New characterizations
of the S topology on the Skorokhod space[*]

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Abstract

The S topology on the Skorokhod space was introduced by the author in 1997 and since then it has proved to be a useful tool in several areas of the theory of stochastic processes. The paper brings complementary information on the S topology. It is shown that the convergence of sequences in the S topology admits a closed form description, exhibiting the locally convex character of the S topology. Morover, it is proved that the S topology is, up to some technicalities, finer than any linear topology which is coarser than Skorokhod's J_1 topology. The paper contains also definitions of extensions of the S topology to the Skorokhod space of functions defined on $[0, +\infty)$ and with multidimensional values.

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1 Introduction

The S topology on the Skorokhod space $\mathbb{D} = \mathbb{D}([0, T])$ of càdlàg functions has emerged as a result of a chain of observations made in eighties and nineties of the twentieth century.

In 1984 Meyer and Zheng [29] considered certain conditions on truncated conditional variations of stochastic processes and proved that these conditions give uniform tightness of the processes in some topology (nowadays called the Meyer-Zheng topology) on $\mathbb{D}$.

A year later Stricker [36] proved that it is possible to relax the Meyer-Zheng conditions to uniform tightness of random variables $\{\|X_n\|_\infty\}$ and $\{N^{a,b}(X_n)\}$, for each pair of levels $a < b$ (where $N^{a,b}(x)$ is the number of up-crossings of levels $a < b$ by function x).

Stricker still operated with the Meyer-Zheng topology. It was clear for Kurtz [20] that such conditions give much more. But an *ad hoc* device constructed by Kurtz *did not have a topological character*.

The S topology was constructed by the author in [13]. This step was final in the sense that Stricker's conditions are *equivalent* to the uniform tightness in the S topology.

It should be emphasized, that the S topology, as considered in [13], is sequential and it is still not known if it is completely regular. So at the moment of its creation there

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was no formalism to deal with it within the Probability Theory. Such formalism has been provided in [14]. And an efficient tool - the almost sure Skorokhod representation in non-metric spaces, given in [12] - made the S topology an operational and useful device in many problems.

The very first application of the S topology was given by the author in [11], where convergence of stochastic integrals was considered. Later the S topology was used in problems related to homogenization of stochastic differential equations (e.g. [1], [2], [5], [25], [30], [35], [32]), diffusion approximation of solutions to the Poisson equation ([31]), stability of solutions to semilinear equations with Dirichlet operator ([19]), martingale transport on the Skorokhod space ([10]), the Skorokhod problem ([23], [24], [28], [34]), econometrics ([7]), control theory ([3], [22]), linear models with heavy-tails ([4]), continuity of semilinear Neumann-Dirichlet problems ([27]), generalized Doob-Meyer decomposition ([15]), modeling stochastic reaction networks ([16]) and even in some considerations of a more general character ([8], [21]).

In the present paper we provide some complementary information related to the S topology. Section 2 restates the definition and basic properties of the S topology. The S topology was defined in [13] by means of so-called $\mathcal{L}$-convergence $\longrightarrow_S$, which leads to the topological convergence $\overset{*}{\longrightarrow}_S$ via the Kantorovich-Vulih-Pinsker-Kisynski recipe (we refer to Section 6 for a primer on sequential spaces).

In Section 3 we provide a closed form formula for the $\overset{*}{\longrightarrow}_S$ convergence.

In Section 4 we find a position for S in the hierarchy of topologies on $\mathbb{D}$, by showing that it is *essentially* finer than any linear topology which is coarser than the Skorokhod J_1 topology.

In Section 5 we extend the notion of the S topology to the case of infinite time horizon and to functions with multidimensional values.

2 Definition of the S topology

All results in this section are taken from [13]. We need some standard notation first.

1. $\mathbb{D} = \mathbb{D}([0,T])$ denotes the Skorokhod space, i.e. a family of functions $x : [0,T] \to \mathbb{R}^1$, which are right-continuous at every $t \in [0,T)$ and admit left-limits at every $t \in (0,T]$.

2. $\mathbb{D}$ is naturally equipped with the *sup-norm* $\|x\|_\infty = \sup_{t \in [0,T]} |x(t)|$.

3. For $a < b$, $N^{a,b}(x)$ is *the number of up-crossings of levels a and b by function* $x \in \mathbb{D}$. In other words, $N^{a,b}(x)$ is the largest integer k such that there are numbers $0 \le t_1 < t_2 < t_3 < \ldots < t_{2k-1} < t_{2k} \le T$ satisfying $x(t_{2i-1}) < a$, $x(t_{2i}) > b$, for each $i = 1, 2, \ldots, k$.

4. For $\eta > 0$, $N_\eta(x)$ is *the number of η-oscillations* of $x \in \mathbb{D}$ on $[0,T]$. This means that $N_\eta(x)$ is the largest ineger k such that there are numbers $0 \le t_1 < t_2 \le t_3 \le \ldots \le t_{2k-1} < t_{2k} \le T$ satisfying $|x(t_{2i}) - x(t_{2i-1})| > \eta$, for each $i = 1, 2, \ldots, k$.

5. $\|v\|(T)$ is *the total variation of v on* $[0,T]$:

$$\|v\|(T) = \sup \big\{ |v(0)| + \sum_{i=1}^{m} |v(t_i) - v(t_{i-1})| \big\},$$

where the supremum is taken over all $0 = t_0 < t_1 < \ldots < t_m = T$, $m \in \mathbb{N}$.

6. $\mathbb{V} = \big\{ x \in \mathbb{D} \, ; \, \|x\|(T) < +\infty \big\}$.

The S topology is defined in terms of S-convergence.

Definition 2.1 (*S*-convergence)**.** *We shall write $x_n \longrightarrow_S x_0$ if for every $\varepsilon > 0$ one can find elements $v_{n,\varepsilon} \in \mathbb{V}$, $n = 0, 1, 2, \ldots$ which are ε-uniformly close to x_n's and weakly-$*$ convergent:*

$$\|x_n - v_{n,\varepsilon}\|_\infty \leq \varepsilon, \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

$$v_{n,\varepsilon} \Rightarrow v_{0,\varepsilon}, \qquad \text{as } n \to \infty. \tag{2.2}$$

Remark 2.2. Recall, that $v_{n,\varepsilon} \Rightarrow v_{0,\varepsilon}$ means that

$$\int_{[0,T]} f(t)\, dv_{n,\varepsilon}(t) \to \int_{[0,T]} f(t)\, dv_{0,\varepsilon}(t), \tag{2.3}$$

for each continuous function $f : [0, T] \to \mathbb{R}^1$. In particular, setting $f(t) \equiv 1$ we get

$$v_{n,\varepsilon}(T) \to v_{0,\varepsilon}(T). \tag{2.4}$$

Moreover, by the Banach-Steinhaus theorem, relation (2.2) implies

$$\sup_n \|v_{n,\varepsilon}\|(T) < +\infty. \tag{2.5}$$

For the sake of brevity of formulation of the next theorem let us list some conditions describing properties of a subset $K \subset \mathbb{D}$.

$$\sup_{x \in K} \|x\|_\infty < +\infty. \tag{2.6}$$

$$\sup_{x \in K} N^{a,b}(x) < +\infty, \quad \text{for all } a < b. \tag{2.7}$$

$$\sup_{x \in K} N_\eta(x) < +\infty, \quad \text{for every } \eta > 0. \tag{2.8}$$

Theorem 2.3. *(Criterion of relative S-compactness) Let $K \subset \mathbb{D}$. We can find in every sequence $\{x_n\}$ of elements of K a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow_S x_0$, as $k \to \infty$, if, and only if, one of the following equivalent statements is satisfied.*

(i) *Conditions 2.6 and 2.7 hold.*

(ii) *Conditions 2.6 and 2.8 hold.*

(iii) *For every $\varepsilon > 0$ and every $x \in K$ there exists $v_{x,\varepsilon} \in \mathbb{V}$ such that*

$$\sup_{x \in K} \|x - v_{x,\varepsilon}\|_\infty \leq \varepsilon, \tag{2.9}$$

and

$$\sup_{x \in K} \|v_{x,\varepsilon}\| < +\infty. \tag{2.10}$$

Definition 2.4 (The S topology)**.** *The S topology is the sequential topology on $\mathbb{D}$ generated by the $\mathcal{L}$-convergence $\longrightarrow_S$.*

Remark 2.5. " $\longrightarrow_S$ " is not an $\mathcal{L}^*$-convergence. In the next section we shall give a closed form of " $\overset{*}{\longrightarrow}_S$ ", obtained from $\longrightarrow_S$ by the KVPK recipe (Theorem 6.3).

3 Closed form definition of $\overset{*}{\longrightarrow}_S$

Definition 3.1. *Let $\mathbb{A} = \mathbb{A}\big([0, T]\big)$ be the family of continuous functions of finite variation ($\mathbb{A} \subset C([0, T]) \cap \mathbb{V}$), satisfying $A(0) = 0$.*

Let $A_n \in \mathbb{A}, n = 0, 1, 2, \ldots$. We will say that $A_n \longrightarrow_\tau A_0$, if

$$\sup_{t \in [0,T]} |A_n(t) - A_0(t)| \to 0, \tag{3.1}$$

and

$$\sup_n \|A_n\|(T) < +\infty. \tag{3.2}$$

Remark 3.2. This is "the mixed topology" on $C([0,T]) \cap \mathbb{V}$.

Theorem 3.3. $x_n \xrightarrow{*}_S x_0$ *if, and only if, $x_n(T) \to x_0(T)$ and*

$$\int_0^T x_n(t) \, dA_n(t) \to \int_0^T x_0(t) \, dA_0(t), \tag{3.3}$$

for each sequence $\{A_n\} \subset \mathbb{A}$ that satisfies $A_n \longrightarrow_\tau A_0$.

Proof. Suppose that $x_n \xrightarrow{*}_S x_0$. Then $x_n(T) \to x_0(T)$ is a consequence of (2.4) and property (2.1) of functions $\{v_{n,\varepsilon}\}$. In order to prove (3.3) one could observe that this convergence is a very particular (deterministic) case of Theorem 1 (or Theorem 5) in [11]. But it is more instructive to give here a direct proof.

So assume that $A_n \longrightarrow_\tau A_0$, choose $\varepsilon > 0$ and let $\{v_{n,\varepsilon}\}_{n=0,1,2,\ldots} \subset \mathbb{V}$ satisfy (2.1) and (2.2). For $n = 0, 1, 2, \ldots$ we have

$$\left| \int_0^T x_n(t) \, dA_n(t) - \int_{[0,T]} v_{n,\varepsilon}(t) \, dA_n(t) \right| \le \varepsilon \sup_n \|A_n\|(T),$$

and therefore it is enough to show that

$$\int_{[0,T]} v_{n,\varepsilon}(t) \, dA_n(t) \to \int_{[0,T]} v_{0,\varepsilon}(t) \, dA_0(t). \tag{3.4}$$

By the integration by parts formula, the continuity of A_n and $A_n(0) = 0$, we obtain that for $n = 0, 1, 2, \ldots$

$$\int_{[0,T]} v_{n,\varepsilon}(t) \, dA_n(t) = v_{n,\varepsilon}(T) A_n(T) - \int_{[0,T]} A_n(t) \, dv_{n,\varepsilon}(t)$$

$$= v_{n,\varepsilon}(T) A_n(T) - \int_{[0,T]} A_0(t) \, dv_{n,\varepsilon}(t)$$

$$+ \int_{[0,T]} \big(A_0(t) - A_n(t) \big) \, dv_{n,\varepsilon}(t)$$

$$= I_1(n) + I_2(n) + I_3(n).$$

By (2.4) and (3.1) $I_1(n) = v_{n,\varepsilon}(T) A_n(T) \to v_{0,\varepsilon}(T) A_0(T) = I_1(0)$. By (2.3) $I_2(n) = \int_{[0,T]} A_0(t) \, dv_{n,\varepsilon}(t) \to \int_{[0,T]} A_0(t) \, dv_{0,\varepsilon}(t) = I_2(0)$. Finally

$$|I_3(n)| = \left| \int_{[0,T]} \big(A_0(t) - A_n(t) \big) \, dv_{n,\varepsilon}(t) \right| \le \sup_{t \in [0,T]} \big| A_0(t) - A_n(t) \big| \sup_n \|v_{n,\varepsilon}\|(T) \to 0$$

by (3.1) and (2.5). Hence (3.4) holds.

Now let us assume that $x_n(T) \to x_0(T)$ and (3.3) holds for every sequence $\{A_n\} \subset \mathbb{A}$, $A_n \longrightarrow_\tau A_0$. We claim that it is enough to establish relative S-compactness of $\{x_n\}$. Indeed, then in every subsequence $\{n'\}$ we can find a further subsequence $\{n''\}$ such that $x_{n''} \longrightarrow_S y_0$, for some $y_0 \in \mathbb{D}$. Take a function $f \in L^1([0,T])$ and define $A_f(t) = \int_0^t f(u) \, du$. Then $A_f \in \mathbb{A}$ and by the first part of the proof,

$$\int_0^T x_{n''}(u) \, dA_f(u) \to \int_0^T y_0(u) \, dA_f(u) = \int_0^T x_0(u) \, dA_f(u).$$

Hence for each integrable f we have

$$\int_0^T y_0(u) f(u) \, du = \int_0^T y_0(u) \, dA_f(u) = \int_0^T x_0(u) \, dA_f(u) = \int_0^T x_0(u) f(u) \, du,$$

and, consequently, $y_0 = x_0$ almost everywhere. Since they are càdlàg functions, $y_0 = x_0$ on $[0, T)$. And $y_0(T) = x_0(T)$ holds by our assumption.

So far we have proved that in every subsequence $\{n'\}$ we can find a further subsequence $\{n''\}$ along which $x_{n''} \longrightarrow_S x_0$. Hence, by the KVPK recipe, $x_n \overset{*}{\longrightarrow}_S x_0$.

In order to prove the relative S-compactness of $\{x_n\}$ it is necessary to adjust integrands A_n in a way suitable for the particular functional determining the relative S-compactness via Theorem 2.3.

First let us consider condition (2.6). Suppose that $\sup_n \|x_n\|_\infty = +\infty$. Then there exists a subsequence n_k and numbers $t_k \in [0, T)$ such that $a_k = |x_{n_k}(t_k)| \to \infty$. Without loss of generality we may assume that $a_k = x_{n_k}(t_k)$ and $t_k < T$. By the right-continuity of x_{n_k} we can find numbers h_k such that $t_k + h_k < T$ and

$$x_{n_k}(t) \geq (1/2) a_k, \quad \text{for } t \in [t_k, t_k + h_k].$$

Let $b_k = \sqrt{a_k} h_k$. Consider function $f_k(u) = (1/b_k) \mathbb{1}_{[t_k, t_k + h_k]}(u)$ and the corresponding function $A_{f_k} \in \mathbb{A}$. We have

$$\begin{aligned}
\int_0^T x_{n_k}(u) \, dA_{f_k}(u) &= \int_0^T x_{n_k}(u) f_k(u) \, du \\
&= \frac{1}{b_k} \int_{t_k}^{t_k + h_k} x_{n_k}(u) \, du \geq \frac{a_k h_k}{2 b_k} = (1/2) \sqrt{a_k} \to +\infty,
\end{aligned}$$

while

$$\|A_{f_k}\|_\infty = \|A_{f_k}\|(T) = \int_0^T f_k(u) \, du = h_k / b_k = 1/\sqrt{a_k} \to 0.$$

It follows that (3.3) cannot be satisfied.

In order to cope efficiently with condition (2.7) we need the following lemma.

Lemma 3.4. *Let $x \in \mathbb{D}$ be such that $N = N^{a,b}(x) \geq 2$ for some $a < b$. Then there exists $A \in \mathbb{A}$ such that:*

$$\int_0^T x(t) \, dA(t) \geq (b - a). \tag{3.5}$$

$$\|A\|(T) = 2. \tag{3.6}$$

$$\|A\|_\infty = 1/(N - 1). \tag{3.7}$$

Proof. Let

$$0 \leq t_1 < t_2 < t_3 < t_4 < \ldots < t_{2N-1} < t_{2N} \leq T$$

be such that $x(t_{2i-1}) < a$, $x(t_{2i}) > b$, $i = 1, 2, \ldots, N$. By the right continuity of x, for each $= 1, 2 \ldots, N - 1$ there are numbers $\delta_i > 0$ such that $t_{2i-1} + \delta_i < t_{2i} < t_{2i} + \delta_i < t_{2i+1}$ and

$$\sup_{t \in [t_{2i-1}, t_{2i-1} + \delta_i]} x(t) \leq a, \quad \inf_{t \in [t_{2i}, t_{2i} + \delta_i]} x(t) \geq b.$$

Let $h_i = 1/\big(\delta_i (N - 1)\big)$ and define

$$f(t) = \sum_{i=1}^{N-1} (-1) h_i \mathbb{1}_{[t_{2i-1}, t_{2i-1} + \delta_i]}(t) + h_i \mathbb{1}_{[t_{2i}, t_{2i} + \delta_i]}(t).$$

Then we have for $A = A_f$

$$
\int_0^T x(t)\, dA_f(t) = \int_0^T x(t) f(t)\, dt
$$
$$
= \sum_{i=1}^{N-1} (-1) h_i \int_{t_{2i-1}}^{t_{2i-1}+\delta_i} x(t)\, dt + h_i \int_{t_{2i}}^{t_{2i}+\delta_i} x(t)\, dt
$$
$$
\geq \sum_{i=1}^{N-1} h_i \delta_i \big(b - a\big) = \big(b - a\big).
$$

Similarly

$$
\|A_f\|(T) = \int_0^T |f(t)|\, dt = \sum_{i=1}^{N-1} 2 h_i \cdot \delta_i = 2.
$$
$$
\|A_f\|_\infty = \sup_{t \in [0,T]} |A_f(t)| = \max_{i=1,2\ldots,N-1} h_i \cdot \delta_i = 1/(N-1). \qquad \square
$$

Now suppose that condition (2.7) does not hold for $K = \{x_n\}$. This means that for some $a < b$ and along some subsequence $\{n'\}$

$$
N^{a,b}(x_{n'}) \to \infty.
$$

Without loss of generality we may assume that $N^{a,b}(x_{n'}) \geq 2$. For each n', let $A_{n'} = A_{f_{n'}}$ be given by Lemma 3.4. Then $A_{n'} \longrightarrow_\tau A_0 = 0$, while $\int_0^T x_{n'}(t)\, dA_{n'}(t) \geq b - a > 0$ cannot converge to $\int_0^T x_0(t)\, dA_0(t) = 0$. This contradicts (3.3). $\qquad \square$

Corollary 3.5. $x_n \xrightarrow{\;*\;}_S x_0$ *if, and only if,* $x_n(T) \to x_0(T)$ *and for each relatively τ-compact set* $\mathcal{A} \subset \mathbb{A}$

$$
\sup_{A \in \mathcal{A}} \Big| \int_0^T \big(x_n(u) - x_0(u)\big)\, dA(u) \Big| \to 0. \tag{3.8}
$$

Proof. Clearly, (3.8) implies (3.3). To prove the converse, assume (3.3) and suppose that (3.8) does not hold for some relatively τ-compact set $\mathcal{A} \subset \mathbb{A}$. This means that for some $\eta > 0$ there exists a subsequence $\{n'\}$ and elements $A_{n'}$ of $\mathcal{A}$ such that for all n'

$$
\Big| \int_0^T \big(x_{n'}(u) - x_0(u)\big)\, dA_{n'}(u) \Big| > \eta.
$$

Passing to a τ-convergent subsequence $A_{n''}$ we obtain a contradiction with (3.3). $\qquad \square$

Remark 3.6. Denote by Σ the locally convex topology on $\mathbb{D}$ given by the seminorm $\rho_1(x) = |x(1)|$ and the seminorms

$$
\rho_{\mathcal{A}}(x) = \sup_{A \in \mathcal{A}} \Big| \int_0^1 x(u)\, dA(u) \Big|,
$$

where $\mathcal{A}$ runs over relatively τ-compact subsets of $\mathbb{A}$.

Then $x_n \xrightarrow{\;*\;}_S x_0$ if, and only if, $x_n \longrightarrow_\Sigma x_0$ and so $S \supset \Sigma$, for S is sequential.

Question 3.7. Is it true that $S \equiv \Sigma$? Positive answer would allow stating that $(\mathbb{D}, S)$ is a locally convex linear topological space.

Remark 3.8. Even if $S \supsetneq \Sigma$, the compact sets are the same in both topologies, as well as classes of sequentially lower-semicontinuous functions. Exploring S-compactness and S-lower-semicontinuity Guo et al. [10] obtained interesting results on martingale optimal transport on the Skorokhod space. It seems that in problems of such type the local convexity related to Σ can be a useful tool as well.

4 S in the hierarchy of topologies

Let us begin with listing some facts on topologies on $\mathbb{D}$.

1. $\mathbb{D}$ with norm $\|\cdot\|_\infty$ is a Banach space, but non-separable.

2. The Skorokhod J_1 topology is metric separable and $(\mathbb{D}, J_1)$ is topologically complete. For definition and properties of J_1 we refer to Billingsley's classic book [6] rather than to its second edition.

3. It is easy to show that $x_n \longrightarrow_{J_1} x_0$ implies $x_n \longrightarrow_S x_0$, hence the S topology is coarser than J_1.

4. It was shown in [4] that the S topology is coarser than Skorokhod's M_1 topology (see [33] for definitions of four Skorokhod's topologies).

5. S is incomparable with Skorokhod's M_2 topology!

6. $(\mathbb{D}, J_1)$ *is not a linear topological space*, for addition is not sequentially J_1-continuous, as Figure 1 shows.

7. On the contrary, the sequence $\{f_n\}$ defined in Figure 1 is S-convergent to 0 and exhibits a typical for S phenomenon of self-cancelling oscillations. Addition is *sequentially continuous* in S!

8. We do not know whether addition is *continuos*, as a function on the product $\mathbb{D} \times \mathbb{D}$ with the *product topology* $S \times S$ (in general sequential continuity does not imply continuity). Therefore we do not know, whether $(\mathbb{D}, S)$ is a linear topological space.
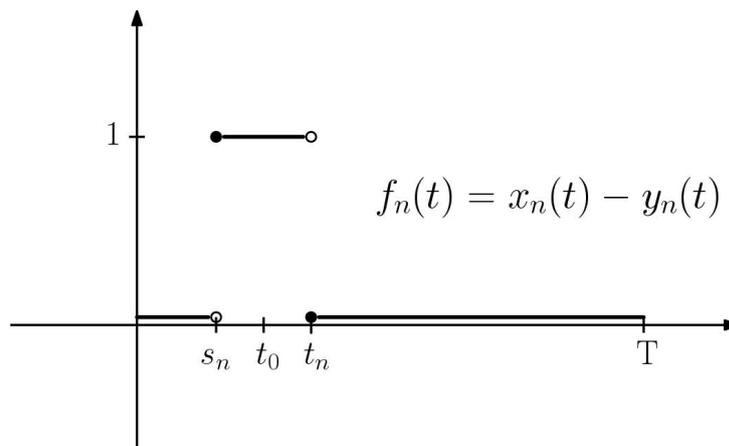

Figure 1: J_1 is not linear ($x_n(t) = \mathbb{1}_{[s_n,T]}(t)$, $y_n(t) = \mathbb{1}_{[t_n,T]}(t)$, $s_n \nearrow t_0, t_n \searrow t_0$)

Theorem 4.1. *Suppose σ is a topology on $\mathbb{D} = \mathbb{D}([0,T])$ which satisfies the following assumptions.*

$(\mathbb{D}, \sigma)$ *is a linear topological space* $\hspace{4cm}$ (4.1)

σ *is coarser than the uniform topology generated by the norm $\|\cdot\|_\infty$.* $\hspace{0.5cm}$ (4.2)

For each $A > 0$ the set $\{a\mathbb{1}_{[u,T]} ; |a| \le A, u \in [0,T]\}$ is relatively σ-compact. $\hspace{0.3cm}$ (4.3)

Then σ is coarser than the S topology.

Proof. We claim that it is enough to prove that $x_n \longrightarrow_S x_0$ implies $x_{n'} \longrightarrow_\sigma x_0$ along some subsequence $\{n'\}$. Indeed, this implies that any σ_s-closed set is also S-closed (for

S is sequential), and so σ_s is coarser than S. Since we always have $\sigma \subset \sigma_s$, our claim follows.

So let us assume that $x_n \longrightarrow_S x_0$. For $\varepsilon > 0$ and $x \in \mathbb{D}$ let us define

$$\tau_0^\varepsilon(x) = 0$$
$$\tau_k^\varepsilon(x) = \inf\{t > \tau_{k-1}^\varepsilon(x) : |x(t) - x(\tau_{k-1}^\varepsilon(x))| > \varepsilon\}, \ k = 1, 2, \dots.$$

(where by convention $\inf \emptyset = +\infty$) and let

$$v_\varepsilon(x)(t) = x(\tau_k^\varepsilon(x)) \ \text{ if } \tau_k^\varepsilon(x) \le t < \tau_{k+1}^\varepsilon(x), \ t \in [0, T], k = 0, 1, 2, \dots.$$

Then by the very definition $\|x - v_\varepsilon(x)\|_\infty \le \varepsilon$. Similarly, if we set

$$M^\varepsilon(x) = \max\{k \,;\, \tau_k^\varepsilon(x) \le T\},$$

then $M^\varepsilon(x) \le N_{\varepsilon/2}(x)$ and by Theorem 2.3 we have

$$\sup_n \|v_\varepsilon(x_n)\|_\infty \le \varepsilon + \sup_n \|x_n\|_\infty =: A_\varepsilon < +\infty, \tag{4.4}$$

$$\sup_n M^\varepsilon(x_n) \le \sup_n N_{\varepsilon/2}(x_n) =: M_\epsilon < +\infty. \tag{4.5}$$

Since $v_\varepsilon(x)$ varies only through $M^\varepsilon(x)$ jumps, it can be represented as a sum of $M^\varepsilon(x) + 1$ terms:

$$v_\varepsilon(x) = \sum_{k=0}^{M^\varepsilon(x)} z_k(x) \mathbb{1}_{[\tau_k^\varepsilon(x), T]}, \tag{4.6}$$

where $z_0(x) = 0$ and

$$z_k(x) = x\big(\tau_k^\varepsilon(x)\big) - \sum_{j=0}^{k-1} x\big(\tau_j^\varepsilon(x)\big) \quad k = 1, 2, \dots, M^\varepsilon(x).$$

By (4.4) and (4.5) we obtain

$$\sup_n \max_k |z_k(x_n)| \le (M_\varepsilon + 1) A_\varepsilon.$$

It follows that the sequence $\{v_\varepsilon(x_n)\}$ lives in the algebraic sum

$$\widetilde{K}_\varepsilon = \underbrace{K_\varepsilon + K_\varepsilon + \dots + K_\varepsilon}_{M_\varepsilon + 1 \text{ times}},$$

where $K_\varepsilon = \{a\mathbb{1}_{[u, T]} \,;\, |a| \le (M_\varepsilon + 1) A_\varepsilon, u \in [0, T]\}$ is relatively σ-compact. Since $(\mathbb{D}, \sigma)$ is linear $\widetilde{K}_\varepsilon$ is relatively σ-compact as well. This means that in every subsequence $\{n'\}$ one can find a further subsequence $\{n''\}$ such that $v_\varepsilon(x_{n''}) \longrightarrow_\sigma v_\varepsilon$, for some $v_\varepsilon \in \mathbb{D}$. But we can say more: by the special form (4.6) of elements $v_\varepsilon(x_n)$ (bounded number of jumps with bounded amplitudes) we may extract a further subsequence $\{n'''\}$ such that $v_\varepsilon(x_{n'''}) \Rightarrow v_\varepsilon$.

Now choose $\varepsilon_m \searrow 0$ and apply the diagonal procedure to extract a subsequence n' such that for each $m \in \mathbb{N}$ we have along $\{n'\}$

$$v_{\varepsilon_m}(x_{n'}) \underset{\sigma}{\longrightarrow} v_m, \qquad v_{\varepsilon_m}(x_{n'}) \Rightarrow v_m$$

for some $v_m = v_{\varepsilon_m} \in \mathbb{D}$. Notice that $v_{\varepsilon_m}(x_{n'}) \Rightarrow v_m$ implies $v_{\varepsilon_m}(x_{n'}) \longrightarrow_S v_m$. By Corollary 2.10 in [13]

$$\|v_m - x_0\|_\infty \le \liminf_{n'} \|v_{\varepsilon_m}(x_{n'}) - x_{n'}\|_\infty \le \varepsilon_m \to 0. \tag{4.7}$$

For later purposes we may write $v_m - x_0 \in B_{\varepsilon_m}$, where $B_r = \{x \in \mathbb{D} \,;\, \|x\|_\infty \leq r\}$ for $r > 0$.

Our final task consists in proving $x_{n'} \longrightarrow_\sigma x_0$. Let V be a σ-open neighborhood of x_0. By the linearity there exists a σ-open neighborhood W of 0 such that $W + W \subset V - x_0$. Since σ is coarser than the uniform topology, there exists $\delta > 0$ such that $B_{2\delta} \subset W$. Let m be such that $\varepsilon_m < \delta$. Then for n' large enough we have

$$x_{n'} = x_{n'} - v_{\varepsilon_m}(x_{n'}) + v_{\varepsilon_m}(x_{n'}) - v_m + v_m - x_0 + x_0$$
$$\in B_{\varepsilon_m} + W + B_{\varepsilon_m} + x_0 \subset W + W + x_0 \subset V. \qquad \square$$

Remark 4.2. Let us consider the space $L^p([0,T]) = L^p([0,T], \mathcal{L}|_{[0,T]}, \ell|_{[0,T]})$, $p \in [0, +\infty]$ of Lebesgue-measurable functions on $[0,T]$ (here ℓ stands for the Lebesgue measure). Of course, for each p we have $\mathbb{D}([0,T]) \subset L^p([0,T])$. Moreover, the induced metric converts $\mathbb{D}([0,T])$ into a normed space (if $p \in [1, +\infty)$) or a metric linear space (if $p \in [0,1]$). Clearly, assumptions (4.1) - (4.3) are satisfied and by our Theorem 4.1 all mentioned metric topologies are coarser than S.

Remark 4.3. If we replace ℓ with another *atomless* finite measure μ on $[0,T]$, then again the metric topologies induced by spaces $L^p([0,T], \mu)$, $p \in [0, +\infty)$, are coarser than the S topology.

This is not so, if we admit atoms for μ, for (4.3) is then violated.

Remark 4.4. In Introduction we suggested that the S topology is *almost* finer than any linear topology which is coarser than Skorokhod's J_1 topology. The delicate point is that condition (4.3) does not hold for J_1. The corresponding typical example is given in Figure 2, with parameters $t_n \to 0$. To overcome this difficulty we shall introduce a variant of the J_1 topology, called mJ_1 (m - for *modified*), which slightly weakens the original topology and for which condition (4.3) is satisfied.
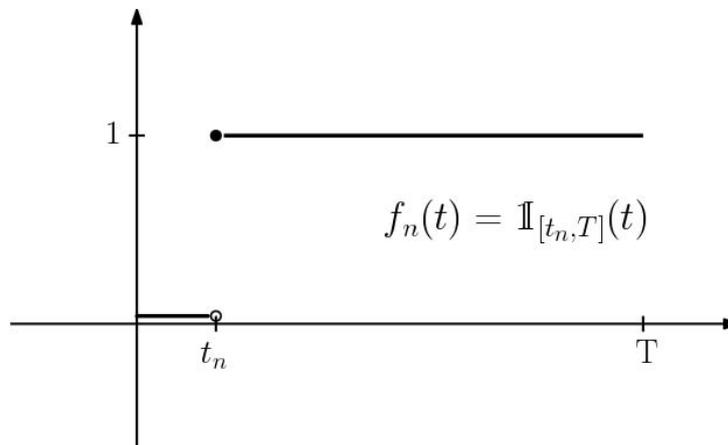

Figure 2: A sequence of single jumps that does not converge in J_1, but converges in mJ_1

Definition 4.5 (The mJ_1 topology). *Fix $\varepsilon > 0$ and consider a one-to-one embedding*

$$\mathbb{D}([0,T]) \ni x \mapsto \widetilde{x} \in \mathbb{D}([-\varepsilon, T + \varepsilon])$$

given by the formula

$$\widetilde{x}(t) = \begin{cases} 0 & \text{if } -\varepsilon \leq t < 0, \\ x(t) & \text{if } 0 \leq t < T, \\ x(T) & \text{if } T \leq t \leq T + \varepsilon. \end{cases} \qquad (4.8)$$

Take the complete metric d_{Sk} on $\mathbb{D}([-\varepsilon, T+\varepsilon])$ (see [6])) and define

$$d(x,y) = d_{Sk}(\widetilde{x}, \widetilde{y}).$$

Then $(\mathbb{D}, d)$ becomes a metric space and the corresponding topology will be called mJ_1.

Theorem 4.6 (Basic facts on mJ_1)**.**

(i) $(\mathbb{D}, d)$ *is a Polish (i.e. complete and separable) metric space.*

(ii) *A subset $K \subset \mathbb{D}$ is relatively mJ_1-compact iff it is uniformly bounded:*

$$\sup_{x \in K} \sup_{t \in [0,T]} |x(t)| < +\infty, \tag{4.9}$$

and

$$\lim_{\delta \to 0} \sup_{x \in K} \sup_{\substack{0- \leq s < t < u \leq T \\ u - s < \delta}} \min\{|x(t) - x(s)|, |x(u) - x(t)|\} = 0, \tag{4.10}$$

where we use the conventions $x(0-) = 0$ and $u - (0-) = u$.

(iii) *For each $A > 0$ the set $\{a\mathbb{1}_{[u,T]} \,;\, |a| \leq A, u \in [0,T]\}$ is relatively mJ_1-compact.*

Proof. It is easy to see that if $\widetilde{x}_n$ converges in $\big(\mathbb{D}([-\varepsilon, T+\varepsilon]), d_{Sk}\big)$ to some z, then the limit is of the form $\widetilde{x}_0$, for some $x_0 \in \mathbb{D}$. Hence $(\mathbb{D}, d)$ is homeomorphic to a closed subset of the Polish space $\big(\mathbb{D}([-\varepsilon, T+\varepsilon]), d_{Sk}\big)$, and so it is Polish itself.

Part (ii) is a specification of Theorem 14.4 in [6].

And part (iii) is a direct consequence of our definition (see Figure 2). $\quad\square$

Taking into account Theorems 4.1 and 4.6 we obtain the following interesting result, positioning the S topology in the hierarchy of topologies on $\mathbb{D}$.

Theorem 4.7 (Maximal character of the S topology)**.** *Every linear topology on $\mathbb{D}$, which is coarser than mJ_1, is coarser than the S topology as well.*

Remark 4.8. Were $\big(\mathbb{D}, S\big)$ a linear topological space, S would be the finest linear topology on $\mathbb{D}$ "below" mJ_1.

5 Extensions

5.1 Infinite time horizon

The problem consists in defining an analog of the S topology on the Skorokhod space $\mathbb{D}\big([0, +\infty)\big)$ of functions $x : \mathbb{R}^+ \to \mathbb{R}^1$, which are right-continuous at every $t \geq 0$ and admit left limits at every $t > 0$. This cannot be achieved by invoking consistency, because the natural projections of $\big(\mathbb{D}([0, T_2]), S\big)$ onto $\big(\mathbb{D}([0, T_1]), S\big)$, $0 < T_1 < T_2$, are not continuous, due to the special role of the end point $T_1 \in (0, T_2)$.

A similar phenomenon was encountered long time ago for Skorokhod's J_1 topology (see [26] and [37] for the ways to overcome this difficulty). The case of the S topology can be handled in a somewhat simpler manner, mainly due to the characterization of $\overset{*}{\longrightarrow}_S$ on $\mathbb{D}$ given in Theorem 3.3 and the fact that we are interested in convergence of sequences only and not in a particular form of a metric.

Definition 5.1. Let $x_n \in \mathbb{D}\big([0, +\infty)\big)$, $n = 0, 1, 2, \ldots$. We will say that $x_n \overset{*}{\longrightarrow}_S x_0$ in $\mathbb{D}\big([0, +\infty)\big)$, if for every $T > 0$

$$\int_0^T x_n(t)\, dA_n(t) \to \int_0^T x_0(t)\, dA_0(t), \tag{5.1}$$

for all sequences $\{A_n\} \subset \mathbb{A}\big([0,T]\big)$ such that $A_n \longrightarrow_\tau A_0$.

The S topology on $\mathbb{D}\big([0,+\infty)\big)$ is the sequential topology generated by the $\mathcal{L}^*$-convergence $x_n \stackrel{*}{\longrightarrow}_S x_0$.

If $x \in \mathbb{D}\big([0,+\infty)\big)$ and $T > 0$, it will be convenient to denote by $x^T \in \mathbb{D}\big([0,T]\big)$ the restriction of x to $[0,T]$:

$$x^T(t) = x(t), \ t \in [0,T].$$

We have the following analog of Theorem 2.3.

Theorem 5.2. *Let $K \subset \mathbb{D}\big([0,+\infty)\big)$.*

We can find in every sequence $\{x_n\}$ of elements of K a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \stackrel{*}{\longrightarrow}_S x_0$, as $k \to \infty$, if, and only if, one of the following equivalent statements (i) *and* (ii) *is satisfied.*

(i)

$$\sup_{x \in K} \|x^T\|_\infty < +\infty, \ \text{for every } T > 0.$$

$$\sup_{x \in K} N^{a,b}(x^T) < +\infty, \ \text{for all } T > 0 \text{ and } a < b.$$

(ii)

$$\sup_{x \in K} \|x^T\|_\infty < +\infty, \ \text{for every } T > 0.$$

$$\sup_{x \in K} N_\eta(x^T) < +\infty, \ \text{for all } T > 0 \text{ and } \eta > 0.$$

Proof. The equivalence of (i) and (ii) is stated in Theorem 2.3, so it is enough to deal with (i) only.

Necessity. Suppose that for some $T > 0$ and along a sequence $\{x_n\} \subset K$ we have either $\lim_{n\to\infty} \sup_{t\in[0,T]} |x_n(t)| = +\infty$ or $\lim_{n\to\infty} N^{a,b}\big(x_n^T\big) = +\infty$ for some $a < b$. Choose $T' > T$. The sequence $\{x_n\}$ contains a subsequence $x_{n_k} \stackrel{*}{\longrightarrow}_S x_0$ and so

$$\int_0^{T'} x_{n_k}(t)\,dA_{n_k}(t) \to \int_0^{T'} x_0(t)\,dA_0(t), \ \ k \to +\infty,$$

for all sequences $\{A_{n_k}\} \subset \mathbb{A}\big([0,T']\big)$ such that $A_{n_k} \longrightarrow_\tau A_0$ in $\mathbb{A}\big([0,T']\big)$. By inspection of the proof of Theorem 3.3 we see that this implies both

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} |x_{n_k}(t)| < +\infty,$$

and

$$\limsup_{k\to\infty} N_{a,b}(x_{n_k}^T) < +\infty.$$

We have arrived to a contradiction.

Sufficiency. Take any sequence $T_r \nearrow +\infty$ and assume (i). By Theorem 2.3 we can find a sequence $\{x_{1,n}\}$ such that $x_{1,n}^{T_1} \longrightarrow_S x_{1,0}$ in $\mathbb{D}\big([0,T_1]\big)$. In $\{x_{1,n}\}$ we can find a subsequence $\{x_{2,n}\}$ such that $x_{2,n}^{T_2} \longrightarrow_S x_{2,0}$ in $\mathbb{D}\big([0,T_2]\big)$. Repeating this process and then applying the diagonal procedure we can find a subsequence $\{x_{n'}\} \subset K$ such that for every $r \in \mathbb{N}$

$$x_{n'}^{T_r} \underset{S}{\longrightarrow} x_{r,0}, \ \text{in } \mathbb{D}\big([0,T_r]\big).$$

We claim that there exists exactly one $x_0 \in \mathbb{D}\big([0,+\infty)\big)$ such that for each $r \in \mathbb{R}$

$$x_0^{T_r}(t) = x_{r,0}(t), \ \ t \in [0,T_r).$$

Let $q < r$. It is enough to verify the consistency of $x_{q,0}$ and $x_{r,0}$ on $[0, T_q)$. By Corollary 2.9 in [13] we can find a further subsequence $\{n''\}$ as well as countable subsets $D_q \subset [0, T_q)$ and $D_r \subset [0, T_r)$ such that

$$x_{n''}(t) \to x_{q,0}(t), \ t \notin D_q, \quad x_{n''}(t) \to x_{r,0}(t), \ t \notin D_r.$$

It follows that if t belongs to the set $[0, T_q) \setminus (D_q \cup D_r)$ that is dense in $[0, T_q)$, then

$$x_{n''}(t) \to x_{q,0}(t) = x_{r,0}(t).$$

Because both $x_{q,0}$ and $x_{r,0}^{T_q}$ are càdlàg, they are equal on $[0, T_q)$.

It remains to show that (5.1) holds for $x_{n'}$ and x_0. Let $T > 0$. Take $T_r \geq T$. By Theorem 3.3

$$\int_0^{T_r} x_{n'}(t) \, dA_{n'}(t) \to \int_0^{T_r} x_{r,0}(t) \, dA_0(t),$$

for all sequences $\{A_{n'}\} \subset \mathbb{A}([0, T_r])$ such that $A_{n'} \longrightarrow_\tau A_0$. Notice that in view of the continuity of A_0, the value of $x_{r,0}$ at $t = T_r$ does not contribute to the value of the integral and therefore the limit integral can be written as $\int_0^{T_r} x_0(t) \, dA_0(t)$. We have thus established (5.1) for $T = T_r$. So let $T < T_r$ and consider a sequence $\{A_{n'}\} \subset \mathbb{A}([0, T])$ such that $A_{n'} \longrightarrow_\tau A_0$. Taking a natural extension

$$\widetilde{A}_{n'}(t) = \begin{cases} A_{n'}(t), & \text{if } t \in [0, T); \\ A_{n'}(T), & \text{if } t \in [T, T_r]; \end{cases}$$

we see that

$$\int_0^T x_{n'}(t) \, dA_{n'}(t) = \int_0^{T_r} x_{n'}(t) \, d\widetilde{A}_{n'}(t),$$

hence the general case is implied by the one already proved. $\qquad\square$

Remark 5.3. The idea to keep the star over the arrow in the above definition of the convergence generating the S topology on $\mathbb{D}([0, +\infty))$ is justified by the fact that also in the infinite time horizon there is a notion corresponding to the $\mathcal{L}$-convergence $\longrightarrow_S$.

Theorem 5.4. $x_n \xrightarrow{\ *\ }_S x_0$ in $\mathbb{D}([0, +\infty))$ if, and only if, in each subsequence $\{x_{n_k}\}$ one can find a further subsequence $\{x_{n_{k_l}}\}$ and a sequence $T_r \nearrow +\infty$ such that in each $\mathbb{D}([0, T_r])$

$$x_{n_{k_l}}^{T_r} \xrightarrow[S]{} x_0^{T_r}, \ \text{as } l \to \infty. \tag{5.2}$$

Proof. Suppose we have the property described by (5.2). $\xrightarrow{\ *\ }_S$ is an $\mathcal{L}^*$-convergence, so it is sufficient to prove that (5.2) implies relation (5.1). But this is done in the final part of the proof of Theorem 5.2.

It remains to show that if $x_n \xrightarrow{\ *\ }_S x_0$, then we can strengthen the construction given in the sufficiency part of the proof of Theorem 5.2 in such a way that

$$x_{r,0}(T_r) = x_0(T_r), \ \ r \in \mathbb{R}.$$

Let us repeat that construction for some $T_r' \nearrow +\infty$ and find a subsequence n' such that

$$x_{n'}^{T_r'} \xrightarrow[S]{} x_{r,0}, \ \text{in } \mathbb{D}([0, T_r']). \ r \in \mathbb{N}.$$

Because (5.1) identifies the limit almost everywhere (as was shown in the proof of Theorem 3.3), we see that $x_{r,0}(t) = x_0(t), \ t \in [0, T_r'), \ \ r \in \mathbb{N}$. Similarly as before, passing to a further subsequence $\{n''\}$ we have for some countable set $D \subset \mathbb{R}^+$

$$x_{n''}(t) \to x_0(t), \ \ t \notin D.$$

Take any $T_r \in (T'_{r-1}, T'_r] \setminus D$. Then $\{x_{n''}^{T_r}\}$ is relatively S-compact in $\mathbb{D}\big([0, T_r]\big)$ and $x_{n''}(t) \to x_0(t)$ on a dense set containing T_r. This implies $x_{n''} \longrightarrow_S x_0$ in $\mathbb{D}\big([0, T_r]\big)$ (for if not, we would be able to show that $\{x_{n''}^{T_r}\}$ *is not* relatively S-compact in $\mathbb{D}\big([0, T_r]\big)$, just as in the proof of Proposition 2.14 in [13]). $\qquad\square$

Definition 5.5. *Let* $x_n \in \mathbb{D}\big([0, +\infty)\big)$, $n = 0, 1, 2, \ldots$. *We will say that* $x_n \longrightarrow_S x_0$ *in* $\mathbb{D}\big([0, +\infty)\big)$, *if one can find a sequence* $T_r \nearrow +\infty$ *such that for every* $r \in \mathbb{N}$

$$x_n^{T_r} \underset{S}{\longrightarrow} x_0^{T_r} \ \text{in } \mathbb{D}\big([0, T_r]\big), \ \text{as } n \to \infty.$$

5.2 Functions with values in $\mathbb{R}^d$

Let $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ be a family of functions $\mathbf{x} : \mathbb{R}^+ \to \mathbb{R}^d$, which are right-continuous at every $t \geq 0$ and admit left limits at every $t > 0$. Taking coordinates, we may identify $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ with the product space $\Big(\mathbb{D}\big([0, +\infty)\big)\Big)^d$. If we equip each space $\mathbb{D}\big([0, +\infty)\big)$ with the sequential S topology, then the natural sequential topology on the product is given by the convergence in the components. In other words, we have

Definition 5.6. *Let* $\mathbf{x}_n \in \mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$, $n = 0, 1, 2, \ldots$, *where*

$$\mathbf{x}_n(t) = \big(\mathbf{x}_n^1(t), \mathbf{x}_n^2(t), \ldots, \mathbf{x}_n^d(t)\big).$$

Then we will say that $\mathbf{x}_n \overset{*}{\longrightarrow}_S \mathbf{x}_0$ *if*

$$\mathbf{x}_n^i \underset{S}{\overset{*}{\longrightarrow}} \mathbf{x}_n^i \ \text{in } \mathbb{D}\big([0, +\infty)\big) \ \text{for each } i = 1, 2, \ldots, d.$$

In a similar way we define convergence $\longrightarrow_S$ *in* $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ *and convergences* $\overset{*}{\longrightarrow}_S$ *and* $\longrightarrow_S$ *in* $\mathbb{D}\big([0, T] : \mathbb{R}^d\big)$.

The S topology on $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ *and on* $\mathbb{D}\big([0, T] : \mathbb{R}^d\big)$ *is the sequential topology generated by* $\overset{*}{\longrightarrow}_S$ *considered in the corresponding space.*

Given this natural definition of the S topology, the criteria of relative S-compactness in the multidimensional setting are obvious.

Theorem 5.7. *Let* $\mathbf{K} \subset \mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ *and let* $\mathbf{K}^i = \{\mathbf{x}^i; \mathbf{x} \in \mathbf{K}\} \subset \mathbb{D}\big([0, +\infty)\big)$.

Then $\mathbf{K}$ *is relatively S-compact in* $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ *if, and only if, each set* $\mathbf{K}^i$, $i = 1, 2, \ldots, d$, *is relatively compact in* $\mathbb{D}\big([0, +\infty)\big)$.

The same equivalence holds, if we replace $\mathbb{D}\big([0, +\infty) : \mathbb{R}^d\big)$ *with* $\mathbb{D}\big([0, T] : \mathbb{R}^d\big)$ *and* $\mathbb{D}\big([0, +\infty)\big)$ *with* $\mathbb{D}\big([0, T]\big)$.

6 Appendix: Sequential topologies generated by $\mathcal{L}$-convergences

Following Fréchet, we say that $\mathcal{X}$ is a space of type $\mathcal{L}$, if among all sequences of elements of $\mathcal{X}$ a class $\mathcal{C}(\to)$ of "convergent" sequences is distinguished in such a way that:

(i) To each convergent sequence (x_n) exactly one point x_0, called "the limit", is attached (symbolically: $x_n \longrightarrow x_0$)

(ii) For every $x \in \mathcal{X}$, the constant sequence $(x, x, \ldots)$ is convergent to x.

(iii) If $x_n \longrightarrow x_0$ and $1 \leq n_1 < n_2 < \ldots$, then the subsequence (x_{n_k}) converges, and to the same limit: $x_{n_k} \longrightarrow x_0$, as $k \to \infty$.

Using the $\mathcal{L}$-convergence $\longrightarrow$ one creates the family of closed sets.

Definition 6.1. *Say that $F \subset \mathcal{X}$ is $\tau(\rightarrow)$-closed if limits of $\longrightarrow$-convergent sequences of elements of F remain in F, i.e. if $x_n \in F$, $n \in \mathbb{N}$ and $x_n \longrightarrow x_0$, then $x_0 \in F$. The topology given by $\tau(\rightarrow)$-closed sets is called the sequential topology generated by the $\mathcal{L}$-convergence $\longrightarrow$ and will be denoted by $\tau(\rightarrow)$.*

Remark 6.2. It must be stressed that for a sequential topology to be defined only the extremely simple properties (ii) and (iii) of convergence $\longrightarrow$ are required. On the other hand, the topology obtained this way has in general extremely poor separation properties. It is only T1 space due to the fact that in view of (ii) above each one-point set $\{x\}$ is $\tau(\rightarrow)$-closed.

But it is enough for the topology $\tau(\rightarrow)$ to define a new (in general) convergence, "$\longrightarrow_{\tau(\rightarrow)}$" say, which, after Urysohn, is called the convergence "a posteriori", in order to distinguish from the original convergence (= convergence "a priori", i.e. "$\longrightarrow$"). So (x_n) converges *a posteriori* to x_0, if for every $\tau(\rightarrow)$-open set U containing x_0 eventually all elements of the sequence (x_n) belong to U.

Kantorovich *et al* [17, Theorem 2.42, p.51] and Kisyński [18] gave a familiar characterization of the convergence *a posteriori* in terms of the convergence *a priori*.

Theorem 6.3 (KVPK recipe)**.** $\{x_n\}$ *converges to x_0 a posteriori if, and only if, each subsequence $\{x_{n_k}\}$ contains a further subsequence $\{x_{n_{k_l}}\}$ convergent to x_0 a priori.*

Remark 6.4. The convergence *a posteriori* is generated by a topology. Suppose an $\mathcal{L}$-convergence $\longrightarrow$ satisfies additionally

(iv) If every subsequence (x_{n_k}) of (x_n) contains a further subsequence $(x_{n_{k_l}})$ $\longrightarrow$-convergent to x_0, then the whole sequence (x_n) is $\longrightarrow$-convergent to x_0.

Then the convergence $\longrightarrow$ is called an $\mathcal{L}^*$-convergence. It is an immediate consequence of Theorem 6.3 that if we start with an $\mathcal{L}^*$-convergence then the convergences *a posteriori* and *a priori* coincide.

Remark 6.5. It follows that given an $\mathcal{L}$-convergence "$\longrightarrow$" we can *weaken* it to an $\mathcal{L}^*$-convergence "$\overset{*}{\longrightarrow}$" which is already the usual convergence of sequences in the topological space $(\mathcal{X}, \tau(\longrightarrow)) \equiv (\mathcal{X}, \tau(\overset{*}{\rightarrow}))$. At least two examples of such a procedure are commonly known.

Example 6.6. If "$\longrightarrow$" denotes the convergence "almost surely" of real random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then "$\overset{*}{\longrightarrow}$" is the convergence "in probability".

Example 6.7. Let $\mathcal{X} = \mathbb{R}^1$ and take a sequence $\varepsilon_n \searrow 0$. Say that $x_n \longrightarrow x_0$, if for each $n \in \mathbb{N}, |x_n - x_0| < \varepsilon_n$, i.e. x_n converges to x_0 at the given rate $\{\varepsilon_n\}$. Then "$\overset{*}{\longrightarrow}$" means the usual convergence of real numbers.

Remark 6.8. It is worth noting that a set $J \subset \mathcal{X}$ is relatively $\longrightarrow$-compact (i.e. in each sequence $\{x_n\} \subset J$ one can find a subsequence $\{n'\}$ such that $x_{n'} \longrightarrow x_0$, for some $x_0 \in \mathcal{X}$) iff it is relatively $\overset{*}{\longrightarrow}$-compact.

Remark 6.9. Let us notice that if $(\mathcal{X}, \tau)$ is a Hausdorff topological space, then

$$\tau \subset \tau_s \equiv \tau(\underset{\tau}{\longrightarrow})$$

and in general this inclusion may be strict (like in the case of the weak topology on an infinite dimensional Hilbert space).

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