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Necessary and sufficient conditions for the r-excessive local martingales to be martingales

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Abstract

We consider the decreasing and the increasing r-excessive functions φ_r and ψ_r that are associated with a one-dimensional conservative regular continuous strong Markov process X with values in an interval with endpoints $\alpha < \beta$. We prove that the r-excessive local martingale $\big(e^{-r(t\wedge T_\alpha)}\varphi_r(X_{t\wedge T_\alpha})\big)$ $\big($resp., $\big(e^{-r(t\wedge T_\beta)}\psi_r(X_{t\wedge T_\beta})\big)\big)$ is a strict local martingale if the boundary point α (resp., β) is inaccessible and entrance, and a martingale otherwise.

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1 Introduction

We consider a one-dimensional conservative regular continuous strong Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X_t;\, t \geq 0,\, x \in \mathcal{I})$ with values in an interval $\mathcal{I} \subseteq [-\infty, \infty]$ with endpoints $\alpha < \beta$ that is open, closed or semi-open. We recall that a Markov process is called conservative if there is no killing and a one-dimensional continuous strong Markov process with state space $\mathcal{I}$ is called regular if

$$\mathbb{P}_x(T_y < \infty) > 0 \quad \text{for all } x \in \mathring{\mathcal{I}} \text{ and } y \in \mathcal{I},$$

where $\mathring{\mathcal{I}} = \,]\alpha, \beta[$ and

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad \text{for } y \in [\alpha, \beta],$$

with the usual convention that $\inf \emptyset = \infty$. We also recall that the boundary point α (resp., β) is called inaccessible if

$$\mathbb{P}_x(T_\alpha < \infty) = 0 \quad \Big(\text{resp., } \mathbb{P}_x(T_\beta < \infty) = 0\Big) \quad \text{for all } x \in \mathring{\mathcal{I}}, \tag{1.1}$$

and accessible otherwise. We denote by p and m the scale function and the speed measure of X (see Definitions VII.3.3 and VII.3.7 in Revuz and Yor [8]). It is worth noting that, although the scale function is defined consistently, different normalisations of the speed measure appear in standard stochastic analysis textbooks. For instance, the speed

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measure in Revuz and Yor [8, Section VII.3] is consistent with the one in Borodin and Salminen [1, Section II.1], but is twice as large as the speed measure in Rogers and Williams [9, Section V.47]. Although our main result, Theorem 2.2, does not depend on such a normalisation, some of the formulas we use for its proof do.

Given any $r > 0$, there exist a continuous decreasing function $\varphi_r : \mathring{\mathcal{I}} \to]0, \infty[$ and a continuous increasing function $\psi_r : \mathring{\mathcal{I}} \to]0, \infty[$ function that are determined uniquely, up to multiplicative constants, by the expressions

$$\varphi_r(y) = \varphi_r(x)\, \mathbb{E}_y \left[e^{-rT_x}\right] \quad \text{and} \quad \psi_r(x) = \psi_r(y)\, \mathbb{E}_x \left[e^{-rT_y}\right] \quad \text{for all } x < y \text{ in } \mathring{\mathcal{I}}. \tag{1.2}$$

These functions are often called r-excessive. Since they are monotone, they can be extended to $[\alpha, \beta]$ by defining

$$\varphi_r(\alpha) = \lim_{x\downarrow\alpha} \varphi_r(x), \quad \psi(\alpha) = \lim_{x\downarrow\alpha} \psi_r(x), \quad \varphi_r(\beta) = \lim_{x\uparrow\beta} \varphi_r(x) \quad \text{and} \quad \psi_r(\beta) = \lim_{x\uparrow\beta} \psi(x).$$

Furthermore,

$$\alpha \Big(\text{resp., } \beta\Big) \text{ is inaccessible if and only if } \varphi_r(\alpha) = \infty \Big(\text{resp., } \psi_r(\beta) = \infty\Big). \tag{1.3}$$

An important property of ψ_r and φ_r is that

$$\text{the processes } \Big(e^{-r(t\wedge T_\alpha)}\varphi_r(X_{t\wedge T_\alpha})\Big) \text{ and } \Big(e^{-r(t\wedge T_\beta)}\psi_r(X_{t\wedge T_\beta})\Big) \text{ are } \mathbb{P}_x\text{-local martingales} \tag{1.4}$$

for all $x \in \mathcal{I}$. Despite their widespread use, these processes still do not have a standard name. In this paper, we refer to them as *r-excessive $\mathbb{P}_x$-local martingales*.

Beyond the central role that they play in the theory of one-dimensional diffusions, the r-excessive functions ψ_r, φ_r and their associated r-excessive $\mathbb{P}_x$-local martingales have been used extensively in the analysis and the solution of numerous optimal stopping and stochastic control problems involving one-dimensional diffusions. This most widespread use has motivated this paper. We refrain from trying to provide any relevant representative references because the use of the r-excessive functions and local martingales in applications of stochastic analysis has become folklore.

We derive necessary and sufficient conditions for the r-excessive $\mathbb{P}_x$-local martingales to be $\mathbb{P}_x$-martingales. If β is accessible, then $\big(e^{-r(t\wedge T_\beta)}\psi_r(X_{t\wedge T_\beta})\big)$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$ because it is a bounded $\mathbb{P}_x$-local martingale. On the other hand, we prove that, if β is inaccessible, then (*i*) $\big(e^{-rt}\psi_r(X_t)\big)$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$ if β is a natural boundary point, and (*ii*) $\big(e^{-rt}\psi_r(X_t)\big)$ is a strict $\mathbb{P}_x$-local martingale for all $x \in \mathcal{I}$ if β is an entrance boundary point, unless α is absorbing and $x = \alpha$, in which case the process $\big(e^{-rt}\psi_r(X_t)\big)$ under $\mathbb{P}_\alpha$ is identically equal to 0. We emphasise that we do not impose any restrictions on the boundary behaviour of α if this is accessible: it can be instantaneously or slowly reflecting as well as absorbing. Symmetric statements hold true for the $\mathbb{P}_x$-local martingale $\big(e^{-r(t\wedge T_\alpha)}\varphi_r(X_{t\wedge T_\alpha})\big)$. We expand on these statements in Theorem 2.2, our main result.

A result of a closely related nature has been established by Kotani [7]: the $\mathbb{P}_x$-local martingale $\big(p(X_{t\wedge T_\alpha\wedge T_\beta})\big)$ is a $\mathbb{P}_x$-martingale if and only if neither α nor β is an entrance boundary point. In fact, Delbaen and Shirakawa [2] had earlier established this result in a special case. The further analysis in Hulley [4, Chapter 3] is also worth mentioning. Furthermore, Gushchin, Urusov and Zervos [3] complemented this result by showing that the $\mathbb{P}_x$-local martingale $\big(p(X_{t\wedge T_\alpha\wedge T_\beta})\big)$ is a $\mathbb{P}_x$-supermaringale (resp., $\mathbb{P}_x$-submartingale) if and only if α (resp., β) is not an entrance boundary point.

2 The main result

Before addressing our main result, we recall that the boundary point β is inaccessible if and only if

$$\int_x^\beta m\big([x,y[\big)\, p(dy) = \infty, \tag{2.1}$$

where $x \in \mathring{\mathcal{I}}$ and $p(dy)$ is the atomless measure on $(\mathring{\mathcal{I}}, \mathcal{B}(\mathring{\mathcal{I}}))$ satisfying $p\big(]a,b]\big) = p(b) - p(a)$ for $\alpha < a < b < \beta$ (see also the definition in (1.1) as well as (1.3)). We note that this characterisation does not depend on the choice of $x \in \mathring{\mathcal{I}}$ because m is a Radon measure. Also, if β is inaccessible, then it is called natural if

$$\lim_{x \downarrow \alpha} \mathbb{P}_x(T_y < t) = 0 \quad \text{for all } y \in \mathring{\mathcal{I}} \text{ and } t > 0 \tag{2.2}$$

and entrance otherwise, namely, if

$$\lim_{x \downarrow \alpha} \mathbb{P}_x(T_y < t) > 0 \quad \text{for some } y \in \mathring{\mathcal{I}} \text{ and } t > 0. \tag{2.3}$$

In terms of an analytic characterisation, if β is inaccessible then it is

$$\text{natural if } \int_x^\beta m\big([y,\beta[\big)\, p(dy) = \infty \tag{2.4}$$

$$\text{and} \quad \text{entrance if } \int_x^\beta m\big([y,\beta[\big)\, p(dy) < \infty, \tag{2.5}$$

where the choice of $x \in \mathring{\mathcal{I}}$ is again arbitrary. Furthermore, we recall that the r-excessive functions φ_r and ψ_r satisfy the second order differential equation

$$\frac{d}{dm}\frac{d^+ f}{dp}(x) = r f(x)$$

in the sense that the limits

$$\frac{d^+ f}{dp}(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x+\varepsilon) - f(x)}{p(x+\varepsilon) - p(x)}$$

exist for all $x \in \mathring{\mathcal{I}}$ and

$$\frac{d^+ f}{dp}(x_2) - \frac{d^+ f}{dp}(x_1) = r \int_{]x_1, x_2]} f(y)\, m(dy) \quad \text{for all } \alpha < x_1 < x_2 < \beta. \tag{2.6}$$

Remark 2.1. All of the claims that we have made about the diffusion X, its boundary classification and its r-excessive functions are standard, and can be found in Itô and McKean [5, Chapter 4], Rogers and Williams [9, Section V.7], Karlin and Taylor [6, Chapter 15], Revuz and Yor [8, Section VII.3], and Borodin and Salminen [1, Chapter II]. In terms of boundary classification, the terminology that we have adopted is the same as the one in Karlin and Taylor [6, Table 15.6.2] and is consistent with the one in Revuz and Yor [8, Section VII.3] and Rogers and Williams [9, Section V.51]. On the other hand, Itô and McKean [5] use the terminology “not exit”, “not entrance, not exit” and “entrance, not exit” in place of “inaccessible”, “natural” and “entrance”, while Borodin and Salminen [1, Section II.1] use the terminology “not exit”, “natural” and “entrance-not-exit” in place of “inaccessible”, “natural” and “entrance”. □

The proof of our main result, which is captured by (A) in the following table, involves establishing first (B)–(D) using (E). We state explicitly all of these cases as well as (F) due to their independent interest as well as for completeness.

Theorem 2.2. *The following statements hold true:*

(I) *If β is accessible, namely, if the conditions in (1.1) and (2.1) fail, then the process $\big(e^{-r(t\wedge T_\beta)}\psi_r(X_{t\wedge T_\beta})\big)$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$.*

(II) *Suppose that β is inaccessible, namely, the conditions in (1.1) and (2.1) hold true. If β is natural, namely, if the conditions in (2.2) and (2.4) hold true, then the process $\big(e^{-rt}\psi_r(X_t)\big)$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$. On the other hand, if β is entrance, namely, if the conditions in (2.3) and (2.5) hold true, then the process $\big(e^{-rt}\psi_r(X_t)\big)$ is a strict $\mathbb{P}_x$-local martingale for all $x \in \mathcal{I}$, unless α is absorbing and $x = \alpha$. Furthermore, the equivalences suggested by the following table hold true (note that all limits appearing here indeed exist).*

	β is natural	β is entrance
(A)	$\forall r>0,\ \forall x\in\mathcal{I},\ \big(e^{-rt}\psi_r(X_t)\big)$ is a $\mathbb{P}_x$-martingale	$\forall r>0,\ \forall x\in\mathring{\mathcal{I}},\ \big(e^{-rt}\psi_r(X_t)\big)$ is a strict $\mathbb{P}_x$-local martingale
(B)	$\forall s>r>0,\ \lim_{x\uparrow\beta}\dfrac{\psi_s(x)}{\psi_r(x)}=\infty$	$\forall s>r>0,\ \lim_{x\uparrow\beta}\dfrac{\psi_s(x)}{\psi_r(x)}\in\,]0,\infty[$
(C)	$\forall r>0,\ \lim_{x\uparrow\beta}\dfrac{\psi_r(x)}{p(x)}=\infty$	$\forall r>0,\ \lim_{x\uparrow\beta}\dfrac{\psi_r(x)}{p(x)}\in\,]0,\infty[$
(D)	$\forall s>r>0,\ \lim_{x\uparrow\beta}\dfrac{\frac{d^+\psi_s}{dp}(x)}{\frac{d^+\psi_r}{dp}(x)}=\infty$	$\forall s>r>0,\ \lim_{x\uparrow\beta}\dfrac{\frac{d^+\psi_s}{dp}(x)}{\frac{d^+\psi_r}{dp}(x)}\in\,]0,\infty[$
(E)	$\forall r>0,\ \lim_{x\uparrow\beta}\dfrac{d^+\psi_r}{dp}(x)=\infty$	$\forall r>0,\ \lim_{x\uparrow\beta}\dfrac{d^+\psi_r}{dp}(x)\in\,]0,\infty[$
(F)	$\forall r>0,\ \forall x\in\mathring{\mathcal{I}},\ \int_{[x,\beta[}\psi_r(y)\,m(dy)=\infty$	$\forall r>0,\ \forall x\in\mathring{\mathcal{I}},\ \int_{[x,\beta[}\psi_r(y)\,m(dy)<\infty$

(III) *Symmetric results hold true for the process $\big(e^{-r(t\wedge T_\alpha)}\varphi_r(X_{t\wedge T_\alpha})\big)$.*

Proof. Statement (I) follows immediately because $\big(e^{-r(t\wedge T_\beta)}\psi_r(X_{t\wedge T_\beta})\big)$ is a bounded $\mathbb{P}_x$-local martingale (see also (1.3)). To prove (II), we assume in what follows that β is inaccessible, which implies that

$$\lim_{x\to\beta}\psi_r(x)=\infty \quad \text{for all } r>0. \tag{2.7}$$

The results in (E) and (F) appear in the fourth and the sixth row of Table 1 in Itô and McKean [5, Section 4.6] (see the third and fourth columns of that table; also, note that (F) follows immediately from (E) and (2.6)). Also, (C) follows from (E) and the calculation

$$\lim_{x\uparrow\beta}\frac{\psi_r(x)}{p(x)}=\lim_{y\uparrow p(\beta)}\frac{\psi_r\big(p^{-1}(y)\big)}{y}=\lim_{y\uparrow p(\beta)}\frac{d^+\psi_r\circ p^{-1}}{dy}(y)=\lim_{x\uparrow\beta}\frac{d^+\psi_r}{dp}(x),$$

in which we have used L'Hôpital's rule.

We now show that

$$\text{the limits } \lim_{x\uparrow\beta}\frac{\psi_r(x)}{\psi_s(x)} \text{ exist in } [0,\infty[\text{ for all } s>r>0 \tag{2.8}$$

as well as that (A) and (B) are equivalent. To this end, we consider an initial condition $x \in \mathring{\mathcal{I}}$, a point $\bar{\beta} \in \,]x, \beta[$ and constant $s > r > 0$, and we use the integration by parts formula to calculate

$$e^{-(s-r)t} M_{t \wedge T_{\bar{\beta}}} = \psi_r(x) - (s-r) \int_0^{t \wedge T_{\bar{\beta}}} e^{-(s-r)u} M_u \, du + \int_0^{t \wedge T_{\bar{\beta}}} e^{-(s-r)u} \, dM_u,$$

where $M_t = e^{-rt} \psi_r(X_t)$. The process $(M_{t \wedge T_{\bar{\beta}}},\ t \geq 0)$ is a $\mathbb{P}_x$-square integrable martingale because it is a bounded $\mathbb{P}_x$-local martingale. Therefore, the stochastic integral in this identity has zero expectation. In view of this observation and the dominated and monotone convergence theorems, we can see that

$$\begin{aligned}
\psi_r(\bar{\beta}) \, \mathbb{E}_x \left[e^{-s T_{\bar{\beta}}} \right] &= \lim_{t \to \infty} \mathbb{E}_x \left[e^{-s(t \wedge T_{\bar{\beta}})} \psi_r(X_{t \wedge T_{\bar{\beta}}}) \right] \\
&= \lim_{t \to \infty} \mathbb{E}_x \left[e^{-(s-r)(t \wedge T_{\bar{\beta}})} M_{t \wedge T_{\bar{\beta}}} \right] \\
&= \psi_r(x) - (s-r) \, \mathbb{E}_x \left[\int_0^{T_{\bar{\beta}}} e^{-(s-r)u} M_u \, du \right] \\
&= \psi_r(x) - (s-r) \, \mathbb{E}_x \left[\int_0^{T_{\bar{\beta}}} e^{-su} \psi_r(X_u) \, du \right].
\end{aligned}$$

Combining this calculation with the definition of ψ_s as in (1.2), we obtain

$$\psi_r(\bar{\beta}) \frac{\psi_s(x)}{\psi_s(\bar{\beta})} = \psi_r(x) - (s-r) \, \mathbb{E}_x \left[\int_0^{T_{\bar{\beta}}} e^{-su} \psi_r(X_u) \, du \right].$$

In view of the monotone convergence theorem and the assumption that β is inaccessible, it follows that

$$\lim_{\bar{\beta} \uparrow \beta} \frac{\psi_r(\bar{\beta})}{\psi_s(\bar{\beta})} = \frac{\psi_r(x)}{\psi_s(x)} - \frac{s-r}{\psi_s(x)} \int_0^\infty e^{-(s-r)u} \, \mathbb{E}_x \left[e^{-ru} \psi_r(X_u) \right] du. \tag{2.9}$$

This identity and the positivity of ψ_r imply that (2.8) is indeed true. Furthermore, since the process $\big(e^{-rt} \psi_r(X_t),\ t \geq 0\big)$ is a positive $\mathbb{P}_x$-local martingale, it is a $\mathbb{P}_x$-supermartingale. Therefore,

$$\mathbb{E}_x \left[e^{-rt} \psi_r(X_t) \right] \leq \psi_r(x) \quad \text{for all } t \geq 0,$$

with equality holding if and only if $\big(e^{-rt} \psi_r(X_t),\ t \geq 0\big)$ is a $\mathbb{P}_x$-martingale. In view of this observation, we can see that (2.9) implies that $\lim_{\bar{\beta} \uparrow \beta} \psi_r(\bar{\beta}) / \psi_s(\bar{\beta}) = 0$ if and only if $(e^{-rt} \psi_r(X_t),\ t \geq 0)$ is a $\mathbb{P}_x$-martingale, and the equivalence of (A) and (B) follows.

To complete the proof, we need to establish (B) and (D). To this end, we note that (2.7), (2.8) and L'Hôpital's rule imply that

$$\lim_{x \uparrow \beta} \frac{\psi_r(x)}{\psi_s(x)} = \lim_{y \uparrow p(\beta)} \frac{\psi_r\big(p^{-1}(y)\big)}{\psi_s\big(p^{-1}(y)\big)} = \lim_{y \uparrow p(\beta)} \frac{\frac{d^+ \psi_r \circ p^{-1}}{dy}(y)}{\frac{d^+ \psi_s \circ p^{-1}}{dy}(y)} = \lim_{x \uparrow \beta} \frac{\frac{d^+ \psi_r}{dp}(x)}{\frac{d^+ \psi_s}{dp}(x)}$$

whenever the last limit exists. If β is an entrance boundary point, then this calculation and the corresponding statement in (E) imply that the corresponding claims in (B) and (D) are indeed true.

On the other hand, if β is a natural boundary point, then we can use (2.6), (2.7) and

(2.8) to see that, given any $x_1 \in \mathring{\mathcal{I}}$,

$$\lim_{x\uparrow\beta} \frac{\psi_r(x)}{\psi_s(x)} = \lim_{x\uparrow\beta} \frac{\frac{d^+\psi_r}{dp}(x)}{\frac{d^+\psi_s}{dp}(x)} = \lim_{x\uparrow\beta} \frac{\frac{d^+\psi_r}{dp}(x_1) + r\int_{]x_1,x]} \psi_r(y)\, m(dy)}{\frac{d^+\psi_s}{dp}(x_1) + s\int_{]x_1,x]} \psi_s(y)\, m(dy)}$$
$$= \frac{r}{s} \lim_{x\uparrow\beta} \frac{\int_{]x_1,x]} \psi_r(y)\, m(dy)}{\int_{]x_1,x]} \psi_s(y)\, m(dy)} = \frac{r}{s} \lim_{x\uparrow\beta} \frac{\psi_r(x)}{\psi_s(x)}.$$

In view of (2.8), all these limits are equal to 0, and the corresponding claims in (B) and (D) follow. □

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