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A Cramér type moderate deviation theorem for the critical Curie-Weiss model

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Abstract

Limit theorems for the magnetization of Curie-Weiss model have been studied extensively by Ellis and Newman. To refine these results, Chen, Fang and Shao prove Cramér type moderate deviation theorems for non-critical cases by using Stein method. In this paper, we consider the same question for the remaining case - the critical Curie-Weiss model. By direct and simple arguments based on Laplace method, we provide an explicit formula of the error and deduce a Cramér type result.

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1 Introduction

Let (X_i) be a sequence of i.i.d. random variables satisfying $\mathbb{E}X_1 = 0$, $\mathrm{Var}(X_1) = 1$. Then the classic Central limit theorem says that the normalized sum $W_n = (X_1 + \ldots + X_n)/\sqrt{n}$ converges in law to a standard normal random variable W. A natural question is to understand the rate of the convergence of the tail probability $\mathbb{P}(W_n > x)$ to $\mathbb{P}(W > x)$ for the largest possible range of x. There are two major approaches to measure the approximation error. The first approach is to study the absolute error by Berry-Esseen type bounds. The other one is to study the relative error of the tail probability. One of the first result in this approach is the following Cramér type moderate deviation theorem. If $\mathbb{E}(e^{\alpha|X_1|^{1/2}}) < \infty$, for some $\alpha > 0$, then

$$\frac{\mathbb{P}(W_n > x)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n},$$

for $0 \leq x \leq n^{1/6}$, with Φ the standard normal distribution function. It has been also shown that the assumptions on the exponential moment of X_1 and the length of range $n^{1/6}$ are optimal. We refer the reader to the book [12] for a proof of this result and a more detailed discussion.

The Cramér type moderate deviation results have been proved to be useful in designing statistical tests since they give a relation between the size and the accuracy of tests, see e.g. [9, 10]. Hence, a lot of attention has been drawn in investigating this

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problem not only for independent variables but also for dependent structures as stationary process [1, 15], self-linear process [11], normalized sums [4, 14], and L-statistics [8]. On the other hand, Cramér type moderate deviation theorems for nonnormal limit distribution are also provided, such as for chi-squared distribution [10], for sub-Gaussian or exponential distribution [2].

In this paper, we study the case of the *critical Curie-Weiss model*, where the spin variables are dependent and the limit distribution is nonnormal. Let us first recall some definitions and existing results for Curie-Weiss model. For $n \in \mathbb{N}$, let $\Omega_n = \{\pm 1\}^n$ be the space of spin configurations. The spin configuration probability is given by Boltzman-Gibbs distribution, i.e. for any $\sigma \in \Omega_n$,

$$\mu_n(\sigma) = Z_n^{-1} \exp\left(\frac{\beta}{n} \sum_{1 \le i < j \le n} \sigma_i \sigma_j + \beta h \sum_{i=1}^n \sigma_i \right),$$

where Z_n is the normalizing factor, $\beta > 0$ and $h \in \mathbb{R}$ are inverse temperature and external field respectively. The Curie-Weiss model has been shown to exhibit a phase transition at $\beta_c = 1$. More precisely, the asymptotic behavior of the total spin (also called the magnetization) $S_n = \sigma_1 + \ldots + \sigma_n$ changes when β crosses the critical value 1. Let us consider the following fixed-point equation

$$m = \tanh(\beta(m + h)). \tag{1.1}$$

Case 1. $0 < \beta < 1$, $h \in \mathbb{R}$ or $\beta \ge 1, h \ne 0$ (*the uniqueness regime of magnetization*). The equation (1.1) has a unique solution m_0, such that $m_0 h \ge 0$. In this case, S/n is concentrated around m_0 and has a Gaussian limit under proper standardization, see [6]. Moreover, in [3] the authors prove the following moderate deviation theorem for the magnetization by using Stein method.

Theorem 1.1. *[3, Proposition 4.3] In case 1, let us define*

$$W_n = \frac{S_n - nm_0}{v_n},$$

where

$$v_n = \sqrt{\frac{n(1 - m_0^2)}{1 - (1 - m_0^2)\beta}}.$$

Then we have

$$\frac{\mu_n(\sigma : W_n > x)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n},$$

for $0 \le x \le n^{1/6}$.

Case 2. $\beta > 1, h = 0$ (*the low temperature regime without external field*). The equation (1.1) has two nonzero solutions $m_1 < 0 < m_2$, where $m_1 = -m_2$. In this case, one has the conditional central limit theorems as follows: conditionally on $S_n < 0$ (resp. $S_n > 0$), S/n is concentrated around m_1 (resp. m_2) and has a Gaussian limit after proper scaling, see [6]. Similarly to case 1, a moderate deviation result has been also proved.

Theorem 1.2. *[3, Proposition 4.4] In case 2, let us define*

$$W_{1,n} = \frac{S_n - nm_1}{v_{1,n}} \quad \text{and} \quad W_{2,n} = \frac{S_n - nm_2}{v_{2,n}},$$

where

$$v_{1,n} = \sqrt{\frac{n(1 - m_1^2)}{1 - (1 - m_1^2)\beta}} \quad \text{and} \quad v_{2,n} = \sqrt{\frac{n(1 - m_2^2)}{1 - (1 - m_2^2)\beta}}.$$

Then we have

$$\frac{\mu_n(\sigma : W_{1,n} > x \mid S_n < 0)}{1-\Phi(x)} = 1 + O(1)(1+x^3)/\sqrt{n},$$

and

$$\frac{\mu_n(\sigma : W_{2,n} > x \mid S_n > 0)}{1-\Phi(x)} = 1 + O(1)(1+x^3)/\sqrt{n},$$

for $0 \le x \le n^{1/6}$.

Case 3. $\beta = 1$ and $h = 0$ (*the critical case*). The equation (1.1) has a unique solution 0 and S/n is concentrated around 0. In this case, $S_n/n^{3/4}$ converges to a nonnormal distribution with density proportional to $e^{-x^4/12}$, see [6, 7]. Moreover, the authors of [2, 5] give Berry-Esseen type bounds for this convergence.

Theorem 1.3. *[2, Theorem 2.1] In case 3, let us define*

$$W_n = \frac{S_n}{n^{3/4}}.$$

Then there exists a positive constant C, such that for all x

$$\limsup_{n\to\infty} \sqrt{n}\left| \mu_n\left(\sigma : W_n \le x\right) - F(x) \right| \le C, \tag{1.2}$$

where

$$F(x) = \frac{\int_{-\infty}^{x} e^{-t^4/12}dt}{\int_{-\infty}^{\infty} e^{-t^4/12}dt}.$$

We remark that in [5], the authors generalize Theorem 1.3 to a near critical regime of inverse temperature $\beta = 1 + O(\frac{1}{\sqrt{n}})$. They also consider a general class of Curie-Weiss model, where the distribution of a single spin is a generic probability measure instead of Bernoulli distribution as in the classical model.

In this paper, we will prove a Cramér type moderate deviation theorem for the total spin in the critical case. Our main result is as follows.

Theorem 1.4. *For the critical case, when $\beta = 1$ and $h = 0$, let us define*

$$W_n = \frac{S_n}{n^{3/4}}.$$

Then there exists a positive constant C, such that for all n large enough and $0 \le x \le n^{1/12}$,

$$\left|\frac{\mu_n(\sigma : W_n > x)}{1-F(x)} - 1 - \frac{G(x)}{\sqrt{n}}\right| \le \frac{C(x^{12} + n^{1/3})}{n}, \tag{1.3}$$

where

$$F(x) = \frac{\int_{\infty}^{x} p_1(t)dt}{\int_{-\infty}^{\infty} p_1(t)dt},$$

and

$$G(x) = \left(\frac{\int_x^\infty p_2(t)dt}{\int_x^\infty p_1(t)dt} - \frac{\int_{-\infty}^\infty p_2(t)dt}{\int_{-\infty}^\infty p_1(t)dt} \right)$$

with

$$p_1(t) = e^{-\frac{t^4}{12}} \qquad \text{and} \qquad p_2(t) = \left(\frac{t^2}{2} - \frac{t^6}{30}\right)e^{-\frac{t^4}{12}}.$$

It is worth noting that Theorem 1.4 gives the exact formula of the error term of order $n^{-1/2}$, while moderate deviation results in Theorems 1.1 and 1.2 only show asymptotic estimates of the error terms. The range of estimate $n^{1/6}$ is replaced by $n^{1/12}$ due to the change of scaling and limit distribution. The proof of Theorem 1.4 is simple and direct, based on Laplace method-like arguments. We have a direct corollary.

Corollary 1.5. *For $0 \leq x \leq n^{1/12}$, we have*

$$\frac{\mu_n(\sigma : W_n > x)}{1 - F(x)} = 1 + O(1)(1 + x^6)/\sqrt{n}.$$

Moreover, for any fixed real number x,

$$\lim_{n\to\infty} \sqrt{n} \Big(\mu_n (\sigma : W_n \leq x) - F(x) \Big) = (F(x) - 1)G(x).$$

The first part of this corollary is a Cramér moderate deviation result in classic form, whereas the second part is an improvement of Theorem 1.3.

The paper is organized as follows. In Section 2, we provide some preliminary results. In Section 3, we prove the main theorem 1.4.

We fix here some notation. If f and g are two real functions, we write $f = O(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x; $f = \Omega(g)$ if $g = O(f)$; and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

2 Preliminaries

2.1 A lemma on the integral approximations

Lemma 2.1. *Let m, q, p be positive real numbers.*

(i) Assume that $f(t)$ is a decreasing function in $[(m-1)/p, (q+1)/p]$. Then

$$\Big| \sum_{\substack{m<\ell<n \\ 2|\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2}\int_{m/p}^{q/p} f(t)dt\Big| \leq \Big|f\left(\frac{m}{p}\right)\Big| + \Big|f\left(\frac{q}{p}\right)\Big|,$$

and

$$\Big| \sum_{\substack{m<\ell<q \\ 2\nmid\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2}\int_{m/p}^{q/p} f(t)dt\Big| \leq \Big|f\left(\frac{m}{p}\right)\Big| + \Big|f\left(\frac{q}{p}\right)\Big|.$$

(ii) Assume that $f(t)$ is a differentiable function on $\mathbb{R}$ and there exists a positive constant K, such that $|f(t)| + |f'(t)| \leq K$. Then

$$\Big| \sum_{\substack{m<\ell<q \\ 2|\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2}\int_{m/p}^{q/p} f(t)dt\Big| \leq \frac{K(q-m)}{p} + 2K,$$

and

$$\Big| \sum_{\substack{m<\ell<q \\ 2\nmid\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2}\int_{m/p}^{q/p} f(t)dt\Big| \leq \frac{K(q-m)}{p} + 2K.$$

Proof. The proof of (i) is simple, so we safely leave it to the reader. For (ii), by using the mean value theorem, we get that for any ℓ,

$$\Big| f\left(\frac{\ell}{p}\right) - \frac{p}{2}\int_{\frac{\ell}{p}}^{\frac{\ell+2}{p}} f(t)dt \Big| \quad \leq \quad \frac{pK}{2}\int_{\frac{\ell}{p}}^{\frac{\ell+2}{p}} \left(t - \frac{\ell}{p}\right) dt = \frac{K}{p}.$$

Therefore, by summing over ℓ we get desired results. ☐

2.2 Estimates on the binomial coefficients

We first recall a version of Stirling approximation (see [13]) that for all $n \geq 1$,

$$\log(\sqrt{2\pi n}) + n\log n - n + \frac{1}{12n+1} \leq \log(n!) \leq \log(\sqrt{2\pi n}) + n\log n - n + \frac{1}{12n}.$$

Using this approximation, we can show that

$$\binom{n}{k} \le e^{nI(k/n)}, \qquad \text{for all } k = 0, \dots, n, \tag{2.1}$$

and

$$\binom{n}{k} = (1 - O(n^{-1}))\sqrt{\frac{n}{2\pi k(n-k)}} \times e^{nI(k/n)}, \qquad \text{for } |k - (n/2)| < n/4, \tag{2.2}$$

where $I(0) = I(1) = 0$ and for $t \in (0,1)$,

$$I(t) = (t-1)\log(1-t) - t\log t.$$

We will see in Section 3.1 that the function $J(t)$ defined by

$$J(t) = I(t) + \frac{(2t-1)^2}{2} \tag{2.3}$$

plays an important role in the expression of the distribution function of W_n. We prove here a lemma to describe the behavior of $J(t)$.

Lemma 2.2. *Let $J(t)$ be the function defined as in* (2.3)*. Then*

(i) $J'(1/2) = J''(1/2) = J'''(1/2) = J^{(5)}(1/2) = J^{(7)}(1/2) = 0$*, and for all* $t \ne 1/2$

$$J''(t) < 0.$$

(ii) $J^{(4)}(1/2) = -32$, $J^{(6)}(1/2) = -1536$*, and for all* $1/4 \le t \le 3/4$

$$-2^{25} < J^{(8)}(t) < 0.$$

Proof. We have

$$J'(t) = \log\left(\frac{1-t}{t}\right) + (4t-2).$$

Hence

$$J''(t) = -\big[t^{-1} + (1-t)^{-1}\big] + 4, \quad J'''(t) = \big[t^{-2} - (1-t)^{-2}\big], \quad J^{(4)}(t) = -2\big[t^{-3} + (1-t)^{-3}\big],$$

$$J^{(5)}(t) = 6\big[t^{-4} - (1-t)^{-4}\big], \quad J^{(6)}(t) = -24\big[t^{-5} + (1-t)^{-5}\big], \quad J^{(7)}(t) = 120\big[t^{-6} - (1-t)^{-6}\big],$$

and

$$J^{(8)}(t) = -720\big[t^{-7} + (1-t)^{-7}\big].$$

Using these equations, we can deduce the desired results. □

3 Proof of Theorem 1.4

3.1 An expression of the distribution function of W_n

Let us denote by $F_n(x)$ the distribution function of W_n, i.e. for $x \in \mathbb{R}$

$$F_n(x) = \mu_n(\sigma : W_n \le x) = \mu_n(\sigma : \sigma_1 + \ldots + \sigma_n \le n^{3/4}x). \tag{3.1}$$

For $\sigma \in \Omega_n$, we define

$$\sigma_+ = \{i : \sigma_i = 1\}.$$

Observe that if $|\sigma_+| = k$, then

$$\frac{1}{n}\sum_{i \le j} \sigma_i\sigma_j = 1 + \frac{1}{n}\sum_{i<j} \sigma_i\sigma_j \quad = \quad 1 + \frac{1}{2n}\left(\left(\sum_{1 \le i \le n} \sigma_i\right)^2 - n\right)$$

$$= \frac{(2k-n)^2}{2n} + \frac{1}{2}.$$

Hence,

$$Z_n = \sum_{\sigma \in \Omega_n} \exp\left(\frac{1}{n}\sum_{i \le j} \sigma_i \sigma_j\right) = \sum_{k=0}^{n} \sum_{\substack{\sigma \in \Omega_n \\ |\sigma_+| = k}} \exp\left(\frac{1}{n}\sum_{i \le j} \sigma_i \sigma_j\right) = \sum_{k=0}^{n} \binom{n}{k} e^{\frac{(2k-n)^2}{2n} + \frac{1}{2}}.$$

Let us define

$$x_{k,n} = \binom{n}{k} e^{\frac{(2k-n)^2}{2n} + \frac{1}{2}}.$$

Then

$$Z_n = \sum_{k=0}^{n} x_{k,n}, \tag{3.2}$$

and

$$\mu_n(\sigma : |\sigma_+| = k) = \frac{x_{k,n}}{Z_n}. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) yields that

$$\begin{aligned} 1 - F_n(x) &= \mu_n(\sigma : \sigma_1 + \ldots + \sigma_n > n^{3/4} x) \\ &= \mu_n(\sigma : 2|\sigma_+| - n > n^{3/4} x) = \mu_n\left(\sigma : |\sigma_+| > \frac{n + n^{3/4} x}{2}\right) \\ &= \frac{1}{Z_n} \sum_{k=0}^{n} x_{k,n} \mathbb{I}\left(k > \frac{n + n^{3/4} x}{2}\right), \end{aligned} \tag{3.4}$$

where $\mathbb{I}(\cdot)$ stands for the indicator function. Using (2.1) and (2.2), we obtain

$$x_{k,n} \le e^{nJ(k/n)+1/2}, \qquad \text{for all } k = 0, \ldots, n, \tag{3.5}$$

and

$$x_{k,n} = (1 - O(n^{-1}))\sqrt{\frac{n}{2\pi k(n-k)}} \times e^{nJ(k/n)+1/2}, \qquad \text{for } |k - (n/2)| < n/4, \tag{3.6}$$

with $J(t)$ the function defined in (2.3).

By Lemma 2.2, we observe that $J(t)$ attains the maximum at the unique point $\frac{1}{2}$. This fact suggests us that the value of Z_n (the sum of $(x_{k,n})$) is concentrated at the middle terms. Let us define

$$y_n = \sqrt{\frac{2}{\pi n}} \times e^{nJ(1/2)+1/2},$$

which is asymptotic to $x_{[n/2],n}$. We define also

$$y_{k,n} = \frac{x_{k,n}}{y_n}.$$

Then the equation (3.4) becomes

$$1 - F_n(x) = \frac{1}{\sum_{k=0}^{n} y_{k,n}} \times \sum_{k=0}^{n} y_{k,n} \mathbb{I}\left(k > \frac{n + n^{3/4} x}{2}\right). \tag{3.7}$$

Moreover, using (3.5) and (3.6), we obtain estimates on $(y_{k,n})$,

$$y_{k,n} \le \sqrt{\frac{\pi n}{2}} \times e^{n\left[J(k/n) - J(1/2)\right]}, \qquad \text{for all } k = 0, \ldots, n, \tag{3.8}$$

and

$$y_{k,n} = (1 - O(n^{-1}))\sqrt{\frac{n^2}{4k(n-k)}} \times e^{n\left[J(k/n) - J(1/2)\right]} \qquad \text{for } |k - \tfrac{n}{2}| < \tfrac{n}{4}. \tag{3.9}$$

We define

$$\begin{aligned} A_n &= \sum_{k=0}^{n} y_{k,n} \mathbb{I}\left(\left|k - \frac{n}{2}\right| \geq \frac{n}{4}\right), \quad B_n = \sum_{k=0}^{n} y_{k,n} \mathbb{I}\left(\left|k - \frac{n}{2}\right| < \frac{n}{4}\right) \\ \hat{A}_n &= \sum_{k=0}^{n} y_{k,n} \mathbb{I}\left(k - \frac{n}{2} \geq \frac{n}{4}\right), \quad B_{n,x} = \sum_{k=0}^{n} y_{k,n} \mathbb{I}\left(\frac{n}{4} > k - \frac{n}{2} > \frac{n^{3/4}x}{2}\right). \end{aligned}$$

Then by (3.7),

$$1 - F_n(x) = \frac{\hat{A}_n + B_{n,x}}{A_n + B_n}. \tag{3.10}$$

3.2 Estimates of A_n and $\hat{A}_n$

Lemma 3.1. *There exists a positive constant c, such that for n large enough,*

$$\hat{A}_n \leq A_n \leq e^{-cn}.$$

Proof. By Lemma 2.2, we have $J'(\frac{1}{2}) = 0$ and $J''(t) \leq 0$ for all $t \in (0,1)$. Therefore,

$$\max_{|x-0.5| \geq 0.25} J(x) = \max\{J(0.75), J(0.25)\} = J(0.25).$$

Hence for all $|k - (n/2)| \geq n/4$,

$$J(k/n) - J(1/2) \leq J(0.25) - J(0.5) < -0.005.$$

Thus for all $|k - (n/2)| \geq n/4$,

$$n\big(J(k/n) - J(1/2)\big) < -0.005n. \tag{3.11}$$

It follows from (3.8) and (3.11) that for $|k - (n/2)| \geq n/4$,

$$y_{k,n} \leq \sqrt{2n} \exp\left(-0.005n\right).$$

Thus

$$\hat{A}_n \leq A_n \leq n\sqrt{2n} \exp\left(-0.005n\right) < \exp\left(-0.004n\right),$$

for all n large enough. □

3.3 Estimates of B_n

By using Lemma 2.2 (i) and Taylor expansion, we get

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) = \frac{J^{(4)}(1/2)}{4!}\left(\frac{k}{n} - \frac{1}{2}\right)^4 + \frac{J^{(6)}(1/2)}{6!}\left(\frac{k}{n} - \frac{1}{2}\right)^6 + \frac{J^{(8)}(\xi_{k,n})}{8!}\left(\frac{k}{n} - \frac{1}{2}\right)^8,$$

with some $\xi_{k,n}$ between k/n and $1/2$. Hence, by Lemma 2.2 (ii),

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) \quad \leq \quad -\frac{(2k-n)^4}{12n^4} - \frac{(2k-n)^6}{30n^6}.$$

and

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) \quad \geq \quad -\frac{(2k-n)^4}{12n^4} - \frac{(2k-n)^6}{30n^6} - \frac{2^{17}(2k-n)^8}{n^8 8!}.$$

Therefore,

$$n(J(k/n)-J(1/2)) \quad \le \quad -\frac{(2k-n)^4}{12n^3}-\frac{(2k-n)^6}{30n^5}$$

and

$$n(J(k/n)-J(1/2)) \quad \ge \quad -\frac{(2k-n)^4}{12n^3}-\frac{(2k-n)^6}{30n^5}-\frac{2^{17}(2k-n)^8}{n^7 8!}.$$

Combining the last two estimates with the inequality that $1-x\le e^{-x}\le 1-x+\frac{x^2}{2}$ for all $x\ge 0$, we get

$$e^{n\left[J(k/n)-J(1/2)\right]} \quad \le \quad \exp\left(\frac{-(2k-n)^4}{12n^3}\right)\left(1-\frac{(2k-n)^6}{30n^5}+\frac{(2k-n)^{12}}{1800n^{10}}\right),$$

and

$$e^{n\left[J(k/n)-J(1/2)\right]} \quad \ge \quad \exp\left(\frac{-(2k-n)^4}{12n^3}\right)\left(1-\frac{(2k-n)^6}{30n^5}\right)\left(1-\frac{2^{17}(2k-n)^8}{n^7 8!}\right).$$

Therefore,

$$e^{n\left[J(k/n)-J(1/2)\right]}=\exp\left(\frac{-(2k-n)^4}{12n^3}\right)\left(1-\frac{(2k-n)^6}{30n^5}+O(1)X_{k,n}\right), \tag{3.12}$$

where

$$X_{k,n}=\frac{(2k-n)^8}{n^7}+\frac{(2k-n)^{12}}{n^{10}}+\frac{(2k-n)^{14}}{n^{12}}.$$

On the other hand, for $|k-(n/2)|<n/4$,

$$\sqrt{\frac{n^2}{4k(n-k)}}=1+\frac{(2k-n)^2}{2n^2}+O\left(\frac{(2k-n)^4}{n^4}\right). \tag{3.13}$$

Combining (3.9), (3.12) and (3.13), we have for $|k-(n/2)|<n/4$,

$$y_{k,n}=\left(1+\frac{(2k-n)^2}{2n^2}-\frac{(2k-n)^6}{30n^5}+O(1)R_{k,n}\right)\exp\left(-\frac{(2k-n)^4}{12n^3}\right),$$

where

$$R_{k,n}=\frac{1}{n}+\frac{(2k-n)^4}{n^4}+\frac{(2k-n)^8}{n^7}+\frac{(2k-n)^{12}}{n^{10}}+\frac{(2k-n)^{14}}{n^{12}}.$$

By letting $\ell=2k-n$, we obtain

$$\begin{aligned}
B_n &= \sum_{n/4<k<3n/4} y_{k,n} \\
&= \sum_{\substack{|\ell|<n/2\\ 2|(\ell+n)}} e^{-\frac{\ell^4}{12n^3}}\left(1+\frac{\ell^2}{2n^2}-\frac{\ell^6}{30n^5}+O\left(\frac{1}{n}+\frac{\ell^4}{n^4}+\frac{\ell^8}{n^7}+\frac{\ell^{12}}{n^{10}}+\frac{\ell^{14}}{n^{12}}\right)\right) \\
&= \sum_{\substack{|\ell|<n/2\\ 2|(\ell+n)}} e^{-\left(\frac{\ell}{n^{3/4}}\right)^4/12}\left[1+\frac{1}{\sqrt{n}}\left(\frac{1}{2}\left(\frac{\ell}{n^{3/4}}\right)^2-\frac{1}{30}\left(\frac{\ell}{n^{3/4}}\right)^6\right)\right. \\
&\qquad \left.+\frac{O(1)}{n}\left(1+\left(\frac{\ell}{n^{3/4}}\right)^4+\left(\frac{\ell}{n^{3/4}}\right)^8+\left(\frac{\ell}{n^{3/4}}\right)^{12}+\frac{1}{\sqrt{n}}\left(\frac{\ell}{n^{3/4}}\right)^{14}\right)\right] \\
&= \sum_{\substack{|\ell|<n/2\\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right)+\frac{1}{\sqrt{n}}\sum_{\substack{|\ell|<n/2\\ 2|(\ell+n)}} p_2\left(\frac{\ell}{n^{3/4}}\right)+\frac{O(1)}{n}\sum_{\substack{|\ell|<n/2\\ 2|(\ell+n)}} r\left(\frac{\ell}{n^{3/4}}\right),
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned} p_1(t) &= e^{-\frac{t^4}{12}} \\ p_2(t) &= \left(\frac{t^2}{2} - \frac{t^6}{30}\right) e^{-\frac{t^4}{12}} \\ r(t) &= (1 + t^4 + t^8 + t^{12} + t^{14}/\sqrt{n}) e^{-\frac{t^4}{12}}. \end{aligned}$$

The proof of the following lemma is simple, so we omit it.

Lemma 3.2. *There exists a positive constant K, such that*

$$\sup_{t\in\mathbb{R}} |p_1(t)| + |p_2(t)| + |r(t)| + |p_1'(t)| + |p_2'(t)| + |r'(t)| \le K.$$

Using Lemma 2.1 (ii) and Lemma 3.2, we obtain that

$$\sum_{\substack{|\ell|\le n^{5/6} \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) = \frac{n^{3/4}}{2} \int_{-n^{1/12}}^{n^{1/12}} p_1(t)dt + O(n^{1/12}).$$

Moreover,

$$\begin{aligned} \sum_{\substack{n^{5/6}<|\ell|<n/2 \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) &\le ne^{-n^{1/3}}, \\ \int_{|t|\ge n^{1/12}} p_1(t)dt &= o(n^{-1}). \end{aligned}$$

Combining the last three estimates gives that

$$\sum_{\substack{|\ell|<n/2 \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) = \frac{n^{3/4}}{2} \int_{-\infty}^{\infty} p_1(t)dt + O(n^{1/12}). \tag{3.15}$$

Similarly,

$$\sum_{\substack{|\ell|<n/2 \\ 2|(\ell+n)}} p_2\left(\frac{\ell}{n^{3/4}}\right) = \frac{n^{3/4}}{2} \int_{-\infty}^{\infty} p_2(t)dt + O(n^{1/12}), \tag{3.16}$$

$$\sum_{\substack{|\ell|<n/2 \\ 2|(\ell+n)}} r_1\left(\frac{\ell}{n^{3/4}}\right) = \frac{n^{3/4}}{2} \int_{-\infty}^{\infty} r_1(t)dt + O(n^{1/12}). \tag{3.17}$$

We now can deduce from (3.14), (3.15), (3.16) and (3.17) an estimate of B_n that

$$B_n = \frac{n^{3/4}}{2} \int_{-\infty}^{\infty} p_1(t) + \frac{n^{1/4}}{2} \int_{-\infty}^{\infty} p_2(t) + O(n^{1/12}). \tag{3.18}$$

3.4 Estimates of $B_{n,x}$

Using the same arguments for (3.14), we also have

$$B_{n,x} = \sum y_{k,n} \mathbb{I}\left(\frac{n}{4} > k - \frac{n}{2} > \frac{n^{3/4}x}{2}\right)$$

$$= \sum_{\substack{n^{3/4}x<\ell<n/2 \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) + \frac{1}{\sqrt{n}} \sum_{\substack{n^{3/4}x<\ell<n/2 \\ 2|(\ell+n)}} p_2\left(\frac{\ell}{n^{3/4}}\right)$$
$$+\frac{O(1)}{n} \sum_{\substack{n^{3/4}x<\ell<n/2 \\ 2|(\ell+n)}} r\left(\frac{\ell}{n^{3/4}}\right). \tag{3.19}$$

In the sequel, we consider two cases: $x > 10$ and $x \le 10$. For the case $x > 10$, we will use Part (i) of Lemma 2.1 to obtain a sharp estimate on $B_{n,x}$, while for the case $x \le 10$, as for B_n, we apply Part (ii) to get a suitable estimate. The choice of the number 10 is flexible. We just need the fact that the functions $p_1(\cdot), p_2(\cdot)$ and $r(\cdot)$ are decreasing in the interval (c, ∞) for a positive constant c (see Lemma 3.3).

3.4.1 Case $x > 10$

Lemma 3.3. *These functions $p_1(t), p_2(t)$ and $r(t)$ are decreasing in $(9, \infty)$.*

The proof of this lemma is elementary, so we omit it. Applying Lemma 2.1 (i) and Lemma 3.3 to the sums in (3.19), we obtain

$$B_{n,x} = \frac{n^{3/4}}{2} \int_x^{\frac{n^{1/4}}{2}} p_1(t)dt + \frac{n^{1/4}}{2} \int_x^{\frac{n^{1/4}}{2}} p_2(t)dt$$
$$+ O(1)\left(\frac{1}{n^{1/4}} \int_x^{\frac{n^{1/4}}{2}} r(t)dt + p_1(x) + \frac{p_2(x)}{\sqrt{n}} + \frac{r(x)}{n} + p_1\left(\tfrac{n^{1/4}}{2}\right) + \frac{p_2\left(\frac{n^{1/4}}{2}\right)}{\sqrt{n}} + \frac{r\left(\frac{n^{1/4}}{2}\right)}{\sqrt{n}}\right).$$

Moreover,

$$p_1\left(\tfrac{n^{1/4}}{2}\right), p_2\left(\tfrac{n^{1/4}}{2}\right), r\left(\tfrac{n^{1/4}}{2}\right), \int_{\frac{n^{1/4}}{2}}^{\infty} p_1(t)dt, \int_{\frac{n^{1/4}}{2}}^{\infty} p_2(t)dt, \int_{\frac{n^{1/4}}{2}}^{\infty} r(t)dt = o(n^{-1}).$$

Therefore,

$$B_{n,x} = \frac{n^{3/4}}{2}\hat{P}_1(x) + \frac{n^{1/4}}{2}\hat{P}_2(x) + O(1)\left(\frac{\hat{R}(x)}{n^{1/4}} + p_1(x) + \frac{p_2(x)}{\sqrt{n}} + \frac{r(x)}{n}\right) + o(1), \tag{3.20}$$

where

$$\begin{aligned} \hat{P}_1(x) &= \int_x^\infty p_1(t)dt, \\ \hat{P}_2(x) &= \int_x^\infty p_2(t)dt, \\ \hat{R}(x) &= \int_x^\infty r(t)dt. \end{aligned} \tag{3.21}$$

3.4.2 Case $x \le 10$

Using the same arguments for (3.18), we can show that

$$B_{n,x} = \frac{n^{3/4}}{2}\hat{P}_1(x) + \frac{n^{1/4}}{2}\hat{P}_2(x) + O(n^{1/12}). \tag{3.22}$$

3.5 Conclusion

We first rewrite (3.18) as

$$B_n = \frac{n^{3/4}}{2}\hat{P}_1(-\infty) + \frac{n^{1/4}}{2}\hat{P}_2(-\infty) + O(n^{1/12}), \tag{3.23}$$

with $\hat{P}_1(x)$ and $\hat{P}_2(x)$ as in (3.21).

3.5.1 Case $x > 10$

Combining (3.10), (3.20), (3.23), we have

$$\begin{aligned}
&1 - F_n(x)\\
=\ & \frac{n^{3/4}\hat{P}_1(x) + n^{1/4}\hat{P}_2(x) + O(1)\left(n^{-1/4}\hat{R}(x) + p_1(x) + n^{-1/2}p_2(x) + n^{-1}r(x)\right)}{n^{3/4}\hat{P}_1(-\infty) + n^{1/4}\hat{P}_2(-\infty) + O(n^{1/12})}\\
=\ & \frac{\hat{P}_1(x) + n^{-1/2}\hat{P}_2(x) + O(1)\left(n^{-1}\hat{R}(x) + n^{-3/4}p_1(x) + p_2(x)n^{-5/4} + n^{-7/4}r(x)\right)}{\hat{P}_1(-\infty) + n^{-1/2}\hat{P}_2(-\infty) + O(n^{-2/3})}.
\end{aligned}$$

Notice that $1 - F(x) = \hat{P}_1(x)/\hat{P}_1(-\infty)$. Therefore,

$$\begin{aligned}
&-1 + \frac{1 - F_n(x)}{1 - F(x)} = -1 + (1 - F_n(x))\frac{\hat{P}_1(-\infty)}{\hat{P}_1(x)}\\
&= -1 + \frac{\hat{P}_1(-\infty) + \frac{\hat{P}_1(-\infty)}{n^{1/2}}\frac{\hat{P}_2(x)}{\hat{P}_1(x)} + O(1)\left(n^{-1}\frac{\hat{R}(x)}{\hat{P}_1(x)} + n^{\frac{-3}{4}}\frac{p_1(x)}{\hat{P}_1(x)} + n^{\frac{-5}{4}}\frac{p_2(x)}{\hat{P}_1(x)} + n^{\frac{-7}{4}}\frac{r(x)}{\hat{P}_1(x)}\right)}{\hat{P}_1(-\infty) + n^{-1/2}\hat{P}_2(-\infty) + O(n^{-2/3})}\\
&= \frac{1}{\sqrt{n}}\left(\frac{\hat{P}_2(x)}{\hat{P}_1(x)} - \frac{\hat{P}_2(-\infty)}{\hat{P}_1(-\infty)}\right) + O(1)\left(\frac{x^{12}}{n} + n^{-3/4}\right),
\end{aligned}$$

where for the last line we have used that

$$\frac{\hat{R}(x)}{\hat{P}_1(x)} = \Theta(x^{12}), \qquad \frac{p_1(x)}{\hat{P}_1(x)} = \Theta(1), \qquad \frac{p_2(x)}{\hat{P}_1(x)} = \Theta(x^6), \qquad \frac{r(x)}{\hat{P}_1(x)} = \Theta(x^{12}).$$

In conclusion,

$$\frac{1 - F_n(x)}{1 - F(x)} = 1 + \frac{G(x)}{\sqrt{n}} + O(1)\left(\frac{x^{12}}{n} + n^{-3/4}\right),$$

with $G(x)$ as in Theorem 1.4.

3.5.2 Case $x \leq 10$

Using (3.22), (3.23) and the same arguments as in the case $x > 10$, we can prove that

$$\frac{1 - F_n(x)}{1 - F(x)} = 1 + \frac{G(x)}{\sqrt{n}} + O(n^{-2/3}).$$

Notice that here the term $O(n^{-2/3})$ comes from the quotient $O(n^{1/12})/n^{3/4}$.

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