}\, s^{n-1}\, (z-ch_K(u))\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\\
&\quad\le (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1}\\
&\qquad\times \int\limits_{s(ch_K(u))}^{1} (1-s)^{N-n}\, s^{n-1}\, (z-ch_K(u))\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\\
&\quad= (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1} \int\limits_{0}^{1} (1-s)^{N-n}\, s^{n-1}\, (z-ch_K(u))\,\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)\\
&\qquad+ (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1}\\
&\qquad\qquad\times \int\limits_{0}^{s(ch_K(u))} (1-s)^{N-n}\, s^{n-1}\, (ch_K(u)-z)\,\mathrm{d}s\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u).
\end{aligned}
$$

This proves the lemma in view of the definitions of I and II. □

We start with the first term.

Lemma 3.6. *For sufficiently large N and sufficiently small $\delta > 0$, we have that*

$$
I \le (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, N^{-\frac{2}{n-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x),
$$

where $a \in (0,\infty)$ is an absolute constant.

Proof of Lemma 3.6. We apply (3.9) and (3.3), to get that for all sufficiently small $\delta > 0$

and sufficiently large N,

$$I \leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} \frac{n}{2} \frac{(n-1)^{\frac{2}{n-1}}}{\omega_{n-1}^{\frac{2}{n-1}}} \Bigg[(1+\delta)^{\frac{2n}{n-1}} \int\limits_{\mathbb{S}^{n-1}} \frac{\kappa_K(x(u))^{-1+\frac{1}{n-1}}}{f(x(u))^{\frac{2}{n-1}}}\, \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u)$$
$$\times \int\limits_0^1 (1-s)^{N-n}\, s^{n-1+\frac{2}{n-1}}\, \mathrm{d}s$$
$$- \left(1-\frac{1}{n}\right) N^{-\frac{2}{n-1}} \frac{(n-1)\,\Gamma\left(n+1+\frac{2}{n-1}\right)}{(n+1)!} \frac{1}{n\,\mathrm{vol}_n(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)$$
$$\times \int\limits_{\mathbb{S}^{n-1}} h_K(u)\kappa_K(x(u))^{-1} \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \int\limits_0^1 (1-s)^{N-n}\, s^{n-1}\, \mathrm{d}s \Bigg].$$

For $u \in \mathbb{S}^{n-1}$, let $x = x(u) \in \partial K$ be such that $N_K(x) = u$. Then, the relation $\mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) = \kappa_K(x)\mathrm{d}\mu_{\partial K}(x)$ gives that

$$n\,\mathrm{vol}_n(K) = \int\limits_{\partial K} \langle x, N_K(x)\rangle \mathrm{d}\mu_{\partial K}(x) = \int\limits_{\partial K} h_K(u(x)) \mathrm{d}\mu_{\partial K}(x) = \int\limits_{\mathbb{S}^{n-1}} \frac{h_K(u)}{\kappa_K(x(u))} \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \tag{3.11}$$

and that

$$\int\limits_{\mathbb{S}^{n-1}} \frac{\kappa_K(x(u))^{-1+\frac{1}{n-1}}}{f(x(u))^{\frac{2}{n-1}}} \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) = \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x).$$

We use those, together with the definition of the Beta function next and arrive at

$$I \leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} \frac{n}{2} \frac{(n-1)^{\frac{2}{n-1}}}{\omega_{n-1}^{\frac{2}{n-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)$$
$$\times \Bigg[(1+\delta)^{\frac{2n}{n-1}} \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma\left(N+1+\frac{2}{n-1}\right)}$$
$$- \left(1-\frac{1}{n}\right) N^{-\frac{2}{n-1}} \frac{(n-1)\Gamma\left(n+1+\frac{2}{n-1}\right)}{(n+1)!} \frac{\Gamma(N-n+1)\Gamma(n)}{\Gamma(N+1)} \Bigg]$$
$$= (1+\delta)^{\frac{3n^2+3n}{2}}\, a \frac{n}{2} \binom{N}{n} \frac{(n-1)^{\frac{2}{n-1}}}{\omega_{n-1}^{\frac{2}{n-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x) \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma\left(N+1+\frac{2}{n-1}\right)}$$
$$\times \Bigg[(1+\delta)^{\frac{2n}{n-1}} - \left(1-\frac{1}{n}\right) N^{-\frac{2}{n-1}} \frac{(n-1)\left(n+\frac{2}{n-1}\right)}{n(n+1)} \frac{\Gamma\left(N+1+\frac{2}{n-1}\right)}{\Gamma(N+1)} \Bigg],$$

where in the last equality we have also used that

$$\Gamma\left(n+1+\frac{2}{n-1}\right) = \Gamma\left(n+\frac{2}{n-1}\right)\left(n+\frac{2}{n-1}\right).$$

Now, observe that for sufficiently large N,

$$\frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma\left(N+1+\frac{2}{n-1}\right)} \sim \frac{1}{\binom{N}{n} n N^{\frac{2}{n-1}}} \quad \text{and} \quad \Gamma(N+1+\frac{2}{n-1}) \sim N^{\frac{2}{n-1}}\Gamma(N+1).$$

Thus, we obtain that for sufficiently large N and sufficiently small δ,

$$\begin{aligned} I &\leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, N^{-\frac{2}{n-1}}\, \frac{(n-1)^{\frac{2}{n-1}}}{\omega_{n-1}^{\frac{2}{n-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x) \\ &\quad \times \left[(1+\delta)^{\frac{2n}{n-1}} - \left(1-\frac{1}{n}\right) \frac{(n-1)\left(n+\frac{2}{n-1}\right)}{n(n+1)} \right] \\ &\leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, N^{-\frac{2}{n-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x), \end{aligned}$$

where in the last inequality we have also used that $\omega_{n-1}^{\frac{2}{n-1}} \sim \frac{1}{n}$ and that $(n-1)^{\frac{2}{n-1}} \leq 2$. □

Now, we deal with the second summand in Lemma 3.5.

Lemma 3.7. *For sufficiently large N and sufficiently small $\delta > 0$, we have that*

$$II \leq (1+\delta)^{\frac{3n^2+3n}{2}} a\, N^{-\frac{2}{n-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x),$$

where $a \in (0,\infty)$ is an absolute constant.

Proof of Lemma 3.7. By definition of II,

$$\begin{aligned} II = (1+\delta)^{\frac{3n^2+3n}{2}}\, a \binom{N}{n} n \int\limits_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1} \\ \times \int\limits_0^{s(ch_K(u))} (1-s)^{N-n}\, s^{n-1}\, (ch_K(u)-z)\, \mathrm{d}s \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u), \end{aligned}$$

where $a \in (0,\infty)$ is an absolute constant. First of all, notice that by (3.2) we get that

$$c \leq a\, \frac{1}{\mathrm{vol}_n(K)\, N^{\frac{2}{n-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x). \tag{3.12}$$

By Lemma 3.4 and again by (3.2),

$$\begin{aligned} s(ch_K(u)) &\leq (1+\delta)^{n+1}\, 2^{\frac{n-1}{2}}\, \kappa_{n-1}\, \frac{f(x(u))\, h_K(u)^{\frac{n-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}}\, c^{\frac{n-1}{2}} \\ &\leq \frac{(1+\delta)^{n+1}\, e^{\frac{1}{12}}\, n}{e\, N}\, \frac{f(x(u))\, h_K(u)^{\frac{n-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}} \left(\frac{\int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)}{n\ \mathrm{vol}_n(K)} \right)^{\frac{n-1}{2}}, \end{aligned} \tag{3.13}$$

since it has been shown in [36] that

$$\left(\frac{n-1}{n+1}\right)^{\frac{2}{n-1}} \sim \frac{1}{e} \quad \text{and} \quad \left(\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{n!} \right)^{\frac{n-1}{2}} \leq e^{\frac{1}{12}} n.$$

Now, we distinguish two cases.

Case 1: $s(ch_K(u)) \leq \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}\, N}$.

The function $(1-s)^{N-n}s^{n-1}$ has its maximum at $\frac{n-1}{N-1}$ and $\frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N} \leq \frac{n-1}{N-1}$. Therefore, $(1-s)^{N-n}s^{n-1}$ is increasing on $\left[0, \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right]$ and thus, since $\binom{N}{n} n \sim \frac{N^n e^n}{\sqrt{2\pi}\sqrt{n}\, n^{n-1}}$,

$$\begin{aligned}
&\binom{N}{n} n \int_0^{s(ch_K(u))} (1-s)^{N-n} s^{n-1} (ch_K(u) - z)\, \mathrm{d}s \\
&\quad \leq c\, h_K(u) \binom{N}{n} n \int_0^{s(ch_K(u))} (1-s)^{N-n} s^{n-1} \mathrm{d}s \\
&\quad \leq c\, h_K(u) \binom{N}{n} n\, s(ch_K(u)) \left(1 - \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right)^{N-n} \left(\frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right)^{n-1} \\
&\quad \leq a\, c\, h_K(u) \frac{N^n e^n}{\sqrt{2\pi}\sqrt{n}\, n^{n-1}} \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N} \left(1 - \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right)^{N-n} \left(\frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right)^{n-1} \\
&\quad \leq a\, c\, h_K(u) \frac{e^n}{n} \exp\left(-\frac{(N-n)(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}\right) \\
&\quad \leq a\, \frac{c\, h_K(u)}{n},
\end{aligned}$$

where we used in the last step that $\frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}} \sim n$. Hence, with (3.12) and (3.11),

$$\begin{aligned}
II &\leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, \frac{c}{n} \int_{\mathbb{S}^{n-1}} \kappa_K(x(u))^{-1}\, h_K(u) \mathrm{d}\mu_{\mathbb{S}^{n-1}}(u) \\
&\leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, N^{-\frac{2}{n-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x),
\end{aligned}$$

which finishes the proof of the lemma in Case 1.

Case 2: $\quad s(ch_K(u)) > \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}$.

The inequality $s(ch_K(u)) > \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}N}$ is in view of (3.13) equivalent to

$$(1+\delta)^{n+1}\, \frac{n\, f(x(u))\, h_K(u)^{\frac{n-1}{2}}}{e^{\frac{11}{12}}\, \kappa_K(x(u))^{\frac{1}{2}}} \left(\frac{\int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)}{n\ \mathrm{vol}_n(K)}\right)^{\frac{n-1}{2}} > \frac{(n-1)^{\frac{n-1}{n}}}{n^{\frac{1}{2(n-1)}}},$$

which is equivalent to

$$h_K(u)\, \frac{\int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)}{n\ \mathrm{vol}_n(K)} > \frac{e^{\frac{11}{6(n-1)}}\, (n-1)^{\frac{2}{n}}}{(1+\delta)^{\frac{2(n+1)}{n-1}}\, n^{\frac{2n-1}{(n-1)^2}}}\, \frac{\kappa_K(x(u))^{\frac{1}{n-1}}}{f(x(u))^{\frac{2}{n-1}}}.$$

We integrate both sides over ∂K with respect to $\mu_{\partial K}$ and get that

$$(1+\delta)^{\frac{2(n+1)}{n-1}} > \frac{e^{\frac{11}{6(n-1)}}\, (n-1)^{\frac{2}{n}}}{n^{\frac{2n-1}{(n-1)^2}}},$$

and arrive at a contradiction, since for all $n \geq 3$ the right hand side is strictly bigger than 1 and δ can be chosen arbitrarily small. This shows that Case 2 is not possible and therefore finishes the proof of the lemma. □

Proof of Theorem 1.1. Lemma 3.6 and Lemma 3.7 imply that for sufficiently large N and sufficiently small $\delta > 0$,

$$\mathbb{E}[\mathrm{vol}_n((1-c)K\Delta P_N)] \leq (1+\delta)^{\frac{3n^2+3n}{2}}\, a\, N^{-\frac{2}{n-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x), \tag{3.14}$$

where $a \in (0,\infty)$ is an absolute constant. Taking into account that we were approximating the body $(1-c)K$ instead of K, we need to multiply the bound (3.14) by $(1-c)^{-n}$. Since

$$(1-c)^n \geq 1 - nc,$$

and $c \to 0$, as $N \to \infty$, we have, for sufficiently large N, that $(1-c)^{-n} \leq a$, where $a \in (0,\infty)$ is an absolute constant. Finally, since the bound (3.14) holds for all $\delta > 0$, the theorem follows. □

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