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A note on Malliavin smoothness on the Lévy space

Eija Laukkarinen*

Abstract

We consider Malliavin calculus based on the Itô chaos decomposition of square integrable random variables on the Lévy space. We show that when a random variable satisfies a certain measurability condition, its differentiability and fractional differentiability can be determined by weighted Lebesgue spaces. The measurability condition is satisfied for all random variables if the underlying Lévy process is a compound Poisson process on a finite time interval.

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1 Introduction

One extension of Malliavin calculus from the Brownian motion to general Lévy processes was made using the Itô chaos decomposition on the L_2-space over the Lévy space. This approach was used for instance by Nualart and Vives [16], Privault [17], Benth, Di Nunno, Løkka, Øksendal and Proske [4], Lee and Shih [13], Solé, Utzet and Vives [18] and Applebaum [2].

The wide interest in Malliavin calculus for Lévy processes in stochastics and applications motivates the study of an accessible characterization of differentiability and fractional differentiability. Fractional differentiability can be defined by real interpolation between the Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $L_2(\mathbb{P})$; we shall recall the definition in Section 4 below. Geiss and Geiss [5] and Geiss and Hujo [11] have shown that Malliavin differentiability and fractional differentiability are in a close connection to discrete-time approximation of certain stochastic integrals when the underlying process is a (geometric) Brownian motion. Geiss et al. [7] proved that this also applies to Lévy processes with jumps. These articles assert that knowing the parameters of fractional smoothness allows to design discretization time-nets such that optimal approximation rates can be achieved. For details, see [5], [11] and [7].

Steinicke [19] and Geiss and Steinicke [10] take advantage of the fact that any random variable Y on the Lévy space can be represented as a functional $Y = F(X)$ of the Lévy process X, where F is a real valued measurable mapping on the Skorohod space of right continuous functions. Let us restrict to the case that $F(X)$ only depends on the jump part of X: Using the corresponding result from Solé, Utzet and Vives [18]

*University of Jyväskylä, Finland. E-mail: eija.laukkarinen@jyu.fi

and Alòs, León and Vives [1] on the canonical space, Geiss and Steinicke [10] show that the condition $F(X) \in \mathbb{D}_{1,2}$ is equivalent with

$$\iint_{[0,\infty)\times\mathbb{R}} \mathbb{E}\left[\left(F(X + x\mathbb{1}_{[t,\infty)}) - F(X)\right)^2\right] \mathrm{d}t\nu(\mathrm{d}x) < \infty,$$

where ν is the Lévy measure of X. On the other hand one gets from Mecke's formula [14] that

$$\iint_A \mathbb{E}[F(X + x\mathbb{1}_{[t,\infty)})]\mathrm{d}t\nu(\mathrm{d}x) = \mathbb{E}[N(A)F(X)]$$

for any nonnegative measurable F and any $A \in \mathcal{B}([0,\infty)\times\mathbb{R}\setminus\{0\})$, where N is the Poisson random measure associated with X as in Section 2. These results raise the following questions: when can Malliavin differentiability be described using a weight function such as $N(A)$, and is there a weight function for fractional differentiability?

In this paper we search for weight functions Λ and measurability conditions on Y such that the criteria

$$\|Y\Lambda\|_{L_2(\mathbb{P})} < \infty \tag{1.1}$$

describes the smoothness of Y. We begin by recalling the orthogonal Itô chaos decomposition

$$Y = \sum_{n=0}^{\infty} I_n(f_n)$$

on $L_2(\mathbb{P})$ and the Malliavin Sobolev space

$$\mathbb{D}_{1,2} = \left\{ Y \in L_2(\mathbb{P}) : \|Y\|^2_{\mathbb{D}_{1,2}} = \sum_{n=0}^{\infty}(n+1)\|I_n(f_n)\|^2_{L_2(\mathbb{P})} < \infty \right\}$$

in Section 2. Then, in Section 3, we obtain an equivalent condition for Malliavin differentiability. The assertion is that

$$Y \in \mathbb{D}_{1,2} \quad \text{if and only if} \quad \left\| Y\sqrt{N(A)+1} \right\|_{L_2(\mathbb{P})} < \infty,$$

whenever Y is measurable with respect to $\mathcal{F}_A$, the completion of the sigma-algebra generated by $\{N(B) : B \subseteq A,\, B \in \mathcal{B}([0,\infty)\times\mathbb{R})\}$, and the set $A \in \mathcal{B}([0,\infty)\times\mathbb{R}\setminus\{0\})$ satisfies $\mathbb{E}[N(A)] < \infty$.

Section 4 treats fractional differentiability and our aim is to adjust the weight function Λ so that the condition (1.1) describes a given degree of smoothness. By fractional differentiability of a random variable Y we mean that Y belongs to an interpolation space $(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ with

$$\mathbb{D}_{1,2} \subsetneq (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q} \subsetneq L_2(\mathbb{P})$$

for $\theta \in (0,1)$ and $q \in [1,\infty]$. We shortly recall the K-method of interpolation. Then we show that when Y is $\mathcal{F}_A$-measurable and $\mathbb{E}[N(A)] < \infty$, then

$$Y \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,2} \quad \text{if and only if} \quad \left\| Y\sqrt{N(A)+1}^{\,\theta} \right\|_{L_2(\mathbb{P})} < \infty.$$

2 Preliminaries

Consider a Lévy process $X = (X_t)_{t\geq 0}$ with càdlàg paths on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the completion of the sigma-algebra generated by X. The

Lévy-Itô decomposition states that there exist $\gamma \in \mathbb{R}$, $\sigma \geq 0$, a standard Brownian motion W and a Poisson random measure N on $\mathcal{B}([0,\infty)\times\mathbb{R})$ such that

$$X_t = \gamma t + \sigma W_t + \iint_{(0,t]\times\{|x|>1\}} xN(\mathrm{d}s,\mathrm{d}x) + \iint_{(0,t]\times\{0<|x|\leq 1\}} x\tilde{N}(\mathrm{d}s,\mathrm{d}x)$$

holds for all $t \geq 0$ a.s. Here $\tilde{N}(\mathrm{d}s,\mathrm{d}x) = N(\mathrm{d}s,\mathrm{d}x) - \mathrm{d}s\nu(\mathrm{d}x)$ is the compensated Poisson random measure and $\nu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ is the Lévy measure of X satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$ and $\nu(B) = \mathbb{E}\left[N((0,1]\times B)\right]$. The triplet (γ,σ,ν) is called the Lévy triplet.

Let us recall the Itô chaos decomposition from [12]: Denote $\mathbb{R}_+ := [0,\infty)$. We consider the following measure $\mathbb{m}$ defined as

$$\mathbb{m} : \mathcal{B}(\mathbb{R}_+\times\mathbb{R}) \to [0,\infty], \quad \mathbb{m}(\mathrm{d}s,\mathrm{d}x) := \mathrm{d}s\left[\sigma^2\delta_0(\mathrm{d}x) + x^2\nu(\mathrm{d}x)\right].$$

For sets $B \in \mathcal{B}(\mathbb{R}_+\times\mathbb{R})$ such that $\mathbb{m}(B) < \infty$, a random measure M is defined by

$$M(B) := \sigma\int_{\{s\in\mathbb{R}_+:(s,0)\in B\}} \mathrm{d}W_s + \lim_{n\to\infty}\iint_{\left\{(s,x)\in B:\frac{1}{n}<|x|<n\right\}} x\,\tilde{N}(\mathrm{d}s,\mathrm{d}x),$$

where the convergence is taken in $L_2(\mathbb{P}) := L_2(\Omega,\mathcal{F},\mathbb{P})$. The random measure M is independently scattered and it holds that $\mathbb{E}[M(B_1)M(B_2)] = \mathbb{m}(B_1\cap B_2)$ for all $B_1, B_2 \in \mathcal{B}(\mathbb{R}_+\times\mathbb{R})$ with $\mathbb{m}(B_1) < \infty$ and $\mathbb{m}(B_2) < \infty$.

For $n = 1,2,\ldots$ write

$$L_2\left(\mathbb{m}^{\otimes n}\right) := L_2\left((\mathbb{R}_+\times\mathbb{R})^n, \mathcal{B}(\mathbb{R}_+\times\mathbb{R})^{\otimes n}, \mathbb{m}^{\otimes n}\right)$$

and set $L_2\left(\mathbb{m}^{\otimes 0}\right) := \mathbb{R}$. A function $f_n : (\mathbb{R}_+\times\mathbb{R})^n \to \mathbb{R}$ is said to be symmetric, if it coincides with its symmetrization $\tilde{f}_n$,

$$\tilde{f}_n((s_1,x_1),\ldots,(s_n,x_n)) := \frac{1}{n!}\sum_{\pi} f_n\left(\left(s_{\pi(1)},x_{\pi(1)}\right),\ldots,\left(s_{\pi(n)},x_{\pi(n)}\right)\right),$$

where the sum is taken over all permutations $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$.

We let I_n denote the multiple integral of order n defined by Itô [12] and shortly recall the definition. For pairwise disjoint $B_1,\ldots,B_n \in \mathcal{B}(\mathbb{R}_+\times\mathbb{R})$ with $\mathbb{m}(B_i) < \infty$ the integral of $\mathbb{1}_{B_1}\otimes\cdots\otimes\mathbb{1}_{B_n}$ is defined by

$$I_n\left(\mathbb{1}_{B_1}\otimes\cdots\otimes\mathbb{1}_{B_n}\right) := M(B_1)\cdots M(B_n). \tag{2.1}$$

It is then extended to a linear and continuous operator $I_n : L_2\left(\mathbb{m}^{\otimes n}\right) \to L_2(\mathbb{P})$. We let $I_0(f_0) := f_0$ for $f_0 \in \mathbb{R}$. For the multiple integral we have

$$I_n(f_n) = I_n(\tilde{f}_n) \text{ and } \mathbb{E}\left[I_n(f_n)I_k(g_k)\right] = \begin{cases} 0, & \text{if } n \neq k \\ n!\left(\tilde{f}_n,\tilde{g_n}\right)_{L_2(\mathbb{m}^{\otimes n})}, & \text{if } n = k \end{cases} \tag{2.2}$$

for all $f_n \in L_2\left(\mathbb{m}^{\otimes n}\right)$ and $g_k \in L_2\left(\mathbb{m}^{\otimes k}\right)$.

According to [12, Theorem 2], for any $Y \in L_2(\mathbb{P})$ there exist functions $f_n \in L_2\left(\mathbb{m}^{\otimes n}\right)$, $n = 0,1,2,\ldots$, such that

$$Y = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{in } L_2(\mathbb{P})$$

and the functions f_n are unique in $L_2(\mathbb{m}^{\otimes n})$ when they are chosen to be symmetric. We have

$$\|Y\|^2_{L_2(\mathbb{P})} = \sum_{n=0}^{\infty} n!\left\|\tilde{f}_n\right\|^2_{L_2(\mathbb{m}^{\otimes n})}.$$

We recall the definition of the Malliavin Sobolev space $\mathbb{D}_{1,2}$ based on the Itô chaos decomposition. We denote by $\mathbb{D}_{1,2}$ the space of all $Y = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ such that

$$\|Y\|^2_{\mathbb{D}_{1,2}} := \sum_{n=0}^{\infty} (n+1)! \left\|\tilde{f}_n\right\|^2_{L_2(\mathbb{m}^{\otimes n})} < \infty.$$

Let us write $L_2(\mathbb{m} \otimes \mathbb{P}) := L_2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}, \mathbb{m} \otimes \mathbb{P})$ and define the Malliavin derivative $D : \mathbb{D}_{1,2} \to L_2(\mathbb{m} \otimes \mathbb{P})$ in the following way: Consider functions $\varphi_n = \mathbb{1}_{B_1} \otimes \cdots \otimes \mathbb{1}_{B_n} \in L_2(\mathbb{m}^{\otimes n})$, where $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are pairwise disjoint and such that $\mathbb{m}(B_i) < \infty$ for all $i = 1, \ldots, n$. Define $DI_n(\varphi_n)$ by

$$D_{t,x} I_n(\varphi_n) := n I_{n-1}\left(\tilde{\varphi_n}(\cdot, (t,x))\right) = \sum_{i=1}^{n} \prod_{j \neq i} M(B_j) \mathbb{1}_{B_i}(t,x).$$

It holds $\|DI_n(\varphi_n)\|_{L_2(\mathbb{m} \otimes \mathbb{P})} = \sqrt{n} \|I_n(\varphi_n)\|_{L_2(\mathbb{P})}$ and the operator is extended to the space $\{I_n(f_n) : f_n \in L_2(\mathbb{m}^{\otimes n})\}$ by linearity and continuity. For $Y = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{D}_{1,2}$ it then holds that

$$D_{t,x} Y := \sum_{n=1}^{\infty} n I_{n-1}\left(\tilde{f}_n(\cdot, (t,x))\right)$$

converges in $L_2(\mathbb{m} \otimes \mathbb{P})$.

Remark 2.1. Note that also for any $u \in L_2(\mathbb{m} \otimes \mathbb{P})$ one finds a chaos representation $u = \sum_{n=0}^{\infty} I_n(g_{n+1})$, where the functions $g_{n+1} \in L_2\left(\mathbb{m}^{\otimes (n+1)}\right)$ are symmetric in the first n variables. For $u, v \in L_2(\mathbb{m} \otimes \mathbb{P})$ with $u = \sum_{n=0}^{\infty} I_n(g_{n+1})$ and $v = \sum_{n=0}^{\infty} I_n(h_{n+1})$ it then holds

$$(u,v)_{L_2(\mathbb{m} \otimes \mathbb{P})} = \sum_{n=0}^{\infty} n! \left(g_{n+1}, h_{n+1}\right)_{L_2\left(\mathbb{m}^{\otimes (n+1)}\right)}. \tag{2.3}$$

For more information, see for example [16], [17], [4], [13], [18] and [2].

3 Differentiability

We shall use the notation $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. For $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ we denote by $\mathcal{F}_A$ the completion of the sigma-algebra $\sigma\left(M(B) : B \subseteq A \text{ and } B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\right)$. In the following theorem we only consider $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$, in which case $\mathcal{F}_A$ is equal to the completion of $\sigma\left(N(B) : B \subseteq A \text{ and } B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\right)$ and it does not depend on the Brownian motion part of the Lévy process.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ be such that $\mathbb{E}[N(A)] = (\mathrm{d}t \otimes \nu)(A) < \infty$ and $Y \in L_2(\mathbb{P})$.*

1. *If $Y \in \mathbb{D}_{1,2}$, then $Y\sqrt{N(A)} \in L_2(\mathbb{P})$ and*

$$\left| \left\| Y\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} - \|Y\|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}[N(A)]} \right| \leq \|DY \mathbb{1}_A\|_{L_2(\mathbb{m} \otimes \mathbb{P})}. \tag{3.1}$$

2. *If $Y\sqrt{N(A)} \in L_2(\mathbb{P})$ and Y is $\mathcal{F}_A$-measurable, then $Y \in \mathbb{D}_{1,2}$ and*

$$\|DY\|_{L_2(\mathbb{m} \otimes \mathbb{P})} \leq \left\| Y\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} + \|Y\|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}[N(A)]}. \tag{3.2}$$

Remark 3.2. (a) If the random variable Y in Theorem 3.1 is $\mathcal{F}_A$-measurable, then by Lemma 3.6 we have $\|DY\mathbb{1}_A\|_{L_2(\mathbb{m} \otimes \mathbb{P})} = \|DY\|_{L_2(\mathbb{m} \otimes \mathbb{P})}$ and $Y \in \mathbb{D}_{1,2}$ if and only if $Y\sqrt{N(A)} \in L_2(\mathbb{P})$.

(b) If Y is not measurable with respect to $\mathcal{F}_A$, then $Y = Y_A + Y'$, where Y_A is $\mathcal{F}_A$-measurable and $Y' \neq 0$ is orthogonal to Y_A in $L_2(\mathbb{P})$ and in $\mathbb{D}_{1,2}$. Since $\|DY\|^2_{L_2(\mathrm{m}\otimes\mathbb{P})} = \|DY_A\|^2_{L_2(\mathrm{m}\otimes\mathbb{P})} + \|DY'\|^2_{L_2(\mathrm{m}\otimes\mathbb{P})}$, it holds by Theorem 3.1 that $Y \in \mathbb{D}_{1,2}$ if and only if both $Y_A\sqrt{N(A)} \in L_2(\mathbb{P})$ and $Y' \in \mathbb{D}_{1,2}$ hold.

(c) Suppose $Y \in L_2(\mathbb{P})$ is measurable with respect to $\mathcal{F}_{A^c}$. This is the case for example if Y depends only on the Brownian motion part of the Lévy process. Then $\left\|Y\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} = \|Y\|_{L_2(\mathbb{P})}\left\|\sqrt{N(A)}\right\|_{L_2(\mathbb{P})}$, so that the left hand side of (3.1) is zero and the right hand side of (3.2) is finite.

In section 5 we will provide two examples where the measurability condition is satisfied. Now we turn to the proof of Theorem 3.1.

We denote by $\mathcal{S}$ the set of random variables Y such that there exists $m \geq 1$, $f \in C_c^\infty(\mathbb{R}^m)$ and $0 \leq t_0 < t_1 < \cdots t_m < \infty$ such that

$$Y = f\left(X_{t_1} - X_{t_0}, \ldots, X_{t_m} - X_{t_{m-1}}\right).$$

Lemma 3.3 (Theorem 4.1, Corollaries 4.1 and 3.1 in [8])**.**

(a) $\mathcal{S}$ is dense in $\mathbb{D}_{1,2}$ and $L_2(\mathbb{P})$.

(b) For $Y, Z \in \mathcal{S}$ it holds $D_{t,x}(YZ) = YD_{t,x}Z + ZD_{t,x}Y + xD_{t,x}YD_{t,x}Z$ $\mathrm{m}\otimes\mathbb{P}$-a.e.

Proposition 3.4. *Let $Y = \sum_{n=0}^\infty I_n(f_n)$ be bounded and $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ be such that $\mathbb{E}[N(A)] = (\mathrm{d}t \otimes \nu)(A) < \infty$. Then $\sum_{n=1}^\infty nI_{n-1}\left(\tilde{f}_n(\cdot, *)\right)\mathbb{1}_A(*)$ converges in $L_2(\mathrm{m}\otimes\mathbb{P})$ and*

$$\begin{aligned}\left|\left\|Y\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} - \|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}[N(A)]}\right| &\leq \left\|\sum_{n=1}^\infty\left(nI_{n-1}\left(\tilde{f}_n\right)\mathbb{1}_A\right)\right\|_{L_2(\mathrm{m}\otimes\mathbb{P})} \\ &\leq \left\|Y\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} + \|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}[N(A)]}.\end{aligned} \tag{3.3}$$

Proof. Assume first that $Y \in \mathcal{S}$. Then also $Y^2 = \sum_{n=0}^\infty I_n(g_n) \in \mathcal{S}$. Letting $h(t,x) := \frac{1}{x}\mathbb{1}_A(t,x)$ we have $I_1(h) = N(A) - \mathbb{E}[N(A)]$ and we get using (2.2) and (2.3) that

$$\mathbb{E}\left[Y^2N(A)\right] - \mathbb{E}\left[Y^2\right]\mathbb{E}[N(A)] = \mathbb{E}\left[Y^2I_1(h)\right] = (g_1, h)_{L_2(\mathrm{m})} = (DY^2, h\otimes\mathbb{1}_\Omega)_{L_2(\mathrm{m}\otimes\mathbb{P})}.$$

From Lemma 3.3 (b) we obtain

$$\begin{aligned}&\mathbb{E}\left[Y^2N(A)\right] \\ &= \mathbb{E}\left[Y^2\right]\mathbb{E}[N(A)] + (DY^2, h\otimes\mathbb{1}_\Omega)_{L_2(\mathrm{m}\otimes\mathbb{P})} \\ &= \mathbb{E}\left[Y^2\right]\mathbb{E}[N(A)] + 2\iint_A \mathbb{E}[YD_{t,x}Y]\,x\mathrm{d}t\nu(\mathrm{d}x) + \iint_A \mathbb{E}\left[(D_{t,x}Y)^2\right]\mathrm{m}(\mathrm{d}t,\mathrm{d}x).\end{aligned}$$

Using Hölder's inequality we get

$$\left|2\iint_A \mathbb{E}[YD_{t,x}Y]\,x\mathrm{d}t\nu(\mathrm{d}x)\right| \leq 2\|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}[N(A)]}\,\|DY\mathbb{1}_A\|_{L_2(\mathrm{m}\otimes\mathbb{P})},$$

so that

$$\begin{aligned}&\left(-\|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}[N(A)]} + \|DY\mathbb{1}_A\|_{L_2(\mathrm{m}\otimes\mathbb{P})}\right)^2 \\ &\leq \mathbb{E}\left[Y^2N(A)\right] \\ &\leq \left(\|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}[N(A)]} + \|DY\mathbb{1}_A\|_{L_2(\mathrm{m}\otimes\mathbb{P})}\right)^2.\end{aligned}$$

Taking the square root yields to the double inequality (3.3).

Using Lemma 3.3 (a) we find for any bounded Y a uniformly bounded sequence $(Y_k) \subset \mathcal{S}$ such that $Y_k \to Y$ a.s. Since inequality (3.3) holds for all random variables $Y_k - Y_m \in \mathcal{S}$, they are uniformly bounded and $Y_k - Y_m \to 0$ a.s. as $k, m \to \infty$, we have by dominated convergence that

$$\|D(Y_k - Y_m)\mathbb{1}_A\|_{L_2(\mathbb{m}\otimes\mathbb{P})} \leq \left\|(Y_k - Y_m)\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} + \|Y_k - Y_m\|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}\left[N(A)\right]} \to 0$$

as $k, m \to \infty$. Thus the sequence $(DY_k\mathbb{1}_A)_{k=1}^{\infty}$ converges in $L_2(\mathbb{m} \otimes \mathbb{P})$ to some mapping $u \in L_2(\mathbb{m} \otimes \mathbb{P})$. Write $Y_k = \sum_{n=0}^{\infty} I_n\left(\tilde{f}_n^{(k)}\right)$. The mapping u has a representation $u = \sum_{n=0}^{\infty} I_n(h_{n+1})$ (see Remark 2.1), where for all $n \geq 0$ we have that

$$\left\|n\tilde{f}_n\mathbb{1}_A - h_n\right\|_{L_2(\mathbb{m}^{\otimes n})} \leq \left\|n\tilde{f}_n\mathbb{1}_A - n\tilde{f}_n^{(k)}\mathbb{1}_A\right\|_{L_2(\mathbb{m}^{\otimes n})} + \left\|n\tilde{f}_n^{(k)}\mathbb{1}_A - h_n\right\|_{L_2(\mathbb{m}^{\otimes n})} \to 0$$

as $k \to \infty$. We obtain (3.3) for the random variable Y using dominated convergence, the convergence $DY_k\mathbb{1}_A \to \sum_{n=0}^{\infty}(DI_n(f_n)\mathbb{1}_A)$ in $L_2(\mathbb{m} \otimes \mathbb{P})$ and the fact that (3.3) holds for all random variables Y_k. □

Lemma 3.5. *If $Y = \sum_{n=0}^{\infty} I_n(f_n\mathbb{1}^{\otimes n}_{\mathbb{R}_+\times\mathbb{R}_0}) \in \mathbb{D}_{1,2}$ and $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, then $g(Y) \in \mathbb{D}_{1,2}$ and*

$$D_{t,x}g(Y) = \frac{g(Y + xD_{t,x}Y) - g(Y)}{x} \quad \text{in } L_2(\mathbb{m} \otimes \mathbb{P}).$$

Proof. The lemma is an immediate consequence of [8, Lemma 5.1 (b)]. □

Lemma 3.6. *Let $Y = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ and $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$. Then*

$$\mathbb{E}\left[Y|\mathcal{F}_A\right] = \sum_{n=0}^{\infty} I_n\left(f_n\mathbb{1}_A^{\otimes n}\right) \text{ in } L_2(\mathbb{P}).$$

Proof. The equality can be shown via the construction of the chaos analogously to the proof of [15, Lemma 1.2.4]. □

Proof of Theorem 3.1. 1. Assume $Y \in \mathbb{D}_{1,2}$ and define $g_m(x) := (-m \vee x) \wedge m$ for $m \geq 1$. From Lemma 3.5 we get $g_m(Y) \in \mathbb{D}_{1,2}$ and $|Dg_m(Y)| \leq |DY|$. Then, using monotone convergence and Proposition 3.4, we obtain

$$\begin{aligned}
&\left|\left\|Y\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} - \|Y\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}\left[N(A)\right]}\right| \\
&= \lim_{m\to\infty}\left|\left\|g_m(Y)\sqrt{N(A)}\right\|_{L_2(\mathbb{P})} - \|g_m(Y)\|_{L_2(\mathbb{P})}\sqrt{\mathbb{E}\left[N(A)\right]}\right| \\
&\leq \limsup_{m\to\infty}\|Dg_m(Y)\mathbb{1}_A\|_{L_2(\mathbb{m}\otimes\mathbb{P})} \\
&\leq \|DY\mathbb{1}_A\|_{L_2(\mathbb{m}\otimes\mathbb{P})} < \infty.
\end{aligned}$$

Hence $Y\sqrt{N(A)} \in L_2(\mathbb{P})$.

2. Assume $\|Y\sqrt{N(A)}\| < \infty$ and define $g_m(Y)$ as above. Write $Y = \sum_{n=0}^{\infty} I_n(f_n)$ and $g_m(Y) = \sum_{n=0}^{\infty} I_n(f_n^{(m)})$. Since $g_m(Y) \to Y$ in $L_2(\mathbb{P})$, it holds $\|\tilde{f}_n^{(m)}\|^2_{L_2(\mathbb{m}^{\otimes n})} \to \|\tilde{f}_n\|^2_{L_2(\mathbb{m}^{\otimes n})}$ as $m \to \infty$. Since $g_m(Y)$ is $\mathcal{F}_A$-measurable, we have $\tilde{f}_n^{(m)} = \tilde{f}_n^{(m)}\mathbb{1}_A^{\otimes n}$ $\mathbb{m}^{\otimes n}$-a.e. by Lemma 3.6 for all $m \geq 1$. By Fatou's Lemma, Proposition 3.4 and monotone convergence we get

$$\begin{aligned}\sqrt{\sum_{n=1}^{\infty} nn! \left\|\tilde{f}_n\right\|^2_{L_2(\mathfrak{m}^{\otimes n})}} &\le \liminf_{m\to\infty} \sqrt{\sum_{n=1}^{\infty} nn! \left\|\tilde{f}_n^{(m)}\right\|^2_{L_2(\mathfrak{m}^{\otimes n})}} \\ &\le \liminf_{m\to\infty} \left(\left\| g_m(Y)\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} + \|g_m(Y)\|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}\left[N(A)\right]} \right) \\ &= \left\| Y\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} + \|Y\|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}\left[N(A)\right]} < \infty.\end{aligned}$$

Thus $Y \in \mathbb{D}_{1,2}$. □

We use the notation $\alpha \sim_c \beta$ for $\frac{1}{c}\beta \le \alpha \le c\beta$ for $c \ge 1$ and $\alpha, \beta \in [0,\infty]$.

Corollary 3.7. *Let $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ be such that $\mathbb{E}\left[N(A)\right] < \infty$ and assume that $Y = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ is $\mathcal{F}_A$-measurable. Then*

$$\|Y\|_{\mathbb{D}_{1,2}} \sim_{\sqrt{2}\left(\sqrt{\mathbb{E}[N(A)]}+1\right)} \left\| Y\sqrt{N(A)+1} \right\|_{L_2(\mathbb{P})},$$

where the norms may be infinite.

Proof. The inequalities (3.1) and (3.2) give the relation

$$\left(\left\| Y\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} + \|Y\|_{L_2(\mathbb{P})} \right) \sim_{\sqrt{\mathbb{E}[N(A)]}+1} \left(\|Y\|_{L_2(\mathbb{P})} + \|DY\|_{L_2(\mathfrak{m}\otimes\mathbb{P})} \right).$$

The claim follows from $\|Y\|_{\mathbb{D}_{1,2}} \le \|Y\|_{L_2(\mathbb{P})} + \|DY\|_{L_2(\mathfrak{m}\otimes\mathbb{P})} \le \sqrt{2}\|Y\|_{\mathbb{D}_{1,2}}$ and

$$\begin{aligned}\left\| Y\sqrt{N(A)+1} \right\|_{L_2(\mathbb{P})} &\le \left\| Y\left(\sqrt{N(A)}+1\right) \right\|_{L_2(\mathbb{P})} \\ &\le \left\| Y\sqrt{N(A)} \right\|_{L_2(\mathbb{P})} + \|Y\|_{L_2(\mathbb{P})} \\ &\le \sqrt{2\left(\left\| Y\sqrt{N(A)} \right\|^2_{L_2(\mathbb{P})} + \|Y\|^2_{L_2(\mathbb{P})} \right)} \\ &= \sqrt{2} \left\| Y\sqrt{N(A)+1} \right\|_{L_2(\mathbb{P})}.\end{aligned}$$

□

4 Fractional differentiability

We consider fractional smoothness in the sense of real interpolation spaces between $L_2(\mathbb{P})$ and $\mathbb{D}_{1,2}$. For parameters $\theta \in (0,1)$ and $q \in [1,\infty]$ the interpolation space $(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ is a Banach space, intermediate between $L_2(\mathbb{P})$ and $\mathbb{D}_{1,2}$.

We shortly recall the K-method of real interpolation. The K-functional of $Y \in L_2(\mathbb{P})$ is the mapping $K(Y,\cdot; L_2(\mathbb{P}), \mathbb{D}_{1,2}) : (0,\infty) \to [0,\infty)$ defined by

$$K(Y,s; L_2(\mathbb{P}), \mathbb{D}_{1,2}) := \inf\{ \|Y_0\|_{L_2(\mathbb{P})} + s\|Y_1\|_{\mathbb{D}_{1,2}} : \ Y = Y_0 + Y_1,\ Y_0 \in L_2(\mathbb{P}),\ Y_1 \in \mathbb{D}_{1,2} \}$$

and we shall use the abbreviation $K(Y,s)$ for $K(Y,s; L_2(\mathbb{P}), \mathbb{D}_{1,2})$. Let $\theta \in (0,1)$ and $q \in [1,\infty]$. The space $(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ consists of all $Y \in L_2(\mathbb{P})$ such that

$$\|Y\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,q}} := \begin{cases} \left[\int_0^\infty \left| s^{-\theta} K(Y,s) \right|^q \frac{\mathrm{d}s}{s}\right]^{\frac{1}{q}}, & q \in [1,\infty) \\ \sup_{s>0} s^{-\theta} K(Y,s), & q = \infty \end{cases}$$

is finite.

The interpolation spaces are nested in a lexicographical order:

$$\mathbb{D}_{1,2} \subset (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\eta,p} \subset (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q} \subseteq (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,p} \subset L_2(\mathbb{P})$$

for $1 \le q \le p \le \infty$ and $0 < \theta < \eta < 1$. For further properties of interpolation we refer to [3] and [20].

Theorem 4.1. *Let $\theta \in (0,1)$, $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ be such that $\mathbb{E}[N(A)] < \infty$ and $Y \in L_2(\mathbb{P})$ be $\mathcal{F}_A$-measurable. Then*

$$Y \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,2} \quad \text{if and only if} \quad \mathbb{E}\left[Y^2 N(A)^\theta\right] < \infty.$$

If $Y \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,2}$, then

$$\|Y\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,2}} \sim_{\sqrt{2}\frac{\sqrt{\mathbb{E}[N(A)]}+1}{\sqrt{\theta(1-\theta)}}} \left\| Y\sqrt{N(A)+1}^{\,\theta} \right\|_{L_2(\mathbb{P})}.$$

Proof. We first show that

$$K(Y,s) \sim_{2\left(\sqrt{\mathbb{E}[N(A)]}+1\right)} \left\| Y \min\left\{1, s\sqrt{N(A)+1}\right\} \right\|_{L_2(\mathbb{P})}. \tag{4.1}$$

From Lemma 3.6 we obtain $\|\mathbb{E}[Y_0|\mathcal{F}_A]\|_{L_2(\mathbb{P})} \le \|Y_0\|_{L_2(\mathbb{P})}$ and $\|\mathbb{E}[Y_1|\mathcal{F}_A]\|_{\mathbb{D}_{1,2}} \le \|Y_1\|_{\mathbb{D}_{1,2}}$ for any $Y_0 \in L_2(\mathbb{P})$ and $Y_1 \in \mathbb{D}_{1,2}$. Hence

$$\begin{aligned} K(Y,s) &= \inf\left\{\|Y_0\|_{L_2(\mathbb{P})} + s\|Y_1\|_{\mathbb{D}_{1,2}} :\ Y_0 + Y_1 = Y,\ Y_0 \in L_2(\mathbb{P}),\ Y_1 \in \mathbb{D}_{1,2}\right\} \\ &= \inf\left\{\|\mathbb{E}[Y_0|\mathcal{F}_A]\|_{L_2(\mathbb{P})} + s\|\mathbb{E}[Y_1|\mathcal{F}_A]\|_{\mathbb{D}_{1,2}} : Y_0 + Y_1 = Y,\ Y_0 \in L_2(\mathbb{P}),\ Y_1 \in \mathbb{D}_{1,2}\right\} \\ &\sim_c \inf\left\{\|Y_0\|_{L_2(\mathbb{P})} + s\left\|Y_1\sqrt{N(A)+1}\right\|_{L_2(\mathbb{P})} : Y_0 + Y_1 = Y,\ Y_0 \in L_2(\mathbb{P}),\ Y_1 \in \mathbb{D}_{1,2}\right\} \end{aligned} \tag{4.2}$$

for $c = \sqrt{2}\left(\sqrt{\mathbb{E}[N(A)]}+1\right)$ by Corollary 3.7. Next we approximate the K-functional from above with the choice $Y_0 := Y\mathbb{1}_{\left\{\sqrt{N(A)+1} > \frac{1}{s}\right\}}$ and get from (4.2) that

$$\begin{aligned} \frac{1}{c}K(Y,s) &\le \left(\left\| Y\mathbb{1}_{\left\{\sqrt{N(A)+1} > \frac{1}{s}\right\}} \right\|_{L_2(\mathbb{P})} + s\left\| Y\sqrt{N(A)+1}\mathbb{1}_{\left\{\sqrt{N(A)+1} \le \frac{1}{s}\right\}} \right\|_{L_2(\mathbb{P})} \right) \\ &\le \sqrt{2}\left\| Y\min\left\{1, s\sqrt{N(A)+1}\right\} \right\|_{L_2(\mathbb{P})}. \end{aligned}$$

Using the triangle inequality and the fact that

$$|Y(\omega) - y| + |y|a \ge |Y(\omega)|\min\{1,a\}$$

for all $\omega \in \Omega$, $y \in \mathbb{R}$ and $a \ge 0$ we obtain from (4.2) the lower bound

$$\begin{aligned} cK(Y,s) &\ge \inf\left\{ \left\| |Y_0| + |Y_1|s\sqrt{N(A)+1} \right\|_{L_2(\mathbb{P})} : Y = Y_0 + Y_1,\ Y_1 \in \mathbb{D}_{1,2} \right\} \\ &\ge \left\| Y\min\left\{1, s\sqrt{N(A)+1}\right\} \right\|_{L_2(\mathbb{P})}. \end{aligned}$$

We have shown that (4.1) holds. From (4.1) we get

$$\|Y\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,2}} \sim_{2\left(\sqrt{\mathbb{E}[N(A)]}+1\right)} \left(\int_0^\infty \left| s^{-\theta} \left\| Y\min\left\{1, s\sqrt{N(A)+1}\right\} \right\|_{L_2(\mathbb{P})} \right|^2 \frac{\mathrm{d}s}{s} \right)^{\frac{1}{2}}.$$

We finish the proof by computing the integral using first Fubini's theorem. We get

$$\begin{aligned} &\int_0^\infty \left| s^{-\theta} \left\| Y\min\left\{1, s\sqrt{N(A)+1}\right\} \right\|_{L_2(\mathbb{P})} \right|^2 \frac{\mathrm{d}s}{s} \\ &= \mathbb{E}\left[Y^2 \int_0^\infty s^{-2\theta}\min\left\{1, s^2(N(A)+1)\right\} \frac{\mathrm{d}s}{s} \right] \\ &= \mathbb{E}\left[Y^2 \frac{1}{2\theta(1-\theta)} (N(A)+1)^\theta \right]. \end{aligned}$$

□

5 Concluding remarks and applications

From Theorem 3.1 assertion *2.* we can conclude that a higher integrability than square integrability can imply Malliavin differentiability. For example, all the spaces $L_p(\Omega, \mathcal{F}_A, \mathbb{P})$ are subspaces of $\mathbb{D}_{1,2}$ when $p > 2$ and $\mathbb{E}[N(A)] < \infty$ as we can deduce from the following corollary.

Corollary 5.1. *Let $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ be such that $\lambda := \mathbb{E}[N(A)] \in (0, \infty)$ so that $N(A) \sim$ Poisson(λ). Then for the space*

$$L_2 \log^+ L_2(\Omega, \mathcal{F}_A, \mathbb{P}) := \left\{ Y \in L_2(\Omega, \mathcal{F}_A, \mathbb{P}) : \mathbb{E}\left[Y^2 \ln^+ Y^2 \right] < \infty \right\},$$

where $\ln^+ x := \max\{\ln x, 0\}$, it holds that

$$L_2 \log^+ L_2(\Omega, \mathcal{F}_A, \mathbb{P}) \subsetneq \mathbb{D}_{1,2} \cap L_2(\Omega, \mathcal{F}_A, \mathbb{P}).$$

Proof. Suppose $\mathbb{E}\left[Y^2 \ln^+ Y^2 \right] < \infty$ and let $\varphi(y) := \ln(y+1)$. The functions Φ and $\Phi^\star$ with

$$\Phi(x) := \int_0^x \varphi(y) \mathrm{d}y = (x+1)\ln(x+1) - x \leq 1 + x \ln^+ x$$

and

$$\Phi^\star(x) := \int_0^x \varphi^{-1}(y) \mathrm{d}y = e^x - x - 1$$

are a complementary pair of Young functions. They satisfy the Young inequality $xy \leq \Phi(x) + \Phi^\star(y)$ for all $x, y \geq 0$ and we get

$$\mathbb{E}\left[Y^2 N(A) \right] \leq \mathbb{E}\left[\Phi\left(Y^2 \right) \right] + \mathbb{E}\left[\Phi^\star(N(A)) \right] \leq \mathbb{E}\left[Y^2 \ln^+ \left(Y^2 \right) \right] + e^{(e-1)\lambda} - \lambda < \infty.$$

Hence $Y \in \mathbb{D}_{1,2}$ by Theorem 3.1.

To see that the inclusion is strict, let $a \in (1, 2]$ and choose a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = f(1) := 0$ and

$$f(n) := \sqrt{e^\lambda \frac{n!}{\lambda^n} \frac{1}{n^2 \ln^a n}} \quad \text{for } n = 2, 3, \ldots.$$

Then, since $\ln n! = \sum_{k=2}^n \ln k \geq \int_1^n \ln x \, \mathrm{d}x = n \ln n - n + 1$ for $n \geq 2$ and $a \leq 2$, we have

$$\begin{aligned}
\mathbb{E}\left[f^2(N(A)) \ln^+ f^2(N(A)) \right] &= \sum_{n=2}^\infty \frac{1}{n^2 \ln^a n} \ln\left(e^\lambda \frac{n!}{\lambda^n} \frac{1}{n^2 \ln^a n} \right) \\
&= \sum_{n=2}^\infty \frac{\ln n!}{n^2 \ln^a n} + \sum_{n=2}^\infty \frac{1}{n^2 \ln^a n} \ln\left(e^\lambda \frac{1}{\lambda^n} \frac{1}{n^2 \ln^a n} \right) \\
&= \infty,
\end{aligned}$$

but

$$\mathbb{E}\left[N(A) f^2(N(A)) \right] = \sum_{n=2}^\infty n f^2(n) e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=2}^\infty \frac{1}{n \ln^a n} < \infty$$

so that $f(N(A)) \in \mathbb{D}_{1,2}$ by Theorem 3.1. □

In the following remark we continue the discussion about the measurability condition of Theorems 3.1 and 4.1, which we started in Remark 3.2.

Remark 5.2. (a) Suppose $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, which means that X is a compound Poisson process (with drift) and

$$X_t = \beta t + \int_{(0,t]\times\mathbb{R}_0} xN(\mathrm{d}s, \mathrm{d}x) \quad \text{for all } t \geq 0 \text{ a.s.}$$

for some $\beta \in \mathbb{R}$. The process $(N_t)_{t\geq 0}$, with $N_t = N((0,t]\times\mathbb{R}_0)$ a.s., is the Poisson process associated to X. Let $T \in (0,\infty)$ and $\mathcal{F}_T$ be the completion of the sigma-algebra generated by $(X_t)_{t\in[0,T]}$. Then $\mathcal{F}_T = \mathcal{F}_{[0,T]\times\mathbb{R}}$ and by Theorems 3.1 and 4.1 for any $\mathcal{F}_T$-measurable random variable Y and any $\theta \in (0,1)$ it holds that

1. $Y \in \mathbb{D}_{1,2}$ if and only if $\left\|Y\sqrt{N_T+1}\right\|_{L_2(\mathbb{P})} < \infty$ and
2. $Y \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,2}$ if and only if $\left\|Y\sqrt{N_T+1}^{\,\theta}\right\|_{L_2(\mathbb{P})} < \infty$.

(b) Assertion 2 of Theorem 3.1 applies also to the case that the jump part of X is of infinite activity. Then $\mathbb{E}[N([0,T]\times\mathbb{R})] = T\nu(\mathbb{R}) = \infty$ but for example $\mathbb{E}[N([0,T]\times\{x : |x| > \varepsilon\})] < \infty$ for all $\varepsilon > 0$. On the other hand, the set A with $\mathbb{E}[N(A)] < \infty$ may cover the whole support of ν for example in the following sense: there are sets $C_1, C_2, \ldots \in \mathcal{B}(\mathbb{R})$ such that $0 < \nu(C_i) < \infty$ and $\mathbb{R}_0 = \bigcup_{i=1}^\infty C_i$. Let $[a_i, b_i] \subset \mathbb{R}_+$ be such that $b_i - a_i \leq \frac{c}{\nu(C_i)} 2^{-i}$ for some $c \in (0,\infty)$ and $A = \bigcup_{i=1}^\infty [a_i, b_i]\times C_i$ so that

$$\mathbb{E}[N(A)] \leq \sum_{i=1}^\infty (b_i - a_i)\nu(C_i) \leq c.$$

In applications, Malliavin fractional smoothness plays a role for example in discrete time hedging in finance. Let S denote the underlying price process and φ be the predictable hedging strategy from the Galtchouk-Kunita-Watanabe decomposition of an option $f(S_T)$. Then the L_2-error with respect to the martingale measure $\mathbb{P}$ between discrete time hedging with $n+1$ trading times $0 = t_0^{(n)} < \cdots < t_n^{(n)} = T$ and the continuous time pricing model is

$$\inf\left\{\left\|\sum_{i=1}^n \int_{t_{i-1}^{(n)}}^{t_i^{(n)}} (v_{i-1} - \varphi_t)\,\mathrm{d}S_t\right\|_{L_2(\mathbb{P})} : \sum_{i=1}^n v_{i-1}\mathbb{1}_{\left(t_{i-1}^{(n)}, t_i^{(n)}\right]} \text{ is predictable}\right\}.$$

The error converges to zero as $\sup_i |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$. What relates our work to this approximation is that the rate of convergence depends on the fractional smoothness of $f(S_T)$. This was shown in the case that S is an L_2-Lévy process or the Doléans-Dade stochastic exponential of a Lévy process: for Brownian motion in, for example, [5] and [11], and for a general L_2-Lévy process in [7]. These results give a correspondence between the fractional smoothness parameter θ of $f(S_T)$ and the convergence rate if equidistant time nets $\left(t_i^{(n)}\right) = \left(\frac{i}{n}\right)$ are used. They also show that one can always optimize the time grid to obtain the best possible convergence rate when $f(S_T)$ has a strictly positive smoothness level θ.

Fractional smoothness is meaningful also when investigating properties of backward stochastic differential equations (BSDEs). The L_p-variation of the solution of a BSDE of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, \bar{Z}_s)\,\mathrm{d}s - \int_t^T Z_{s,x}\, M(\mathrm{d}s, \mathrm{d}t)$$

(where $\bar{Z}_s = \sigma Z_{s,0} + \int_{\mathbb{R}} Z_{s,x} x^2 \nu(\mathrm{d}x)$) with Lipschitz generator f, depends on the Malliavin fractional smoothness of the terminal condition $\xi \in L_2(\mathbb{P})$. This was shown for Brownian motion in [6] and for $p = 2$ for a Lévy process with jumps in [9].

For (geometric) Brownian motion, fractional smoothness of $f(S_T)$ means that f is in a weighted Besov space (see [11], for example), but what does fractional smoothness mean for a functional of a Lévy process with jumps? Suppose that $(X_t)_{t\geq 0}$ is a subordinator with $\nu(\mathbb{R}) < \infty$. Then, by Theorem 4.1 the fractional smoothness of $f(X_T)$ is determined by the property $\mathbb{E}[f(X_T)^2 N_T^\theta] < \infty$. This means that efficiency of discrete time hedging does not depend on the *smoothness* of f, but only on how rapidly $f(s)$ increases as s increases. When $\nu(\mathbb{R}) = \infty$, then Malliavin (fractional) smoothness of $f(X_1)$ requires certain smoothness properties of f which depend on intensity properties of ν. This relation will be a topic of a forthcoming publication.

References

[1] E. Alós, J. A. León, and J. Vives, *An anticipating Itô formula for Lévy processes*, ALEA Lat. Am. J. Probab. Math. Stat. **4** (2008), 285–305. MR-2456970

[2] D. Applebaum, *Universal Malliavin calculus in Fock and Lévy-Itô spaces*, Commun. Stoch. Anal. **3** (2009), no. 1, 119–141. MR-2541741

[3] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988. MR-928802

[4] F. E. Benth, G. Di Nunno, A. Løkka, B. Øksendal, and F. Proske, *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*, Math. Finance **13** (2003), no. 1, 55–72, Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). MR-1968096

[5] C. Geiss and S. Geiss, *On approximation of a class of stochastic integrals and interpolation*, Stoch. Stoch. Rep. **76** (2004), no. 4, 339–362. MR-2075477

[6] C. Geiss, S. Geiss, and E. Gobet, *Generalized fractional smoothness and L_p-variation of BSDEs with non-Lipschitz terminal condition*, Stochastic Process. Appl. **122** (2012), no. 5, 2078–2116. MR-2921973

[7] C. Geiss, S. Geiss, and E. Laukkarinen, *A note on Malliavin fractional smoothness for Lévy processes and approximation*, Potential Anal. **39** (2013), no. 3, 203–230. MR-3102985

[8] C. Geiss and E. Laukkarinen, *Denseness of certain smooth Lévy functionals in $\mathbb{D}_{1,2}$*, Probab. Math. Statist. **31** (2011), no. 1, 1–15. MR-2804974

[9] C. Geiss and A. Steinicke, *L_2-variation of Lévy driven BSDEs with non-smooth terminal conditions*, Bernoulli **22** (2016), no. 2, 995–1025. MR-3449806

[10] C. Geiss and A. Steinicke, *Malliavin derivative of random functions and applications to Lévy driven BSDEs*, Electron. J. Probab. **21** (2016), Paper No. 10, 28. MR-3485352

[11] S. Geiss and M. Hujo, *Interpolation and approximation in $L_2(\gamma)$*, J. Approx. Theory **144** (2007), no. 2, 213–232. MR-2293387

[12] K. Itô, *Spectral type of the shift transformation of differential processes with stationary increments*, Trans. Amer. Math. Soc. **81** (1956), 253–263. MR-0077017

[13] Y.-J. Lee and H.-H. Shih, *The product formula of multiple Lévy-Itô integrals*, Bull. Inst. Math. Acad. Sinica **32** (2004), no. 2, 71–95. MR-2069183

[14] J. Mecke, *Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1967), 36–58. MR-0228027

[15] D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, New York, 1995. MR-1344217

[16] D. Nualart and J. Vives, *Anticipative calculus for the Poisson process based on the Fock space*, Séminaire de Probabilités, XXIV, 1988/89, Lecture Notes in Math., vol. 1426, Springer, Berlin, 1990, pp. 154–165. MR-1071538

[17] N. Privault, *An extension of stochastic calculus to certain non-Markovian processes*, Preprint 49, Universite d'Evry, 1997, `https://www.maths.univ-evry.fr/prepubli/49.ps`.

[18] J. Solé, F. Utzet, and J. Vives, *Canonical Lévy process and Malliavin calculus*, Stochastic Process. Appl. **117** (2007), no. 2, 165–187. MR-2290191

[19] A. Steinicke, *Functionals of a Lévy process on canonical and generic probability spaces*, J. Theoret. Probab. **29** (2016), no. 2, 443–458. MR-3500406

[20] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Publishing Co., Amsterdam-New York, 1978. MR-0503903

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