

First passage percolation on the exponential of two-dimensional branching random walk*

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Abstract

We consider the branching random walk $\{\mathcal{R}_z^N : z \in V_N\}$ with Gaussian increments indexed over a two-dimensional box V_N of side length N , and we study the first passage percolation where each vertex is assigned weight $e^{\gamma \mathcal{R}_z^N}$ for $\gamma > 0$. We show that for $\gamma > 0$ sufficiently small but fixed, the expected FPP distance between the left and right boundaries is at most $O(N^{1-\gamma^2/10})$.

Keywords: first passage percolation (FPP); branching random walk (BRW); Gaussian free field (GFF); Liouville quantum gravity (LQG).

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1 Introduction

For $n \in \mathbb{N}$ and $N = 2^n$, let $V_N \subseteq \mathbb{Z}^2$ be a box of side length N whose lower left corner is located at the origin. Let \mathbb{BD}_j denote the collection of boxes of the form $([0, 2^j - 1] \cap \mathbb{Z})^2 + (i_1 2^j, i_2 2^j)$ where i_1, i_2 and j are nonnegative integers. For $z \in \mathbb{Z}^2$, let $\mathbb{BD}_j(z)$ be the (unique) box in \mathbb{BD}_j that contains v . Let $\{a_B\}_{k \geq 0, B \in \mathbb{BD}_k}$ be a collection of i.i.d. standard Gaussian variables. We define a *branching random walk* $\{\mathcal{R}_z^N : z \in V_N\}$ with Gaussian increments by

$$\mathcal{R}_z^N = \sum_{k=0}^{n-1} a_{\mathbb{BD}_k(z)}. \quad (1.1)$$

For any rectangle $V \subseteq \mathbb{Z}^2$, we denote by Π_{LR}^V the collection of all paths π in V (considered as a subgraph of \mathbb{Z}^2) that connect the left and right boundaries of V . We refer to such a path as a (left-right) *crossing* of V . For a fixed $\gamma > 0$ we define

$$D_{\gamma, LR}(V_N) = \min_{\pi \in \Pi_{LR}^{V_N}} \sum_{z \in \pi} e^{\gamma \mathcal{R}_z^N},$$

to be the first passage percolation (FPP) distance between the two boundaries of V_N where we assign each vertex z a weight of $e^{\gamma \mathcal{R}_z^N}$.

Theorem 1.1. *For a fixed γ_0 that is small, and all $\gamma < \gamma_0$, we have that*

$$\mathbb{E} D_{\gamma, LR}(V_N) = O(N^{1-\gamma^2/10}).$$

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Remark 1.2. Our proof in fact gives an upper bound on the expected weight for the geodesic crossing through a rectangle. As a result, one can show that the expected weight for the geodesic connecting two fixed vertices has exponent strictly less than 1, by constructing a sequence of ($\log N$ many) rectangles with geometrically growing size that connect these two vertices.

1.1 A brief discussion on motivation and background

In a broad context, our work is motivated by studying first passage percolation on random media with heavy correlation. The main contribution of our work is to demonstrate an instance that in a hierarchical random field the exponent of the FPP distance can be strictly less than 1 (despite the fact that the a straight line has weight with exponent strictly larger than 1, and that the majority of the vertices are of values $O(\sqrt{\log N})$). This is, of course, rather different from the classical FPP where the edge/vertex weights are independent and identically distributed. We refer to [2, 12] and references therein for a review of the classical first passage percolation.

In the specific instance of branching random walk, our work is closely related to the first passage percolation problem when the vertex weight is given by exponentiating a two-dimensional discrete Gaussian free field (GFF). Inspired by the current work, a better upper bound on the FPP distance in this case has been recently obtained in [7] (see also [6]). The FPP on two-dimensional discrete GFF, as we expect, is of fundamental importance in understanding the random metric associated with the Liouville quantum gravity (LQG) [17, 11, 18]. We remark that the random metric of LQG is a major open problem, even “just” to make rigorous sense of it. In a recent series of works of Miller and Sheffield, much understanding has been obtained (more on the continuum set up) in the special case of $\gamma = \sqrt{8/3}$; see [16, 15] and references therein. Another recent work [13] has provided some bounds on the scaling exponent for a type of LQG metric, though we emphasize that their definition is of very different flavor from the FPP perspective considered in our paper (in particular, no mathematical connection can be drawn between these works at the moment). We refrain ourselves from an extensive discussion on background of LQG in this article. As a final remark to the connection of LQG metric, we note that there are other candidate discrete metrics (associated with LQG or GFF) that have been proposed, and it is possible that these metrics with suitable normalization would give a more desirable scaling limit than that of FPP. However, we feel that in the level of precision of the present article and that of [7], it is quite possible that the fundamental mathematical structures (and thus obstacles) are common for these related notions of metrics.

BRW is perhaps the simplest construction that approximates a log-correlated Gaussian field, of which the two-dimensional GFF is a special instance. A number of other properties, especially those related to the extreme values of the field, have been proved to exhibit universal behavior among the log-correlated Gaussian fields (c.f., [14, 8]). Indeed, it is typical that properties were first proved for BRW, and then later for GFF; and it is also typical that the proof in the case of GFF were substantially enlightened by the understanding on BRW. However, caution is required when drawing heuristic conclusion on GFF based on BRW for FPP: as demonstrated in [9], there exists a family of log-correlated Gaussian fields whose exponent for FPP can be made arbitrarily close to 1 (as K grows to ∞) if one is allowed to perturb the covariance entry-wise by an additive amount that is at most K (and thus the scaling exponent for FPP is non-universal among log-correlated fields). That being said, we remark that while the mathematical details in [7] and [6] are substantially more involved, the proofs in these two papers share the same very basic framework of multi-scale analysis.

1.2 Main ideas of our approach

Our approach is inspired by the rescaling argument employed in the proof of connectivity for the fractal percolation process [3, 5, 4], the idea of which went back to [1]. For the purpose of induction, we will in fact work with crossings through a rectangle rather than a square. In what follows, we should consider γ as a small positive number (but fixed as $N \rightarrow \infty$).

We wish to carry out an inductive construction for a light path crossing the rectangle: we will take advantage of the hierarchical structure of BRW and will construct the BRW along the way as we construct light paths in different scales. In order to properly describe our procedure, we need a number of definitions. Let $\Gamma = \Gamma(\gamma) \in [\alpha/\gamma^2, \alpha/\gamma^2 + 2]$ be an odd, positive integer for some α to be selected, and let $V_N^\Gamma = ([0, \Gamma N - 1] \times [0, N - 1]) \cap \mathbb{Z}^2$. The reason for choosing Γ odd is mere technical convenience which will become apparent later. We will also work under the assumption that $\gamma \ll 1/\alpha \ll 1$ (small or large enough for our bounds or inequalities to hold although we keep this implicit in our discussions). We define a Gaussian process $\{\mathcal{R}_z^N : z \in V_N^\Gamma\}$ on V_N^Γ as follows:

- $\{\mathcal{R}_z^N : z \in V_{N;j}^\Gamma\}$ is a BRW on $V_{N;j}^\Gamma = ((j - 1)N, jN - 1] \times [0, N - 1]) \cap \mathbb{Z}^2$ where $j \in [\Gamma] := \{1, \dots, \Gamma\}$.
- $\{\mathcal{R}_z^N : z \in V_{N;j}^\Gamma\}$'s are independent of each other for $j \in [\Gamma]$.

So $\{\mathcal{R}_z^N : z \in V_N^\Gamma\}$ is basically a concatenation of Γ independent BRWs placed side by side. We can view V_N^Γ as a $\Gamma N \times N$ rectangle divided into Γ cells of dimension $N \times N$. Similarly, V_{2N}^Γ can be viewed as a $2\Gamma N \times 2N$ rectangle divided into 4 sub-rectangles of dimension $\Gamma N \times N$ each of which is a copy of V_N^Γ (see Figure 1 below for an illustration).

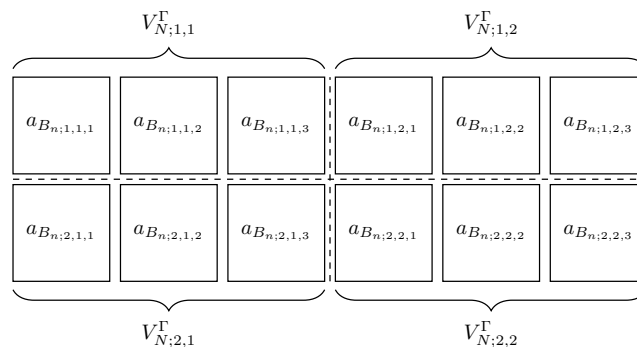


Figure 1: $(n + 1)$ -th stage of the recursive construction of BRW on V_{2N}^Γ . Here $\Gamma = 3$.

We wish to construct a light path in V_{2N}^Γ based on constructions in V_N^Γ . As the path crosses through the rectangle of V_{2N}^Γ , we will make a choice of whether the path will stay in the upper or lower sub-rectangles based on the Gaussian values associated with the boxes of dimension $N \times N$, and thus we will switch back and forth between top and bottom layers. Our goal is to show that the expected weight of the crossing we constructed expands by a factor less than 2 when the size of the rectangles doubles. Apart from doubling of dimension at each level, there are a number of reasons that the weight will expand: (1) The expected weight assigned to a vertex z i.e. $\mathbb{E}e^{\gamma R_z^N}$ grows by a factor bigger than 1; (2) We will need to make a construction to connect the paths on the left half and the right half of the rectangle; (3) Every time we switch between top and bottom layer, we will have to add a top-down crossing in the switching location. What we would possibly gain is from the “variation” of these Gaussian variables associated with $N \times N$ boxes and thus choosing the favorable layer would reduce the weight of our path.

The aforementioned scheme involves several subtleties. One has to be rather strategic in switchings as there is a cost for that. Eventually, this is reduced to a question of computing the following *regularized total variation* for the Brownian motion, defined by

$$\mathbb{E} \max_k \max_{0=t_0 < t_1 < t_2 < \dots < t_k < t_{k+1}=1} \sum_{i=1}^{k+1} |B_{t_i} - B_{t_{i-1}}| - \lambda k. \tag{1.2}$$

Here $B_{[0,1]}$ is a standard Brownian motion, and $\lambda > 0$ is the term that measures the switching cost. We will use a recent result of [10] on the asymptotic value of (1.2) as $\lambda \rightarrow 0$.

Notation convention. In the rest of the paper we refer to the vertices of \mathbb{Z}^2 as *points*. For any point $z \in \mathbb{Z}^2$, the horizontal and vertical coordinates of z are denoted by z_x and z_y respectively. If $z_y = 0$, we represent z simply as z_x , which should be clear from the context. A (finite) *path* π is a finite sequence (v_0, v_1, \dots, v_m) of points such that for each $i \geq 0$, v_i and v_{i+1} are neighbors in \mathbb{Z}^2 . In each of the pairs of non-negative integers (n, N) , (ℓ, L) , (ℓ', L') and (ℓ'', L'') , the second element is always assumed to be 2 raised to the power of the first element. For functions $F(\cdot)$ and $G(\cdot)$, we write $F = O(G)$ if there exists an absolute constant $C > 0$ such that $F \leq CG$ everywhere in the domain. We write $F = \Omega(G)$ if $G = O(F)$.

2 Preliminaries

2.1 Coarsening of paths and \mathcal{L} -segments

The L -coarsening of a path $\pi = (v_0, v_1, \dots, v_m)$ is defined as follows. Define a sequence of integers m_0, m_1, \dots recursively as $m_0 = 0$ and,

$$m_j = \inf\{m \geq i \geq m_{j-1} : v_i \notin \text{BD}_\ell(v_{m_{j-1}})\} \text{ for } j \geq 1.$$

Here we adopt the standard convention that infimum of an empty set is ∞ . Let $p_{L,\pi}$ be the last integer j such that m_j is finite. The L -coarsening of π , denoted as $\pi_{L,\text{coarse}}$, can now be defined as the sequence of points $(c(\text{BD}_\ell(v_{m_0})), c(\text{BD}_\ell(v_{m_1})), \dots, c(\text{BD}_\ell(v_{m_{p_{L,\pi}}}))$) where $c(B)$ is the center of the box B . We can also treat $\pi_{L,\text{coarse}}$ as a path in \mathbb{R}^2 that connects successive points in this list by a line segment. Notice that the coarsening of a simple (i.e. self-avoiding) path is not necessarily simple. However if the L -coarsening of a (left-right) crossing π^* of V_{2L}^Γ is simple, then the $2L$ -coarsening of π^* is also simple. We will use this fact when we carry out the induction step. An \mathcal{L} -segment at level ℓ is a path π whose 2^ℓ -coarsening is one of two shapes shown in Figure 2.

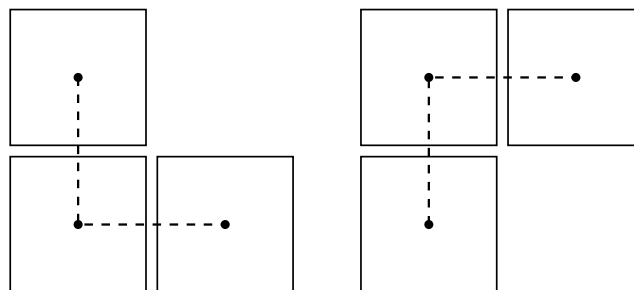


Figure 2: **Two possible 2^ℓ -coarsenings of an \mathcal{L} -segment at level ℓ .**

2.2 Switched sign construction for branching random walk

We first introduce a few more notations. As described in the introduction, we view V_{2N}^Γ as a $2\Gamma N \times 2N$ rectangle divided into 4 sub-rectangles of dimension $\Gamma N \times N$ each

of which is a copy of V_N^Γ (see Figure 1 for an illustration). Denote these 4 subrectangles (or the copies of V_N^Γ) as $\{V_{N;i,k}^\Gamma\}_{i \in [2], k \in [2]}$ in the usual order. We can also shift the origin by $u = (\Gamma i 2^{n+1}, j 2^{n+1})$ for $i, j \in \mathbb{N} \cup \{0\}$ and denote the corresponding subrectangles (respectively rectangles) by $\{V_{N;i,k}^{\Gamma,u}\}_{i \in [2], k \in [2]}$ (respectively $V_{2N}^{\Gamma,u}$). Similarly we can define the fields $\{\mathcal{R}_{z;i,k}^{N,u} : z \in V_{N;i,k}^{\Gamma,u}\}$ or $\{\mathcal{R}_z^{N,u} : z \in V_N^{\Gamma,u}\}$. The j -th constituent cell in $V_{N;i,k}^{\Gamma,u}$ (which is box of side length N) will be denoted as $B_{n;i,k,j}^u$ where $j \in [\Gamma]$. We observe that the Gaussian field $\{\mathcal{R}_z^{2N,u} : z \in V_{2N}^{\Gamma,u}\}$ is obtained by adding $a_{B_{n;i,k,j}^u}$ to the field $\{\mathcal{R}_{z;i,k}^{N,u} : z \in V_{N;i,k}^{\Gamma,u}\}$ in the box $B_{n;i,k,j}^u$ for $i \in [2], k \in [2]$ and $j \in [\Gamma]$ (See Figure 1). We will omit the additional superscript u in all these notations when $u = 0$.

We will represent BRW by a construction in the fashion of switching-signs, which is tailored to our inductive construction for light crossings. To this end, we denote the collection of all rectangles of the form $([0, 2^\ell - 1] \times [0, 2^{\ell+1} - 1]) \cap \mathbb{Z}^2 + (i 2^\ell, j 2^\ell)$ by $\text{BD}_{\ell,2}$ where i, j, ℓ are nonnegative integers. Like $\text{BD}(z)$, we use $\text{BD}_{\ell,2}(z)$ to denote the unique rectangle in $\text{BD}_{\ell,2}$ containing $z \in \mathbb{Z}^2$. Let $A_{\ell,\Gamma}$ be the collection of all points of the form $(\Gamma i 2^\ell, j 2^\ell)$. Denote by $R_{\ell;k,j}^u$ the rectangle formed by the boxes $B_{\ell;1,k,j}^u$ and $B_{\ell;2,k,j}^u$ when $u \in A_{\ell+1,\Gamma}$ (see Figure 3). Note that $R_{\ell;k,j}^u \in \text{BD}_{\ell,2}$. Also let $\{a_B\}_{B \in \text{BD}_{k,2}, k \geq 0}$ be a collection of i.i.d. standard normal random variables, that is independent of $\{a_B\}_{B \in \text{BD}_k, k \geq 0}$. We will now construct a family of Gaussian fields $\{\chi_z^{L,u} : z \in V_L^{\Gamma,u}\}$ for $u \in A_{\ell,\Gamma}$ and $\ell = 0, 1, 2, \dots, n$ recursively as follows:

- $\chi_z^{1,u} = 0$ for all $z \in V_1^{\Gamma,u}$.
- Let $u \in A_{\ell+1,\Gamma}$ and $\{\chi_{z;i,k}^{L,u} : z \in V_{L;i,k}^{\Gamma,u}\}$ be the field on $V_{L;i,k}^{\Gamma,u}$ constructed in level ℓ . For $z \in B_{\ell;i,k,j}^u$, define

$$\chi_z^{2L,u} = \chi_{z;i,k}^{L,u} + (-1)^{(k-1)\Gamma+j} b_\ell a_{\text{BD}_{\ell+1}(z)} + (-1)^i c_\ell a_{\text{BD}_{\ell,2}(z)}, \tag{2.1}$$

where $b_\ell = \sqrt{(1 - 4^{-\ell})/3}$ and $c_\ell = \sqrt{2(1 - 4^{-\ell})/3}$.

A few remarks about the above construction might be helpful. For $z \in B_{\ell;i,k,j}^u$, the box $\text{BD}_{\ell+1}(z)$ is simply $V_{2L;[(k-1)\Gamma+j]/2}^{\Gamma,u} = u + V_{2L;[(k-1)\Gamma+j]/2}^\Gamma$ whereas $\text{BD}_{\ell,2}(z)$ is $R_{\ell;k,j}^u$. Thus we add the same variable, namely

$$Z_{\ell;i,k,j} = (-1)^{(k-1)\Gamma+j} b_\ell a_{V_{2L;[(k-1)\Gamma+j]/2}^{\Gamma,u}} + (-1)^i c_\ell a_{R_{\ell;k,j}^u}, \tag{2.2}$$

to $\chi_{z;i,k}^{L,u}$ on $B_{\ell;i,k,j}^u$. $Z_{\ell;1,k,j}$ and $Z_{\ell;2,k,j}$ differ only by the sign of $c_\ell a_{R_{\ell;k,j}^u}$. Also notice that $\text{BD}_{\ell+1}(z)$ is same on the $2m + 1$ -th and $2m + 2$ -th rectangles (i.e. the $R_{\ell;.,.}^u$'s) in $V_{2L}^{\Gamma,u}$. However the sign of $b_\ell a_{\text{BD}_{\ell+1}(z)}$ changes inside $Z_{\ell;i,k,j}$. See Figure 3 for an illustration of this construction at level $\ell + 1$ (we drop the superscript u whenever $u = 0$).

Since $c_\ell^2 = 2b_\ell^2$, the distribution of the field $\{\chi_z^N : z \in V_N^\Gamma\}$ is symmetric with respect to symmetries of the box V_N^Γ . This fact will be used later when we construct crossings through $V_{N;j}^\Gamma$'s in the vertical direction. Finally, we define

$$\tilde{\chi}_z^{N,u} = b_n a_{\text{BD}_{\ell+1;2}(z)} + \chi_z^{N,u} \text{ for } u \in A_{n,\Gamma} \text{ and } z \in V_{N;j}^{\Gamma,u} = u + V_{N;j}^\Gamma.$$

Lemma 2.1. *The Gaussian fields $\{\tilde{\chi}_z^N : z \in V_N^\Gamma\}$ and $\{\mathcal{R}_z^N : z \in V_N^\Gamma\}$ are identically distributed.*

Proof. It is not difficult to check that the fields $\{\tilde{\chi}_z^N : z \in V_{N;j}^\Gamma\}$'s are independent and identically distributed for $j \in [\Gamma]$. Hence it suffices to verify that we have the correct covariances between field values at all pairs of vertices inside V_N . Toward this end take a pair (u, v) of vertices in V_N which were separated until level ℓ , i.e., $\text{Cov}(\mathcal{R}_u^N, \mathcal{R}_v^N) = n - \ell$. The covariance between χ_u^N and χ_v^N is given by

$$\text{Cov}(\chi_u^N, \chi_v^N) = -(1 - 4^{-\ell})/3 + \sum_{\ell+1 \leq m \leq n} (1 - 4^{-m}) = n - \ell - (1 - 4^{-n})/3.$$

Consequently $\text{Cov}(\tilde{\chi}_u^N, \tilde{\chi}_v^N) = \text{Cov}(\chi_u^N, \chi_v^N) + (1 - 4^{-n})/3 = n - \ell$. Hence the lemma follows. \square

Remark 2.2. It is clear that it suffices to work with $\{\chi_z^N : z \in V_N^\Gamma\}$ rather than the BRW in what follows. Using this construction, in every scale of our optimization, we crucially have effective variance $c_\ell^2 \approx \frac{2}{3}$ that is larger than $1/2$ (which would be the variance we effectively used if we work with the more canonical construction of BRW).

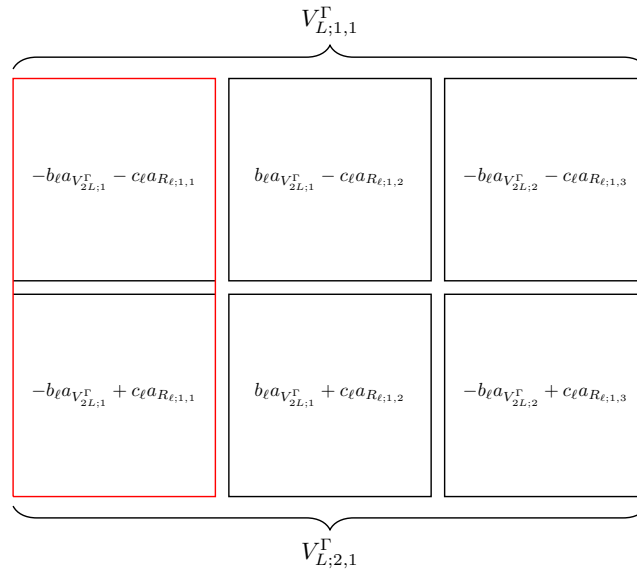


Figure 3: **Construction of the field χ_z^{2L} on $V_{L;i,1}^\Gamma$.** Here $\Gamma = 3$. The red lines indicate the boundary of $R_{\ell;1,1}$.

2.3 Regularized total variation of Brownian motion

In this subsection we present a result from [10] that provides a lower bound on the functional defined in (1.2). This is only one part of the main result in [10] which also provides an upper bound.

Theorem 2.3. Let $\{W_t\}_{0 \leq t \leq 1}$ be a Standard Brownian Motion and $1 > \lambda > 0$. For any partition $P = [t_0, t_1, \dots, t_{k+1}]$ of $[0, 1]$ where $0 = t_0 < t_1 < \dots < t_{k+1} = 1$, define the quantity $\Phi_{P,\lambda}$ as

$$\Phi_{P,\lambda} = \sum_{0 \leq i \leq k} |W_{t_{i+1}} - W_{t_i}| - \lambda k.$$

Then there exists a (random) partition $\mathcal{P} = [t_0^*, t_1^*, \dots, t_{k^*+1}^*]$ of $[0, 1]$ such that the following conditions hold:

- (a) The random vector $(t_1^*, \dots, t_{k^*}^*, k^*)$ is measurable relative to $\sigma(\{W_t\}_{0 \leq t \leq 1})$.
- (b) $\mathbb{E}\Phi_{P,\lambda} \geq \frac{1}{\lambda}$.
- (c) $W_{t_1^*} - W_{t_0^*}, W_{t_2^*} - W_{t_1^*}, \dots, W_{t_{k^*+1}^*} - W_{t_{k^*}^*}$ have alternating signs. Furthermore $\mathbb{P}(W_{t_1^*} - W_{t_0^*} > 0) = 0.5$.
- (d) $|W_{t_{j+1}^*} - W_{t_j^*}| \geq \lambda$ for all $1 \leq j \leq k^* - 1$.

3 Inductive construction for light crossings

3.1 The induction hypotheses

We will construct a light (left-right) crossing through V_L^Γ (or equivalently $V_L^{\Gamma,u}$) inductively for $L = 2^\ell$ and $\ell = 0, \dots, n$. Throughout this section whenever we refer to

a crossing of $V_L^{\Gamma,u}$, it is implicitly assumed that the underlying field is $\{\chi_z^{L,u} : z \in V_L^{\Gamma,u}\}$ unless stated otherwise. For technical convenience, let us assume that for each $\ell \geq 0$ and $u \in A_{\ell+1,\Gamma}$ we have an independent standard Brownian motion $\{W_t^{u,\ell}\}_{0 \leq t \leq 2\Gamma}$ that is also independent of $\{a_B\}_{B \in \mathbb{B}D_k, k \geq 0}$. We sample $a_{R_{\ell,k,j}^u}$'s as increments of this process at appropriate time points. We are now ready to state our induction hypotheses. For $\ell = 0$, we have only one option, i.e., to take the straight line as our crossing. This crossing has weight exactly Γ . Suppose that for each $1 \leq \ell' \leq \ell$ and $u \in A_{\ell',\Gamma}$, we have a crossing $\pi^{*,u,\ell'}$ of $V_{2^{\ell'}}^{\Gamma,u}$, identically distributed for different u , that satisfies the following properties:

1. $\pi_{L'/2,coarse}^{*,u,\ell'}$ is simple and measurable relative to $\sigma(\{W_t^{u,\ell'-1}\}_{0 \leq t \leq 2\Gamma})$.
2. The Law of $\pi_{L'/2,coarse}^{*,u,\ell'}$ is invariant with respect to reflection about the line $y = u_y + 2^{\ell'-1} - 0.5$, i.e. the horizontal line passing through the center of $V_{2^{\ell'}}^{\Gamma,u}$.
3. For $j \in [\Gamma]$, denote the line $x = u_x + jL' - 1$ as $\mathbb{L}_{jL'}$. Let $z_{j,\ell'}^u$ be the point in the set $\{u, u + (0, L'/2), u + (\Gamma L'/2, 0), u + (\Gamma L'/2, L'/2)\}$ such that the point where $\pi^{*,u,\ell'}$ hits $\mathbb{L}_{jL'}$ for the last time lies within $V_{L'/2}^{\Gamma,z_{j,\ell'}^u}$. Also for $j \in \mathbb{N}^+$, let $v_{j,L',L'}^u$ be the last point along $\pi_{L'/2,coarse}^{*,u,\ell'}$ that lies to the left of $\mathbb{L}_{jL'}$. Then $v_{j,L',L'}^u = v_{2j,L'/2,L'_{100}}^{z_{j,\ell'}^u}$, i.e., the crossings $\pi^{*,u,\ell'}$ and $\pi^{*,z_{j,\ell'}^u,\ell'-1}$ hit $\mathbb{L}_{jL'}$ for the last time in the same $L'_{100} \times L'_{100}$ box. Here $L'_{100} = \max(2^{-100}L', 1)$.
4. $\mathbb{E}(\sum_{z \in \pi^{*,u,\ell'}} e^{\gamma \chi_z^{\ell'}}) \leq 2\Gamma(2 - 0.2\gamma^2)^{\ell'}$.

One of the consequences of our induction hypotheses, which is crucial for our analysis, is that $v_{j,L,\delta L}^u$ is uniformly distributed for all $\ell' \leq \ell$ as given by the following lemma.

Lemma 3.1. *Let $R_{\ell,j}^u$ be the unique $L_{100} \times L$ rectangle that shares its right boundary with $V_{L;j}^u$. Also let $\{B_{\ell;j,m}^u\}_{1 \leq m \leq L/L_{100}}$ be the collection of boxes in $\mathbb{B}D_{\log_2 L_{100}}$ that comprise $R_{\ell;j}^u$. Now if the induction hypotheses 1, 2 and 3 hold for all $\ell' \leq \ell$, then $v_{j,L,L_{100}}^u$ is distributed uniformly on the set $\{c_{\ell;j,m}^u\}_{1 \leq m \leq L/L_{100}}$, where $c_{\ell;j,m}^u$ is the center of the box $B_{\ell;j,m}^u$.*

Proof. Notice that $v_{j,L,L/2}^u$ is determined by whether $\pi_{L/2,coarse}^{*,u,\ell}$ exits the box $V_{L;j}^u$ through the segment $\{u_x + jL - 1\} \times [u_y, u_y + L/2 - 1]$ or through the segment $\{u_x + jL - 1\} \times [u_y + L/2, u_y + L - 1]$. Hypothesis 2 tells us that these choices have probability 1/2 each. Conditional on such a choice, we then look at $v_{2j,L/2,L/4}^{z_{j,\ell}^u}$ (see hypothesis 3). By hypothesis 1, $v_{2j,L/2,L/4}^{z_{j,\ell}^u}$ is independent of $v_{j,L,L/2}^u$. Consequently by hypothesis 2, the two possible choices for $v_{2j,L/2,L/4}^{z_{j,\ell}^u}$ have probability 1/2 each given the choice of $v_{j,L,L/2}^u$. Now noting that $v_{2j,L/2,L/4}^{z_{j,\ell}^u} = v_{j,L,L/4}^u$ by hypothesis 3, we get that $v_{j,L,L/4}^u$ is uniformly distributed over the set of all 4 possible choices. Iterating this argument $\log_2 L/L_{100}$ times gives us the lemma. \square

3.2 The induction step

Now we will carry out the induction step. It suffices to produce a crossing for $u = 0$. As usual, we will drop u from all the superscripts when u is origin. In the following paragraph we give a broad overview of our strategy. This will be explained in more detail later, as we move to the different cases.

We will first decide the L -coarsening of $\pi^{*,\ell+1}$. It would consist of a sequence H_1, H_2, \dots, H_{R+1} of horizontal segments (possibly of length 0) from left to right such that every two successive segments are connected by a vertical segment. See Figure 4

for an illustration. Notice that this is the only possible option for $\pi_{L,coarse}^{*,\ell+1}$, if it has to be simple as required by induction hypothesis 1. So we can “encode” $\pi_{L,coarse}^{*,\ell+1}$ as a sequence of $\{1, 2\}$ valued random variables $\{i_{j,k}\}_{j \in [\Gamma], k \in [2]}$, where $i_{k,j} = 1$ or 2 accordingly as $\pi_{L,coarse}^{*,\ell+1}$ enters the rectangle $R_{\ell;k,j}$ through $B_{\ell;1,k,j}$ or $B_{\ell;2,k,j}$ respectively. We refer to this as a *switching strategy*. After defining $\pi_{L,coarse}^{*,\ell+1}$, we will construct the path $\pi^{*,\ell+1}$. From the previous level we have four crossings $\{\pi_{i,k}^{*,\ell}\}_{i \in [2], k \in [2]}$ respectively in $\{V_{L;i,k}^\Gamma\}_{i \in [2], k \in [2]}$. We will “join” the paths $\pi_{i,1}^{*,\ell}$ and $\pi_{i,2}^{*,\ell}$ into a crossing $\pi_i^{*,\ell+1}$ of V_{2L}^Γ for $i \in [2]$. Now let $H_1^* = H_1$ and for $2 \leq r \leq R + 1$, let H_r^* be the subsegment of H_r consisting of all but the left endpoint of H_r . Thus the points in $\cup_{r \in [R+1]} H_r^*$ span the entire horizontal range of points in $\cup_{r \in [R+1]} H_r$ while no two points in the former share the same horizontal coordinate. Since L -coarsenings of the crossings at level ℓ are also simple, it follows that the vertices of $\pi_i^{*,\ell+1}$ lying within the $L \times L$ boxes intersecting H_r^* define a subpath of $\pi_i^{*,\ell+1}$, say, $h_r^{*,\ell+1}$. Here i is 1 or 2 accordingly as H_r lies in the top or bottom row of V_{2L}^Γ (See Figure 4 and the discussions preceding Figure 1). Finally we will connect $h_r^{*,\ell+1}$ and $h_{r+1}^{*,\ell+1}$ by an appropriate \mathcal{L} -segment $v_r^{*,\ell+1}$ at level ℓ . The paths $h_1^{*,\ell+1}, v_1^{*,\ell+1}, h_2^{*,\ell+1}, \dots, v_R^{*,\ell+1}, h_{R+1}^{*,\ell+1}$ define the crossing $\pi^{*,\ell+1}$. Note that the L -coarsening of $\pi^{*,\ell+1}$ is the very one that we started with.

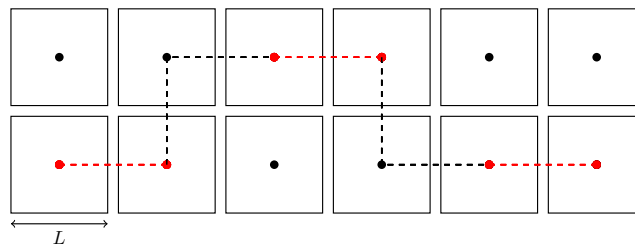


Figure 4: L -coarsening of $\pi^{*,\ell+1}$. Here $\Gamma = 3$ and $R = 2$. The three red segments are $h_1^{*,\ell+1}, h_2^{*,\ell+1}$ and $h_3^{*,\ell+1}$ from left to right.

We also need some notations in order to track the change in expected weight of crossings between two levels. To this end define,

$$D_{\gamma,\ell} = \sum_{v \in \pi^{*,\ell}} e^{\gamma \chi_v^L} \text{ and } d_{\gamma,\ell} = \mathbb{E} D_{\gamma,\ell},$$

$$D_{\gamma,\ell,j} = \sum_{v \in \pi^{*,\ell} \cap V_{L,j}} e^{\gamma \chi_v^L} \text{ and } d_{\gamma,\ell,j} = \mathbb{E} D_{\gamma,\ell,j} \text{ for } j \in [\Gamma].$$

Also define for $k \in [2]$,

$$\tilde{D}_{\gamma,\ell;k} = \sum_{j \in [\Gamma]} \sum_{v \in \pi_{i_j,k}^{*,\ell}} e^{\gamma \chi_v^{2L}},$$

where $\{i_{j,k}\}_{j \in [\Gamma], k \in [2]}$ is the switching strategy that we use. Notice that if $\pi^{*,\ell+1}$ is constructed according to the general strategy described in the last paragraph, then $\tilde{D}_{\gamma,\ell;k}$ gives an upper bound on the total weight of vertices in $\pi^{*,\ell+1}$ coming from $\pi_{i,k}^{*,\ell}$'s. Recalling from (2.1) and (2.2) that $\chi_z^{2L} = \chi_{z;i,k}^L + Z_{\ell;i,k,j}$ for all $z \in B_{\ell;i,k,j}$, we can then write

$$\begin{aligned} \mathbb{E} \tilde{D}_{\gamma,\ell;k} &= \sum_{j \in [\Gamma]} \mathbb{E} \left(e^{\gamma Z_{\ell;i,k,j}} \sum_{v \in \pi_{i_j,k}^{*,\ell}} e^{\gamma \chi_{v;i,k}^L} \right) = \sum_{j \in [\Gamma]} \mathbb{E} e^{\gamma Z_{\ell;i,k,j}} \mathbb{E} \left(\sum_{v \in \pi_{1,k}^{*,\ell}} e^{\gamma \chi_{v;1,k}^L} \right) \\ &= \sum_{j \in [\Gamma]} d_{\gamma,\ell,j} e^{\gamma Z_{\ell;i,k,j}}, \end{aligned}$$

where for the second step we assumed that the switching strategy is independent of $\{\chi^{L'}\}$ for $L' \leq L$ (induction hypothesis 1). Writing

$$e^{\gamma Z_{\dots}} = \left(1 + \gamma Z_{\dots} + \frac{a_{\ell+1}^2 \gamma^2}{2}\right) + \frac{\gamma^2}{2} (Z_{\dots}^2 - a_{\ell+1}^2) + \left(e^{\gamma Z_{\dots}} - \left(1 + \gamma Z_{\dots} + \frac{\gamma^2 Z_{\dots}^2}{2}\right)\right)$$

for $a_{\ell+1}^2 = b_{\ell+1}^2 + c_{\ell+1}^2 = 1 - 4^{-(\ell+1)}$, we get

$$\mathbb{E} \tilde{D}_{\gamma, \ell; k} = \mathbb{E} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} \left(1 + \gamma Z_{\ell; i_j, k, k, j} + \frac{a_{\ell+1}^2 \gamma^2}{2}\right) + \mathbb{E} \text{Err}_{\gamma, \ell; k, 1} + \mathbb{E} \text{Err}_{\gamma, \ell; k, 2}, \quad (3.1)$$

where, $\text{Err}_{\gamma, \ell; k, 1} = \frac{\gamma^2}{2} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} (Z_{\ell; i_j, k, k, j}^2 - a_{\ell+1}^2)$ and

$$\text{Err}_{\gamma, \ell; k, 2} = \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} \left(e^{\gamma Z_{\ell; i_j, k, k, j}} - \left(1 + \gamma Z_{\ell; i_j, k, k, j} + \frac{\gamma^2 Z_{\ell; i_j, k, k, j}^2}{2}\right)\right).$$

It is easy to see that $\mathbb{E}|\text{Err}_{\gamma, \ell; k, 2}| = d_{\gamma, \ell} O(\gamma^3)$. On the other hand, we get the following from expanding the terms in $\text{Err}_{\gamma, \ell; k, 1}$ (see (2.2)):

$$\begin{aligned} \text{Err}_{\gamma, \ell; k, 1} &= \frac{b_{\ell}^2 \gamma^2}{2} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} (a_{V_{2L; \lceil ((k-1)\Gamma+j)/2 \rceil}}^2 - 1) + \frac{c_{\ell}^2 \gamma^2}{2} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} (a_{R_{\ell; k, j}}^2 - 1) \\ &\quad + \gamma^2 b_{\ell} c_{\ell} (-1)^{(k-1)\Gamma} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} a_{V_{2L; \lceil ((k-1)\Gamma+j)/2 \rceil}} (-1)^{j+i_j, k} a_{R_{\ell; k, j}}. \end{aligned}$$

The expectations of the first two terms in the expansion above are obviously 0. As to the third term, recall that our switching mechanism is independent of the random variables $a_{V_{2L; \lceil ((k-1)\Gamma+j)/2 \rceil}}$'s by induction hypothesis 1. Consequently $\mathbb{E} \text{Err}_{\gamma, \ell; k, 1} = 0$. Thus we only need to reckon with the first term in the right hand side of (3.1) so far as the contributions from $\pi_{i, k}^{*, \ell}$'s are concerned. This will, of course, depend on the particular switching strategy we adopt. Finally we need to add the contributions from $v_r^{*, \ell+1}$'s and the extra crossings that we make in order to link $\pi_{i, 1}^{*, \ell}$ and $\pi_{i, 2}^{*, \ell}$ for $i \in [2]$.

Now we carry out the detailed computations. It is expected that $d_{\gamma, \ell, j}$ is approximately uniform over $j \in [\Gamma]$. However, proving this requires some effort. It turns out easier to treat the "hypothetical" case that $d_{\gamma, \ell}$ is dominated by a small number of $d_{\gamma, \ell, j}$'s. We also include a case to deal with small values of ℓ when c_{ℓ}^2 is not big enough for our "main" strategy to work.

Case 1: $\sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} \mathbf{1}_{\{d_{\gamma, \ell, j} \geq \Gamma^{-2/3} d_{\gamma, \ell}\}} \geq d_{\gamma, \ell} \Gamma^{-1/10}$.

It is easy to see that

$$\sum_{j=1}^{\Gamma} d_{\gamma, \ell, j}^2 \geq d_{\gamma, \ell}^2 \Gamma^{-13/15}. \quad (3.2)$$

In this case our switching strategy is simple. For each $k = 1, 2$, we choose the rectangle that minimizes $\sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} Z_{\ell; i, k, j}$. Recall from (2.2) that

$$Z_{\ell; i, k, j} = (-1)^{(k-1)\Gamma+j} b_{\ell} a_{V_{2L; \lceil ((k-1)\Gamma+j)/2 \rceil}} + (-1)^i c_{\ell} a_{R_{\ell; k, j}}.$$

Since $a_{V_{2L; \lceil ((k-1)\Gamma+j)/2 \rceil}}$'s are centered, we have

$$\mathbb{E} \left(\min_{i=1, 2} \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} Z_{\ell; i, k, j} \right) = -c_{\ell} \mathbb{E} \left| \sum_{j=1}^{\Gamma} d_{\gamma, \ell, j} a_{R_{\ell; i, k, j}} \right|.$$

But $\sum_{j=1}^{\Gamma} d_{\gamma,\ell,j} a_{R_{\ell;i,k,j}}$ is a centered Gaussian variable with variance $\sum_{j=1}^{\Gamma} d_{\gamma,\ell,j}^2$. Thus from (3.2) we get that for $k \in [2]$,

$$\begin{aligned} \mathbb{E} \tilde{D}_{\gamma,\ell;k} &\leq d_{\gamma,\ell}(1 + \gamma^2/2 + O(\gamma^3)) + \mathbb{E} \min_{i=1,2} \sum_{j=1}^{\Gamma} d_{\gamma,\ell,j} \gamma Z_{\ell;i,k,j} \\ &\leq d_{\gamma,\ell}(1 + O(\gamma^2) - \Omega(\gamma)\Gamma^{-13/30}) \leq d_{\gamma,\ell}(1 - \Omega(\gamma^{1.9})), \end{aligned}$$

where the last inequality follows from our assumption that $\Gamma \geq \alpha/\gamma^2$. Let $u_k \in A_{\ell,\Gamma}$ be such that V_L^{Γ,u_k} is the minimizer. In order to connect $\pi^{*,u_1,\ell}$ and $\pi^{*,u_2,\ell}$ into a crossing of V_{2L}^{Γ} , we need some additional paths. To this end we add two vertical crossings through the rectangles $V_{\text{left}} = ([(\Gamma - 1)L, \Gamma L - 1] \times [0, 2L - 1]) \cap \mathbb{Z}^2$ and $V_{\text{right}} = ([\Gamma L, (\Gamma + 1)L - 1] \times [0, 2L - 1]) \cap \mathbb{Z}^2$. We also add two horizontal crossings through the rectangles $V_{\text{up}} = ([(\Gamma - 1)L, (\Gamma + 1)L - 1] \times [L, 2L - 1]) \cap \mathbb{Z}^2$ and $V_{\text{down}} = ([(\Gamma - 1)L, (\Gamma + 1)L - 1] \times [0, L - 1]) \cap \mathbb{Z}^2$. All these crossings have minimum weights. We can now form $\pi^{*,\ell+1}$ by concatenating $\pi^{*,u_1,\ell}$, $\pi^{*,u_2,\ell}$ and these four crossings. See Figure 5 for an illustration. It is obvious from the construction that $\pi_{L,\text{coarse}}^{*,\ell+1}$ is simple, symmetric in law with respect to the line $y = L - 0.5$ and entirely determined by the random variables $\{a_{R_{\ell;k,j}}\}_{k \in [2], j \in [\Gamma]}$. Thus the hypotheses 1 and 2 hold. The only possible case where the hypothesis 3 could fail would be the line $\mathbb{L}_{\Gamma L}$. But we do not need to consider this line as Γ is odd. Finally notice that, since $\Gamma \gg 1$ and minimum crossing weights through adjacent rectangles are super-additive, each of the four additional crossings will have expected weight bounded by, say, $\frac{2.1}{\Gamma} d_{\gamma,\ell}$. Using the fact that $\alpha \gg 1$, we then deduce that in this case

$$d_{\gamma,\ell+1} \leq (2 - \gamma^2/2)d_{\gamma,\ell} \leq 2\Gamma(2 - 0.2)^{\ell+1}.$$

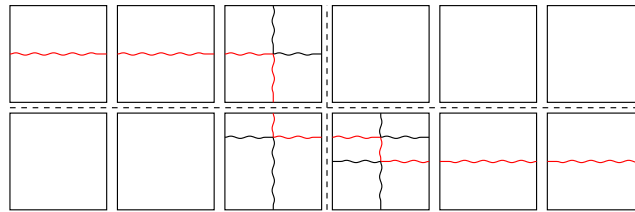


Figure 5: **The construction of $\pi^{*,\ell+1}$ for Case 1.** Here $\Gamma = 3$. The segments of $\pi^{*,\ell+1}$ have been colored in red.

Case 2: $\ell < 60$.

Here we use the strategy described in Case 1. It then follows from the previous discussions that

$$d_{\gamma,\ell+1} \leq (2 + 2\gamma^2)d_{\gamma,\ell}.$$

Since $d_{\gamma,0} = \Gamma$, we get by repeated application of the previous inequality

$$d_{\gamma,\ell+1} \leq \Gamma(2 + 2\gamma^2)^{\ell+1} \leq 2\Gamma(2 - 0.2\gamma^2)^{\ell+1}.$$

Case 3. $\sum_{j=1}^{\Gamma} d_{\gamma,\ell,j} \mathbf{1}_{\{d_{\gamma,\ell,j} \geq \Gamma^{-2/3} d_{\gamma,\ell}\}} \leq d_{\gamma,\ell} \Gamma^{-1/10}$ and $\ell \geq 60$.

This is the main and real case. We will also assume that

$$d_{\gamma,\ell} \geq 2\Gamma(2 - 0.2\gamma^2)^{\ell}(1 - \gamma^2),$$

for otherwise we can simply follow the strategy in Case 1. In this case our switching strategy will in fact be informed by the way we plan to construct the \mathcal{L} -segments $v_r^{*,\ell+1}$'s.

So we first give a detailed description of the latter. We refer the reader to Fig 6 for an illustration of the construction.

For each $r \leq R$, let $V_{L;U(r)}$ and $V_{L;D(r)}$ respectively denote the $L \times L$ boxes in the top and bottom row of V_{2L}^Γ containing the endpoints of H_r and H_{r+1} . Call the $L \times 2L$ rectangle formed by $V_{L;U(r)}$ and $V_{L;D(r)}$ as $V_{L;r}$. Suppose the right boundary of $V_{L;r}$ lies along the line $\mathbb{L}_{j_r,L}$ where $j_r \in [2\Gamma - 1]$. It suffices to consider the case when $j_r \leq \Gamma$. Let $v_{j_r,L,L_{100}}^u$ and $v_{j_r,L,L_{100}}^d$ respectively denote the last points along $\pi_{L_{100},coarse}^{*,(0,L),\ell}$ and $\pi_{L_{100},coarse}^{*,\ell}$ that lie to the left of $\mathbb{L}_{j_r,L}$ where $L_{100} = \max(2^{-100}L, 1)$. Now if $L_{100} = 1$, we simply set $v_r^{*,\ell+1}$ to be the (discrete) straight line joining $v_{j_r,L,L_{100}}^u$ and $v_{j_r,L,L_{100}}^d$. Otherwise $v_{j_r,L,L_{100}}^u$ and $v_{j_r,L,L_{100}}^d$ are the centers of two $L_{100} \times L_{100}$ boxes, say, $A_{j_r,L}^u$ and $A_{j_r,L}^d$ respectively. Let $\tilde{A}_{j_r,L}^u$ (respectively $\tilde{A}_{j_r,L}^d$) be the box around $A_{j_r,L}^u$ (respectively $A_{j_r,L}^d$) of side length $2L_{100}$. Then we construct light contours in each of the annuli $\tilde{A}_{j_r,L}^u \setminus A_{j_r,L}^u$ and $\tilde{A}_{j_r,L}^d \setminus A_{j_r,L}^d$. These can easily be achieved by concatenating four minimum weight crossings in four rectangles. In addition, we let \tilde{A}_r be a rectangle whose left top corner is $v_{j_r,L,L_{100}}^u + (-0.5, L_{100} - 0.5)$ and right bottom corner is $v_{j_r,L,L_{100}}^d + (L_{100}/2 - 0.5, -L_{100} + 0.5)$. We construct a minimum weight vertical crossing through \tilde{A}_r . Finally we concatenate these light crossings into $v_r^{*,\ell+1}$. The total expected weight of these crossings can be bounded by our induction hypotheses. Notice that we can always concatenate these paths in a way so that our construction of $v_r^{*,\ell+1}$ obeys induction hypothesis 3.

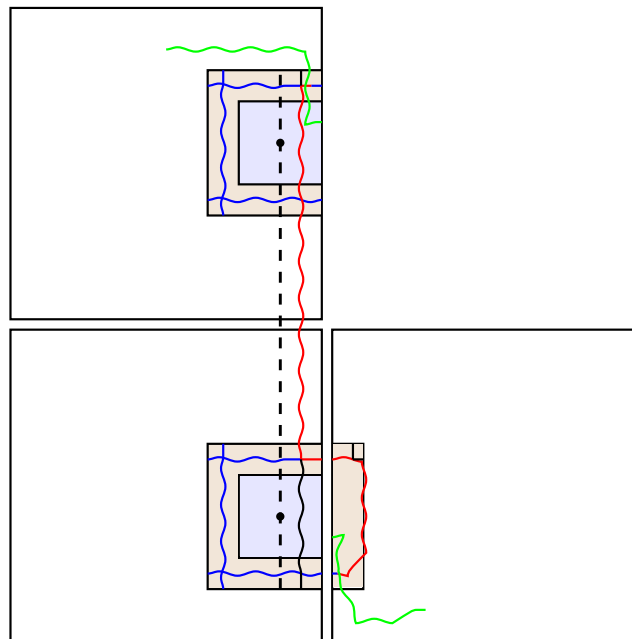


Figure 6: **The construction of $v_r^{*,\ell+1}$.** The two big boxes on the left define the rectangle $V_{L;r}$, the top and bottom one being $V_{L;U(r)}$ and $V_{L;D(r)}$ respectively. The two boxes filled with light blue are $A_{j_r,L}^u$ (top) and $A_{j_r,L}^d$ (bottom). The annuli $\tilde{A}_{j_r,L}^u \setminus A_{j_r,L}^u$ and $\tilde{A}_{j_r,L}^d \setminus A_{j_r,L}^d$ (or portions thereof) are indicated by light brown. The path colored in red is $v_r^{*,\ell+1}$. The green paths represent $h_r^{*,\ell+1}$ and $h_{r+1}^{*,\ell+1}$.

Let us now try to bound the expected weight of $v_r^{*,\ell+1}$ where we assign vertex weight based on the variables associated with boxes of side length $\leq L$. Notice that this does not require the knowledge of the particular switching strategy we employ. Denote by D_r the

total weight of the light crossings in the two annuli where we assign vertex weight based on variables associated with the boxes of side length $\leq \frac{L_{100}}{2} \vee 1$. By iterating the induction hypotheses 3 and 1 (see also the argument presented in the proof of Lemma 3.1) we get that the L_{100} coarsenings of $\pi^{*,u,\ell}$'s are independent of all the Gaussians at level lower than L_{100} , which implies

$$\mathbb{E}D_r \leq \frac{33d_{\gamma,(\ell-\log_2(2L/L_{100}))^+}}{\Gamma}, \tag{3.3}$$

where $x^+ = \max(x, 0)$. Let \tilde{D}_r denote the weight of the vertical crossing through \tilde{A}_r or the weight of $v_r^{*,\ell+1}$ when $L_{100} = 1$ (where again, we assign the vertex weight based on Gaussian variables associated with boxes of side length $\leq \frac{L_{100}}{2} \vee 1$). Clearly we have

$$\mathbb{E}\tilde{D}_r \leq \frac{(3L_{100} + \mathbb{E}|v_{j_r,L,L_{100}}^u - v_{j_r,L,L_{100}}^d|)}{(L_{100}/2)\Gamma} d_{\gamma,(\ell-\log_2(2L/L_{100}))^+}. \tag{3.4}$$

In both (3.3) and (3.4), we have silently used the fact that expected minimum weights of crossings through V_L^m 's form a superadditive sequence in m . The following lemma says that we do not lose much in terms of the expected costs of these crossings after we take into account the contributions from Gaussians appearing in higher levels.

Lemma 3.2. *Let X_1, \dots, X_m i.i.d. standard Gaussian variables, where $m \leq (1/\gamma)^{100}$ and U be a nonnegative random variable bounded by C . Then*

$$\mathbb{E}(U \max_i e^{\gamma X_i}) \leq \mathbb{E}U(1 + O(\gamma \log(1/\gamma))) + Ce^{-\Omega(\log(1/\gamma))}.$$

Proof. Let $M = \max_i X_i$. It is clear that $\mathbb{E}M \leq \sqrt{200 \log(1/\gamma)}$, and also that $\mathbb{P}(M \geq \mathbb{E}M + u) \leq e^{-u}$. The conclusion follows by a straightforward computation. \square

Combining (3.3), (3.4), Lemmas 3.1 and 3.2, we get that the expected weight for each “switching gadget” is bounded by $(1 + 10 \times 2^{-100})(2 - 0.2\gamma^2)^\ell \leq \frac{1+20 \times 2^{-100}}{\Gamma} d_{\gamma,\ell}$, where we use the fact that $\gamma \ll 1$. Recall the expression for $\mathbb{E}\tilde{D}_{\gamma,\ell;k}$ as given in (3.1) and the fact that

$$Z_{\ell;i,k,j} = (-1)^{(k-1)\Gamma+j} b_\ell a_{V_{2L;[(k-1)\Gamma+1/2]}} + (-1)^i c_\ell a_{R_{\ell;k,j}}.$$

It is then natural to consider switching strategies $\{i_{j,k}\}_{j \in [\Gamma], k \in [2]}$ that give a small value of the following quantity:

$$c_\ell \gamma \sum_{j=1}^{\Gamma} d_{\gamma,\ell,j} (-1)^{i_{j,k}} a_{R_{\ell;k,j}} + R \frac{1 + 20 \times 2^{-100}}{\Gamma} d_{\gamma,\ell}. \tag{3.5}$$

To this end we consider a similar optimization problem for a process that is related to this quantity. Denote by $J = \{j \in [\Gamma] : d_{\gamma,\ell,j} \leq \Gamma^{-2/3} d_{\gamma,\ell}\}$. By Cauchy-Schwartz inequality, we see that

$$\sum_{j \in J} d_{\gamma,\ell,j}^2 \geq \frac{(1 - \Gamma^{-1/5}) d_{\gamma,\ell}^2}{\Gamma}. \tag{3.6}$$

Write $J = \{j_1, \dots, j_{|J|}\}$ with $1 \leq j_1 < j_2 < \dots < j_{|J|} \leq \Gamma$, and define the sequence

$$g_{\gamma,i} = c_\ell^2 \gamma^2 \sum_{k \leq i} d_{\gamma,\ell,j_k}^2 \text{ for } 1 \leq i \leq 2|J|,$$

where $j_i = \Gamma + j_{i-|J|}$ and $d_{\gamma,\ell,j_i} = d_{\gamma,\ell,j_{i-|J|}}$ for $i > |J|$. Consider the process

$$S_{t;\ell} = c_\ell \gamma \sum_{0 \leq k \leq i} d_{\gamma,\ell,j_{k+1}} (W_{t_i \wedge j_{k+1}}^\ell - W_{t_i \wedge (j_{k+1}-1)}^\ell) \text{ for } g_{\gamma,i} \leq t \leq g_{\gamma,i+1},$$

where $t_i = j_{i+1} - 1 + \frac{t - g_{\gamma,i}}{c_\ell^2 \gamma^2 d_{\gamma,\ell,j_{i+1}}^2}$ and $g_{\gamma,0} = 0$. Thus $\{S_{t;\ell}\}_{0 \leq t \leq g_{\gamma,2|J|}}$ is a standard Brownian motion whose increment between the time points $g_{\gamma,i-1}$ and $g_{\gamma,i}$ is given by

$c_\ell \gamma d_{\gamma,\ell,j_i} a_{R_\ell,k,j_i}$ for $1 \leq i \leq 2|J|$. Now suppose that $[T_0, T_1, \dots, T_{R'+1}]$ is the partition of $[0, g_{\gamma,2|J|}]$ given by Theorem 2.3 when applied to $\{S_{t;\ell}\}_{0 \leq t \leq g_{\gamma,2|J|}}$ for $\lambda = \frac{(1+20 \times 2^{-100})d_{\gamma,\ell}}{\Gamma}$ (with appropriate change of scales). Using this we will define L -coarsening of the path that we will construct. Since this has to be simple by our induction hypothesis, it suffices to assign an element in $\{-1, 1\}^{2|J|}$ to $j_1, j_2, \dots, j_{2|J|}$. For this purpose define the random variables $\text{sgn}_{1;\ell}, \dots, \text{sgn}_{2|J|;\ell}$ as follows:

$$\text{sgn}_{i;\ell} = \begin{cases} -1 & \text{if } T_r < g_{\gamma,i} \leq T_{r+1} \text{ such that } S_{T_r;\ell} < S_{T_{r+1};\ell} \\ 1 & \text{if } T_r < g_{\gamma,i} \leq T_{r+1} \text{ such that } S_{T_r;\ell} > S_{T_{r+1};\ell}. \end{cases}$$

It is clear that $(\text{sgn}_{1;\ell}, \dots, \text{sgn}_{2|J|;\ell})$ is identically distributed as $(-\text{sgn}_{1;\ell}, \dots, -\text{sgn}_{2|J|;\ell})$. As a consequence the distribution of $\pi_{L,coarse}^{*,\ell+1}$ is invariant relative to reflection about the line $y = L - 0.5$.

Now we collect the contributions to the weight of $\pi^{*,\ell+1}$ from all the sources. We use crossings through $V_{\text{up}}, V_{\text{down}}, V_{\text{left}}$ and V_{right} , as in Case 1, to link the crossings $\pi_{i,1}^{*,\ell}$ and $\pi_{i,1}^{*,\ell}$ for $i \in [2]$. As discussed in Case 1, this will not tamper with induction hypothesis 3. We also need a bound on the error we made by interpolating $(S_{t;\ell})$ as a Brownian motion. Notice that for each (T_r, T_{r+1}) that is contained in some interval $(g_{\gamma,i-1}, g_{\gamma,i})$, we lose at most M_{dis} due to “discretization” of the optimal partition obtained from Theorem 2.3, where

$$M_{\text{dis}} = \max_{1 \leq i \leq 2|J|} \max_{g_{\gamma,i-1} \leq s, t \leq g_{\gamma,i}} |S_{t;\ell} - S_{s;\ell}|.$$

Thus the total loss we incur due to discretization is at most $M_{\text{dis}}R'$. We use a simple estimate for M_{dis} as follows:

$$\mathbb{E}M_{\text{dis}} \leq O(\gamma \Gamma^{-2/3} \log(\Gamma))d_{\gamma,\ell}, \text{ and } \mathbb{P}(|M_{\text{dis}} - \mathbb{E}M_{\text{dis}}| \geq u\gamma \Gamma^{-2/3} \log(\Gamma)d_{\gamma,\ell}) \leq e^{-\Omega(u^2)}, \tag{3.7}$$

where we use the fact that $d_{\gamma,\ell,j} \leq \Gamma^{-2/3}d_{\gamma,\ell}$ on J . Finally we need to make amends for the fact that what we pass as “penalty” term to (3.5) is actually based on the Gaussians up to level ℓ . But the multiplier at level $\ell + 1$ can be at most $e^{M_{\ell+1}}$, where $M_{\ell+1}$ is the maximum of all Gaussians added at level $\ell + 1$. Altogether, we deduce that

$$d_{\gamma,\ell+1} \leq 2d_{\gamma,\ell} \left(1 + \gamma^2/2 + O(\gamma^3) + \frac{10}{\Gamma}\right) - \mathcal{I}_\ell + \sqrt{\mathbb{E}R'^2} \sqrt{\mathbb{E}M_{\text{dis}}^2} + \frac{(1 + 20 \times 2^{-100})d_{\gamma,\ell}}{\Gamma} \sqrt{\mathbb{E}R'^2} \sqrt{\mathbb{E}(e^{\gamma M_{\ell+1}} - 1)^2}, \tag{3.8}$$

where

$$\mathcal{I}_\ell = \mathbb{E} \left(\sum_{i=1}^{R'+1} |S_{T_{i;\ell}} - S_{T_{i-1;\ell}}| - R' \frac{(1 + 20 \times 2^{-100})d_{\gamma,\ell}}{\Gamma} \right),$$

It follows from condition (d) in Theorem 2.3 and a straightforward computation that

$$\mathbb{E}R'^2 \leq O \left(\left(\frac{\Gamma^2 \gamma^2 \sum_{j \in J} d_{\gamma,\ell,j}^2}{d_{\gamma,\ell}^2} \right)^2 \right). \tag{3.9}$$

In addition, by Theorem 2.3, we deduce that for all $\alpha > 0$

$$\mathcal{I}_\ell \geq \left(\frac{4}{3} - 2^{-100} \right) \gamma^2 \Gamma \frac{\sum_{j \in J} d_{\gamma,\ell,j}^2}{d_{\gamma,\ell}}.$$

Here we used the fact that $\ell \geq 60$. Plugging the bounds from (3.7), (3.9) and Lemma 3.2 into (3.8) we obtain

$$d_{\gamma,\ell+1} \leq (2 - 0.6\gamma^2)2\Gamma(2 - 0.2\gamma^2)^\ell \leq 2\Gamma(2 - 0.2\gamma^2)^{\ell+1}.$$

This completes the proof of Theorem 1.1.

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