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Boundary rules and breaking of self-organized criticality in 2D frozen percolation

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Abstract

We study frozen percolation on the (planar) triangular lattice, where connected components stop growing ("freeze") as soon as their "size" becomes at least N, for some parameter $N \geq 1$. The size of a connected component can be measured in several natural ways, and we consider the two particular cases of diameter and volume (i.e. number of sites).

Diameter-frozen and volume-frozen percolation have been studied in previous works ([5, 15] and [6, 4], resp.), and they display radically different behaviors. These works adopt the rule that the boundary of a frozen cluster stays vacant forever, and we investigate the influence of these "boundary rules" in the present paper. We prove the (somewhat surprising) result that they strongly matter in the diameter case, and we discuss briefly the volume case.

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1 Introduction

1.1 Frozen percolation

In statistical physics, the phenomenon of self-organized criticality (or SOC for short) refers, roughly speaking, to the spontaneous (approximate) arising of a critical regime without any fine-tuning of a parameter. Numerous works have been devoted to it, mostly in physics (see e.g. [2, 12] and the references therein) but also on the rigorous mathematical side. The critical regime of independent percolation is of particular interest, and arises (or seems to arise in some sense) in models of forest fires [8, 3], displacement of oil by water in a porous medium [27, 7], diffusion fronts [22, 20], and in frozen percolation, the topic of the present paper.

Frozen percolation is a percolation-type growth process introduced by Aldous [1], inspired by sol-gel transitions [26]. In [1], it is shown that in the particular case of the binary tree, frozen percolation displays a striking exact form of SOC: at any time $p \geq p_c = \frac{1}{2}$, the finite ("non-frozen") clusters have the same distribution as critical clusters, while the infinite ("frozen") clusters all look like incipient infinite clusters. In two dimensions, it was shown in [15] that diameter-frozen percolation also displays

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a form of SOC: all frozen clusters freeze in a near-critical window around p_c, and consequently, they all look similar to critical percolation clusters. Here, we prove the somewhat unexpected result (Theorem 1.1 below) that, in two dimensions, the particular mechanism to freeze clusters (what we call "boundary rules") matters strongly, and can lead to a partial breaking of SOC. As we explain later, this result is based on a rather subtle geometric argument showing the existence of narrow passages that can be used to create highly supercritical frozen clusters.

In order to understand the precise *meaning* of Theorem 1.1, no pre-knowledge of percolation theory is needed (this introduction suffices). However, for a detailed understanding of the *proof*, acquaintance with near-critical percolation techniques at the level of e.g. [19] is recommended.

We now focus on a specific version of frozen percolation in two dimensions, defined in terms of site percolation on the triangular lattice $\mathbb{T}$. This lattice has vertex set

$$V(\mathbb{T}) = \left\{x + ye^{i\pi/3} \in \mathbb{C} \,:\, x, y \in \mathbb{Z}\right\},$$

and its edge set $E(\mathbb{T})$ is obtained by connecting all vertices $v, v' \in V(\mathbb{T})$ at Euclidean distance 1 apart (in this case, we say that v and v' are neighbors, and we denote it by $v \sim v'$). For a subset $A \subseteq V(\mathbb{T})$, we consider the following two ways of measuring its "size". We call *diameter of* A, denoted by $\mathrm{diam}(A)$, its diameter for the supremum norm $\|.\| := \|.\|_\infty$ (where A is seen as a subset of $\mathbb{C} \simeq \mathbb{R}^2$): $\mathrm{diam}(A) = \sup_{v,v' \in A} \|v - v'\|$. On the other hand, the *volume of* A is simply the number of vertices that it contains.

Let us consider a family $(\tau_v)_{v \in V(\mathbb{T})}$ of i.i.d. random variables uniformly distributed on $[0,1]$. For each $p \in [0,1]$, we declare a vertex v to be p-black (resp. p-white) if $\tau_v \leq p$ (resp. $\tau_v > p$). Then, p-black and p-white vertices are distributed according to independent site percolation with parameter p: vertices are black or white with probability p and $1-p$, respectively, independently of each other. In the following, the corresponding probability measure is denoted by $\mathbb{P}_p$, while we use the notation $\mathbb{P}$ for events involving the whole collection of random variables $(\tau_v)_{v \in V(\mathbb{T})}$. It is now classical (Section 3.4 in [13]) that site percolation on $\mathbb{T}$ displays a phase transition at the critical parameter $p_c = \frac{1}{2}$: for all $p \leq p_c$, there is a.s. no infinite p-black cluster, while for $p > p_c$, there exists a.s. an infinite p-black cluster, which is moreover unique. For an introduction to percolation theory, the reader can consult [11].

The diameter- and volume-frozen percolation processes are defined in terms of the same family $(\tau_v)_{v \in V(\mathbb{T})}$. These processes have a parameter $N \geq 1$. At time $t = 0$, we start with the initial configuration where all the vertices in $V(\mathbb{T})$ are white. As time t increases from 0 to 1, each vertex v can become black only at time $t = \tau_v$, *iff* all the black clusters adjacent to v have a diameter (resp. volume) $< N$. Note that if v is not allowed to turn black at time τ_v, then it stays white until time $t = 1$. Hence, black clusters grow until their diameter (resp. volume) becomes $\geq N$, and then their growth is stopped. In this case, the cluster (and the vertices that it contains) is said to be *frozen*. When referring to this process, we use the notation $\mathbb{P}_N^{\mathrm{diam}}$ (resp. $\mathbb{P}_N^{\mathrm{vol}}$). As noted in [5], this process is essentially (after a time change, using exponentially instead of uniformly distributed τ_v's) a finite-range interacting particle system (indeed, the rate at which a vertex changes its state depends only on the configuration within distance N of that vertex). Therefore, the process is well-defined (see e.g. Section 2 in [9]).

These processes were studied in the previous works [5, 15] (diameter-frozen percolation) and [6, 4] (volume-frozen percolation). With this definition, a cluster freezes when it becomes large, and all the vertices along its outer boundary then stay white until the end. However, depending on the particular mechanism of the underlying real-world process (for instance sol-gel transitions), one may ask whether these boundary rules are always the most natural, and if tweaking them would lead to a different macroscopic

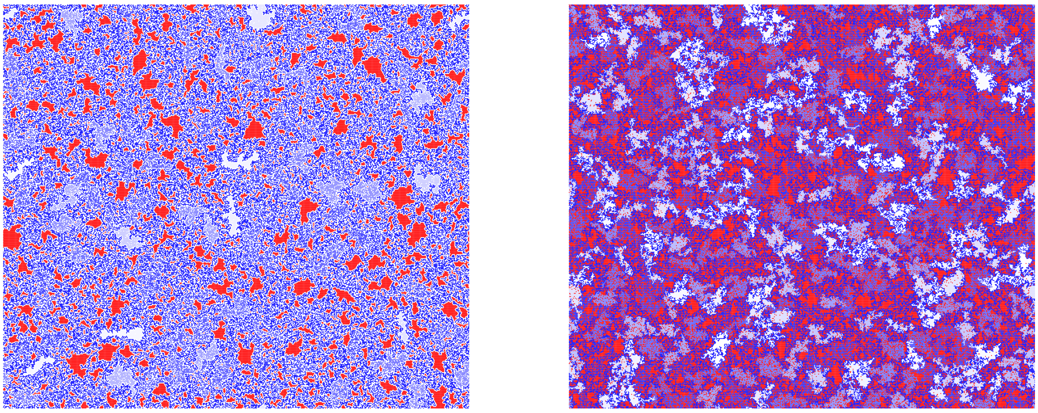

Figure 1: Final configuration for diameter-frozen percolation with parameter $N = 30$ (Fig. Demeter Kiss): "original" process (left) and "modified" process (right). The blue sites are occupied and frozen (where a lighter blue corresponds to a later time of freezing), the red sites are occupied and non-frozen (i.e. trapped in a hole with diameter $< N$), and the white sites (only present in the original process) are vacant (they lie along the boundary of frozen clusters).

behavior. This leads us to discuss *modified* (diameter- and volume-) frozen percolation processes, where, informally speaking, the sites adjacent to a frozen cluster become black (and may freeze) at a later time.

More precisely, these processes are defined as follows. Again, we use the collection of random variables $(\tau_v)_{v \in V(\mathbb{T})}$, and we start with all vertices white. Now, a vertex $v \in V(\mathbb{T})$ can be in *three* possible states: either white, black (unfrozen), or *frozen*. As time t increases, each vertex v changes state at time $t = \tau_v$. Just before this time, it is white, and it then becomes either black or frozen, depending on the configuration around it: let $\mathcal{B}_{t^-}(v) \subseteq V(\mathbb{T})$ be the union of v and all the black clusters adjacent to v at time t^-. If $\mathcal{B}_{t^-}(v)$ has a diameter (resp. volume) $\geq N$, then all the vertices in $\mathcal{B}_{t^-}(v)$ change state, from black to frozen. Otherwise, v just becomes black (and may become frozen at a later time). The laws of these modified processes are denoted by $\tilde{\mathbb{P}}_N^{\mathrm{diam}}$ and $\tilde{\mathbb{P}}_N^{\mathrm{vol}}$ respectively; they are well-defined by the same arguments as for ordinary frozen percolation.

1.2 Effect of boundary rules

In the case of diameter-frozen percolation, we show that boundary rules do have a strong effect. We first discuss briefly the results of [15] for the original process, i.e. when the vertices along the boundary of a frozen cluster stay white forever. In that paper, it is proved that frozen clusters only arise in a near-critical window around p_c: for every fixed $K > 0$,

$$\liminf_{N \to \infty} \mathbb{P}_N^{\mathrm{diam}}\big(\text{some vertex in } [-KN, KN]^2 \text{ freezes outside } [p_{-\lambda}(N), p_\lambda(N)]\big) \overset{\lambda \to \infty}{\longrightarrow} 0, \tag{1.1}$$

where $p_\lambda(N) = p_c + \frac{\lambda}{N^{3/4+o(1)}}$ refers to the usual near-critical parameter scale (a precise definition requires the introduction of more percolation notation, and is postponed to Section 2.3). Also, macroscopic non-frozen clusters asymptotically have full density:

$$\liminf_{N \to \infty} \mathbb{P}_N^{\mathrm{diam}}\big(\operatorname{diam}(\mathcal{C}_1(0)) \in [\varepsilon N, (1-\varepsilon)N]\big) \overset{\varepsilon \to 0^+}{\longrightarrow} 1, \tag{1.2}$$

where $\mathcal{C}_1(0)$ denotes the black cluster of 0 at time 1 (which we consider to be $\emptyset$ if 0 is not black). In particular, $\mathbb{P}_N^{\mathrm{diam}}(0 \text{ freezes}) \to 0$ as $N \to \infty$.

We prove the following results for the modified process described in the end of Section 1.1. First, the probability that 0 freezes is still bounded away from 1 (see (3.1)). However, in contrast with the original process, this probability is now also bounded away from 0. In particular, some "very dense" (see Remark 1.2 b) below) frozen clusters form at a late time (close to 1).

Theorem 1.1. *For the modified diameter-frozen percolation process on* $\mathbb{T}$*,*

$$\liminf_{N\to\infty} \tilde{\mathbb{P}}_N^{\mathit{diam}}(0 \textit{ freezes}) > 0. \tag{1.3}$$

Remark 1.2.

(a) Actually, the following more precise property holds:

$$\text{for all } \varepsilon > 0, \quad \liminf_{N\to\infty} \tilde{\mathbb{P}}_N^{\mathrm{diam}}\big(0 \text{ freezes in } \big(1 - N^{-\frac{3}{4}+\varepsilon}, 1\big)\big) > 0. \tag{1.4}$$

(b) The construction used in the proof of Theorem 1.1 also provides more information about the final configuration. In the scenario that we give, and which occurs with a probability bounded away from 0 (as $N \to \infty$), 0 lies in a macroscopic "chamber" (with diameter smaller than N but of order N) which is "protected from the outside" until time $1 - N^{-\frac{3}{4}+\varepsilon}$. In that scenario, 0 lies in a highly supercritical cluster which freezes at some time $p^* \in \big(1 - N^{-\frac{3}{4}+\varepsilon}, 1\big)$. In particular, our proof implies that in the ball around 0 of radius $\frac{N}{8}$, with probability bounded away from 0, the fraction of vertices that belong to this frozen cluster is larger than $1 - \delta(N)$, for some function $\delta(N) \to 0$ as $N \to \infty$.

1.3 Organization of the paper

We first discuss independent percolation in Section 2. After fixing notations, we collect tools from critical and near-critical percolation which are central in our proofs. We then study the modified diameter-frozen percolation process in Section 3, where we prove Theorem 1.1. For that, we use an "ad-hoc" configuration of near-critical clusters (see Figure 4 below). We prove that such a configuration occurs with reasonable probability, and (combined with some extra features, see Figure 6) gives a scenario where 0 freezes. Finally, in Section 3.3, we pose some open questions and make some remarks about *volume*-frozen percolation (which seems to be more "robust" with respect to the modification of boundary rules).

2 Preliminary: independent percolation

2.1 Setting and notations

In this section, we first fix some notations regarding site percolation on $\mathbb{T}$. For a subset $A \subseteq V(\mathbb{T})$, we denote by $\partial^{\mathrm{in}} A := \{v \in A \ : \ v \sim v' \text{ for some } v' \in A^c\}$ its inner boundary, and by $\partial^{\mathrm{out}} A := \partial^{\mathrm{in}}(V(\mathbb{T}) \setminus A)$ its outer boundary. These definitions can be extended easily to arbitrary $A \subseteq \mathbb{C}$ by taking $\partial^{\mathrm{in}} A := \partial^{\mathrm{in}}(A \cap V(\mathbb{T}))$ and $\partial^{\mathrm{out}} A := \partial^{\mathrm{in}}(\mathbb{C} \setminus A)$. For $v \in V(\mathbb{T})$, we write $\partial v := \partial^{\mathrm{out}}\{v\}$.

A path of length k ($k \geq 1$) is a sequence of vertices $v_0 \sim v_1 \sim \ldots \sim v_k$. Two vertices $v, v' \in V(\mathbb{T})$ are said to be connected if for some $k \geq 1$, there exists a path of length k from v to v' (i.e. such that $v_0 = v$ and $v_k = v'$) containing only black sites: we denote this event by $v \leftrightarrow v'$. More generally, two subsets $A, A' \subseteq V(\mathbb{T})$ are said to be connected if there exist $v \in A$ and $v' \in A'$ such that $v \leftrightarrow v'$, which we denote by $A \leftrightarrow A'$. Note that we sometimes consider white paths: in this case, the color is always specified explicitly, and we use the notation $\leftrightarrow^*$.

A horizontal (resp. vertical) crossing of a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ is "a black path in R connecting the left and right (resp. top and bottom) sides" (we wrote it between quotation marks because the rectangle R does not "fit" the lattice $\mathbb{T}$, and so the definition needs to be made more accurate; this can be done easily, see for instance Definition 1 in Section 3.3 of [13]). The event that there exists such a crossing is denoted by $\mathcal{C}_H(R)$ (resp. $\mathcal{C}_V(R)$), and we also write $\mathcal{C}_H^*(R)$ and $\mathcal{C}_V^*(R)$ for the similar events with white paths.

Let $B_n := [-n,n]^2$ be the ball of radius $n > 0$ around 0 for the norm $\|.\|$, and for $0 < m < n$, let $A_{m,n} := B_n \setminus B_m$. For $z \in \mathbb{C}$, we write $B_n(z) := z + B_n$, and $A_{m,n}(z) := z + A_{m,n}$. For such an annulus $A = A_{m,n}(z)$, we denote by $\mathcal{O}(A)$ (resp. $\mathcal{O}^*(A)$) the event that there exists a black (resp. white) circuit in A surrounding $B_m(z)$. If $k \geq 1$ and $\sigma \in \mathfrak{S}_k := \{b, w\}^k$ (where we write b for black, and w for white), we also define the event $\mathcal{A}_\sigma(A)$ that there exist k disjoint paths $(\gamma_i)_{1 \leq i \leq k}$ in A with respective colors σ_i, in counter-clockwise order, each "crossing" A (i.e. connecting $\partial^{\text{in}} B_n(z)$ and $\partial^{\text{out}} B_m(z)$). We also use the notations

$$\mathcal{A}_\sigma(m,n) := \mathcal{A}_\sigma(A_{m,n}) \quad \text{and} \quad \pi_\sigma(m,n) := \mathbb{P}_{p_c}\big(\mathcal{A}_\sigma(m,n)\big), \tag{2.1}$$

and we simply write $\pi_\sigma(n) := \pi_\sigma(1,n)$. For $k \geq 1$, we also use the shorthand notations $\mathcal{A}_k$ and π_k in the particular case when $\sigma = (bwb\ldots) \in \mathfrak{S}_k$ is alternating (i.e the color sequence ends with $\sigma_k = b$ or w according to the parity of k).

We will use repeatedly the usual Harris inequality for monotone events, and in some cases, we will need the slightly more general version below (see Lemma 3 in [14]), for "locally monotone" events.

Lemma 2.1. *Consider $\mathcal{E}^+$, $\tilde{\mathcal{E}}^+$ two increasing events, $\mathcal{E}^-$, $\tilde{\mathcal{E}}^-$ two decreasing events, and assume that for some disjoint subsets $A, A^+, A^- \subseteq V(\mathbb{T})$, these events depend only on the sites in $A \cup A^+$, A^+, $A \cup A^-$, and A^-, respectively. Then*

$$\mathbb{P}\big(\tilde{\mathcal{E}}^+ \cap \tilde{\mathcal{E}}^- \cap \mathcal{E}^+ \cap \mathcal{E}^-\big) \geq \mathbb{P}\big(\tilde{\mathcal{E}}^+\big)\mathbb{P}\big(\tilde{\mathcal{E}}^-\big)\mathbb{P}\big(\mathcal{E}^+ \cap \mathcal{E}^-\big).$$

This result follows easily by first conditioning on the configuration in A, and then applying twice the Harris inequality (to the configuration in A^+ and in A^-).

2.2 Critical and near-critical percolation

Our results are based on a precise description of the behavior of percolation through its phase transition, i.e. at and near criticality. We now collect classical properties of near-critical percolation which are used throughout the proofs. We define the characteristic length L by: for $p < p_c = \frac{1}{2}$,

$$L(p) = \min\big\{n > 0 \ : \ \mathbb{P}_p\big(\mathcal{C}_V([0,2n] \times [0,n])\big) \leq 0.01\big\}, \tag{2.2}$$

and $L(p) = L(1-p)$ for $p > p_c$. We also set $L(p_c) = \infty$.

(i) *Russo-Seymour-Welsh (RSW) bounds.* For all $k \geq 1$, there exists a universal constant $\delta_k > 0$ such that: for all $p \in (0,1)$, and $n \leq L(p)$,

$$\mathbb{P}_p\big(\mathcal{C}_H([0,kn] \times [0,n])\big) \geq \delta_k \quad \text{and} \quad \mathbb{P}_p\big(\mathcal{C}_H^*([0,kn] \times [0,n])\big) \geq \delta_k \tag{2.3}$$

(see (2.16) in [14], or (3.4) in [19]).

(ii) *Exponential decay with respect to L(p).* There exist universal constants $c_i, c_i' > 0$ ($i \in \{1,2\}$) such that: for all $p < p_c$, and $n \geq 1$,

$$\mathbb{P}_p\big(\mathcal{C}_V([0,2n]\times[0,n])\big) \leq c_1 e^{-c_2 \frac{n}{L(p)}} \quad \text{and} \quad \mathbb{P}_p\big(\mathcal{C}_H([0,2n]\times[0,n])\big) \geq c_1' e^{-c_2' \frac{n}{L(p)}} \tag{2.4}$$

(see Lemma 39 in [19]). Note that by symmetry (see the definition of L), similar results hold for white paths when $p > p_c$.

(iii) *A-priori bound on* 4*-arm events.* Using the well-known fact that the arm exponent for $\mathcal{A}_5$ equals 2 (see e.g. Theorem 24 (3) in [19]), the BKR inequality and (2.3), it follows from standard arguments that for some universal constant $c' > 0$: for all $p \in (0,1)$, and $0 < m < n \leq L(p)$,

$$\mathbb{P}_p\big(\mathcal{A}_4(m,n)\big) \geq c' \left(\frac{m}{n}\right)^{2-\beta}. \tag{2.5}$$

(iv) *Asymptotic equivalence.* For the functions π_4 and L, the following estimate holds:

$$\big|p - p_c\big| L(p)^2 \pi_4\big(L(p)\big) \asymp 1 \quad \text{as } p \to p_c \tag{2.6}$$

(see (4.5) in [14], or Proposition 34 in [19]).

(v) *Arm events at criticality.* Due to conformal invariance at criticality (see [24]), many arm exponents for site percolation on $\mathbb{T}$ could be computed [18, 25] using the Schramm-Loewner Evolution (SLE) processes [23, 16, 17]. We will only use the 4-arm exponent:

$$\pi_4(n) = n^{-\frac{5}{4}+o(1)} \quad \text{as } n \to \infty. \tag{2.7}$$

2.3 Near-critical parameter scale

For the constructions in the rest of this paper, the following near-critical parameter scale (already mentioned in Section 1.2) is convenient to work with.

Definition 2.2. *For $\lambda \in \mathbb{R}$ and $N \geq 1$, let*

$$p_\lambda(N) := p_c + \frac{\lambda}{N^2 \pi_4(N)}. \tag{2.8}$$

This particular choice has turned out to be quite suitable to study near-critical percolation and related phenomena, see e.g. [21, 10, 15]. Note that for every fixed λ, $p_\lambda(N) \to p_c$ as $N \to \infty$ (using the a-priori lower bound on 4-arm events (2.5)). In particular, $p_\lambda(N) \in (0,1)$ for N large enough. We use the following properties.

(i) For every fixed $\lambda \in \mathbb{R}$,

$$L\big(p_\lambda(N)\big) \asymp N \quad \text{as } N \to \infty. \tag{2.9}$$

(ii) On the other hand,

$$\limsup_{N\to\infty} \frac{L\big(p_\lambda(N)\big)}{N} \longrightarrow 0 \quad \text{as } \lambda \to \pm\infty. \tag{2.10}$$

(iii) *RSW bounds.* For all $\lambda \geq 0$ and $k \geq 1$, there exists a constant $\bar{\delta}_k = \bar{\delta}_k(\lambda) > 0$ such that: for all $N \geq 1$, $n \leq N$, and $p \in \big[p_{-\lambda}(N), p_\lambda(N)\big]$,

$$\mathbb{P}_p\big(\mathcal{C}_H([0,kn]\times[0,n])\big) \geq \bar{\delta}_k \quad \text{and} \quad \mathbb{P}_p\big(\mathcal{C}^*_H([0,kn]\times[0,n])\big) \geq \bar{\delta}_k. \tag{2.11}$$

Properties (i) and (ii) follow, using standard arguments, from (2.8), (2.6), and (2.5). Property (iii) then follows from (i) and (multiple use of) (2.3).

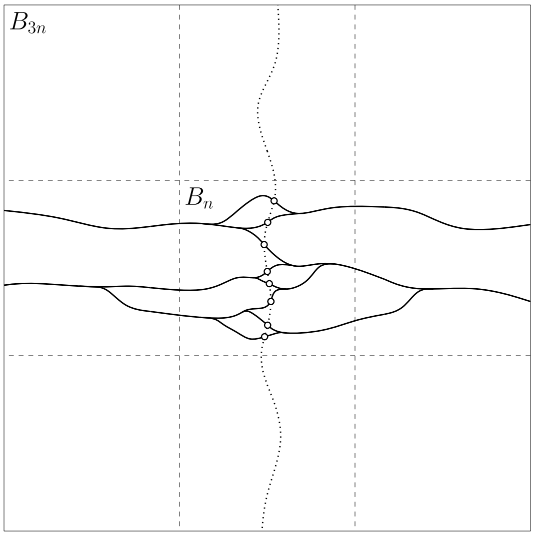

Figure 2: This figure depicts the event $\Gamma_p^{n,\delta}$ used in the proof of Theorem 1.1. The solid paths are p-black, the dotted ones are p_c-white, and there are at least $\delta n^2 \pi_4(n)$ "passage sites", i.e. p_c-white vertices with neighbors connected by p-black paths to the left and right sides of B_{3n}.

2.4 Additional results

In our proofs, we also make use of two more specific results that we now state. We first introduce an event which is instrumental in the proof of Theorem 1.1.

Definition 2.3. *Let $n \geq 1$, $\delta > 0$, and $p \in (0, p_c]$. We define the event $\Gamma_p^{n,\delta}$, depending on the sites in the box B_{3n}, that there exists a vertical crossing γ of $[-n, n] \times [-3n, 3n]$ with the following two properties (see Figure 2).*

(i) *γ is p_c-white.*

(ii) *There are at least $\delta n^2 \pi_4(n)$ sites $v \in B_n$ along γ which are "passage sites": each such v possesses two neighbors v_1 and v_2 which are connected in $[-3n, 3n] \times [-n, n]$ by p-black paths to the left and right sides of B_{3n}, respectively.*

Lemma 2.4. *For every $\lambda \geq 0$, there exists $\delta = \delta(\lambda) > 0$ such that: for all $N \geq 1$ and $n \leq N$,*

$$\mathbb{P}\big(\Gamma_{p_{-\lambda}(N)}^{n,\delta}\big) \geq \delta. \tag{2.12}$$

Proof of Lemma 2.4. Consider $\lambda \geq 0$. For $1 \leq n \leq N$, let $X_n = X_n^{\lambda,N}$ be the number of vertices $v \in B_{n/2}$ satisfying

(a) v is p_c-white,

(b) and there exist four paths γ_i ($1 \leq i \leq 4$), in counterclockwise order, connecting ∂v to the right, top, left, and bottom sides of B_{3n}, respectively, and such that

- γ_1 and γ_3 are $p_{-\lambda}(N)$-black and stay in $[-3n, 3n] \times [-n, n]$,
- γ_2 and γ_4 are p_c-white and stay in $[-n, n] \times [-3n, 3n]$.

By standard arguments, $\mathbb{E}[X_n] \geq c_1 n^2 \pi_4(n)$ and $\mathbb{E}[X_n^2] \leq c_2 (n^2 \pi_4(n))^2$ for some $c_i = c_i(\lambda) > 0$ ($i = 1, 2$), from which Lemma 2.4 follows by applying the second-moment method to X_n (see e.g. Proposition 2 in [21] for a similar proof, with 2 arms). $\square$

We also make use of the following geometric construction.

Definition 2.5. *For $1 \leq m \leq n$, we consider all the horizontal and vertical rectangles of the form*

$$B_m(2mx) \cup B_m(2mx'), \quad \text{with } x, x' \in B_{\lceil n/2m \rceil + 1},\ x \sim x'$$

(covering the ball B_{n+2m}), and for $p \in (0,1)$, we denote by $\mathcal{N}_p(m,n)$ the event that in each of these rectangles, there exists a p-black crossing in the long direction.

Note that the event $\mathcal{N}_p(m,n)$ implies the existence of a p-black cluster $\mathcal{N}$ such that all the p-black clusters and all the p-white clusters that intersect B_n, except $\mathcal{N}$ itself, have a diameter at most $4m$. Such a cluster $\mathcal{N}$ is called a *net with mesh* m.

Lemma 2.6. *There exist universal constants $c_1, c_2 > 0$ such that: for all $1 \leq m \leq n$ and $p > p_c$,*

$$\mathbb{P}\big(\mathcal{N}_p(m,n)\big) \geq 1 - c_1 \Big(\frac{n}{m}\Big)^2 e^{-c_2 \frac{m}{L(p)}}. \tag{2.13}$$

Proof of Lemma 2.6. This is an immediate consequence of the exponential decay property (2.4), since the definition of $\mathcal{N}_p(m,n)$ involves of order $\left(\frac{n}{m}\right)^2$ rectangles, each with side lengths $4m$ and $2m$. □

3 Diameter-frozen percolation

We now turn our attention to diameter-frozen percolation. We recall in Section 3.1 the proof of the main result in [5], which also applies to the modified diameter-frozen percolation process. This proof uses a construction which is the starting point of the more complicated construction in the proof of Theorem 3.1, in Section 3.2. Finally, we state some remarks and open questions in Section 3.3.

3.1 Construction of macroscopic chambers

First, we note that the main construction from [5] (see Theorem 1.1 and Figure 1 in that paper) works in exactly the same way for the process with modified boundary rules. Hence, we obtain the analog of the main result in [5]: for all $0 < a < b < 1$,

$$\liminf_{N \to \infty} \tilde{\mathbb{P}}_N^{\mathrm{diam}}\big(\operatorname{diam}(\mathcal{C}_1(0)) \in [aN, bN]\big) > 0. \tag{3.1}$$

In other words, macroscopic non-frozen clusters (with diameter $< N$ but of order N) have a positive density.

Since we are using this construction as a building block for the proof of Theorem 1.1, we remind it (and the argument behind (3.1)) quickly to the reader on Figure 3. For that, we choose $\eta, \ell \in (0,1)$ such that

(i) $a + 6\eta \leq b$,

(ii) $a + 7\eta < 1$ and $\ell + 4\eta < 1$,

(iii) and $\ell + a + 4\eta > 1$.

We can choose for instance $\ell = 1 - a$, and then $\eta > 0$ small enough.

Note that by (i), the inner chamber has a diameter between aN and bN. By (ii), $\mathcal{C}$ and the crossing in r_1 cannot freeze separately. By (iii), the big structure, that contains both $\mathcal{C}$ and part of the crossing, freezes before time p_c (note that the $p_{-\lambda}(N)$-white crossing of r_1' prevents that part of $\mathcal{C}$ freezes already with the crossing of r_1 before every site of $\mathcal{C}$ is black).

This construction creates a cluster which freezes at some time in $[p_{-\lambda}(N), p_c]$, and completely surrounds $B_{\frac{a}{2}N}$ (without intersecting it). In this "chamber" with diameter $< N$, no connected component can freeze and we just observe an independent percolation

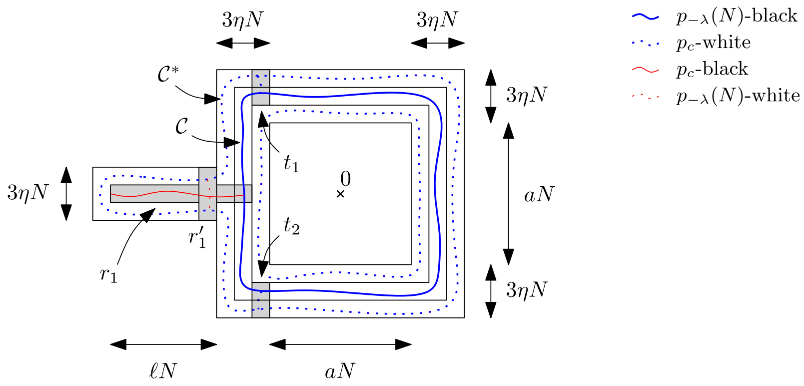

Figure 3: Construction used to prove (3.1), where all the "corridors" have width ηN. When the big structure (containing $\mathcal{C}$ and part of the crossing in r_1) freezes, it leaves a hole whose boundary lies in $A_{\frac{a}{2}N,(\frac{a}{2}+3\eta)N}$. Note that the p_c-white crossings in t_1 and t_2 prevent the appearance of big clusters other than the ones that we want to be created.

configuration. In particular, all the sites are black at time 1, which produces a non-frozen cluster with a diameter between aN and bN. This gives (3.1).

3.2 Existence of highly supercritical frozen clusters

We now prove Theorem 1.1 about the appearance of clusters freezing at a time very close to 1. We actually prove the (obviously stronger) result below. The claim in Remark 1.2 a) (i.e. (1.4)) easily follows from it by using (2.7).

Theorem 3.1. *Consider the modified diameter-frozen percolation process on $\mathbb{T}$. For every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \tilde{\mathbb{P}}_N^{diam}\left(0 \text{ freezes in } \left(1 - \frac{\varepsilon}{N^2\pi_4(N)}, 1\right)\right) > 0. \tag{3.2}$$

Proof of Theorem 3.1. In the following, we fix some small value $\eta > 0$: to fix ideas, we can take $\eta = \frac{1}{40}$. We provide a scenario under which two stages of freezing occur: a first stage in the near-critical window around p_c (more precisely, in the time interval $[p_{-\lambda}(N), p_c]$, for some well-chosen λ large enough), and then a second stage much later, at some time very close to 1.

To avoid long and cumbersome definitions, many geometric objects in the proof are defined by Figure 4. We use the construction depicted in this figure to create two "chambers": as we will show, they have the property that each of them has a diameter $< N$, but the diameter of their union is $\geq N$. Note that the left and right parts in Figure 4 are each similar to the construction in Figure 3.

Let us fix $\varepsilon > 0$ as in the statement of Theorem 3.1. For $\lambda \geq 0$, and $\delta = \delta(\lambda) > 0$ associated with λ by Lemma 2.4, we introduce the three events $\tilde{\Gamma}_i = \tilde{\Gamma}_i(N, \lambda, \varepsilon)$ ($1 \leq i \leq 3$): $\tilde{\Gamma}_i$ is the event that in the square s_i, $\Gamma_{p_{-\lambda}(N)}^{\frac{\eta}{2}N,\delta}$ (translated, and rotated in the case of s_3) holds, and at least one of the (more than $\delta N^2\pi_4(N)$) passage points in the "inner square" (with side length ηN) is still white at time $1 - \frac{\varepsilon}{N^2\pi_4(N)}$ (i.e. at the beginning of the time interval in (3.2)).

By using Lemma 2.4, and conditioning on the percolation configuration at time p_c, we obtain that for every $\lambda \geq 0$: for all $N \geq 1$,

$$\mathbb{P}(\tilde{\Gamma}_i) \geq \delta' = \delta'(\lambda, \varepsilon) > 0 \quad (1 \leq i \leq 3).$$

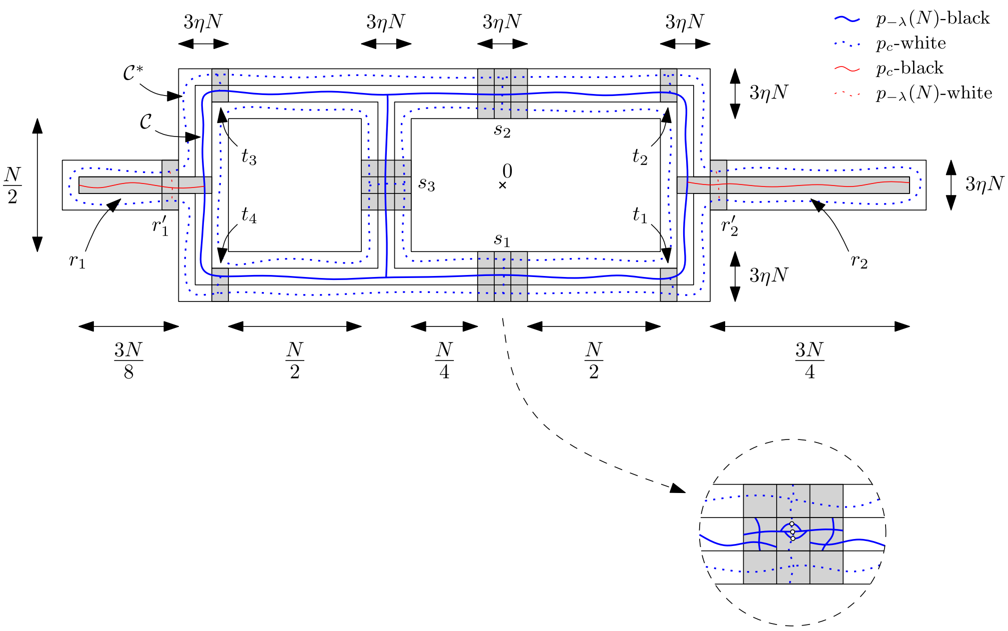

Figure 4: Construction used to create two "chambers", with diameters (approximately) $\frac{N}{2}$ and $\frac{3N}{4}$. On this figure, the various corridors have width ηN. In each of the three gray squares s_i $(1 \leq i \leq 3)$, we consider the event $\Gamma^{\frac{\eta}{2}N,\delta}_{p_{-\lambda}(N)}$ (properly translated, and also rotated by an angle $\frac{\pi}{2}$ in the case of s_3), where $\delta = \delta(\lambda) > 0$ is produced by Lemma 2.4.

We denote by $\mathcal{E}_N^\lambda$ the event (for the underlying percolation configuration) that the various paths depicted on Figure 4 exist, and the three events $\tilde{\Gamma}_i$ $(1 \leq i \leq 3)$ occur. We first establish the following result.

Claim: By choosing λ large enough, we can ensure that

$$\liminf_{N\to\infty} \mathbb{P}\big(\mathcal{E}_N^\lambda\big) > 0. \tag{3.3}$$

Proof of the Claim. The rectangles r_1 and r_2 have lengths $\big(\frac{3}{8} + 2\eta\big)N$ and $\big(\frac{3}{4} + 2\eta\big)N$, respectively, and both have width ηN. Since they each have constant aspect ratio, it follows from RSW that

$$\mathbb{P}_{p_c}\big(\mathcal{C}_H(r_1)\big) \geq \mathbb{P}_{p_c}\big(\mathcal{C}_H(r_2)\big) \geq c_1, \tag{3.4}$$

for some constant $c_1 = c_1(\eta) > 0$ independent of N (recall that we consider η to be fixed). We can then choose $\lambda > 0$ large enough so that, for all sufficiently large N,

$$\mathbb{P}_{p_{-\lambda}(N)}\big(\mathcal{C}_V^*(r_1')\big) = \mathbb{P}_{p_{-\lambda}(N)}\big(\mathcal{C}_V^*(r_2')\big) \geq 1 - \frac{c_1}{2} \tag{3.5}$$

(by combining (2.10) with (2.4)). In the remainder of the proof, we fix such a value λ, and we consider the constant $\delta = \delta(\lambda) > 0$ associated with it by Lemma 2.4.

We now consider the event $\mathcal{F}_N^\lambda$ that

(i) all the $p_{-\lambda}(N)$-black and p_c-white paths on Figure 4, except possibly the short vertical connections in t_i $(1 \leq i \leq 4)$, exist,

(ii) and the events $\tilde{\Gamma}_i$ $(1 \leq i \leq 3)$ hold.

It follows from RSW (2.11) and Lemma 2.4 (recall our choice of δ), using Lemma 2.1 in a similar way as in [14] (for a detailed example of application of Lemma 2.1, see e.g. (2.41)

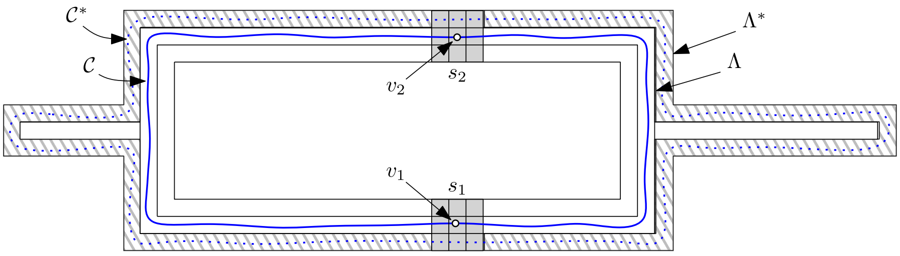

Figure 5: This figure presents the "annuli" and the circuits involved in the proof of the Claim. v_1 and v_2 are two p_c-white "passage sites", in s_1 and s_2 respectively. We condition on the innermost circuit $\mathcal{C}$ in Λ, and the outermost circuit $\mathcal{C}^*$ in Λ^*, where $\mathcal{C}$ and $\mathcal{C}^*$ are as on the figure: $\mathcal{C}^*$ is p_c-white, and $\mathcal{C}$ is $p_{-\lambda}(N)$-black except at two sites, one in each of s_1 and s_2, which are p_c-white.

and the preceding explanation in that paper), that

$$\mathbb{P}\big(\mathcal{F}_N^\lambda\big) \geq c_2 \tag{3.6}$$

for some $c_2 = c_2(\eta, \lambda) > 0$ independent of N.

Using the notations of Figure 5, we then condition on the innermost circuit $\mathcal{C}$ in Λ, and on the outermost circuit $\mathcal{C}^*$ in Λ^*, having the properties that $\mathcal{C}^*$ is p_c-white, and $\mathcal{C}$ is $p_{-\lambda}(N)$-black except on two p_c-white vertices $v_1 \in s_1$ and $v_2 \in s_2$ (note that the existence of $\mathcal{C}$ is not immediately obvious; however, by the restriction on the locations of the "defects" v_1 and v_2, it can be proved in essentially the same way as for entirely black circuits). Now, consider the sites that lie between these two circuits *and* outside of the squares s_i: the percolation configuration in this region is "fresh". We make the following observations.

- In each t_i $(1 \leq i \leq 4)$, there exists a vertical p_c-white connection between $\mathcal{C}$ and $\mathcal{C}^*$ with a probability $\geq c_3 > 0$, for some universal constant c_3 independent of N (by RSW).
- The paths in r_1 and r'_1 (in red on Figure 4), respectively p_c-black and $p_{-\lambda}(N)$-white, exist with a probability $\geq c_1 - \frac{c_1}{2} = \frac{c_1}{2}$ (by combining (3.4) and (3.5)).
- For the same reason, the red paths in r_2 and r'_2 exist with a probability $\geq \frac{c_1}{2}$.

Moreover, all these events are conditionally independent, so that the conditional probability of their intersection is at least $c_3^4 \big(\frac{c_1}{2}\big)^2$. We deduce

$$\mathbb{P}\big(\mathcal{E}_N^\lambda\big) \geq c_2 \cdot c_3^4 \Big(\frac{c_1}{2}\Big)^2 > 0,$$

which completes the proof of the Claim. □

We now assume that the event $\mathcal{E}_N^\lambda$ holds, and we examine consequences of it for the modified diameter-frozen percolation process itself. First, note that all the $p_{-\lambda}(N)$-black and p_c-white paths in this event (in blue on Figure 4) are present throughout the time interval $[p_{-\lambda}(N), p_c]$. The circuit $\mathcal{C}$ can be divided into two parts, to the left and to the right of the passage sites in s_1 and s_2, and we denote by $\mathcal{C}_L$ and $\mathcal{C}_R$ the connected components containing them.

- On the time interval $[p_{-\lambda}(N), p_c]$, there cannot be any other black connected component with diameter $\geq N$ inside $\mathcal{C}^*$, thanks to the p_c-white paths in the t_i's $(1 \leq i \leq 4)$.

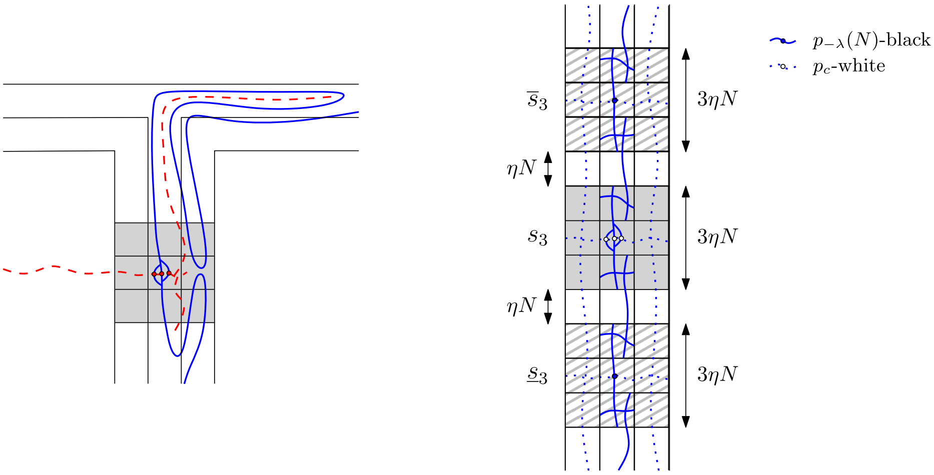

Figure 6: *Left:* If the (frozen) boundary of the right chamber goes back and forth too much, then some connected component with a diameter larger than N (in red dashed lines) may arise without intersecting the net inside the chamber. *Right:* To circumvent this issue, we introduce two extra four-arm events in the boxes $\underline{s}_3$ and $\overline{s}_3$.

- $\mathcal{C}_L$ has a diameter $\leq 4\eta N + \frac{N}{2} + 3\eta N + \frac{N}{4} + 2\eta N < N$ at time $p_{-\lambda}(N)$, because of the $p_{-\lambda}(N)$-white vertical path in r_1' (here we also use our choice of η).
- $\mathcal{C}_L$ has a diameter $\geq \frac{3N}{8} + \frac{N}{2} + \frac{N}{4} > N$ at time p_c (using the p_c-black horizontal path in r_1). Hence, in the frozen percolation process, it freezes at some time in $[p_{-\lambda}(N), p_c]$.
- Similarly, $\mathcal{C}_R$ has a diameter $\leq 2\eta N + \frac{N}{2} + 4\eta N < N$ at time $p_{-\lambda}(N)$, and $\geq \frac{N}{2} + \frac{3N}{4} > N$ at time p_c. Hence, in the frozen percolation process, it freezes at some time in $[p_{-\lambda}(N), p_c]$.

When these two clusters freeze, they create two chambers as desired, which are separated by a sequence of at least $\delta n^2 \pi_4(n)$ p_c-white "passage sites" in s_3.

Intuitively, it is now tempting to conclude the proof as follows. In the right chamber, at time close to 1, there exists a net $\mathcal{N}$ with mesh $\frac{\eta}{4} N$ with very high probability (using Lemma 2.6), so that any connected component with diameter $> \eta N$ inside this chamber has to intersect $\mathcal{N}$. Hence, $\mathcal{N}$ freezes, at the latest when it gets connected to the left chamber (but possibly earlier, due to connections to the outside through s_1 or s_2).

However, we have to take into account the possibility that after the time T_3 when all the passage sites in s_3 have become black, a large cluster with diameter $\geq N$ may occur without touching the net. Indeed, the boundary of the right chamber may go back and forth, thus leaving "bubbles" with large diameter, as shown on Figure 6 (left). In order to prevent this undesirable behavior, we introduce yet another event depicted on Figure 6 (right), ensuring some regularity for the boundary. With this additional event, the situation depicted on Figure 6 (left) cannot occur, and it is guaranteed that a cluster with diameter $\geq N$ emerging after time T_3, and containing passage sites in s_3, has to contain the net.

We are now in a position to conclude. Indeed, the extra cost of the configurations in $\underline{s}_3$ and $\overline{s}_3$ is just a positive uniform constant (this can be obtained in a similar way as Lemma 2.4, but also in an elementary fashion by successive applications of RSW (2.3) and conditionings). So, if we let $\tilde{\mathcal{E}}_N^\lambda$ denote the event that $\mathcal{E}_N^\lambda$ holds, as well as the additional event on Figure 6 (right) that we mentioned, we have

$$\mathbb{P}\big(\tilde{\mathcal{E}}_N^\lambda\big) \geq c_4 \cdot \mathbb{P}\big(\mathcal{E}_N^\lambda\big), \tag{3.7}$$

for some $c_4 = c_4(\lambda, \eta) > 0$.

For $\tilde{\varepsilon} = \frac{\varepsilon}{N^2 \pi_4(N)}$ (so that $1 - \tilde{\varepsilon}$ is the beginning of the time interval in (3.2)), we introduce the following two events $E_i = E_i(\eta, N, \varepsilon)$ $(i = 1, 2)$.

- $E_1 := \mathcal{N}_{1-\tilde{\varepsilon}}(\frac{\eta}{4}N, 2N)$ (i.e. at time $1 - \tilde{\varepsilon}$, there is a net with mesh $\frac{\eta}{4}N$ in the box B_{2N}). It follows from Lemma 2.6 that

$$\mathbb{P}(E_1) \to 1 \quad \text{as } N \to \infty. \tag{3.8}$$

- $E_2 := \{$there exists a $(1 - \tilde{\varepsilon})$-black path from 0 to $\partial B_{2\eta N}\}$. We have, since $\tilde{\varepsilon} \to 0$,

$$\mathbb{P}(E_2) \geq \mathbb{P}_{1-\tilde{\varepsilon}}(0 \text{ belongs to an infinite black cluster}) \to 1 \quad \text{as } N \to \infty. \tag{3.9}$$

If $\tilde{\mathcal{E}}_N^\lambda$, E_1 and E_2 occur, then the event in the left-hand side of (3.2) occurs as well. Hence, the latter event has probability at least

$$\mathbb{P}\big(\tilde{\mathcal{E}}_N^\lambda \cap E_1 \cap E_2\big) \geq \mathbb{P}\big(\tilde{\mathcal{E}}_N^\lambda\big) - \mathbb{P}(E_1^c) - \mathbb{P}(E_2^c),$$

which, for λ sufficiently large, is bounded away from 0 as $N \to \infty$, using (3.3), (3.7), (3.8), and (3.9). This completes the proof of Theorem 3.1. □

3.3 Remarks and open questions

As mentioned earlier, the boundary rules are essential for the results of [15], which contrast with Theorem 3.1. It should be noted that the first important observation in this paper, namely that for every fixed $K > 1$, the number of frozen clusters in B_{KN} is tight in N (Lemma 3.3 in [15]) already breaks down for the modified model. Indeed, its proof makes use of the existence of white paths at some time $p = p_{-\lambda}(N)$, which are provided by the boundaries of frozen clusters. This observation is crucial for the arguments in [15] since it then allows one to study the frozen clusters "one by one" (and ensures that the procedure ends after a finite number of steps).

Our results leave open the question whether clusters freeze at times bounded away from both p_c and 1. In particular, is it true that for every fixed finite connected C with $0 \in C$,

$$\liminf_{N \to \infty} \tilde{\mathbb{P}}_N^{\text{diam}}\big(\mathcal{C}_1(0) = C\big) > 0,$$

i.e. that microscopic clusters (with diameter of order 1) have a positive density as well?

We conclude the paper by making a few remarks about *volume-frozen* percolation. This process seems to be much more robust with respect to the modification of the boundary rules, and we have partial proof of this robustness. In [6] we studied the original volume-frozen process on a finite box $B_{m(N)}$, with $m(N)$ a function of the parameter N. We showed that there exists a sequence of "exceptional scales" $(m_k(N))_{k \geq 1}$ such that the following dichotomy holds.

- For all $c > 1$ and $k \geq 2$, if $m(N)$ satisfies $c^{-1} m_k(N) \leq m(N) \leq c m_k(N)$ for all N large enough, then

$$\liminf_{N \to \infty} \mathbb{P}_N^{\text{vol}}(0 \text{ freezes for the process in } B_{m(N)}) > 0.$$

- For all $\varepsilon > 0$ and $k \geq 1$, there exists $c > 1$ such that: if $c m_k(N) \leq m(N) \leq c^{-1} m_{k+1}(N)$ for all N large enough, then

$$\limsup_{N \to \infty} \mathbb{P}_N^{\text{vol}}(0 \text{ freezes for the process in } B_{m(N)}) \leq \varepsilon.$$

These results also hold (with practically the same proof) for the modified model (i.e. with $\tilde{\mathbb{P}}_N^{\mathrm{vol}}$ instead of $\mathbb{P}_N^{\mathrm{vol}}$).

In [4] we proved for the original model on the full lattice that

$$\mathbb{P}_N^{\mathrm{vol}}(0 \text{ freezes}) \to 0 \quad \text{as } N \to \infty,$$

and we strongly believe, even if we have not worked out all the necessary details, that this also holds for $\tilde{\mathbb{P}}_N^{\mathrm{vol}}$.

References

[1] David J. Aldous, *The percolation process on a tree where infinite clusters are frozen*, Math. Proc. Cambridge Philos. Soc. **128** (2000), no. 3, 465–477. MR-1744108

[2] Per Bak, *How nature works: the science of self-organized criticality*, Copernicus, New York, 1996. MR-1417042

[3] Jacob van den Berg and Rachel Brouwer, *Self-organized forest-fires near the critical time*, Comm. Math. Phys. **267** (2006), no. 1, 265–277. MR-2238911

[4] Jacob van den Berg, Demeter Kiss, and Pierre Nolin, *Two-dimensional volume-frozen percolation: deconcentration and prevalence of mesoscopic clusters*, Ann. Sci. Éc. Norm. Supér. (4), to appear.

[5] Jacob van den Berg, Bernardo N. B. de Lima, and Pierre Nolin, *A percolation process on the square lattice where large finite clusters are frozen*, Random Structures Algorithms **40** (2012), no. 2, 220–226. MR-2877564

[6] Jacob van den Berg and Pierre Nolin, *Two-dimensional volume-frozen percolation: exceptional scales*, Ann. Appl. Probab. **27** (2017), no. 1, 91–108. MR-3619783

[7] Michael Damron, Artëm Sapozhnikov, and Bálint Vágvölgyi, *Relations between invasion percolation and critical percolation in two dimensions*, Ann. Probab. **37** (2009), no. 6, 2297–2331. MR-2573559

[8] Barbara Drossel and Franz Schwabl, *Self-organized critical forest-fire model*, Phys. Rev. Lett. **69** (1992), 1629–1632.

[9] Rick Durrett, *Ten lectures on particle systems*, Lectures on probability theory (Saint-Flour, 1993), Lecture Notes in Math., vol. 1608, Springer, Berlin, 1995, pp. 97–201. MR-1383122

[10] Christophe Garban, Gábor Pete, and Oded Schramm, *The scaling limits of near-critical and dynamical percolation*, J. Eur. Math. Soc., to appear.

[11] Geoffrey Grimmett, *Percolation*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 321, Springer-Verlag, Berlin, 1999. MR-1707339

[12] Henrik J. Jensen, *Self-organized criticality: emergent complex behavior in physical and biological systems*, Cambridge Lecture Notes in Physics, vol. 10, Cambridge University Press, Cambridge, 1998. MR-1689042

[13] Harry Kesten, *Percolation theory for mathematicians*, Progress in Probability and Statistics, vol. 2, Birkhäuser, Boston, 1982. MR-0692943

[14] Harry Kesten, *Scaling Relations for 2D-percolation*, Comm. Math. Phys. **109** (1987), 109–156. MR-0879034

[15] Demeter Kiss, *Frozen percolation in two dimensions*, Probab. Theory Related Fields **163** (2015), nos 3–4, 713–768. MR-3418754

[16] Gregory F. Lawler, Oded Schramm, and Wendelin Werner, *Values of Brownian intersection exponents. I. Half-plane exponents*, Acta Math. **187** (2001), no. 2, 237–273. MR-1879850

[17] Gregory F. Lawler, Oded Schramm, and Wendelin Werner, *Values of Brownian intersection exponents. II. Plane exponents*, Acta Math. **187** (2001), no. 2, 275–308. MR-1879851

[18] Gregory F. Lawler, Oded Schramm, and Wendelin Werner, *One-arm exponent for critical 2D percolation*, Electron. J. Probab. **7** (2002), no. 2, 13 pp. MR-1887622

[19] Pierre Nolin, *Near-critical percolation in two dimensions*, Electron. J. Probab. **13** (2008), no. 55, 1562–1623. MR-2438816

[20] Pierre Nolin, *SLE(6) and the geometry of diffusion fronts*, Preprint arXiv:0912.3770, 2009.

[21] Pierre Nolin and Wendelin Werner, *Asymmetry of near-critical percolation interfaces*, J. Amer. Math. Soc. **22** (2009), no. 3, 797–819. MR-2505301

[22] Bernard Sapoval, Michel Rosso, and Jean-François Gouyet, *The fractal nature of a diffusion front and the relation to percolation*, J. Physique Lett. **46** (1985), no. 4, 149–156.

[23] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR-1776084

[24] Stanislav Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 3, 239–244. MR-1851632

[25] Stanislav Smirnov and Wendelin Werner, *Critical exponents for two-dimensional percolation*, Math. Res. Lett. **8** (2001), nos 5–6, 729–744. MR-1879816

[26] Walter H. Stockmayer, *Theory of molecular size distribution and gel formation in branched-chain polymers*, Journal of Chemical Physics **11** (1943), 45–55.

[27] David Wilkinson and Jorge F. Willemsen, *Invasion percolation: a new form of percolation theory*, Journal of Physics A: Mathematical and General **16** (1983), 3365–3376. MR-0725616

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